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Non-Markovianity is not a resource for quantum spatial search on a star graph subject to generalized percolation

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Abstract: Continuous-time quantum walks may be exploited to enhance spatial search, i.e., for finding a marked element in a database structured as a complex network. However, in practical implementations, the environmental noise has detrimental effects, and a question arises on whether noise engineering may be helpful in mitigating those effects on the performance of the quantum algorithm. Here we study whether time-correlated noise inducing non-Markovianity may represent a resource for quantum search. In particular, we consider quantum search on a star graph, which has been proven to be optimal in the noiseless case, and analyze the effects of independent random telegraph noise (RTN) disturbing each link of the graph. Upon exploiting an exact code for the noisy dynamics, we evaluate the quantum non-Markovianity of the evolution, and show that it cannot be considered as a resource for this algorithm, since its presence is correlated with lower probabilities of success of the search.

1 Introduction

There is a close connection between quantum metrological precision bounds and quantum computation speed-up limits, e.g. the search time in a database [1]. In turn, the interest in quantum computation relies on its ability to outperform standard classical computation in solving some peculiar tasks. Among these is the problem of finding a certain element with a given property in a disordered database of N items. Grover's quantum algorithm [2] retrieves the specified target at time of order $T = O(\sqrt{N})$ instead of the classical $T = O(N)$. Moreover, this quantum speed-up has been proven to be optimal [3]. Quantum spatial search [4] is the generalization of this problem to a database characterized by a complex structure, i.e., a database whose elements are distributed in space and connected by links according to a certain topology. Such a database can be described by a graph.

Different methods of solving the problem of quantum spatial search have been proposed [4–8]. Here we focus on the algorithm based on continuous-time quantum walks (CTQWs) [9], introduced by Childs and Goldstone [5], that can achieve the optimal speed-up $T = O(\sqrt{N})$ on certain topologies, such as the complete graph or the hypercube. Many other graphs are suitable for quantum spatial search using this algorithm: For instance, the star graph was recently proven to be optimal [10]. However, despite a great theoretical effort in characterizing the networks that are suitable for the search and to find more efficient versions of the algorithm, only few studies address the effects of noise on quantum spatial search via CTQWs.

The presence of broken links in complex networks has been investigated in [11], and it has been shown that the coupling of the system to a thermal bath may improve the performance of the algorithm affected by static disorder [12]. Changing the complex structure of the graph after a time interval τ, i.e., creating random temporal networks, can lead to a dynamical topology suitable for search as well [13], while the first study of a fully-dynamical description of the noise has been recently presented [10], in particular introducing classical random telegraph noise (RTN) affecting the hopping rate of the links of the network. The effect of RTN on the dynamics of quantum systems has been widely studied in the literature, being a typical model for noise affecting solid state devices [14–16] and used as

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a building block of $1/f$ noise [17, 18]. Many works have focused on one- or two-qubit systems [14, 19–21], with studies of its effect on CTQWs appearing in recent literature [22–25].

A key concept in the field of open quantum systems is non-Markovianity. Depending on different points of view, it expresses the divisibility of the quantum map describing the evolution of the system [26] or the backflow of quantum information going from the environment to the system [27]. A crucial point in the study of non-Markovianity relies on understanding when its presence is a resource, i.e., when non-Markovianity enhances the results of the particular task we want to achieve using the quantum system. Quantum non-Markovianity was proven to be a resource in different scenarios for quantum information processing [28, 29], teleportation [30], computation [31], metrology [32]. Systems affected by RTN can exhibit either Markovian and non-Markovian quantum dynamics, depending on the parameters and on the type of interaction with the environment [20, 24, 33]: the latter has been shown to allow for recoherence effects in qubit systems [33], while it induces localization in quantum walks on lattices [22, 24].

In this paper, we address the role of quantum non-Markovianity in the computational task of quantum spatial search. We answer the questions: is non-Markovianity a resource for quantum spatial search via CTQW? Does its presence improve the performance of the algorithm?

As a matter of fact, the dynamics of the CTQW on graphs subject to dynamical noise is obtained by Montecarlo simulation of the noise [10]. However, the study of non-Markovianity requires higher precision and numerical stability, therefore in this paper we employ a numerically exact technique to obtain the state of the walker at a generic time t. This technique, valid for any system subject to classical dynamical noise, was first proposed in [34] and specifically used to study the dynamics of small quantum systems, such as one or two qubits perturbed by random telegraph noise [16, 20]. Here, we develop a fast code that allows us to scale up the technique to larger quantum systems. We discuss the general technique in Sec. 4, while the code we used to implement it is available on GitHub [35].

The paper is structured as follows: in Sec. 2 we review the quantum spatial search algorithm based on CTQW and we discuss the noise model. In Sec. 3 we review the concept of quantum non-Markovianity and introduce the measures we employ to study the noisy evolution. In Sec. 4 we present the analytical method we have used to calculate the evolution of the quantum walk subject to dynamical noise. In Sec. 5 we discuss the results, while Sec. 6 closes the paper with some concluding remarks.

2 Noisy quantum spatial search

We model our structured database as a given graph G composed of N nodes, and we want to find the marked element w, called target node. Any graph is characterized by an adjacency matrix A, whose elements are defined as

$$A_{ij} = \begin{cases} 1 & \text{if nodes } i, j \text{ connected} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We want to run a CTQW on this graph in order to find w. The Hilbert space of the walker is $\mathcal{H} = \text{span}\{|j\rangle\}$ with $j = 1, \ldots, N$, where $|j\rangle$ is the single-particle localized state associated to the node j. According to the original definition [9], the Hamiltonian of the walk is proportional to the Laplacian matrix of the graph L, defined as $L = D - A$, where D is the degree matrix, a diagonal matrix such that D_{jj} is the number of links connected to node j. To perform the spatial search, we add to the original Hamiltonian a projector onto the target node, in order to localize the walker there. Therefore, the Hamiltonian of the algorithm reads

$$H = \gamma L + H_w = \gamma L - |w\rangle\langle w| \,, \tag{2}$$

where $H_w = -|w\rangle\langle w|$ is called *oracle Hamiltonian*, γ is a suitable coupling constant and L is the Laplacian matrix associated to G.

