Discrete Mathematics and Theoretical Computer Science DMTCS vol. **19**:3, 2017, #4

Post-surjectivity and balancedness of cellular automata over groups

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received 12th May 2016, *revised 18th July 2017*, *accepted 30th Aug. 2017*.

We discuss cellular automata over arbitrary finitely generated groups. We call a cellular automaton post-surjective if for any pair of asymptotic configurations, every pre-image of one is asymptotic to a pre-image of the other. The well known dual concept is pre-injectivity: a cellular automaton is pre-injective if distinct asymptotic configurations have distinct images. We prove that pre-injective, post-surjective cellular automata are reversible. Moreover, on sofic groups, post-surjectivity alone implies reversibility. We also prove that reversible cellular automata over arbitrary groups are balanced, that is, they preserve the uniform measure on the configuration space.

Keywords: cellular automata, reversibility, sofic groups.

1 Introduction

Cellular automata (briefly, CA) are parallel synchronous systems on regular grids where the next state of a point depends on the current state of a finite neighborhood. The grid is determined by a finitely generated group and can be visualized as the Cayley graph of the group. In addition to being a useful tool for simulations, CA are studied as models of massively parallel computers, and as dynamical systems on symbolic spaces. From a combinatorial point of view, interesting questions arise as to how the properties of the global transition function (obtained by synchronous application of the local update rule at each point) are linked to one another.

One such relation is provided by Bartholdi's theorem (Bartholdi, 2010), which links surjectivity of cellular automata to the preservation of the product measure on the space of global configurations: the latter implies the former, but is only implied by it if the grid is an *amenable* group. In the amenable setting, the *Garden of Eden theorem* equates surjectivity with *pre-injectivity*, that is, the property that two asymptotic configurations (*i.e.*, two configurations differing on at most finitely many points) with the

*This research was supported by the ERDF funded project Coinduction, the Estonian Ministry of Education and Research institutional research grant IUT33-13, and the Estonian Science Foundation grant no. 9398.
†Research supported by the Academy of Finland grant 296018.
‡The work of ST is supported by ERC Advanced Grant 267356-VARIS of Frank den Hollander.

ISSN 1365–8050

same image must be equal. In the same setting, by (Fiorenzi, 2003, Theorem 4.7), the Garden of Eden theorem still holds for CA on subshifts that are of finite type and are strongly irreducible. Counterexamples with general subshifts are known already in dimension 1. In the general case, the preservation of the product measure can be expressed combinatorially by the so-called *balancedness* property. Furthermore, bijectivity is always equivalent to reversibility, that is, the existence of an inverse that is itself a CA.

A parallel to pre-injectivity is *post-surjectivity*, which is described as follows: given a configuration e and its image c, every configuration c' asymptotic to c has a pre-image e' asymptotic to e. While pre-injectivity is *weaker* than injectivity, post-surjectivity turns out to be *stronger* than surjectivity. It is natural to ask whether such trade-off between injectivity and surjectivity preserves bijectivity.

In this paper, which expands the work presented at the conferences Automata 2015 and Automata 2016, we discuss the two properties of balancedness and post-surjectivity, and their links with reversibility. First, we prove that post-surjectivity and pre-injectivity together imply reversibility; that is, the trade-off above actually holds over all groups. Next, we show that, in a context so broad that no counterexamples are currently known (*i.e.*, when the grid is a sofic group), post-surjectivity actually implies pre-injectivity. Finally, we prove that a reversible cellular automaton over any group is balanced, hence giving an "almost positive" answer to a conjecture proposed in (Capobianco et al., 2013).

2 Background

Given a set X, we indicate by $\mathcal{PF}(X)$ the collection of all finite subsets of X. If X is finite, we indicate by $|X|$ the number of its elements.

Let $\mathbb{G}$ be a group and let $U, V \subseteq \mathbb{G}$. We put $UV = \{x \cdot y \mid x \in U, y \in V\}$, and $U^{-1} = \{x^{-1} \mid x \in U\}$. If $U = \{g\}$ we write gV for $\{g\}V$.

A *labeled graph* is a triple (V, L, E) where V is a set of *vertices*, L is a set of *labels*, and $E \subseteq V \times L \times V$ is a set of *labeled edges*. A *labeled graph isomorphism* from (V_1, L, E_1) to (V_2, L, E_2) is a bijection $\phi : V_1 \to V_2$ such that, for every $x, y \in V_1$ and $\ell \in L$, $(x, \ell, y) \in E_1$ if and only if $(\phi(x), \ell, \phi(y)) \in E_2$. We may say that (V, E) is an L-labeled graph to mean that (V, L, E) is a labeled graph.

A subset B of $\mathbb{G}$ is a *set of generators* for $\mathbb{G}$ if every $g \in \mathbb{G}$ can be written as $g = x_0 \cdots x_{n-1}$ for suitable $n \geq 0$ and $x_0, \ldots, x_{n-1} \in B \cup B^{-1}$. The group $\mathbb{G}$ is *finitely generated* (briefly, f.g.) if B can be chosen finite.

Let B be a finite set of generators for the group $\mathbb{G}$. The *Cayley graph* of $\mathbb{G}$ with respect to B is the labeled graph $(\mathbb{G}, L, E)$ where $L = (B \cup B^{-1})$ and $E = \{(g, x, h) \mid gx = h\}$. The *length* of $g \in \mathbb{G}$ with respect to B is the *minimum* length $n = \|g\|_B$ of a representation $g = x_0 \cdots x_{n-1}$. The *distance* between g and h with respect to B is $d_B(g, h) = \|g^{-1} \cdot h\|_B$, *i.e.*, the length of the shortest path from g to h in the Cayley graph of $\mathbb{G}$ with respect to B. With respect to such distance, multiplications to the left by a fixed element, *i.e.*, the functions $x \mapsto gx$ where $g \in \mathbb{G}$ is fixed, are isometries. The *disk* of center g and radius r with respect to B is the set $D_{B,r}(g)$ of those $h \in \mathbb{G}$ such that $d_B(g, h) \leq r$. We omit g if it is the identity element $1_{\mathbb{G}}$ of $\mathbb{G}$ and write $D_{B,r}$ for $D_{B,r}(1_{\mathbb{G}})$. The distance between two subsets U, V of $\mathbb{G}$ with respect to B is $d_B(U, V) = \inf\{d_B(u, v) \mid u \in U, v \in V\}$. We omit B if irrelevant or clear from the context.

A group $\mathbb{G}$ is *amenable* if for every $K \in \mathcal{PF}(\mathbb{G})$ and every $\varepsilon > 0$ there exists $F \in \mathcal{PF}(\mathbb{G})$ such that $|F \cap kF| > (1 - \varepsilon)|F|$ for every $k \in K$. The groups $\mathbb{Z}^d$ are amenable, whereas the *free groups* on two or more generators are not. For an introduction to amenability see, *e.g.*, (Ceccherini-Silberstein and Coornaert, 2010, Chapter 4).

Let S be a finite set and let $\mathbb{G}$ be a group. The elements of the set $S^{\mathbb{G}}$ are called *configurations*.

The space $S^{\mathbb{G}}$ is given the *prodiscrete topology* by considering S as a discrete set. This makes $S^{\mathbb{G}}$ a compact space by Tychonoff's theorem. In the prodiscrete topology, two configurations are "near" if they coincide on a "large" finite subset of $\mathbb{G}$. If $\mathbb{G}$ is countable, then the prodiscrete topology is metrizable: indeed, if $\mathbb{G} = \{g_n\}_{n \geq 0}$, then $d(c,e) = 2^{-n}$ for all $c, e \in S^{\mathbb{G}}$, where $n \geq 0$ is the smallest index such that $c(g_n) \neq e(g_n)$, is a distance that induces the product topology. If, in addition, B is a finite set of generators for $\mathbb{G}$, then setting $d_B(c,e) = 2^{-n}$, for all $c, e \in S^{\mathbb{G}}$, where n is the smallest non-negative integer such that c and e differ on $D_{B,n}$, also defines a distance that induces the prodiscrete topology. Given $c, c' \in S^{\mathbb{G}}$, we call $\Delta(c,c') = \{g \in \mathbb{G} \mid c(g) \neq c'(g)\}$ the *difference set* of c and c'. Two configurations are *asymptotic* if they differ at most on finitely many points of $\mathbb{G}$. A *pattern* is a function $p : E \to S$ where E is a finite subset of $\mathbb{G}$.

