BARRLUND'S DISTANCE FUNCTION AND QUASICONFORMAL MAPS

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ABSTRACT. Answering a question about triangle inequality suggested by R. Li, A. Barrlund [2] introduced a distance function which is a metric on a subdomain of $\mathbb{R}^n$. We study this Barrlund metric and give sharp bounds for it in terms of other metrics of current interest. We also prove sharp distortion results for the Barrlund metric under quasiconformal maps.

1. INTRODUCTION

For a given domain $G \subset \mathbb{R}^n$ with $G \neq \mathbb{R}^n$, for a number $p \geq 1$, and for points $z_1, z_2 \in G$, let

$$b_{G,p}(z_1, z_2) = \sup_{z \in \partial G} \frac{|z_1 - z_2|}{\sqrt[p]{|z_1 - z|^p + |z - z_2|^p}} . \tag{1.1}$$

A. Barrlund [2] studied this expression for the case $G = \mathbb{R}^n \setminus \{0\}$ and proved, answering a question of R.-C. Li [16], that it is a metric. These facts motivated, in part, P. Hästö's papers [12, 13], where he proved that $b_{G,p}$ is a metric in a general domain and studied also some other metrics.

The *triangular ratio metric* s_G of a given domain $G \subset \mathbb{R}^n$ defined as follows

$$s_G(z_1, z_2) = \sup_{z \in \partial G} \frac{|z_1 - z_2|}{|z_1 - z| + |z - z_2|} , z_1, z_2 \in G , \tag{1.2}$$

was recently studied in [5, 11]. As shown in [11], this metric is closely related to the quasi-hyperbolic metric [10, 8, 24] and several other metrics of current interest [14, 20, 19, 9].

We study *the Barrlund metric* $b_{G,p}$ and compare it to $s_G = b_{G,1}$. For the cases of a ball or a half-plane we give in our main theorems 3.27 and 3.24 explicit formulas for $b_{G,2}$. To this end, we first recall some properties of s_G . By compactness, the suprema in (1.1) and (1.2) are attained. If G is convex, it is simple to see that the extremal point z_0 for (1.2) is a point of contact of the boundary with an ellipse contained in G with foci at z_1, z_2 .

We prove the following sharp inequality between the above two metrics.

Theorem 1.3. *Let G be a domain in $\mathbb{R}^n$ and let $p \geq 1$. Then for all points $z_1, z_2 \in G$*

$$s_G(z_1, z_2) \leq b_{G,p}(z_1, z_2) \leq 2^{1-1/p} s_G(z_1, z_2) .$$

Clearly, this inequality holds as an identity if $p = 1$. But perhaps more interesting is that the right hand side holds as an equality for all $p \geq 1$ if $G = \{z \in \mathbb{C} : \mathrm{Im}\,(z) > 0\}$, and $z_1, z_2 \in G$ with $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

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Anders Barrlund 1962-2000 was a Swedish mathematician and [2] was his last paper.

The metric $s_{\mathbb{D}}$ is also connected with a classical problem of optics. The well-known Ptolemy-Alhazen problem reads [21]: "Given a light source and a spherical mirror, find the point on the mirror where the light will be reflected to the eye of an observer." We consider now the following two-dimensional version of the problem when two points z_1, z_2 are in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and its circumference $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ is a reflecting curve. The problem is to find all points $u \in \partial\mathbb{D}$ such that

$$\measuredangle(z_1, u, 0) = \measuredangle(0, u, z_2)\,. \tag{1.4}$$

Here $\measuredangle(z, u, w)$ denotes the radian measure in $(-\pi, \pi]$ of the oriented angle with initial side $[u, z]$ and final side $[u, w]$. This condition says that the angles of incidence and reflection are equal, a light ray from z_1 to u is reflected at u and goes through the point z_2.

The equality (1.4) shows that the ellipse with foci z_1, z_2, passing through u, is tangent at u to the unit circle. A point $u = e^{i\theta_0} \in \partial\mathbb{D}$ satisfies (1.4) if and only if θ_0 is a critical point of $f(\theta) := \left|e^{i\theta} - z_1\right| + \left|e^{i\theta} - z_2\right|$, $\theta \in \mathbb{R}$. Note that $f'(\theta) = \mathrm{Im}\,(z\overline{w})$, where $z = e^{i\theta}$ and $w = \frac{e^{i\theta}-z_1}{\left|e^{i\theta}-z_1\right|} + \frac{e^{i\theta}-z_2}{\left|e^{i\theta}-z_2\right|}$, therefore $f'(\theta) = 0$ if and only if the radius of the unit circle terminating at z is the bisector of the angle formed by segments joining z_1, z_2 to z.

Now for the case of the unit disk $G = \mathbb{D}$ and $z_1, z_2 \in \mathbb{D}$ and the extremal point $z_0 \in \partial\mathbb{D}$, for the definition (1.2), the connection between the triangular ratio metric

$$s_{\mathbb{D}}(z_1, z_2) = \frac{|z_1 - z_2|}{|z_1 - z_0| + |z_2 - z_0|}$$

and the Ptolemy-Alhazen problem is clear: $u = z_0$ satisfies (1.4). This connection was recently pointed out in [7].