The initial state of the quantum walk is the fully delocalized state $|s\rangle$:

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} |j\rangle \,, \tag{3}$$

and the state at time t reads

$$|\psi(t)\rangle = e^{-iHt} |s\rangle \,. \tag{4}$$

If, at time t, we measure the walker in the node basis, the probability of obtaining the target node is given by $p(t) = |\langle w|\psi(t)\rangle|^2$. We assume that we can choose to measure at the time T for which the above probability is maximal, and we define the success probability of the algorithm as

$$p_{\text{succ}} = |\langle w|\psi(T)\rangle|^2 \tag{5}$$

We want to maximize p_{succ} keeping T as short as possible. The algorithm is optimal on the given graph G if there exists a time $T = O(\sqrt{N})$ and a suitable constant γ for which the probability of success is close to 1.

We now describe how to introduce dynamical noise on the algorithm, following the approach of [10]: a pictorial representation of the model is shown in Fig. 1. We insert independent random telegraph noise (RTN) perturbing the hopping rate of each link of the graph, where the RTN is a classical dynamical noise that can assume only two values, say $g(t) = \pm 1$, and the probability of switching value n times in a time t follows the Poisson distribution

$$p_\mu(n,t) = e^{-\mu t}\frac{(\mu t)^n}{n!}, \tag{6}$$

where μ is called switching rate.

Therefore, RTN describes a stationary stochastic process with autocorrelation function

$$\langle g(\tau)g(0)\rangle = e^{-2\mu|\tau|}, \tag{7}$$

corresponding to a Lorentzian spectrum.

CTQWs affected by RTN have been studied in the recent past for one-dimensional lattices [22–24], and for quantum spatial search on graphs [10]. Here we consider independent random telegraph noise perturbing each link of the complex network with the same switching rate μ, and we accordingly modify the Laplacian matrix in Eq. (2) as follows.

The noise is described by the $N \times N$ matrix $\mathbf{g}(t)$, where N is the number of nodes in the graph and $g_{jk}(t)$ is the stochastic process describing the noise on the link connecting j to k. The matrix $\mathbf{g}(t)$ is thus symmetric, zero-diagonal and has only l independent entries, where l is the number of links in the graph. Keeping in mind that the noise realizations on different links are uncorrelated, we have the following autocorrelation function, for the non-zero elements of $\mathbf{g}(t)$

$$\langle g_{jk}(\tau)g_{j'k'}(0)\rangle = e^{-2\mu|\tau|}(\delta_{jj'}\delta_{kk'} + \delta_{jk'}\delta_{kj'})\,. \tag{8}$$

The noisy Laplacian $L^{(\mathbf{g})}(t)$ thus reads

$$L^{(\mathbf{g})}_{jk}(t) = \begin{cases} -\left[1 + \nu g_{jk}(t)\right] & \text{if } (j,k) \text{ connected} \\ D_{jk} + \nu\sum_{i=1}^N g_{ik}(t) & \text{if } j = k \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

where $\nu \in [0,1]$ is the relative noise strength, assumed to be the same for all the links. The Hamiltonian of the noisy walk, replacing the one in Eq. (2), is now a function of the stochastic process $\mathbf{g}(t)$ and reads

$$H^{(\mathbf{g})}(t) = \gamma L^{(\mathbf{g})}(t) - |w\rangle\langle w|\,. \tag{10}$$

Using the language of open quantum systems, we describe the state of the system at time t as a density matrix $\rho(t)$. Starting from the initial state $\rho_0 = |s\rangle\langle s|$,

$$\rho(t) = \langle U[\mathbf{g}(t)]\rho_0 U^\dagger[\mathbf{g}(t)]\rangle_{\{\mathbf{g}(t)\}}, \tag{11}$$

where $\langle\ldots\rangle_{\{\mathbf{g}(t)\}}$ denotes the average over all possible realizations of the stochastic process $\mathbf{g}(t)$, while $U[\mathbf{g}(t)]$ is the unitary evolution operator that drives the evolution associated to a particular realization of the noise, given by

$$U[\mathbf{g}(t)] = \mathcal{T}\exp\left\{-i\int_0^t ds\, H^{(\mathbf{g})}(s)\right\}, \tag{12}$$

where $\mathcal{T}$ is the time-ordering operator.

Equation (11) describes a quantum map sending a density matrix into a density matrix. Considering the initial time $t_0 = 0$, for each time t we denote such a map as $\mathcal{E}(t,0)$, defined as

$$\mathcal{E}(t,0)\rho_0 = \rho(t). \tag{13}$$

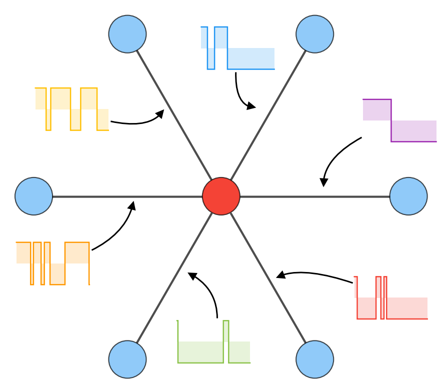

Figure 1: Pictorial representation of the model described in Section 2: the links between the nodes of a graph (in this work, we focus on the star graph) are affected by independent sources of RTN, all characterized by the same switching rate μ and noise strength ν. The red node corresponds to the marked node in the Hamiltonian, $|w\rangle$.

3 Measures of non-Markovianity

Extending the concept of non-Markovianity for stochastic processes to the quantum world is not trivial, since the classical definition of Markovianity is based on probability distributions evaluated at different times, while in quantum mechanics measuring the state of the system affects it, thus it is not meaningful anymore to define a general quantity like quantum non-Markovianity using classical objects such as probability distributions [26, 36].