For $g \in \mathbb{G}$, the *translation* by g is the function $\sigma_g : S^{\mathbb{G}} \to S^{\mathbb{G}}$ that sends an arbitrary configuration c into the configuration $\sigma_g(c)$ defined by

$$\sigma_g(c)(x) = c(g \cdot x) \quad \forall x \in \mathbb{G} \,. \tag{1}$$

A *shift subspace* (briefly, *subshift*) is a subset X of $S^{\mathbb{G}}$ which is closed (equivalently, compact) and invariant by all translations σ_g with $g \in \mathbb{G}$. The set $S^{\mathbb{G}}$ itself is referred to as the *full shift*. It is well known (see e.g. (Lind and Marcus, 1995)) that every subshift X is determined by a set of *forbidden patterns* $\mathcal{F}$ in the sense that the elements of the subshift X are precisely those configurations in which translations of patterns in $\mathcal{F}$ do not occur. If $\mathcal{F}$ can be chosen finite, X is called a *shift of finite type* (briefly, SFT). A pattern $p : E \to S$ is said to be *admissible* for X if there exists a configuration $c \in X$ such that $c|_E = p$. The set of patterns that are admissible for X is called the *language* of X, indicated as $\mathcal{L}_X$.

A *cellular automaton* (briefly, CA) on a group $\mathbb{G}$ is a triple $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ where the *set of states* S is finite and has at least two elements, the *neighborhood* $\mathcal{N}$ is a finite subset of $\mathbb{G}$, and the *local update rule* is a function that associates to every pattern $p : \mathcal{N} \to S$ a state $f(p) \in S$. The *global transition function* of $\mathcal{A}$ is the function $F_{\mathcal{A}} : S^{\mathbb{G}} \to S^{\mathbb{G}}$ defined by

$$F_{\mathcal{A}}(c)(g) = f\left((\sigma_g(c))|_{\mathcal{N}} \right) \quad \forall g \in \mathbb{G} \; : \tag{2}$$

that is, if $\mathcal{N} = \{n_1, \ldots, n_m\}$, then $F_{\mathcal{A}}(c)(g) = f\left(c(g \cdot n_1), \ldots, c(g \cdot n_m)\right)$. Observe that (2) is continuous in the prodiscrete topology and commutes with the translations, *i.e.*, $F_{\mathcal{A}} \circ \sigma_g = \sigma_g \circ F_{\mathcal{A}}$ for every $g \in \mathbb{G}$. The *Curtis-Hedlund-Lyndon theorem* states that the continuous and translation-commuting functions from $S^{\mathbb{G}}$ to itself are precisely the CA global transition functions.

We shall use the following notation to represent the application of the local rule on patterns. If $p : E \to S$ and $q : C \to S$ are two patterns, we write $p \xrightarrow{f} q$ to indicate that $C\mathcal{N} \subseteq E$ and $q(g) = f\left((\sigma_g(p))|_{\mathcal{N}} \right)$ for each $g \in C$.

If X is a subshift and $F_{\mathcal{A}}$ is a cellular automaton, it is easy to see that $F_{\mathcal{A}}(X)$ is also a subshift. If, in addition, $F_{\mathcal{A}}(X) \subseteq X$, we say that $\mathcal{A}$ is a CA on the subshift X. From now on, when we speak of cellular automata on $\mathbb{G}$ without specifying any subshift, we will imply that such subshift is the full shift.

We may refer to injectivity, surjectivity, etc. of the cellular automaton $\mathcal{A}$ on the subshift X meaning the corresponding properties of $F_{\mathcal{A}}$ when restricted to X. Since X is a compact metric space, it follows from the Curtis-Hedlund-Lyndon theorem that the inverse of the global transition function of a bijective cellular automaton $\mathcal{A}$ is itself the global transition function of some cellular automaton. In this case, we say that $\mathcal{A}$ is *reversible*. A group $\mathbb{G}$ is *surjunctive* if for every finite set S, every injective cellular automaton on the full shift $S^{\mathbb{G}}$ is surjective. Currently, there are no known examples of non-surjunctive groups.

Conjecture 1 (Gottschalk, 1973)**.** *Every injective* CA *on a full shift is surjective.*

If $\mathbb{G}$ is a subgroup of a group Γ and $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ is a cellular automaton on $\mathbb{G}$, the cellular automaton $\mathcal{A}^\Gamma$ *induced* by $\mathcal{A}$ on Γ has the same set of states, neighborhood, and local update rule as $\mathcal{A}$, and maps S^Γ (instead of $S^\mathbb{G}$) into itself via $F_{\mathcal{A}^\Gamma}(c)(\gamma) = f\left(c(\gamma \cdot n_1), \ldots, c(\gamma \cdot n_m)\right)$ for every $\gamma \in \Gamma$. We also say that $\mathcal{A}$ is the *restriction* of $\mathcal{A}^\Gamma$ to $\mathbb{G}$. In addition, if $X \subseteq S^\mathbb{G}$ is a subshift defined by a set $\mathcal{F}$ of forbidden patterns on $\mathbb{G}$, then the subshift $X^\Gamma \subseteq S^\Gamma$ obtained from the same set $\mathcal{F}$ of forbidden patterns satisfies the following property: if $\mathcal{A}$ is a CA on X, then $\mathcal{A}^\Gamma$ is a CA on X^Γ, and vice versa. (Here, it is fundamental that all the forbidden patterns have their supports in $\mathbb{G}$.) It turns out (see (Capobianco, 2009, Lemma 4.3)) that induction of subshifts does not depend on the choice of $\mathcal{F}$, and that injectivity and surjectivity are preserved by both induction and restriction (see also (Ceccherini-Silberstein and Coornaert, 2010, Section 1.7) and (Capobianco, 2009, Theorem 5.3)).

Let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a CA on a subshift X, let $p : E \to S$ be an admissible pattern for X, and let $E\mathcal{N} \subseteq M \in \mathcal{PF}(\mathbb{G})$. A *pre-image* of p on M under $\mathcal{A}$ is a pattern $q : M \to S$ that is admissible for X and is such that $q \xrightarrow{f} p$. An *orphan* is an admissible pattern that has no admissible pre-image, or equivalently, a pattern that is admissible for X but not admissible for $F_\mathcal{A}(X)$. Similarly, a configuration which is not in the image of X by $F_\mathcal{A}$ is a *Garden of Eden* for $\mathcal{A}$. By a compactness argument, every Garden of Eden contains an orphan. We call this fact the *orphan pattern principle*. A cellular automaton $\mathcal{A}$ is *pre-injective* if every two asymptotic configurations c, e satisfying $F_\mathcal{A}(c) = F_\mathcal{A}(e)$ are equal. The *Garden of Eden theorem* (see (Ceccherini-Silberstein et al., 1999)) states that, for CA on amenable groups, pre-injectivity is equivalent to surjectivity; on non-amenable groups, the two properties are independent of each other (see (Bartholdi, 2010) and (Bartholdi, 2016)).

Let $\mathbb{G}$ be a finitely generated group, let B be a finite set of generators for $\mathbb{G}$, and let S be a finite set. A subshift $X \subseteq S^\mathbb{G}$ is *strongly irreducible* if there exists $r \geq 1$ such that, for every two admissible patterns $p_1 : E_1 \to S, p_2 : E_2 \to S$ such that $d_B(E_1, E_2) \geq r$, there exists $c \in X$ such that $c|_{E_1} = p_1$ and $c|_{E_2} = p_2$. We then say that r is a *constant of strong irreducibility* for X with respect to B. The notion of strong irreducibility does not depend on the choice of the finite set of generators, albeit the associated constant of strong irreducibility usually does. If no ambiguity is possible, we will suppose B fixed once and for all, and always speak of r relative to B. For $\mathbb{G} = \mathbb{Z}$, strong irreducibility is equivalent to existence of $r \geq 1$ such that, for every two $u, v \in \mathcal{L}_X$, there exists $w \in S^r$ satisfying $uwv \in \mathcal{L}_X$. Clearly, every full shift is strongly irreducible.