Theorem 1.5 ([7]). *The point u in (1.4) is given as a solution of the equation*

$$\overline{z_1 z_2}u^4 - (\overline{z_1} + \overline{z_2})u^3 + (z_1 + z_2)u - z_1 z_2 = 0. \tag{1.6}$$

This quartic equation can be solved by symbolic computation programs. This method was used in [7] to compute the values of $s_{\mathbb{D}}(z_1, z_2)$.

We also study the limiting case $p = \infty$ of the Barrlund metric. As pointed out by P. Hästö [12], it was proved by D. Day in a short note [6] that the p-relative distance with $p = \infty$ is a metric in G, for $G = \mathbb{R}^n \setminus \{0\}$.

We conclude our paper by studying the behavior of the Barrlund distance under Möbius transformations and quasiconformal mappings defined on the upper half plane $\mathbb{H}$ and prove the following theorem.

Theorem 1.7. *Let $f : \mathbb{H} \to \mathbb{H}$ be a K-quasiconformal map and $z_1, z_2 \in \mathbb{H}$. Then for $p \geq 1$*

$$b_{\mathbb{H},p}(f(z_1), f(z_2)) \leq 2^{1-1/p}4^{1-1/K}b_{\mathbb{H},p}(z_1, z_2)^{1/K}\,.$$

Observe that this theorem is sharp.

We also formulate two conjectures.

Remark 1.8. After the publication of [7], we have learned more about the history of the Ptolemy-Alhazen problem: e.g. the book of A.M. Smith [21] provides a historical account of Alhazen's work on optics. Dr. F.G. Nievinski has kindly informed us about the papers of P.M. Neumann [17] and J.D. Smith [22], which also study this problem. The equation (1.6) appears also in [17, (1), p. 525] and [22, p.194 line 1]. Note that in [7] we study this topic from a different point of view.

2. Preliminaries

We recall the definition of the hyperbolic distance $\rho_{\mathbb{D}}(z_1, z_2)$ between two points $z_1, z_2 \in \mathbb{D}$ [3, Thm 7.2.1, p. 130]:

$$\tanh \frac{\rho_{\mathbb{D}}(z_1, z_2)}{2} = \frac{|z_1 - z_2|}{\sqrt{|z_1 - z_2|^2 + (1 - |z_1|^2)(1 - |z_2|^2)}}\,. \tag{2.1}$$

The triangular ratio metric can be estimated in terms of the hyperbolic metric as follows. By [11, 2.16] for $z_1, z_2 \in \mathbb{D}$

$$\tanh \frac{\rho_{\mathbb{D}}(z_1, z_2)}{4} \le s_{\mathbb{D}}(z_1, z_2) \le \tanh \frac{\rho_{\mathbb{D}}(z_1, z_2)}{2}\,. \tag{2.2}$$

Conjecture 2.3. *The function*

$$\operatorname{artanh} s_{\mathbb{D}}(z_1, z_2)$$

satisfies the triangle inequality.

We have checked this conjecture using the aforementioned formula [7] for $s_{\mathbb{D}}(z_1, z_2)$ based on Theorem 1.5 and found no counterexamples. Experiments also show that for points $0 < r < s < t < 1$ we have the following addition formula

$$\operatorname{artanh} s_{\mathbb{D}}(r, t) = \operatorname{artanh} s_{\mathbb{D}}(r, s) + \operatorname{artanh} s_{\mathbb{D}}(s, t)$$

and this equality statement also follows from formula (2.7) below.

Let $G \subset \mathbb{R}^n$ be a proper open subset of $\mathbb{R}^n$. As in [5], we define the point pair function p_G as follows for $z_1, z_2 \in G$:

$$p_G(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{|z_1 - z_2|^2 + 4\, d_G(z_1)\, d_G(z_2)}}\,,$$

where $d_G(x) = \operatorname{dist}(x, \partial G)$. By [5, Lemma 3.4 (1)] if G is convex and $z_1, z_2 \in G \subset \mathbb{R}^n$, then

$$s_G(z_1, z_2) \le p_G(z_1, z_2)\,. \tag{2.4}$$

Theorem 2.5. *If $z_1, z_2 \in \mathbb{D}$,*

$$s_{\mathbb{D}}(z_1, z_2) \le m_{\mathbb{D}}(z_1, z_2) := \frac{|z_1 - z_2|}{2 - |z_1 + z_2|}. \tag{2.6}$$

Here equality holds if and only if $z_1, 0, z_2$ are collinear.