A very well-known class of *master equations*, i.e., evolution equations for quantum states, is the one written in the *Gorini-Kossakowski-Sudarshan-Lindblad* (GKSL) form [37, 38]. The quantum maps described by the GKSL master equation (or, more generally, its time-local generalization) are said to be Markovian because of their *divisibility* property [26]: if, considering also time-inhomogeneous processes, $\mathcal{E}(t_2, t_1)$ is the quantum map generating the evolution of a quantum state from t_1 to t_2, and if this evolution follows the GKSL master equation, then the quantum map is *divisible* in the sense that

$$\mathcal{E}(t_3, t_1) = \mathcal{E}(t_3, t_2)\mathcal{E}(t_2, t_1) \quad \forall\, t_1 < t_2 < t_3. \tag{14}$$

This property can be seen as a sort of quantum analogue of the Chapman-Kolmogorov equation characterizing a Markovian stochastic process. Furthermore, the GKSL master equation is obtained by imposing some approximations upon the coupling between system and environment [39]. In particular, weak coupling, Born approximation, and fast decay of the environment's correlation functions (compared to the typical time-scale of the evolution of the quantum state) are required. These conditions can be seen as reflecting a memoryless evolution of the system, thus strengthening the idea of "quantum Markovianity" of the quantum map. We refer the reader to a standard textbook for a more rigorous explanation of the GKSL master equation [39].

Further definitions of quantum non-Markovianity have been proposed and used. In particular, a really common definition is the one based on the *backflow of quantum information* between system and environment [27]. The choice of a definition rather than another one depends on the specific purposes for which we want to evaluate quantum non-Markovianity. The main results and proposals on the topic are reviewed in [26, 36, 40]. Moreover, it is known that different definitions follow a hierarchy, i.e., some classes of definitions are contained in other ones; this aspect, first discovered in [41], has been deeply investigated in a very recent paper [42]. In addition to the detection of quantum non-Markovianity of a quantum process, we would like to quantify the amount of non-Markovianity of a quantum map. Various measures of non-Markovianity have been introduced in the literature to achieve this goal. In what follows we explore two measures of quantum non-Markovianity that we will use in our work, chosen for their significance in the literature and the possibility to compute them with the problem at hand.

3.1 Divisibility measure

The first measure we analyze is strictly related to the definition of non-Markovianity based on the divisibility of the quantum map. We will employ a variation of the one proposed in [43], which has already been used in [22] for quantum walks on lattices.

Suppose that $\mathcal{E}$ is the quantum map describing the evolution of a quantum state starting at $t = 0$, and suppose to take ρ_0 as the initial state. We evaluate the quantity

$$\Gamma(\tau, \tau_1) = D(\mathcal{E}(\tau, 0)\rho_0,\ \mathcal{E}(\tau, \tau_1)\mathcal{E}(\tau_1, 0)\rho_0), \tag{15}$$

where $0 \le \tau_1 \le \tau$, and D is the trace distance between two states, defined as:

$$D(\rho_1, \rho_2) = \frac{1}{2}|\rho_1 - \rho_2|, \tag{16}$$

with $|A| = \mathrm{Tr}\sqrt{A^\dagger A}$ for a square matrix A.

Obviously, in the case of time-homogeneous processes, $\mathcal{E}(\tau, \tau_1) = \mathcal{E}(\tau - \tau_1)$. Eq. (15) is basically evaluating how distant the final state obtained through the complete evolution is, compared to the one for which the evolution has been stopped and restarted at a certain time t_1; it is thus detecting how $\mathcal{E}$ deviates from divisibility. $\Gamma(\tau, \tau_1)$ is clearly zero for any τ and τ_1 if $\mathcal{E}$ is described with a master equation in the GKSL form.

In order to get a number quantifying the deviation from divisibility, one takes the maximal deviation from the property of divisible quantum map, i.e., the maximum over all τ and τ_1 up to infinity. Therefore, the measure of non-Markovianity that we employ is

$$\mathcal{N}_{\mathrm{M}} = \max_{\tau, \tau_1} \Gamma(\tau, \tau_1). \tag{17}$$

It is evident that Eq. (17) does not define a measure of the non-Markovianity of the quantum map, but only of the evolution of a particular initial state. Indeed, in [43] the trace distance in Eq. (15) is replaced with a distance in the quantum maps space. However, in the case at hand the initial state of the system is fixed by the prescription of the spatial search algorithm, thus Eq. (17) is both easier to calculate and appropriate for our purposes.

3.2 BLP measure

Probably the most famous measure of non-Markovianity is the *BLP measure* [27], based on the backflow of quantum information between system and environment.

The trace distance between two states is contractive under the action of a quantum channel, i.e. a completely positive and trace preserving map [44]. It is straightforward to prove [27] that, if $\mathcal{E}$ is a divisible quantum map Eq. (14), then the trace distance of two evolved states (with initial state $\rho_1(0)$ and $\rho_2(0)$) is monotonically decreasing in time: namely, if $\rho_1(t) = \mathcal{E}(t, 0)\rho_1(0)$ and $\rho_2(t) = \mathcal{E}(t, 0)\rho_2(0)$,

$$D(\rho_1(t + \tau), \rho_2(t + \tau)) \le D(\rho_1(t), \rho_2(t)) \quad \forall\, t, \tau > 0. \tag{18}$$

This may not be true anymore if the dynamics is non-Markovian. Therefore, let us define the quantity

$$\sigma(t, \rho_{1,2}(0)) = \frac{d}{dt} D(\rho_1(t), \rho_2(t)). \tag{19}$$

If $\sigma(t, \rho_{1,2}(0))$ is positive for certain time intervals, then the quantum map is non-Markovian and, in particular, during those time intervals we are observing a backflow of quantum information. Indeed, the trace distance expresses our ability to distinguish the states ρ_1 and ρ_2 [44], therefore when it increases we are acquiring more *quantum information* about the two states.