As a consequence of the definition, strongly irreducible subshifts are *mixing*: given two open sets $U, V \subseteq X$, the set of those $g \in \mathbb{G}$ such that $U \cap \sigma_g^{-1}(V) = \emptyset$ is, at most, finite. In addition to this, as by (Lind and Marcus, 1995, Theorem 8.1.16), the Garden of Eden theorem is still valid on strongly irreducible subshifts of finite type. We remark that for one-dimensional subshifts of finite type, strong irreducibility is equivalent to the mixing property.

Another property of strongly irreducible subshifts, which will have a crucial role in the next section, is that they allow a "cut and paste" technique which is very common in proofs involving the full shift, but may be inapplicable for more general shifts.

Proposition 1. *Let $X \subseteq S^\mathbb{G}$ be a strongly irreducible subshift, let $c \in X$, and let $p : E \to S$ be an admissible pattern for X. There exists $c' \in X$ asymptotic to c such that $c'|_E = p$.*

Proof: It is not restrictive to suppose $E = D_n$ for suitable $n \geq 0$. Let $r \geq 1$ be a constant of strong irreducibility for X. Writing $E_k = D_{n+r+k} \setminus D_{n+r}$ for $k \geq 1$, we have of course $d(E, E_k) = r$. Set

$p_k = c|_{E_k}$. By strong irreducibility, there exists $c_k \in X$ such that $c_k|_E = p$ and $c_k|_{E_k} = p_k$. Then every limit point c' of $\{c_k\}_{k \geq 1}$, which exists and belongs to X because of compactness, satisfies the thesis. $\square$

Induction and restriction do not affect strong irreducibility.

Proposition 2. *Let $\mathbb{G}$ and Γ be finitely generated groups, where $\mathbb{G}$ is a subgroup of Γ, and let S be a finite set. Let $X \subseteq S^{\mathbb{G}}$ be a subshift and let $X^\Gamma \subseteq S^\Gamma$ be the subshift induced by X. If one between X and X^Γ is strongly irreducible, so is the other.*

Proof: To fix ideas, let $B_{\mathbb{G}}$ and B_Γ be two finite sets of generators for $\mathbb{G}$ and Γ, respectively, let J be a set of representatives of the left cosets of $\mathbb{G}$ in Γ, so that $\Gamma = \bigsqcup_{j \in J} j\mathbb{G}$, and let $\mathcal{F}$ be a set of forbidden patterns that determines X.

Suppose that X^Γ is strongly irreducible and $r \geq 1$ a constant of strong irreducibility for X^Γ. Take $r' \geq 1$ such that $D_{B_\Gamma, r-1} \cap \mathbb{G} \subseteq D_{B_{\mathbb{G}}, r'-1}$, which exists because the left-hand side is finite. Let $E_1, E_2 \subseteq \mathbb{G}$ satisfy $d_{B_{\mathbb{G}}}(E_1, E_2) \geq r'$. Then, by construction, $d_{B_\Gamma}(E_1, E_2) \geq r$ too. Given two admissible patterns $p_1 : E_1 \to S$, $p_2 : E_2 \to S$, take $c \in X^\Gamma$ such that $c|_{E_1} = p_1$ and $c|_{E_2} = p_2$. Then $c|_{\mathbb{G}} \in X$ has the same property.

Next, suppose that X is strongly irreducible and $r \geq 1$ is a constant of strong irreducibility for X. Let $M \geq 1$ be such that every element of $B_{\mathbb{G}}$ can be written as a product of at most M elements of B_Γ. Then Mr is a constant of strong irreducibility for X^Γ. Indeed, let $p_1 : E_1 \to S, p_2 : E_2 \to S$ be two admissible patterns such that $d_{B_\Gamma}(E_1, E_2) \geq Mr$. For $i = 1, 2$, there exist at most finitely many $j \in J$ such that $E_{i,j} = E_i \cap j\mathbb{G} \neq \emptyset$. If for a given j both $E_{1,j}$ and $E_{2,j}$ are nonempty, then $d_{B_\Gamma}(E_{1,j}, E_{2,j}) \geq d_{B_\Gamma}(E_1, E_2) \geq Mr$, hence, since $B_{\mathbb{G}} \subseteq B_\Gamma$ and multiplications on the left are isometries, $d_{B_{\mathbb{G}}}(j^{-1}E_{1,j}, j^{-1}E_{2,j}) \geq r$ by definition of M. We can then construct a configuration $c \in S^\Gamma$ such that $c|_{E_1} = p_1$ and $c|_{E_2} = p_2$ as follows:

- If $x \in j\mathbb{G}$ and both $E_{1,j}$ and $E_{2,j}$ are nonempty, let $c(x) = c_j(j^{-1}x)$, where $c_j \in X$ is such that $c_j(j^{-1}x) = p_1(x)$ if $x \in E_{1,j}$ and $c_j(j^{-1}x) = p_2(x)$ if $x \in E_{2,j}$.
- If $x \in j\mathbb{G}$ and, of $E_{1,j}$ and $E_{2,j}$, one is nonempty and the other is empty, then, calling E the nonempty one and p the corresponding pattern, let $c(x) = c_j(j^{-1}x)$, where $c_j \in X$ is such that $c_j(j^{-1}x) = p(x)$ for every $x \in E$.
- If $x \in j\mathbb{G}$ and $E_{1,j}$ and $E_{2,j}$ are both empty, let $c(x) = \bar{c}(j^{-1}x)$ where $\bar{c} \in X$ is fixed.

It is easy to see that no pattern from $\mathcal{F}$ can have any occurrences in c, so that $c \in X^\Gamma$. $\square$

3 Post-surjectivity

The notion of post-surjectivity is a sort of "dual" to pre-injectivity: it is a strengthening of surjectivity, in a similar way that pre-injectivity is a weakening of injectivity. The maps that are both pre-injective and post-surjective were studied in (Kari and Taati, 2015) under the name of complete pre-injective maps.

Definition 1. Let $\mathbb{G}$ be a group, S a finite set, and $X \subseteq S^{\mathbb{G}}$ a strongly irreducible subshift. A cellular automaton $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ on X is *post-surjective* if, however given $c \in X$ and a predecessor $e \in X$ of c, every configuration $c' \in X$ asymptotic to c has a predecessor $e' \in X$ asymptotic to e.

When $X = S^{\mathbb{G}}$ is the full shift, if no ambiguity is present, we will simply say that the CA is post-surjective.