Proof. Fix $z_1, z_2 \in \mathbb{D}$, and let $u \in \partial\mathbb{D}$. Then by the triangle inequality we have

$$\frac{|z_1 - z_2|}{|z_1 - u| + |z_2 - u|} \le \frac{|z_1 - z_2|}{|2u - (z_1 + z_2)|} \le \frac{|z_1 - z_2|}{\big|2|u| - |z_1 + z_2|\big|} = \frac{|z_1 - z_2|}{2 - |z_1 + z_2|}\,.$$

Hence the inequality follows. The equality statement follows from the equality statement for the triangle inequality. □

Note that the equality statement in (2.6) implies for $0<r<s<1$ that

$$\operatorname{artanh} s_{\mathbb{D}}(r,s)=\frac{1}{2}\log\frac{1-r}{1-s}\,. \tag{2.7}$$

Remark 2.8. The inequalities (2.4) and (2.6) are not comparable. We always have

$$s_{\mathbb{D}}(z_1,z_2)\le p_{\mathbb{D}}(z_1,z_2)\le \tanh\frac{\rho_{\mathbb{D}}(z_1,z_2)}{2}<1\,.$$

Sometimes $p_{\mathbb{D}}(z_1,z_2)>m_{\mathbb{D}}(z_1,z_2)$. On the other hand the function $m_{\mathbb{D}}$ is unbounded. Finally, for $r,t\in(0,1)$ we have $p_{\mathbb{D}}(r,t)=m_{\mathbb{D}}(r,t)$. It is easily seen that $m_{\mathbb{D}}(t,it)>m_{\mathbb{D}}(0,t)+m_{\mathbb{D}}(0,it)$ for $t\in(0.85,1)$ and hence $m_{\mathbb{D}}$ is not a metric.

3. On Barrlund's metric

In this section we will give explicit formulas for the Barrlund metric (1.1) when $p=2$ and the domain is either the unit disk or the upper half plane and study some properties of the Barrlund metric for $1\le p\le\infty$.

3.1. Basic properties of the Barrlund metric.

Suppose that G is a proper subdomain of the complex plane and $p\ge 1$. Because $s_G(z_1,z_2)=b_{G,1}(z_1,z_2)$ for all $z_1,z_2\in G$, it is natural to expect that some properties of s_G might have a counterpart also for $b_{G,p}$, $p>1$. We list a few immediate observations and recall first the notion of midpoint convexity.

Definition 3.2. [4, p.60] A domain $G\subset\mathbb{R}^n$ is *midpoint convex* if for $x,y\in G$ also the midpoint $(x+y)/2\in G$.

Obviously, every convex set is midpoint convex. If a midpoint convex set in $\mathbb{R}^n$ is closed or is open, then the set is convex. In particular, every midpoint convex domain is also convex.

(1) If $\lambda>0, a\in\mathbb{C}$, and $h(z)=\lambda z+a$, then $b_{G,p}$ is invariant under h , i.e. for all $z_1,z_2\in G$,
$$b_{h(G),p}(h(z_1),h(z_2))=b_{G,p}(z_1,z_2)\,.$$
(2) $b_{G,p}$ is monotone with respect to the domain: If G_1 is a midpoint convex subdomain of G and $z_1,z_2\in G_1$, then $b_{G,p}(z_1,z_2)\le b_{G_1,p}(z_1,z_2)$, see Lemma 3.4. In particular, if G is midpoint convex,
$$b_{G,p}(z_1,z_2)\ge\sup\{b_{\mathbb{C}\setminus\{z\},p}(z_1,z_2): z\in\partial G\}\,.$$
(3) $b_{G,p}$ satisfies the triangle inequality, i.e. it is a metric.

Remark 3.3. *Replacing ∂G by $\mathbb{R}^n\setminus G$ in Definition* (1.1) *we obtain a modified Barrlund function that is is monotone with respect to the domain.*

We show here that for $p=2$ and $n=2$ the monotonicity with respect to the domain (3) does not hold for all domains $G_1\subset G\subsetneq\mathbb{R}^n$.

(1) *We first observe that by elementary geometry (Stewart's theorem) for all $x,y,w\in\mathbb{R}^n$*
$$|w-x|^2+|w-y|^2=2|w-\frac{1}{2}\,(x+y)\,|^2+\frac{1}{2}|x-y|^2\,.$$

(2) *The formula in (1) implies that for a domain $D \subsetneq \mathbb{R}^n$ and for $x, y \in D$*

$$b_{D,2}(x,y) = \frac{|x-y|}{\sqrt{2d_D^2(\frac{1}{2}(x+y)) + \frac{1}{2}|x-y|^2}} .$$

(3) *For $a > 0$ let $S_a = \{z \in \mathbb{C} : \mathrm{Re}\,(z), \mathrm{Im}\,(z) \in (-a,a)\}$ be a square and $G = S_4 \setminus \overline{S}_1$ and $G_1 = S_4 \setminus \overline{S}_2$. With $z_1 = 3$, $z_2 = -3$ we have $z_1, z_2 \in G_1 \subset G$, but by part (2)*

$$\frac{6}{\sqrt{26}} = b_{G_1,2}(z_1, z_2) < b_{G,2}(z_1, z_2) = \frac{6}{\sqrt{20}} .$$