The BLP measure is defined by integrating over all the time intervals in which we are gaining quantum information, i.e., in which Eq. (19) is positive, and then taking the maximum upon all the possible pairs of initial states:

$$\mathcal{N}_{\text{BLP}}(\mathcal{E}) = \max_{(\rho_1(0),\rho_2(0))} \int_{\sigma>0} dt\, \sigma(t, \rho_{1,2}(0)). \tag{20}$$

Following the hierarchy of non-Markovianity measures, there are some dynamical maps for which the BLP measure is zero, despite being non-Markovian with respect to the divisibility definition [26]. Nonetheless, Eq. (20) is a true measure of non-Markovianity of a quantum map (and not only of a specific evolution), and it provides the quantifier of a useful resource (the backflow of information).

Due to the maximization upon all the possible initial states, the evaluation of Eq. (20) is, in general, a formidable task, only slightly mitigated by the fact, proven in [45], that the states of the optimal pair must lie on the boundary of the space of the density matrices and must be orthogonal. Given the problem at hand, we choose to fix $\rho_1(0)$ as the initial state of the spatial search algorithm, and we optimize over the state $\rho_2(0)$ only.

4 Analytical solution of the noisy dynamics

The solution of Eq. (11) is usually computed numerically, because of the cumbersome expression that arises in Eq. (12) and of the huge number of possible realizations of the noise. Exact analytical solutions are possible only in certain cases in which the Hamiltonian commutes with itself at different times, such as in the case of pure dephasing of qubits [22].

Joynt et al. have proposed an exact method of solving the dynamics of a quantum system coupled to a classical environment modeled as a Markovian stochastic process, and particularly effective for RTN [16, 34]. The method allows for analytical results only for a single qubit [16], while it requires numerical matrix diagonalization for higher dimensions, [20]. The strengths of this method are that it gives exact results up to machine-precision, and it avoids fluctuations typical of Montecarlo simulations: the drawback, however, is the exponential complexity in terms of the number of noise fluctuators.

We have implemented the method in Julia [46], with particular emphasis on optimization for the problem at hand: the code is available on GitHub [35]. Based on this code we are able to solve the dynamics of a CTQW subject to dynamical noise for graphs with up to $N = 10$ links. While this number is still quite small, it allows for gaining intuition on the effects of noise on the spatial search algorithm and the relation to non-Markovianity.

In this section, we briefly explain how to obtain the exact dynamics of the system with the method introduced in [34], leaving the full explanation and proof to the original paper. Suppose to have a N_q-dimensional quantum state, described at time t by the $N_q \times N_q$ density matrix $\rho(t)$, and a classical system made of N_c states, representing the possible values of the noise. For example, if the classical noise is a single fluctuator, $N_c = 2$; if it consists of N independent RTN sources, $N_c = 2^N$.

We start at $t = 0$ with $\rho(0)$ and the classical probability distribution $\mathbf{P}(0)$, describing the initial state of the stochastic process associated to the classical noise. The Hamiltonian of the quantum system is $H[g(t)]$, i.e., a function of the stochastic process describing the noise. At every time instant, to every particular configuration of the noise, which we label with the index $c \in \{1, \dots, N_c\}$, corresponds a particular form of the Hamiltonian H_c.

Since we assume a Markovian classical environment, the probability of the different states is described by the master equation

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{V}\mathbf{P}(t), \tag{21}$$

where the element $\mathbf{V}_{c,c'}$ of the matrix $\mathbf{V}$ dictates the transition rates between the states c and c' of the environment. Notice that $\mathbf{V}$ is time-independent if the stochastic process describing the environment is homogeneous, as is the case in this work. In the case of a single RTN with a switching rate μ, we have that $N_c = 2$ and

$$\mathbf{V}_\mu = \begin{pmatrix} -\mu & \mu \\ \mu & -\mu \end{pmatrix}. \tag{22}$$

For a collection of N independent fluctuators, the matrix $\mathbf{V}$ becomes

$$\mathbf{V} = \sum_{i=1}^{N} \mathbf{V}_\mu^{(i)}, \quad \mathbf{V}_\mu^{(i)} = \mathbf{I}_2^{\otimes i-1} \otimes \mathbf{V}_\mu \otimes \mathbf{I}_2^{\otimes N-i}. \tag{23}$$

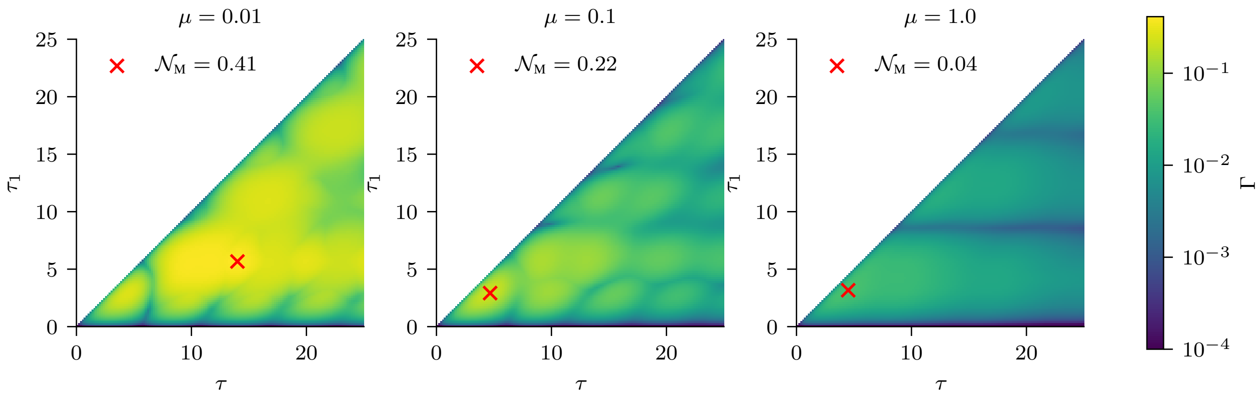

Figure 2: $\Gamma(\tau, \tau_1)$ as a function of τ and $0 \le \tau_1 \le \tau$, for several values of switching rate μ and noise strength $\nu = 1$, in logarithmic scale. The plots are for the spatial search dynamics on the star graph with $N = 7$ and central target node. The red cross marks the maximum value, i.e., the measure $\mathcal{N}_{\mathrm{M}}$. We can observe that slow strong noise (deleterious for the algorithm) leads to a higher value of non-Markovianity.