Example 1. Every reversible cellular automaton is post-surjective. If $R \geq 0$ is such that the neighborhood of the inverse CA is included in D_R, and $N \geq 0$ is such that c and c' coincide outside D_N, then their unique pre-images e and e' must coincide outside D_{N+R}. ○

Example 2. The XOR CA with the right-hand neighbor (the one-dimensional elementary CA with rule 102) is surjective, but not post-surjective. As the XOR function is a permutation of each of its arguments given the other, every $c \in \{0,1\}^{\mathbb{Z}}$ has two pre-images, uniquely determined by their value in a single point. However (actually: because of this!) $\ldots 000 \ldots$ is a fixed point, but $\ldots 010 \ldots$ only has pre-images that take value 1 infinitely often. ○

The qualification "post-surjective" is well earned:

Proposition 3. *Let $X \subseteq S^{\mathbb{G}}$ be a strongly irreducible subshift. Every post-surjective* CA *on X is surjective.*

Proof: Let $r \geq 1$ be the constant of strong irreducibility of X, *i.e.*, let every two admissible patterns whose supports have distance at least r be jointly subpatterns of some configuration. Take an arbitrary $e \in X$ and set $c = F(e)$. Let $p : E \to S$ be an admissible pattern for X. By Proposition 1, there exists $c' \in X$ asymptotic to c such that $c'|_E = p$. By post-surjectivity, such c' has a pre-image in X, which means p has a pre-image admissible for X. The thesis follows from the orphan pattern principle. □

From Proposition 3 together with (Fiorenzi, 2003, Theorem 4.7) follows:

Proposition 4. *Let $\mathbb{G}$ be a finitely generated amenable group and let $X \subseteq S^{\mathbb{G}}$ be a strongly irreducible* SFT. *Every post-surjective* CA *on X is pre-injective.*

In addition, via a reasoning similar to the one employed in (Ceccherini-Silberstein and Coornaert, 2010, Section 1.7) and (Capobianco et al., 2013, Remark 18), we can prove:

Proposition 5. *Let $\mathbb{G}$ and Γ be finitely generated groups where $\mathbb{G}$ is a subgroup of Γ. Let $X \subseteq S^{\mathbb{G}}$ be a strongly irreducible subshift and let $X^{\Gamma} \subseteq S^{\Gamma}$ be the induced subshift. Let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a cellular automaton on X and $\mathcal{A}^{\Gamma}$ the induced cellular automaton on X^{Γ}. Then $\mathcal{A}$ is post-surjective if and only if $\mathcal{A}^{\Gamma}$ is post-surjective.*

In particular, post-surjectivity of arbitrary CA is equivalent to post-surjectivity on the subgroup generated by the neighborhood.

Proof: Suppose $\mathcal{A}$ is post-surjective. Let J be a set of representatives of the left cosets of $\mathbb{G}$ in Γ, *i.e.*, let $\Gamma = \bigsqcup_{j \in J} j\mathbb{G}$. Let $c, c' \in X^{\Gamma}$ be two asymptotic configurations and let e be a pre-image of c. For every $j \in J$ and $g \in \mathbb{G}$ set

$$\begin{array}{rcl} c_j(g) & = & c(jg)\,; \\ c'_j(g) & = & c'(jg)\,; \\ e_j(g) & = & e(jg)\,. \end{array}$$

By construction, each c_j belongs to X, is asymptotic to c'_j and has e_j, which also belongs to X, as a pre-image according to $\mathcal{A}$. Moreover, as c and c' are asymptotic in the first place, $c'_j \neq c_j$ only for finitely many $j \in J$. For every $j \in J$ let $e'_j \in X$ be a pre-image of c'_j according to $\mathcal{A}$ asymptotic to e_j, if $c'_j \neq c_j$, and e_j itself if $c'_j = c_j$. Then,

$$e'(\gamma) = e'_j(g) \quad \Longleftrightarrow \quad \gamma = jg$$

defines a pre-image of c' according to $\mathcal{A}^\Gamma$ which belongs to X^Γ and is asymptotic to e.

The converse implication is immediate. □

Proposition 6. *Let $X \subseteq S^{\mathbb{Z}}$ be a strongly irreducible* SFT *and let $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ be a post-surjective* CA *on X. Then $\mathcal{A}$ is reversible.*

Proof: Suppose $F = F_{\mathcal{A}}$ is not a bijection. For CA on one-dimensional strongly irreducible SFT, reversibility is equivalent to injectivity on periodic configurations. Namely, if two distinct configurations with the same image exist, then one can construct two distinct *periodic* configurations with the same image. Let then $u, v, w \in S^*$ be such that $e_u = \ldots uuu \ldots$, the configuration obtained by extending u periodically in both directions, and $e_v = \ldots vvv \ldots$ are different and have the same image $c = \ldots www \ldots$. It is not restrictive to suppose $|u| = |v| = |w|$. Without loss of generality, we also assume that X is defined by a set of forbidden words of length at most $|u|$.

Let $r \geq 1$ be a strong irreducibility constant for X and let $p, q \in S^r$ be such that $upv, vqu \in \mathcal{L}_X$. The two configurations $c_{u,v} = F(\ldots uupvv \ldots)$ and $c_{v,u} = F(\ldots vvquu \ldots)$ are both asymptotic to c. By post-surjectivity, there exist $x, y \in \mathcal{L}_X$ such that $e_{u,v} = \ldots uuxvv \ldots$ and $e_{v,u} = \ldots vvyuu \ldots$ satisfy $F(e_{u,v}) = F(e_{v,u}) = c$. Again, it is not restrictive to suppose that $|x| = |y| = m \cdot |u|$ for some $m \geq 1$, and that x and y start in $e_{u,v}$ and $e_{v,u}$ at the same point $i \in \mathbb{Z}$.

Let us now consider the configuration $e' = \ldots uuxv^N yuu \ldots$. By our previous discussion, for N large enough (*e.g.*, so that x and y do not have overlapping neighborhoods) $F_{\mathcal{A}}(e')$ cannot help but be c. Now, recall that e_u is also a pre-image of c and note that e_u and e' are asymptotic but distinct. Then $\mathcal{A}$ is surjective (by Proposition 3) but not pre-injective, contradicting the Garden of Eden theorem (Lind and Marcus, 1995, Theorem 8.1.16) as well as Proposition 4.

A graphical description of the argument is provided by Figure 1. □

Proposition 6 depends critically on the group being $\mathbb{Z}$, where CA that are injective on periodic configurations are reversible. Moreover, in our final step, we invoke the Garden of Eden theorem, which we know from (Ceccherini-Silberstein et al., 1999) not to hold for CA on generic groups. Not all is lost, however: maybe, by explicitly adding the pre-injectivity requirement, we can recover Proposition 6 on more general groups.

It turns out that it is so, at least for CA on full shifts. To see this, we need some preparations.

Lemma 1. *Let $\mathcal{A}$ be a post-surjective* CA *on a finitely generated group $\mathbb{G}$ and let F be its global transition function. There exists $N \geq 0$ such that, given any three configurations c, c', e with $c = F(e)$ and $\Delta(c, c') = \{1_{\mathbb{G}}\}$, there exists a pre-image e' of c' which coincides with e outside D_N.*

Proof: By contradiction, assume that for every $n \geq 0$, there exist $c_n, c'_n \in S^{\mathbb{G}}$ and $e_n \in F^{-1}(c_n)$ such that $\Delta(c_n, c'_n) = \{1_{\mathbb{G}}\}$, but every $e'_n \in F^{-1}(c'_n)$ differs from e_n on some point outside D_n. By compactness, there exits a sequence n_i such that the limits $c = \lim_{i \to \infty} c_{n_i}, c' = \lim_{i \to \infty} c'_{n_i}$, and $e = \lim_{i \to \infty} e_{n_i}$, all exist. Then $F(e) = c$ by continuity. By construction, c differs from c' only at $1_{\mathbb{G}}$. By post-surjectivity, there exists a pre-image e' of c' such that $\Delta(e, e') \subseteq D_m$ for some $m \geq 0$. Take $\ell \gg m$ and choose k large enough such that $c'_{n_k}\big|_{D_\ell} = c'|_{D_\ell}$ and $e_{n_k}|_{D_\ell} = e|_{D_\ell}$. Define $\tilde{e}$ so that it agrees with