Lemma 3.4. *Let $1 \le p \le \infty$. If $G_1 \subset G \subsetneq \mathbb{R}^n$ are domains, such that G_1 is midpoint convex, then $b_{G_1,p}(x,y) \ge b_{G,p}(x,y)$ for all $x, y \in G_1$.*

Proof. Fix $x, y \in G_1$. There exists $a = a(p) \in \partial G$ such that

$$b_{G,p}(x,y) = \frac{|x-y|}{\sqrt[p]{|x-a|^p + |y-a|^p}}$$

if $1 \le p < \infty$, respectively

$$b_{G,\infty}(x,y) = \frac{|x-y|}{\max\{|x-a|, |y-a|\}}.$$

Since G_1 is midpoint convex, G_1 contains $m = \frac{1}{2}(x+y)$. The intersection of the segment $[m,a]$ with the boundary ∂G_1 contains at least one point, which we denote by d.

We prove that

$$\max\{|x-d|, |y-d|\} \le \max\{|x-a|, |y-a|\}$$

and that

$$|x-d|^p + |y-d|^p \le |x-a|^p + |y-a|^p$$

if $1 \le p < \infty$.

Then

$$b_{G_1,\infty}(x,y) \ge \frac{|x-y|}{\max\{|x-d|, |y-d|\}} \ge b_{G,\infty}(x,y)$$

and

$$b_{G_1,p}(x,y) \ge \frac{|x-y|}{\sqrt[p]{|x-d|^p + |y-d|^p}} \ge b_{G,p}(x,y)$$

if $1 \le p < \infty$.

Let $\lambda \in [0,1)$ such that $d = (1-\lambda)a + \lambda m$. For every $z \in \mathbb{R}^n$, $(z-d) = (1-\lambda)(z-a) + \lambda(z-m)$, hence

$$|z-d| \le (1-\lambda)|z-a| + \lambda|z-m| . \tag{3.5}$$

If $p = \infty$, note that (3.5) implies

$$\max\{|x-d|, |y-d|\} \le (1-\lambda)\max\{|x-a|, |y-a|\} + \lambda\max\{|x-m|, |y-m|\} .$$

But

$$|x-m| = |y-m| = \frac{1}{2}|x-y| \le \frac{1}{2}(|x-a| + |y-a|) \le \max\{|x-a|, |y-a|\} . \tag{3.6}$$

Then $\max\{|x-d|, |y-d|\} \le \max\{|x-a|, |y-a|\}$.

If $1 \le p < \infty$, inequality (3.5) and the convexity of the function $t \mapsto t^p$ on $(0,\infty)$ imply $|z-d|^p \le (1-\lambda)\,|z-a|^p + \lambda\,|z-m|^p$. Adding the inequalities for $z = x$ and $z = y$ we obtain

$$|x-d|^p + |y-d|^p \le (1-\lambda)\left(|x-a|^p + |y-a|^p\right) + \lambda\left(|x-m|^p + |y-m|^p\right).$$

Again by convexity, inequality (3.6) implies $|x-m|^p + |y-m|^p \le |x-a|^p + |y-a|^p$. The latter two inequalities yield $|x-d|^p + |y-d|^p \le |x-a|^p + |y-a|^p$. □

Remark 3.7. *In the case $p = 1$ we do not need to assume that G_1 is midpoint convex. Let c be a point belonging to the intersection $[x,a] \cap \partial G_1$. Then $|x-a| = |x-c| + |c-a|$, hence $|x-a| + |y-a| \ge |x-c| + |y-c|$, by the triangle inequality. Then*

$$s_{G_1}(x,y) \ge \frac{|x-y|}{|x-c|+|y-c|} \ge \frac{|x-y|}{|x-a|+|y-a|} = s_G(x,y).$$

Proposition 3.8. *The Barrlund distance satisfies the triangle inequality.*

Proof. The proof follows from a more general argument in [12, Lemma 6.1], but for the reader's convenience, we give a short argument here. Denote $b_p = b_{\mathbb{R}^n\setminus\{0\},p}$. Let $x, y, z \in G$. Because b_p is a metric by [2], for $u \in \partial G$,

$$b_p(x-u, y-u) \le b_p(x-u, z-u) + b_p(z-u, y-u) \le b_{G,p}(x,z) + b_{G,p}(z,y),$$

hence

$$b_p(x-u, y-u) \le b_{G,p}(x,z) + b_{G,p}(z,y).$$

Taking the supremum over $u \in \partial G$, it follows that

$$b_{G,p}(x,y) \le b_{G,p}(x,z) + b_{G,p}(z,y).$$ □

Theorem 3.9. *The Barrlund metric is monotone with respect to the parameter p: given a domain $G \subsetneq \mathbb{R}^n$, for $z_1,\ z_2 \in G$ and $p > r \ge 1$,*

$$b_{G,r}(z_1,z_2) \le b_{G,p}(z_1,z_2) \le 2^{\frac{1}{r}-\frac{1}{p}}\, b_{G,r}(z_1,z_2). \tag{3.10}$$