We need to represent the density matrix as a vector, and we do so by employing the generalized Bloch vector $\mathbf{n}(t)$, a vector of dimension $N_q^2 - 1$, with real components

$$n_i(t) = \frac{\sqrt{N_q}}{2} \operatorname{Tr} \lambda_i \rho(t), \tag{24}$$

where λ_j are the generators of $SU(N_q)$, and they are $N_q \times N_q$ matrices chosen to satisfy

$$\operatorname{Tr} \lambda_j = 0, \quad \lambda_j^\dagger = \lambda_j, \quad \operatorname{Tr}\left(\lambda_j \lambda_k\right) = 2\delta_{jk}. \tag{25}$$

We can go back to the density matrix $\rho(t)$ from the Bloch vector $\mathbf{n}(t)$ by means of the equation

$$\rho(t) = \frac{1}{N_q} \left[\mathbf{I}_{N_q} + \sqrt{N_q} \sum_{j=1}^{N_q^2-1} n_j(t) \lambda_j \right], \tag{26}$$

where $\mathbf{I}_{N_q}$ denotes the identity in the Hilbert space of the quantum system.

The action of a unitary operator U onto the density matrix $\rho(t)$ is translated into the multiplication of the Bloch vector $\mathbf{n}(t)$ by a transfer matrix T defined as

$$\mathbf{T}_{ij} = \frac{1}{2} \operatorname{Tr}\left[\lambda_i U \lambda_j U^\dagger\right]. \tag{27}$$

Consider now a short time interval Δt in which the environment is in a fixed state c; during Δt, the unitary evolution is generated by the Hamiltonian H_c: $U_c(\Delta t) = \exp[-iH_c \Delta t]$. The corresponding transfer matrix $\mathbf{T}_c$ is generated by the matrix

$$G_c = i \lim_{\Delta t \to 0} \frac{\mathbf{T}_c - \mathbf{I}_{N_q}}{\Delta t} = \frac{i}{2} \sum_{i,j=1}^{N_q} \operatorname{Tr}\left([\lambda_i, \lambda_j] H_c\right). \tag{28}$$

In their paper [34], Joynt *et al.* introduced the *quasi-Hamiltonian* matrix

$$\mathbf{H}_q = i\mathbf{V} \otimes \mathbf{I}_{N_q^2-1} + \bigoplus_{i=c}^{N_c} G_c, \tag{29}$$

where the second term is a direct sum of all the generators defined in (28), and showed that the dynamics of the system, averaged all the possible realizations of the stochastic process describing the noise (as defined in Eq. (12)), is given by

$$\mathbf{n}(t) = \langle \mathbf{1} | \exp(-i\mathbf{H}_q t) | p_0 \rangle \cdot \mathbf{n}(0). \tag{30}$$

In Eq. (30), $|p_0\rangle$ and $|\mathbf{1}\rangle$ are vectors belonging to the space of the classical configurations: $|\mathbf{1}\rangle$ is a vector with all components set to 1, while $|p_0\rangle \equiv \mathbf{P}(0)$ is the initial probability distribution of the configurations of the noise. In the case at hand, where we assume stationary noise, all the configurations are equally probable and so

$$|p_0\rangle = \frac{1}{N_c} |\mathbf{1}\rangle. \tag{31}$$

The expression $\langle \mathbf{1} | A | p_0 \rangle$ where A is a $N_c(N_q^2 - 1) \times N_c(N_q^2 - 1)$ matrix, denotes a partial inner product in the space of classical configurations: the result is a $(N_q^2 - 1) \times (N_q^2 - 1)$ matrix acting on the Bloch vector of the quantum system.

Now let us focus on the study of continuous-time noisy quantum walk on the star graph. If N is the number of nodes in the graph, there are $N - 1$ links and thus $N - 1$ independent RTN sources: the number of possible states of the noise is $N_c = 2^{N-1}$. The number of real parameters of the quantum system is $N^2 - 1$, and hence the number of rows of the matrix $\mathbf{H}_q$ is $2^{N-1}(N^2 - 1)$, growing more than exponentially with N.

Evaluation of (30) thus looks like a formidable task, considering that matrix exponentiation is a costly function. However, the matrices V and Q_E are largely sparse, with the number of nonzero elements growing sub-exponentially with N: this allows us to resort to various numerical techniques that ease the computational cost of (30). While the matrix exponential of a sparse matrix is dense, and thus its evaluation is still extremely costly, its action on a vector v can be evaluated just in terms of matrix-vector product operations [47, 48].

By using the above techniques, we can evaluate a single exact dynamics for multiple time instants for $N = 10$ within seconds on a laptop. However, due to the exponential scaling of the dimensions of $\mathbf{H}_q$, we cannot reach values of N much higher than that, so this excludes, for example, the study of complete graphs of more than 6-7 nodes. Nevertheless, this method allows us to gain insight into the dynamics of quantum walks affected by RTN on small graphs. Further optimizations of the algorithm, using techniques of matrix compression and distributed computation, may allow to reach even higher dimensions.