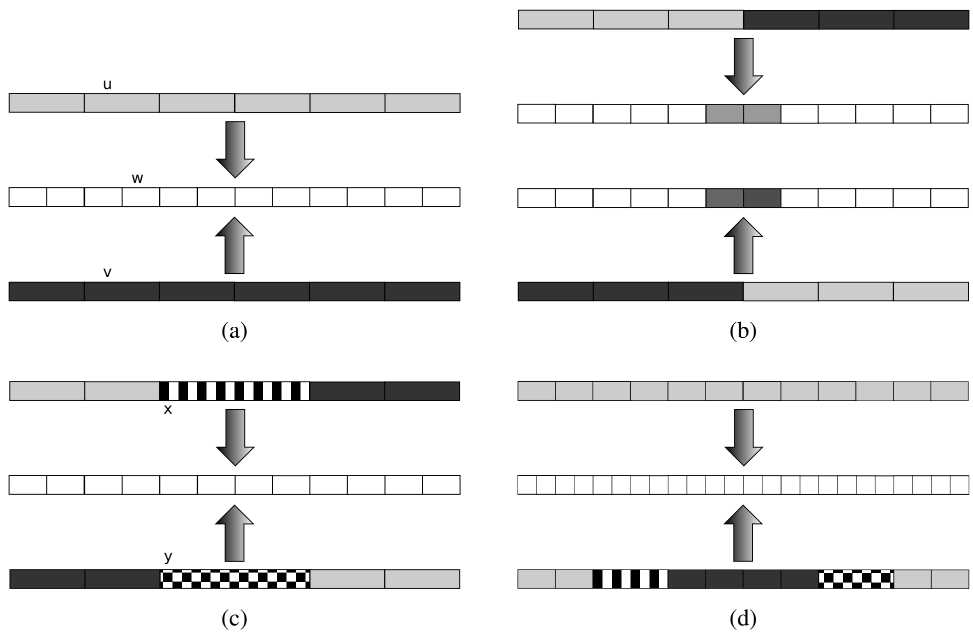

Figure 1: A graphical description of the argument in Proposition 6 for the full shift. (a) Let a 1D periodic configuration w have two different (periodic) preimages u and v. (b) By swapping the right-hand halves of the preimages, the new images only differ from the initial one in finitely many points. (c) By post-surjectivity, we can change them in finitely many points, and get two preimages of the initial configuration. (d) Then a violation of the Garden of Eden theorem occurs.

e' on D_ℓ and with e_{n_k} outside D_m. Such $\tilde{e}$ is well defined, because e', e, and e_{n_k} agree on $D_\ell \setminus D_m$. Then $\tilde{e}$ is a pre-image of c'_{n_k} which is asymptotic to e_{n_k} and agrees with e_{n_k} outside D_{n_k}, thus contradicting our assumption. □

By repeatedly applying Lemma 1 we get:

Proposition 7. *Let $\mathcal{A}$ be a post-surjective* CA *on a finitely generated group $\mathbb{G}$ and let F be its global transition function. There exists $N \geq 0$ such that, for every $r \geq 0$, however given three configurations c, c', e with $c = F(e)$ and $\Delta(c, c') \subseteq D_r$, there exists a pre-image e' of c' such that $\Delta(e, e') \subseteq D_{N+r}$.*

Assuming also pre-injectivity, we get the following stronger property:

Corollary 1. *Let $\mathcal{A}$ be a pre-injective, post-surjective* CA *on a finitely generated group $\mathbb{G}$ and let F be its global transition function. There exists $M \in \mathcal{PF}(\mathbb{G})$ with the following property: for every pair (e, e') of asymptotic configurations, $\Delta(e, e') \subseteq \Delta(F(e), F(e)')M$.*

We are now ready to prove:

Theorem 1. *Every pre-injective, post-surjective cellular automaton on a full shift is reversible.*

Proof: By Proposition 5, it is sufficient to consider the case where $\mathbb{G}$ is finitely generated.

Let $\mathcal{A}$ be a pre-injective and post-surjective CA on the group $\mathbb{G}$, let S be its set of states, and let F be its global transition function. Let M be as in Corollary 1. We construct a new CA with neighborhood $\mathcal{N} = M^{-1}$. Calling H the global transition function of the new CA, we first prove that H is a *right* inverse of F. We then show that H is also a *left* inverse for F, thus completing the proof.

To construct the local update rule $h : S^{\mathcal{N}} \to S$, we proceed as follows. Fix a constant configuration u and let $v = F(u)$. Given $g \in \mathbb{G}$ and $p : \mathcal{N} \to S$, for every $i \in \mathbb{G}$, put

$$y_{g,p}(i) = \begin{cases} p(g^{-1}i) & \text{if } i \in g\mathcal{N} \\ v(i) & \text{otherwise} \end{cases} \tag{3}$$

that is, let $y_{g,p}$ be obtained from v by cutting away the piece with support $g\mathcal{N}$ and pasting p as a "patch" for the "hole". By post-surjectivity and pre-injectivity combined, there exists a unique $x_{g,p} \in S^{\mathbb{G}}$ asymptotic to u such that $F(x_{g,p}) = y_{g,p}$. Let then

$$h(p) = x_{g,p}(g)\,. \tag{4}$$

Observe that (4) does *not* depend on g: if $g' = i \cdot g$, then $y_{g',p} = \sigma_i(F(x_{g,p})) = F(\sigma_i(x_{g,p}))$, so that $x_{g',p} = \sigma_i(x_{g,p})$ by pre-injectivity, and $x_{g',p}(g') = x_{g,p}(g)$.

Let now y be *any* configuration asymptotic to v such that $y|_{g\mathcal{N}} = p$, and let x be the unique pre-image of y asymptotic to u. We claim that $x(g) = h(p)$. To prove this, we observe that, as y and $y_{g,p}$ are both asymptotic to v and they agree on $g\mathcal{N} = gM^{-1}$, the set $K = \Delta(y, y_{g,p})$ is finite and is contained in $\mathbb{G} \setminus gM^{-1}$. By Corollary 1, their pre-images x and $x_{g,p}$ can disagree only on $KM \subseteq \left(\mathbb{G} \setminus gM^{-1}\right) M$. The set KM does not contain g, because if $g \in \left(\mathbb{G} \setminus gM^{-1}\right) M$, then for some $m \in M$, $gm^{-1} \in \left(\mathbb{G} \setminus gM^{-1}\right)$, which is not the case! Therefore, $x(g) = x_{g,p}(g) = h(p)$, as we claimed.

The argument above holds whatever the pattern $p : \mathcal{N} \to S$ is. By applying it finitely many times to arbitrary finitely many points, we determine the following fact: if y is any configuration which is asymptotic to v, then $F(H(y)) = y$. But the set of configurations asymptotic to v is dense in $S^{\mathbb{G}}$, so it follows from continuity of F and H that $F(H(y)) = y$ for every $y \in S^{\mathbb{G}}$.

We have thus shown that H is a right inverse of F. We next verify that H is also a left inverse of F.

Let x be a configuration asymptotic to u, and set $y = F(x)$. Note that y is asymptotic to v. The two configurations x and $H(y)$ are both asymptotic to u, and furthermore, $F(x) = y = F(H(y))$. Therefore, by the pre-injectivity of F, x and $H(y)$ must coincide, that is, $H(F(x)) = x$. The continuity of F and H now implies that the equality $H(F(x)) = x$ holds even if x is not asymptotic to u. Hence, H is a left inverse for F. □

Corollary 2. *A cellular automaton on an amenable group (in particular, a d-dimensional* CA*) is post-surjective if and only if it is reversible.*

4 Post-surjectivity on sofic groups

After proving Theorem 1, we might want to find a post-surjective cellular automaton that is not pre-injective. However, the standard examples of surjective CA which are not pre-injective fail when post-surjectivity is sought instead. The next example illustrates how.

Example 3. Let $\mathbb{G} = \mathbb{F}_2$ be the free group on two generators a, b, *i.e.*, the group of *reduced words* on the alphabet $B = \{a, b, a^{-1}, b^{-1}\}$. Let $\mathcal{N} = B \cup \{1_{\mathbb{G}}\} = D_1$, and for every $x, y, z, w, v \in \{0, 1\}$ let $f(x, y, z, w, v)$ be 1 if $x + y + z + w + v \geq 3$, and 0 otherwise. Then $\mathcal{A} = \langle \mathbb{G}, \{0, 1\}, \mathcal{N}, f \rangle$ is the *majority* CA on $\mathbb{F}_2$.