In particular,

$$s_G(z_1,z_2) \le b_{G,p}(z_1,z_2) \le 2^{1-1/p} s_G(z_1,z_2). \tag{3.11}$$

Moreover, if $n = 2$, then

$$\sup\{b_{G,p}(z_1,z_2) : z_1, z_2 \in G\} = 2^{1-1/p}.$$

Proof. The functions $p \mapsto ((a^p+b^p)/2)^{1/p}$ and $p \mapsto (a^p+b^p)^{1/p}$ are increasing and decreasing, respectively, on $(1,\infty)$ for fixed $a, b > 0$. The monotonicity and (3.10) follow from these basic facts and (3.11) is the special case $r = 1$ of (3.10). For the proof of the last statement fix $x \in G$ and $z \in \partial G$ with $d(x) = d(x,\partial G) = |x-z|$ and denote $w = (x+z)/2$. Then for $\alpha \in (0,\pi/6)$ choose points u_α, v_α with

$$|u_\alpha - w| = |v_\alpha - w| = d(x)/2, \quad |u_\alpha - v_\alpha| = 2d(x)\sin\alpha\cos\alpha,$$
$$|x-u_\alpha| = |x-v_\alpha| = d(x)\cos\alpha, \quad |z-u_\alpha| = |z-v_\alpha| = d(x)\sin\alpha.$$

Applying the definition (1.1) to the triple u_α, v_α, z we have

$$b_{G,p}(u_\alpha, v_\alpha) \ge \frac{2d(x)\sin\alpha\cos\alpha}{d(x)\sqrt[p]{\sin^p\alpha + \sin^p\alpha}} = 2^{1-1/p}\cos\alpha \to 2^{1-1/p},$$

when $\alpha \to 0$. This convergence together with (3.11) proves the claim. $\square$

Remark 3.12.

(1) The supremum in Theorem 3.9 is attained for some domains, as shown below.

Let $p \geq 1$. Let $G = \mathbb{D}\setminus\{0\}$, $t \in (0,1)$ and $z_1 = t$, $z_2 = -t$. For every $z \in \partial\mathbb{D}$, $|z_1 - z|^p + |z - z_2|^p \geq 2^{1-p}(|z_1 - z| + |z - z_2|)^p \geq 2^{1-p}|z_1 - z_2|^p$ and both inequalities hold as equalities for $z = 0$, hence $b_{\mathbb{D},p}(z_1, z_2) = 2^{1-\frac{1}{p}}$. The same argument shows that this holds in a more general case: if G is a proper subdomain of $\mathbb{R}^n$ and there exist $z_1, z_2 \in G$, $z_0 \in \partial G$ such that $z_0 = (z_1 + z_2)/2$, then $b_{G,p}(z_1, z_2) = 2^{1-\frac{1}{p}}$. It follows that

$$\sup\left\{\sup_{z_1,z_2\in G} b_{G,p}(z_1,z_2) : G \subsetneq \mathbb{R}^n \text{ is a domain}\right\} = 2^{1-\frac{1}{p}}.$$

(2) We will see below in Theorem 3.38 that the second inequality in (3.11) holds as equality for all $p \geq 1$ if $G = \mathbb{H}$, $z_1, z_2 \in \mathbb{H}$ with $\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)$.

Several upper and lower bounds for s_G are given in [11]. Using these bounds and Theorem 3.9 one could find bounds also for the Barrlund metric.

3.13. The proof of Theorem 1.3. The proof follows from Theorem 3.9. $\square$

We will next study a few problems which lead us to a formula for the Barrlund metric when the domain is either the disk or the half-plane.

Problem A. For given $z_1, z_2 \in \mathbb{D}$, find the contact points and the corresponding parameter value $c > 0$ of "power p ellipses" $\{|z_1 - u|^p + |z_2 - u|^p = c^p\}$ and the unit circle.

This Problem A is closely related to the following Problems A'.

Problem A'. For $z_1, z_2 \in \mathbb{D}$ and $p \geq 1$, find the points u on the unit circle $\partial\mathbb{D}$ such that $\sqrt[p]{|z_1 - u|^p + |z_2 - u|^p}$ is minimal.

Lemma 3.14. *Any point u in Problem A' is given as a solution of*

$$(3.15)\quad \Big((z_1\overline{z_1}+1)u - \overline{z_1}u^2 - z_1\Big)^{\frac{p}{2}-1}(\overline{z_1}u^2 - z_1) + \Big((z_2\overline{z_2}+1)u - \overline{z_2}u^2 - z_2\Big)^{\frac{p}{2}-1}(\overline{z_2}u^2 - z_2) = 0\,,$$

where we consider the principal branch of the complex power function.

Proof. We need to find the point u on $\partial\mathbb{D}$ such that $|z_1 - u|^p + |z_2 - u|^p$ is minimal. Let

$$G(\theta) = \Big((z_1 - e^{i\theta})(\overline{z_1} - e^{-i\theta})\Big)^{\frac{p}{2}} + \Big((z_2 - e^{i\theta})(\overline{z_2} - e^{-i\theta})\Big)^{\frac{p}{2}}.$$

We remark that G is a real-valued periodic function that is differentiable on the real line. Therefore, $G(\theta)$ attains a global minimum at one point, which has to be a critical point of G. For $G'(\theta) = 0$, setting $u = e^{i\theta}$, we obtain (3.15). $\square$

The above equation (3.15) is no longer an algebraic equation for a general real number $p > 1$.