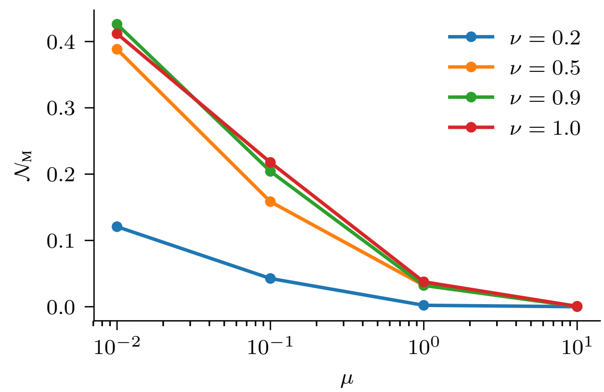

Figure 3: Divisibility measure of non-Markovianity $\mathcal{N}_M$ for the evolution of the initial state $|s\rangle\langle s|$ through the noisy algorithm of quantum spatial search, as a function of the switching rate μ, for several values of the noise strength ν. Non-Markovianity increases with the strength of the noise and decreases with the switching rate: strong, slow noise, which is the most detrimental, shows the greatest memory effects.

5 Results

Both measures of non-Markovianity, defined in Eq. (17) and Eq. (20), are highly sensitive to numerical errors in the evaluation of the dynamics of the quantum walk, meaning that small fluctuations can lead to completely wrong results (see the discussion in [22]). Hence, the need to employ the exact method presented in Sec. 4, instead of the Montecarlo simulation used in [10]. Due to the numerical complexity of the above method, we are restricted to a small number of RTN sources. For this reason, we here consider quantum spatial search on the star graph with central node as target, proven to be optimal in [10], where it is also shown that the random telegraph noise with fast switching rate μ has almost no effects on the probability of success of the search, while decreasing μ leads to worse and worse results, proving that semi-static noise jeopardizes the performance of the algorithm. Obviously, higher noise strength ν implies lower success probability.

In this section we investigate if the presence of non-Markovianity is a resource for quantum spatial search, i.e if it correlates with better performance of the noisy algorithm. To do so, we employ both the measures of non-Markovianity presented in Sec. 3.

5.1 Non-Markovianity of the evolution according to the divisibility measure

Considering the dynamics of the algorithm on a star graph with $N = 7$ nodes and central node as target, we have calculated $\Gamma(\tau, \tau_1)$ as defined in Eq. (15), for the map defined in Eq. (11), considering the starting state of the algorithm $\rho(0) = |s\rangle\langle s|$. The maximum of $\Gamma(\tau, \tau_1)$ appears for finite τ and τ_1 because the dynamics has the maximally mixed state as fixed point (as can be easily checked from Eq. (11)). The actual values for τ and τ_1 vary with the parameters of the dynamics, but accurate analysis has shown that, for $N \leq 10$, we can restrict to the region $\tau, \tau_1 \leq 25$.

The results for $\Gamma(\tau, \tau_1)$ are depicted in Fig. 2, for several values of μ and ν. Fig. 3 shows the value of the measure $\mathcal{N}_M$, obtained after taking the maximum of all the values of $\Gamma(\tau, \tau_1)$ in Fig. 2. Apart from a slight bend from $\nu = 0.9$ to $\nu = 1$ for $\mu = 0.01$, we obtain higher values of $\mathcal{N}_M$ for slower and stronger random telegraph noise, leading to bad performance of the algorithm. Therefore, using such measure of non-Markovianity and in this specific case, the presence of non-Markovianity is correlated with inefficient quantum spatial search. In Fig. 4 we show the success probability of the spatial search algorithm p_{succ} as a function of the non-Markovianity measure $\mathcal{N}_M$ of the dynamics, for the same values of μ and ν of Fig. 3. At fixed noise strength, the success probability increases as the non-Markovianity decreases. However, no clear correlation between $\mathcal{N}_M$ and p_{succ} may be seen.

5.2 Non-Markovianity of the evolution according to the BLP measure

To strengthen our results, we calculate the BLP measure of Eq. (20) as a second indicator of quantum non-Markovianity, again for the algorithm on a star graph with $N = 7$ and central node as target. The optimization over

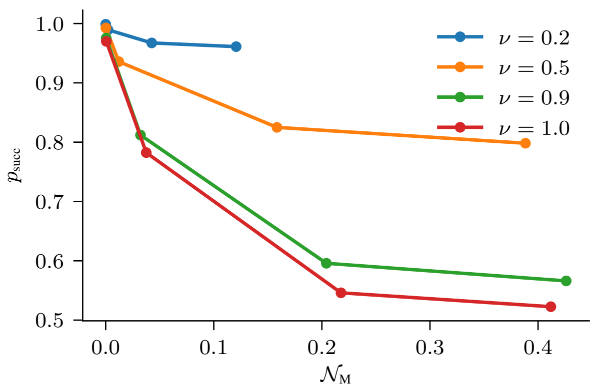

Figure 4: Success probability of the spatial search algorithm p_{succ} as a function of the non-Markovianity measure $\mathcal{N}_M$ of the dynamics, for the same values of μ and ν of Fig. 3. At fixed noise strength, the success probability increases as the non-Markovianity decreases; however, there is no clear correlation between $\mathcal{N}_M$ and p_{succ}.