The CA $\mathcal{A}$ is clearly not pre-injective; however, it is surjective. Indeed, a preimage of an arbitrary pattern p on D_n, for $n \geq 1$, can be obtained from a preimage of the restriction of p to D_{n-1} by exploiting the fact that every element of length n has three neighbors of length $n + 1$. We can tweak the procedure a little bit and see that every configuration c has a (not unique) *critical* preimage e where, for every $g \in \mathbb{G}$, exactly three between $e(g)$, $e(ga)$, $e(gb)$, $e(ga^{-1})$, and $e(gb^{-1})$ have value $c(g)$. An example is provided in Figure 2.

Let c be a configuration such that $c(1_{\mathbb{G}}) = c(a) = c(b) = 0$, $c(a^{-1}) = c(b^{-1}) = 1$, and for every $n \geq 1$, each point of length n has at least one neighbor of length $n + 1$ with value 0, and at least one neighbor of length $n + 1$ with value 1. Let e be a critical preimage for c which coincides with c on D_1, and let c' only differ from c in $1_{\mathbb{G}}$. Suppose, for the sake of contradiction, that there exists a preimage e' of c' which is asymptotic to e. Let x be a point of maximum length $n = \|x\|$ where e and e' differ. Call $e(x) = s$ and $e'(x) = t \neq s$. Two cases are possible:

1. $n = 0$. Then $s = 0$, $t = 1$, and $e'(g) = e(g)$ for every $g \neq 1_{\mathbb{G}}$. But as e is critical and $c(a) = e(a) = e(1_{\mathbb{G}}) = 0$, exactly two between $e'(a^2)$, $e'(ab)$, and $e'(ab^{-1})$ have value 1. As $e'(1_{\mathbb{G}}) = 1$ too, it must be $c'(a) = 1$, against the hypothesis that c and c' only differ at $1_{\mathbb{G}}$.

2. $n \geq 1$. Let u, v, and w be the three neighbors of x of length $n+1$. As e is critical, $c'(u) = c'(v) = c'(w) = t$. But by construction, either $c(u) = s$, or $c(v) = s$, or $c(w) = s$. This contradicts that c and c' only differ at $1_{\mathbb{G}}$.

This proves that $\mathcal{A}$ is not post-surjective. ○

The reason behind this failure is that, as we shall see below, finding such a counterexample amounts to finding a group which is not *sofic*, and that appears to be a difficult open problem.

The notion of a sofic group was originally introduced in (Gromov, 1999), but was later reformulated, for finitely generated groups, in (Weiss, 2000) in combinatorial, rather than geometric, terms.

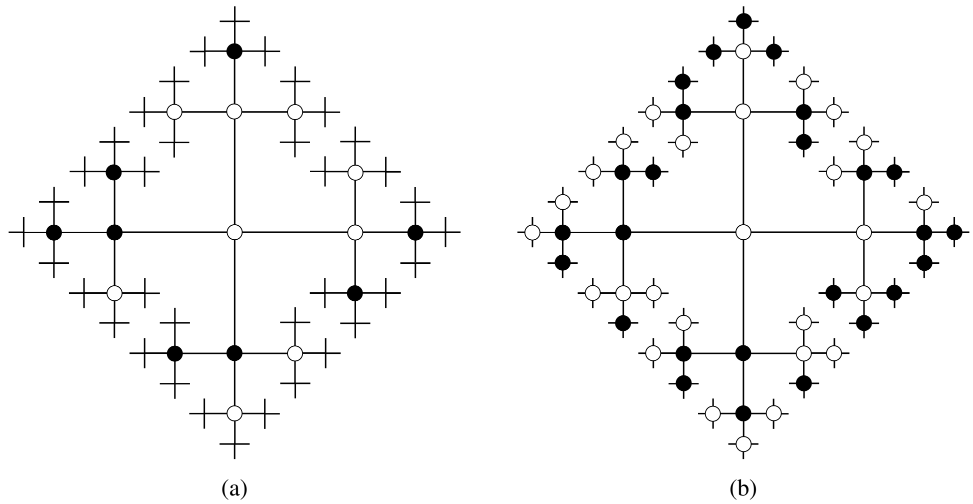

Figure 2: (a) A configuration on the free group on two generators, restricted to D_2. (b) A critical preimage of the configuration of point a, restricted to D_3.

Definition 2. Let $\mathbb{G}$ be a finitely generated group and let B be a finite symmetric set of generators for $\mathbb{G}$. Let $r \geq 0$ be an integer and $\varepsilon > 0$ a real. An (r, ε)-*approximation* of $\mathbb{G}$ (relative to B) is a B-labeled graph (V, E) along with a subset $U \subseteq V$ such that the following hold:

1. For every $u \in U$, the neighborhood of radius r of u in (V, E) is isomorphic to $D_{B,r}$ as a labeled graph.
2. $|U| > (1 - \varepsilon)|V|$.

The group $\mathbb{G}$ is *sofic* (relative to B) if for every choice of $r \geq 0$ and $\varepsilon > 0$, there is an (r, ε)-approximation of $\mathbb{G}$ (relative to B).

As explained in (Weiss, 2000), the notion of soficity does not depend on the generating set B. For this reason, in the rest of this section, we will suppose B given once and for all. It is easy to see that finitely generated residually finite groups and finitely generated amenable groups are all sofic.

The importance of sofic groups is threefold: firstly, as per (Weiss, 2000, Section 3), sofic groups are surjunctive; secondly, no examples of non-sofic groups are currently known. We add a third reason:

Theorem 2. *Let $\mathbb{G}$ be a sofic group. Every post-surjective cellular automaton on $\mathbb{G}$ is pre-injective (and therefore reversible).*

As a corollary, cellular automata which are post-surjective, but not pre-injective, could only exist over non-sofic groups!

To prove Theorem 2, we need two auxiliary lemmas. Observe that if $f : S^{D_R} \to S$ is the local rule of a cellular automaton $\mathcal{A}$ on a group $\mathbb{G}$ with a finite generating set B, and (V, E) is a B-labeled graph, then f is applicable in an obvious fashion to patterns on V at every point $v \in V$ whose R-neighborhood in (V, E) is isomorphic to the disk of radius R in the Cayley graph of $\mathbb{G}$ with generating set B. Therefore, we extend our notation, and for two patterns $p : H \to S$ and $q : C \to S$ with $H, C \subseteq V$, we write $p \xrightarrow{f} q$ if for every $v \in C$, the R-neighborhood $D_R(v)$ is a subset of H and is isomorphic to the disk of radius R, and furthermore $f\big(p|_{D_R(v)} \big) = q(v)$. Note that even when $\mathcal{A}$ is surjective, the induced maps $S^H \to S^C$ are not necessarily surjective.

Example 4. Let $\mathcal{A}$ be the elementary CA with rule 102 (same as in Example 2). Let (V, E) be a cycle on four nodes. The 1-neighborhood of each node is isomorphic to $D_1 \subseteq \mathbb{Z}$. Let then $H = C = V$. As each bit is counted twice during the update (one as a center, the other as a right neighbor) and the rule is linear, the image in S^C of an element of S^H must have an even number of 1s. Then $0001 \in S^C$ has no preimage in S^H. ○

Lemma 2. *Let $\mathcal{A}$ be a post-surjective* CA *on a sofic group $\mathbb{G}$. Let $\mathcal{A}$ have state set S, neighborhood $\mathcal{N} \subseteq D_R$ and local rule f, and let N be given by Lemma 1. Consider an (r, ε)-approximation given by a graph (V, E) and a set $U \subseteq V$, where $\varepsilon > 0$ and $r \geq N + 2R$. For every pattern $q : U \to S$, there is a pattern $p : V \to S$ such that $p \xrightarrow{f} q$.*

Proof: Take arbitrary $p_0 : V \to S$ and $q_0 : U \to S$ such that $p_0 \xrightarrow{f} q_0$. Let $q_0, q_1, \ldots, q_m = q$ be a sequence of patterns with support U such that, for every i, q_i and q_{i+1} only differ in a single $k_i \in U$. Since the r-neighborhood of k_i is isomorphic to the disk of the same radius from the Cayley graph of $\mathbb{G}$, we can apply Lemma 1 and deduce the existence of a sequence $p_0, p_1, \ldots, p_m$ with common support V such that each p_i is a pre-image of q_i and, for every i, p_i differs from p_{i+1} at most in $D_N(k_i)$. Then $p = p_m$ satisfies the thesis. □

The next lemma is an observation made in (Weiss, 2000).