Next we give a counterpart of the above lemma for the upper half space.

Lemma 3.16. *Let $z_1, z_2 \in \mathbb{H}$ and $p \geq 1$. The function $S_p : \mathbb{R} \to \mathbb{R}$ defined by $S_p(t) = |t - z_1|^p + |t - z_2|^p$ has a unique minimum point t_0. If $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$, then $t_0 = \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$, otherwise $\min\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\} < t_0 < \max\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\}$ and $t = t_0$ is the unique real solution of the equation.*

$$(t - \mathrm{Re}\,(z_1))\,|t - z_1|^{p-2} = (\mathrm{Re}\,(z_2) - t)\,|t - z_2|^{p-2}\,. \tag{3.17}$$

Proof. For every $t \in \mathbb{R}$ we have

$$S_p'(t) = p\sum_{k=1}^{2}\big(t - \mathrm{Re}\,(z_k)\big)\,|t - z_k|^{p-2}$$

and

$$S_p''(t) = p\sum_{k=1}^{2}\Big[|t - z_k|^{p-2} + (p-2)\big(t - \mathrm{Re}\,(z_k)\big)^2\,|t - z_k|^{p-4}\Big].$$

Since $S_p''(t) > 0$ for every $t \in \mathbb{R}$, the derivative S_p' is increasing on $\mathbb{R}$. Then S_p is strictly convex on $\mathbb{R}$, hence, as $\lim_{t\to\pm\infty} S_p(t) = +\infty$, it follows that S_p has a unique minimum point [18, Theorems 3.4.4 and 3.4.5].

Note that $a < \min\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\}$ implies $S_p'(a) < 0$, while $b > \max\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\}$ implies $S_p'(b) > 0$. Then the derivative S_p' has a unique zero t_0, which is the unique minimum point of S_f. It follows that

$$b_{\mathbb{H},p}\,(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt[p]{|t_0 - z_1|^p + |t_0 - z_2|^p}}.$$

Case 1. $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$

The derivative $S_p'(t) = \big(t - \mathrm{Re}\,(z_1)\big)\big(|t - z_1|^{p-2} + |t - z_2|^{p-2}\big)$, $t \in \mathbb{R}$ has the unique zero $t_0 = \mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$. Then

$$b_{\mathbb{H},p}\,(z_1, z_2) = \frac{|\mathrm{Im}\ (z_1) - \mathrm{Im}\ (z_2)|}{\sqrt[p]{\mathrm{Im}\ (z_1)^p + \mathrm{Im}\ (z_2)^p}}.$$

Case 2. $\mathrm{Re}\,(z_1) \neq \mathrm{Re}\,(z_2)$.

In this case,

$$\min\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\} < t_0 < \max\{\mathrm{Re}\,(z_1), \mathrm{Re}\,(z_2)\}\,.$$

Here t_0 is the unique real solution of the equation

$$(t - \mathrm{Re}\,(z_1))\,|t - z_1|^{p-2} = (\mathrm{Re}\,(z_2) - t)\,|t - z_2|^{p-2}\,. \tag{3.18}$$

In the following we will assume that $\mathrm{Re}\,(z_1) < \mathrm{Re}\,(z_2)$, the case $\mathrm{Re}\,(z_2) < \mathrm{Re}\,(z_1)$ being analogous. For every $t \in \mathbb{R}$ there exists a unique $\lambda = \lambda(t) \in \mathbb{R}$ such that $t = (1-\lambda)\,\mathrm{Re}\ (z_1) + \lambda\mathrm{Re}\ (z_2)$, and $\mathrm{Re}\,(z_1) < t < \mathrm{Re}\,(z_2)$ if and only if $0 < \lambda(t) < 1$. Then $\lambda = \lambda_0 := \lambda(t_0)$ is the unique solution of the equation

$$\lambda\,|\lambda\mathrm{Re}\ (z_2 - z_1) - i\mathrm{Im}\,(z_1)|^{p-2} = (1-\lambda)\,|(1-\lambda)\,\mathrm{Re}\ (z_2 - z_1) + i\mathrm{Im}\,(z_2)|^{p-2}\,. \tag{3.19}$$ □

Remark 3.20. For $p = 2$ we have $S_p'(t) = 4t - 2\mathrm{Re}\ (z_1 + z_2)$ hence $t_0 = \frac{1}{2}\mathrm{Re}\ (z_1 + z_2)$ and we obtain an alternative proof of Theorem 3.24.

In the general case, we can use (3.19) for numerical computation of λ_0.