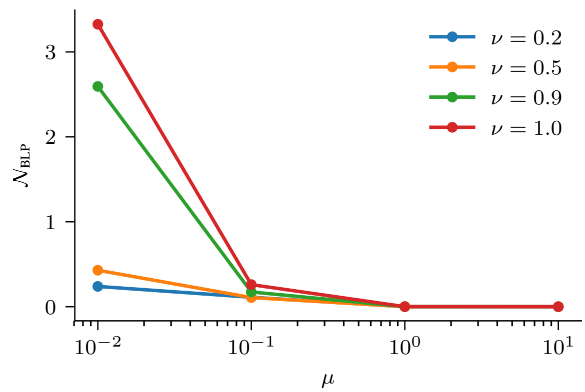

Figure 5: BLP measure of non-Markovianity for the evolution of the initial state $|s\rangle\langle s|$ through the noisy algorithm of quantum spatial search, versus the switching rate μ, for various values of the noise strength ν. In computing the value of the BLP, we have considered as initial pair $|s\rangle\langle s|$ and $|r\rangle\langle r|$, as defined in Eq. (32). The presence of information backflow between system and environment is correlated with slow strong noise, i.e., with poorer performance of the algorithm. The stronger the noise the higher the non-Markovianity of the map. The measure is basically zero for switching rates above $\mu \simeq 1$, but the map is still non-Markovian, according to the divisibility measure $\mathcal{N}_M$.

all the possible initial states $\rho_1(0)$ and $\rho_2(0)$ is difficult to compute efficiently, but for our purposes we just need to study the non-Markovianity of the evolution of the CTQW, therefore we have kept fixed one of the two states, say $\rho_1(0)$, as the initial state $|s\rangle\langle s|$, and we have optimized the measure only over all the possible $\rho_2(0)$.

Numerical investigation showed that, keeping $\rho_1(0) = |s\rangle\langle s|$, we obtain the maximum in Eq. (20) by choosing as $\rho_2(0)$ the eigenstate $|r\rangle$ of the Laplacian of the star graph, defined as:

$$|r\rangle = -(N-1)\,|1\rangle + \sum_{k=2}^{N} |k\rangle\,, \tag{32}$$

where N is the number of nodes in the graph and $\{|k\rangle\}_{k=1}^{N}$ is the node basis.

The results for the BLP measure are shown in Fig. 5, and they perfectly confirm the correlation between presence of non-Markovianity in the evolution and lower success probability of quantum spatial search.

Notice that this is one of the cases in which the divisibility measure proves to be"higher" in the hierarchy of quantum non-Markovianity [42]. Indeed, the divisibility measure detects the presence of non-Markovianity, although small, for $\mu = 10$ and $\mu = 1$, while the BLP measure does not.

5.3 Dependence on the size of the graph

The analysis above focused on the star graph with a central target node and $N = 7$. Here we address the dependence of non-Markovianity on the size of the graph, by studying the two measures for different values of N, up to $N = 10$, so that the dynamics can be still evaluated with the exact method.

We found that the two quantities $\mathcal{N}_M$ and $\mathcal{N}_{BLP}$ have a very similar behavior as functions of N, and we show the former in Fig. 6, for different values of the switching rate and for the maximum noise strength ($\nu = 1$). We see that the non-Markovianity decreases with N for fast noise, while it is inappreciably increasing for slow noise.

While the computational complexity does not allow us to explore higher values of N, we can expect non-Markovianity to maintain the same trend. This correlates with the dependence of p_{succ} on N, which is slightly decreasing for strong, slow noise, and increasing for fast noise, as shown in [10], further confirming the link between non-Markovianity and poorer performance of the algorithm.

6 Concluding remarks

In this paper we have addressed spatial search implemented by CTQW on a star graph and in the presence of RTN affecting the links between the nodes. In particular, we have discussed the role of non-Markovianity of the quantum dynamical map of the walker in determining the performance of the algorithm.

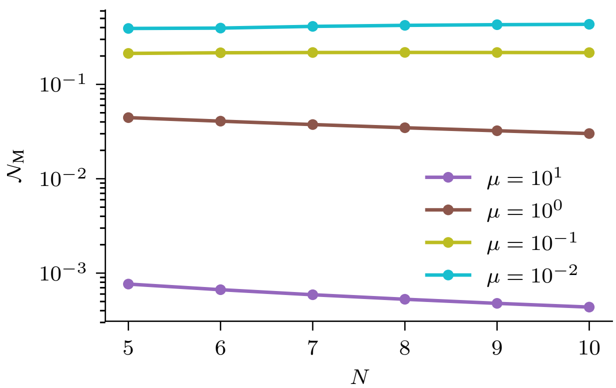

Figure 6: Non-Markovianity measure $\mathcal{N}_M$ as a function of the size N of the graph for different values of the switching rate μ, for noise strength $\nu = 1.0$. The non-Markovianity measure slightly depends on N (notice that the y axis is in logarithmic scale), with $\mathcal{N}_M$ decreasing for fast noise, and basically constant for slow noise. A qualitatively identical plot could be made for the BLP measure $\mathcal{N}_{BLP}$.

In order to address the above problem, we have developed fast and optimized code, not based on Montecarlo generation of stochastic trajectories, to achieve a numerically exact solution of the dynamics of the walker. Avoiding stochasticity allows one to increase the accuracy of the result and to reduce fluctuations, a key requirement for evaluating most quantifiers of non-Markovianity. The code is available online and can be applied to a general quantum system affected by any number of RTN sources.

Our results show that, unlike many other scenarios in which non-Markovianity can be seen as a resource for various quantum information tasks, in the case at hand spatial search performs better when the noise is fast, i.e., Markovian, as opposed to slow noise, which induces a non-Markovian dynamics and is detrimental for the algorithm. A possible intuitive explanation of the results above lies in the fact that the typical recoherence effect due to the non-Markovianity of the quantum map, happens on timescales that are much larger than the typical running time of the algorithm. Notice also that there exists different physical platforms in which state-of-the-art experiments are available with a considerable dynamical control, and where this phenomena may be, in principle, demonstrated.

It is still unknown whether these conclusions are specific to the particular statistics of the RTN, or if they are valid in a more general sense. Also, the topology of the graph might play a role in the interplay between memory effects and the localization of the walker in the target node. Further investigation should hence address other graphs layouts, as well as other types of classical or quantum noise that induce non-Markovian dynamics, and their effect on the quantum spatial search algorithm.