Lemma 3 (Packing lemma)**.** *Let $\mathbb{G}$ be a group with a finite generating set B. Let (V, E) be a B-labeled graph and $U \subseteq V$ a subset with $|U| \geq \frac{1}{2}|V|$ such that, for every $u \in U$, the 2ℓ-neighborhood of u in (V, E) is isomorphic to the disk of radius 2ℓ in the Cayley graph of $\mathbb{G}$. Then, there is a set $W \subseteq U$ of size at least $\frac{|V|}{2|D_{2\ell}|}$ such that the ℓ-neighborhoods of the elements of W are disjoint.*

Proof: Let $W \subseteq U$ be a maximal set such that the ℓ-neighborhoods of the elements of W are disjoint. Then, for every $u \in U$, the neighborhood $D_\ell(u)$ must intersect the set $\bigcup_{w \in W} D_\ell(w)$. Therefore, $U \subseteq D_{2\ell}(W)$, which gives $|U| \leq |D_{2\ell}| \cdot |W|$. □

Proof of Theorem 2: Let $\mathbb{G}$ be a sofic group and assume that $\mathcal{A} = \langle S, D_R, f \rangle$ is a cellular automaton on $\mathbb{G}$ that is post-surjective, but not pre-injective. For brevity, set $|S| = s \geq 2$. Let N be as in Lemma 1.

Since the CA is not pre-injective, there are two asymptotic configurations $x, x' : \mathbb{G} \to S$ such that $F_\mathcal{A}(x) = F_\mathcal{A}(x')$. Take m such that the disk D_m contains $\Delta(x, x')$. It follows that there are two mutually erasable patterns on D_{m+2R}, that is, two patterns $p, p' : D_{m+2R} \to S$ such that on any configuration z, replacing an occurrence of p with p' or vice versa does not change the image of z under $F_\mathcal{A}$.

Take $r \geq \max\{N, m\} + 2R$ and $\varepsilon > 0$ small. We shall need ε small enough so that

$$s^\varepsilon \cdot \left(1 - s^{-|D_r|}\right)^{\frac{1}{2|D_{2r}|}} < 1 \,.$$

Such a choice is possible, because the second factor on the left-hand side is a constant smaller than 1. Since $\mathbb{G}$ is sofic, there is a $(2r, \varepsilon)$-approximation of $\mathbb{G}$ given by a graph (V, E) and a set $U \subseteq V$. Let

$\varphi : S^V \to S^U$ be the map given by $\varphi(p) = q$ if $p \xrightarrow{f} q$. Such φ is well defined, because the R-neighborhood of each $u \in U$ is isomorphic to the disk of radius R in $\mathbb{G}$.

By Lemma 2, the map φ is surjective, hence

$$|\varphi(S^V)| = s^{|U|} \,. \tag{5}$$

On the other hand, by Lemma 3, there is a collection $W \subseteq U$ of $|W| \geq \frac{|V|}{2|D_{2r}|}$ points in U whose r-neighborhoods are disjoint. Each of these r-neighborhoods is isomorphic to the disk $D_r \supseteq D_{m+2R}$ in $\mathbb{G}$. The existence of the mutually erasable patterns on D_r thus implies that there are at most

$$|\varphi(S^V)| \leq (s^{|D_r|} - 1)^{|W|} \cdot s^{|V| - |W| \cdot |D_r|}$$

patterns on V with distinct images. However,

$$\begin{aligned}
(s^{|D_r|} - 1)^{|W|} \cdot s^{|V| - |W| \cdot |D_r|} &= \left(1 - s^{-|D_r|}\right)^{|W|} \cdot s^{|V|} \\
&\leq \left(1 - s^{-|D_r|}\right)^{\frac{|V|}{2|D_{2r}|}} \cdot s^{|V|} \\
&< s^{-\varepsilon|V|} \cdot s^{|V|} \\
&= s^{(1-\varepsilon)|V|} \\
&< s^{|U|} \,,
\end{aligned}$$

which contradicts (5). □

Corollary 3. *Let $\mathbb{G}$ be a sofic group and $\mathcal{A}$ a cellular automaton on $\mathbb{G}$. Then, $\mathcal{A}$ is post-surjective if and only if it is reversible.*

Do post-surjective cellular automata on full shifts which are not pre-injective exist at all? By Theorem 2, such examples might exist only if non-sofic groups exist. We thus make the following “almost dual” to Gottschalk’s conjecture:

Conjecture 2. *Let $\mathbb{G}$ be a group and $\mathcal{A}$ a cellular automaton on $\mathbb{G}$. If $\mathcal{A}$ is post-surjective, then it is pre-injective.*

5 Balancedness

Definition 3. Let $\mathbb{G}$ be a group and let $E \in \mathcal{PF}(\mathbb{G})$. A cellular automaton $\mathcal{A} = \langle S, \mathcal{N}, f \rangle$ on a group $\mathbb{G}$ is *E-balanced* if for every $M \in \mathcal{PF}(\mathbb{G})$ such that $E\mathcal{N} \subseteq M$, every pattern $p : E \to S$ has $|S|^{|M|-|E|}$ pre-images on M. $\mathcal{A}$ is *balanced* if it is E-balanced for every $E \in \mathcal{PF}(\mathbb{G})$.

If $\mathbb{G}$ is finitely generated, and $r \geq 0$ is such that $\mathcal{N} \subseteq D_r$, it is easy to see that Definition 3 is equivalent to the following property: for every $n \geq 0$ every pattern on D_n has exactly $|S|^{|D_{n+r}|-|D_n|}$ pre-images on D_{n+r}. In addition (see (Capobianco et al., 2013, Remark 18)) balancedness is preserved by both induction and restriction: hence, it can be determined by only checking it on the subgroup generated by the neighborhood. Balancedness does not depend on the choice of the neighborhood, because it is equivalent to preservation by the CA global function of the *uniform product measure* on $S^{\mathbb{G}}$ (see (Capobianco et al., 2013, Proposition 17)). Finally, as $|S|^{|M|-|E|} \geq 1$ when $E\mathcal{N} \subseteq M$, every balanced CA is surjective by the orphan pattern principle.

The notion of balancedness given in Definition 3 is meaningful for CA on the full shift, but not for CA on proper subshifts. The reason is that, with proper subshifts, it may happen that the number of patterns on a given set is not a divisor of the number of patterns on a larger set.

Example 5. Let $X \subseteq \{0,1\}^{\mathbb{Z}}$ be the *golden mean shift* of all and only bi-infinite words where the factor 11 does not appear. It is easy to see (see (Lind and Marcus, 1995, Example 4.1.4)) that $|\mathcal{L}_X \cap \{0,1\}^n| = f_{n+2}$, where f_n is the nth Fibonacci number. Any two consecutive Fibonacci numbers are relatively prime. ○

Lemma 4. *Let $\mathbb{G}$ be a group, let S be a finite set, and let $F, H : S^{\mathbb{G}} \to S^{\mathbb{G}}$ be* CA *global transition functions.*

1. *If F and H are both balanced, then so is $F \circ H$.*
2. *If F and $F \circ H$ are both balanced, then so is H.*
3. *If H and $F \circ H$ are both balanced, and in addition H is reversible, then F is balanced.*

In particular, a reversible CA and its inverse are either both balanced or both unbalanced.

Proof: It is sufficient to consider the case when $\mathbb{G}$ is finitely generated, *e.g.*, by the union of the neighborhoods of the two CA. Let $r \geq 0$ be large enough that the disk D_r includes the neighborhoods of both F and H.