3.21. Barrlund's metric for $p=1$.

3.21.1. **The domain $G=\mathbb{H}$.** The upper half space $\{z\in\mathbb{C} : \mathrm{Im}(z)>0\}$ is denoted by $\mathbb{H}$. Recall that the hyperbolic metric in $\mathbb{H}$ is defined by the formula [3, Thm 7.2.1, p. 130]

$$\cosh\rho_{\mathbb{H}}(z_1,z_2)=1+\frac{|z_1-z_2|^2}{2\mathrm{Im}(z_1)\mathrm{Im}(z_2)}\,,\qquad z_1,z_2\in\mathbb{H}\,.$$

Equivalently [3, Thm 7.2.1, p. 130],

$$\tanh\left(\frac{\rho_{\mathbb{H}}(z_1,z_2)}{2}\right)=\frac{|z_1-z_2|}{|z_1-\overline{z_2}|}\,.$$

In the case $p=1$, (3.17) in Lemma 3.16 is equivalent to

$$\frac{\mathrm{Re}\,(t-z_1)}{|t-z_1|}=\frac{\mathrm{Re}\,(z_2-t)}{|z_2-t|}.$$

Assume that $\mathrm{Re}\,(z_1)<\mathrm{Re}\,(z_2)$. The above equality holds for $t=(1-\lambda)\mathrm{Re}\,(z_1)+\lambda\mathrm{Re}\,(z_2)$, $\lambda\in(0,1)$, if and only if the triangles $\Delta(z_1,t,\mathrm{Re}\,(z_1))$ and $\Delta(z_2,t,\mathrm{Re}\,(z_2))$ are similar, that is, if and only if

$$\frac{\mathrm{Re}\,(t-z_1)}{\mathrm{Re}\,(z_2-t)}=\frac{\mathrm{Im}\,(z_1)}{\mathrm{Im}\,(z_2)}=\frac{|t-z_1|}{|z_2-t|}=\frac{\lambda}{1-\lambda}.$$

For $p=1$ we get $\lambda_0=\mathrm{Im}\,(z_1)/(\mathrm{Im}\,(z_1)+\mathrm{Im}\,(z_2))$, hence

$$|t_0-z_1|=\lambda_0\,|z_1-\overline{z_2}|\qquad\text{and}\qquad |t_0-z_2|=(1-\lambda_0)\,|z_1-\overline{z_2}|$$

hence we recover the formula

$$s_{\mathbb{H}}(z_1,z_2)=b_{\mathbb{H},1}(z_1,z_2)=\frac{|z_1-z_2|}{|z_1-\overline{z_2}|}\,.$$

3.21.2. **The domain $G=\mathbb{D}$.**

Remark 3.22. *Substituting $p=1$ into* (3.15) *and canceling the denominators, we have*

$$(\overline{z_1}u^2-z_1)\sqrt{(z_2\overline{z_2}+1)u-\overline{z_2}u^2-z_2}=-(\overline{z_2}u^2-z_2)\sqrt{(z_1\overline{z_1}+1)u-\overline{z_1}u^2-z_1}\,.$$

Squaring the both sides and factorizing, we have

$$F\cdot\Big((\overline{z_1}-\overline{z_2})u^2-(\overline{z_1}z_2-z_1\overline{z_2})u+z_2-z_1\Big)=0\,.$$

The factor F coincides with the left hand side of the quartic equation (1.6)*, and one of the roots gives the minimum.*

3.23. Barrlund's metric for $p=2$.

The power 2 ellipse is a circle. In fact, an equation of a power 2 ellipse $|z_1-w|^2+|z_2-w|^2=r^2$, $r>\frac{|z_1-z_2|}{2}$ is expressed as $|2w-(z_1+z_2)|=\sqrt{2r^2-|z_1-z_2|^2}\,$.

3.23.1. **The domain $G = \mathbb{H}$.**

Theorem 3.24. *For $z_1, z_2 \in \mathbb{H}$ we have*

$$b_{\mathbb{H},2}(z_1,z_2) = \frac{\sqrt{2}|z_1-z_2|}{\sqrt{|z_1-z_2|^2+|\mathrm{Im}(z_1+z_2)|^2}} = \frac{|z_1-z_2|}{\sqrt{|z_1-m|^2+|z_2-m|^2}}\,,$$

where $m = \mathrm{Re}(z_1+z_2)/2$.

Proof. Fix $z_1, z_2 \in \mathbb{H}$ and write $z = (z_1+z_2)/2$. We will find

$$\min\{(|z_1-u|^2+|z_2-u|^2) : u \in \partial\mathbb{H}\}.$$

By Remark 3.3 (1),

$$|u-z_1|^2+|u-z_2|^2 = 2|u-z|^2+\frac{1}{2}|z_1-z_2|^2\,.$$

Then $|u-z_1|^2+|u-z_2|^2$ attains its minimum if and only if $|u-z|$ does, i.e. if and only if $u = m = \mathrm{Re}(z_1+z_2)/2$. In conclusion,

$$\min\{(|z_1-u|^2+|z_2-u|^2) : u \in \partial\mathbb{H}\} = \frac{1}{2}(|z_1-z_2|^2+|\mathrm{Im}(z_1+z_2)|^2)$$

and the desired formula follows. □

Remark 3.25. By the definition of $s_{\mathbb{H}}$, for $z_1, z_2 \in \mathbb{H}$

$$s_{\mathbb{H}}(z_1,z_2) = \frac{|z_1-z_2|}{|z_1-z|+|z_2-z|} = \frac{|z_1-z_2|}{|z_1-\overline{z_2}|} = \tanh\frac{\rho_{\mathbb{H}}(z_1,z_2)}{2}$$

where $\{z\} = [z_1, \overline{z_2}] \cap \mathbb{R}$ [15, Prop. 4.2].