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References

[1] R. Demkowicz-Dobrzański and M. Markiewicz, Phys. Rev. A **91**, 062322 (2015).
[2] L. K. Grover, in *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing - STOC '96* (1996) pp. 212–219.
[3] C. Zalka, Phys. Rev. A **60**, 2746 (1999).
[4] S. Aaronson and A. Ambainis, in *Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science* (2003) pp. 200–209.
[5] A. M. Childs and J. Goldstone, Phys. Rev. A **70**, 022314 (2004).
[6] A. Ambainis, J. Kempe, and A. Rivosh, in *Proceedings of the sixteenth annual ACM-SIAM symposium on Discrete algorithms* (2005) pp. 1099–1108.
[7] A. Tulsi, Phys. Rev. A **78**, 012310 (2008).
[8] S. Chakraborty, L. Novo, and J. Roland, arXiv preprint arXiv:1807.05957 (2018).
[9] E. Farhi and S. Gutmann, Phys. Rev. A **58**, 915 (1998).
[10] M. Cattaneo, M. A. C. Rossi, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **98**, 052347 (2018).
[11] L. Novo, S. Chakraborty, M. Mohseni, H. Neven, and Y. Omar, Sci. Rep. **5**, 13304 (2015).
[12] L. Novo, S. Chakraborty, M. Mohseni, and Y. Omar, Phys. Rev. A **98**, 022316 (2018).
[13] S. Chakraborty, L. Novo, S. Di Giorgio, and Y. Omar, Phys. Rev. Lett. **119**, 220503 (2017).
[14] Y. M. Galperin, B. L. Altshuler, J. Bergli, and D. V. Shantsev, Phys. Rev. Lett. **96**, 097009 (2006).
[15] B. Abel and F. Marquardt, Phys. Rev. B **78**, 201302 (2008).
[16] B. Cheng, Q.-H. Wang, and R. Joynt, Phys. Rev. A **78**, 022313 (2008).
[17] E. Paladino, L. Faoro, G. Falci, and R. Fazio, Phys. Rev. Lett. **88**, 228304 (2002).
[18] E. Paladino, Y. M. Galperin, G. Falci, and B. Altshuler, Rev. Mod. Phys. **86**, 361 (2014).
[19] C. Benedetti, F. Buscemi, P. Bordone, and M. G. A. Paris, Phys. Rev. A **87**, 052328 (2013).
[20] M. A. C. Rossi and M. G. A. Paris, J. Chem. Phys. **144**, 024113 (2016).
[21] S. Cialdi, M. A. C. Rossi, C. Benedetti, B. Vacchini, D. Tamascelli, S. Olivares, and M. G. A. Paris, Appl. Phys. Lett. **110**, 081107 (2017).
[22] C. Benedetti, F. Buscemi, P. Bordone, and M. G. A. Paris, Phys. Rev. A **93**, 042313 (2016).

[23] I. Siloi, C. Benedetti, E. Piccinini, J. Piilo, S. Maniscalco, M. G. A. Paris, and P. Bordone, Phys. Rev. A **95**, 022106 (2017).
[24] M. A. C. Rossi, C. Benedetti, M. Borrelli, S. Maniscalco, and M. G. A. Paris, Phys. Rev. A **96**, 040301 (2017).
[25] C. Benedetti, M. A. C. Rossi, and M. G. A. Paris, EPL **124**, 60001 (2018).
[26] Á. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. **77**, 094001 (2014).
[27] H.-P. Breuer, E.-M. Laine, and J. Piilo, Phys. Rev. Lett. **103**, 210401 (2009).
[28] S. F. Huelga, A. Rivas, and M. B. Plenio, Phys. Rev. Lett. **108**, 160402 (2012).
[29] B. Bylicka, D. Chruściński, and S. Maniscalco, Sci. Rep. **4**, 5720 (2014).
[30] E.-M. Laine, H.-P. Breuer, and J. Piilo, Scientific Reports **4** (2014), 10.1038/srep04620.
[31] Y. Dong, Y. Zheng, S. Li, C.-C. Li, X.-D. Chen, G.-C. Guo, and F.-W. Sun, npj Quantum Inf. **4**, 3 (2018).
[32] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **109**, 233601 (2012).
[33] C. Benedetti, M. G. A. Paris, and S. Maniscalco, Phys. Rev. A **89**, 012114 (2014).
[34] R. Joynt, D. Zhou, and Q.-H. Wang, Int. J. Mod. Phys. B **25**, 2115 (2011).
[35] M. A. C. Rossi, "QuasiHamiltonianRTN.jl: A Julia package for solving the dynamics of a quantum system affected by RTN," (2018).
[36] H.-P. Breuer, E.-M. Laine, J. Piilo, and B. Vacchini, Rev. Mod. Phys. **88**, 021002 (2016).
[37] V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, J. Math. Phys. **17**, 821 (1976).
[38] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).
[39] H.-P. Breuer and F. Petruccione, *The theory of open quantum systems* (Oxford University Press, 2002).
[40] I. de Vega and D. Alonso, Rev. Mod. Phys. **89**, 015001 (2017).
[41] D. Chruściński and S. Maniscalco, Phys. Rev. Lett. **112**, 120404 (2014).
[42] L. Li, M. J. Hall, and H. M. Wiseman, Phys. Rep. (2018), 10.1016/j.physrep.2018.07.001.
[43] S. C. Hou, S. L. Liang, and X. X. Yi, Phys. Rev. A **91**, 012109 (2015).
[44] M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, 2010).
[45] S. Wißmann, A. Karlsson, E.-M. Laine, J. Piilo, and H.-P. Breuer, Phys. Rev. A **86**, 062108 (2012).
[46] J. Bezanson, A. Edelman, S. Karpinski, and V. B. Shah, SIAM Rev. **59**, 65 (2017).
[47] R. B. Sidje, ACM Trans. Math. Softw. **24**, 130 (1998).
[48] A. H. Al-Mohy and N. J. Higham, SIAM J. Sci. Comput. **33**, 488 (2011).