First, suppose F and H are both balanced. Let $p : D_n \to S$ By balancedness, p has exactly $|S|^{|D_{n+r}|-|D_n|}$ pre-images over D_{n+r} according to H. In turn, every such pre-image has $|S|^{|D_{n+2r}|-|D_{n+r}|}$ pre-images over D_{n+2r} according to F, again by balancedness. All the pre-images of p on D_{n+2r} by $F \circ H$ have this form, so p has $|S|^{|D_{n+2r}|-|D_n|}$ pre-images on D_{n+2r} according to $F \circ H$. This holds for every $n \geq 0$ and $p : D_n \to S$, thus, $F \circ H$ is balanced.

Now, suppose F is balanced but H is not. Take $n \geq 0$ and $p : D_n \to S$ having $M > |S|^{|D_{n+r}|-|D_n|}$ pre-images according to H. By balancedness of F, each of these M pre-images has exactly $|S|^{|D_{n+2r}|-|D_{n+r}|}$ pre-images according to F. Then p has overall $M \cdot |S|^{|D_{n+2r}|-|D_{n+r}|} > |S|^{|D_{n+2r}|-|D_n|}$ pre-images on D_{n+2r} according to $F \circ H$, which is thus not balanced.

Finally, suppose H and $F \circ H$ are balanced and H is reversible. As the identity CA is clearly balanced, by the previous point (with H taking the role of F and H^{-1} that of H) H^{-1} is balanced. By the first point, as $F \circ H$ and H^{-1} are both balanced, so is their composition $F = F \circ H \circ H^{-1}$. □

As we observed after Definition 3, a balanced CA gives at least one pre-image to each pattern, thus is surjective. On amenable groups (see (Bartholdi, 2010)) the converse is also true; on non-amenable groups (ibid.) some surjective cellular automata are not balanced. In the last section of (Capobianco et al., 2013), we ask ourselves the question whether *injective* cellular automata are balanced. The answer is that, at least in all cases currently known, it is so.

Theorem 3. *Reversible CA are balanced.*

Proof: It is not restrictive to suppose that $\mathbb{G}$ is finitely generated. Let $\mathcal{A}$ be a reversible cellular automaton on $\mathbb{G}$ with state set S and global transition function $F = F_{\mathcal{A}}$. Let $r \geq 0$ be large enough so that the disk

D_r includes the neighborhoods of both F and F^{-1}. Then for every $c \in S^{\mathbb{G}}$ the states of both $F(c)$ and $F^{-1}(c)$ on D_n are determined by the state of c in D_{n+r}.

Let $p_1, p_2 : D_n \to S$ be two patterns. It is not restrictive to suppose $n \geq r$. We exploit reversibility of F to prove that they have the same number of pre-images on D_{n+r} by constructing a bijection $T_{1,2}$ between the set of the pre-images of p_1 and that of the pre-images of p_2. As this will hold whatever n, p_1, and p_2 are, F will be balanced.

For $i = 1, 2$ let Q_i be the set of the pre-images of p_i on D_{n+r}. Given $q_1 \in Q_1$, and having fixed a state $0 \in S$, we proceed as follows:

1. First, we extend q_1 to a configuration e_1 by setting $e_1(g) = 0$ for every $g \notin D_{n+r}$.
2. Then we apply F to e_1 and set $c_1 = F(e_1)$. By construction, $c_1|_{D_n} = p_1$.
3. Next, from c_1 we construct c_2 by replacing p_1 with p_2 inside D_n.
4. Then we set $e_2 = F^{-1}(c_2)$.
5. Finally, we call q_2 the restriction of e_2 to D_{n+r}.

Observe that $q_2 = e_2|_{D_{n+r}} \in Q_2$. This follows immediately from $\mathcal{A}$ being reversible: by construction, if we apply F to e_2, and restrict the result to D, we end up with p_2. We call $T_{1,2} : Q_1 \to Q_2$ the function computed by performing the steps from 1 to 5, and $T_{2,1} : Q_2 \to Q_1$ the one obtained by the same steps with the roles of p_1 and p_2 swapped. The procedure is illustrated in Figure 3.

Now, by construction, c_1 and c_2 coincide outside D_n, and their updates e_1 and e_2 by F^{-1} coincide outside D_{n+r}. But e_1 is 0 outside D_{n+r}, so that updating c_2 to e_2 is the same as extending q_2 with 0 outside D_{n+r}. This means that $T_{2,1}$ is the inverse of $T_{1,2}$. Consequently, Q_1 and Q_2 have the same number of elements. As p_1 and p_2 are arbitrary, any two patterns on D_n have the same number of pre-images on D_{n+r}. As $n \geq 0$ is also arbitrary, $\mathcal{A}$ is balanced. □

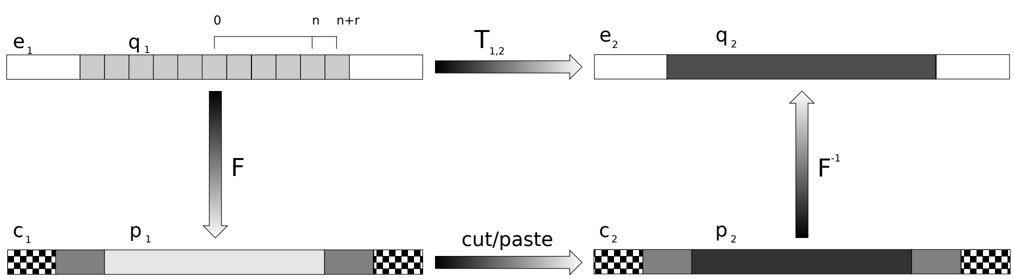

Figure 3: An illustration of the argument for Theorem 3.

Corollary 4. *Injective cellular automata over surjunctive groups are balanced. In particular, injective* CA *over sofic groups are balanced.*

Corollary 5. *Gottschalk's conjecture is equivalent to the statement that every injective* CA *on a full shift is balanced.*

Proof: If Gottschalk's conjecture is true, then every injective CA is reversible, thus balanced because of Theorem 3. If Gottschalk's conjecture is false, then there exists a CA which is injective, but not surjective. Such CA cannot be balanced, because balanced CA have no orphans. □

References

L. Bartholdi. Gardens of Eden and amenability on cellular automata. *Journal of the European Mathematical Society*, 12:241–248, 2010.

L. Bartholdi. Amenability of groups is characterized by Myhill's theorem. `http://arxiv.org/abs/1605.09133v2`, 2016.

S. Capobianco. On the induction operation for shift subspaces and cellular automata as presentations of dynamical systems. *Information and Computation*, 207:1169–1180, 2009.

S. Capobianco, P. Guillon, and J. Kari. Surjective cellular automata far from the Garden of Eden. *Discrete Mathematics and Theoretical Computer Science*, 15:41–60, 2013.

T. Ceccherini-Silberstein and M. Coornaert. *Cellular Automata and Groups*. Springer, 2010.

T. Ceccherini-Silberstein, A. Machì, and F. Scarabotti. Amenable groups and cellular automata. *Annales de l'Institut Fourier*, 49:673–685, 1999.

F. Fiorenzi. Cellular automata and strongly irreducible shifts of finite type. *Theoretical Computer Science*, 299:477–493, 2003.

W. Gottschalk. Some general dynamical notions. In *Recent Advances in Topological Dynamics*, volume 318 of *Lecture Notes in Mathematics*, pages 120–125. Springer, 1973.

M. Gromov. Endomorphisms of symbolic algebraic varieties. *Journal of the European Mathematical Society*, 1:109–197, 1999.

J. Kari. Theory of cellular automata: A survey. *Theoretical Computer Science*, 334:3–33, 2005.

J. Kari and S. Taati. Statistical mechanics of surjective cellular automata. *Journal of Statistical Physics*, 160:1198–1243, 2015.

D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

B. Weiss. Sofic groups and dynamical systems. *Sankhyā: Indian Journal of Statistics*, 62A:350–359, 2000.