We have by Theorem 3.9

$$s_{\mathbb{H}}(z_1,z_2) \le b_{\mathbb{H},2}(z_1,z_2) \le \sqrt{2}s_{\mathbb{H}}(z_1,z_2) = \sqrt{2}\tanh\frac{\rho_{\mathbb{H}}(z_1,z_2)}{2} = \sqrt{2}p_{\mathbb{H}}(z_1,z_2)$$

(see also [12, Remark 6.2]).

Moreover, $b_{\mathbb{H},2}(z_1,z_2) = \sqrt{2}s_{\mathbb{H}}(z_1,z_2)$ if and only if $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

3.23.2. **The domain $G = \mathbb{D}$.**

Remark 3.26. *Substituting $p = 2$ into (3.15), we have*

$$(\overline{z_1}u^2 - z_1) + (\overline{z_2}u^2 - z_2) = (\overline{z_1+z_2})u^2 - (z_1+z_2) = 0\,,$$

and $u = \pm\dfrac{z_1+z_2}{|z_1+z_2|}$. Clearly, $u = \dfrac{z_1+z_2}{|z_1+z_2|}$ gives the minimum.

Theorem 3.27. *For $z_1, z_2 \in \mathbb{D}$*

$$b_{\mathbb{D},2}(z_1,z_2) = \frac{|z_1-z_2|}{\sqrt{2+|z_1|^2+|z_2|^2-2|z_1+z_2|}}\,. \tag{3.28}$$

In particular, $\lim_{(0,1)\ni r\to 1} b_{\mathbb{D},2}(r,t) = 1$ for $t \in (-1,1)$.

Proof.

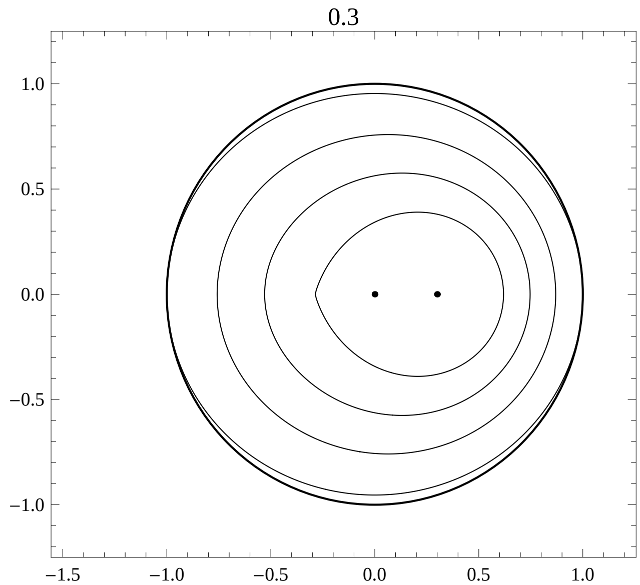

FIGURE 1. Level sets $\{x+iy : b_{\mathbb{D},2}(0.3, x+iy) = c\}$ for $c = 0.4, 0.6, 0.8, 1.0$ and the unit circle. Note that for $c = 1.0$ the level set meets the points $(\pm 1, 0)$ in accordance with Theorem 3.27.

Case 1. $z_1 + z_2 \neq 0$.

Writing $u = (z_1+z_2)/|z_1+z_2|$ we see that $\overline{u}(z_1+z_2) = |z_1+z_2|$ and

$$\begin{aligned} |z_1-u|^2 + |z_2-u|^2 &= 2 + |z_1|^2 + |z_2|^2 - u(\overline{z_1}+\overline{z_2}) - \overline{u}(z_1+z_2) \\ &= 2 + |z_1|^2 + |z_2|^2 - 2|z_1+z_2|\,. \end{aligned}$$

Applying Remark 3.26 and substituting into

$$b_{\mathbb{D},2}(z_1,z_2) = \frac{|z_1-z_2|}{\sqrt{|z_1-u|^2+|z_2-u|^2}}$$

yields the desired formula.

Case 2. $z_1 + z_2 = 0$.

For every $z \in \partial\mathbb{D}$, the segment joining z to 0 is a median in the triangle $\Delta(z, z_1, z_2)$, therefore

$$|z-z_1|^2 + |z-z_2|^2 = 2 + \frac{1}{2}|z_1-z_2|^2\,.$$

Then $b_{\mathbb{D},2}(z_1,z_2) = |z_1-z_2|/\sqrt{2+\frac{1}{2}|z_1-z_2|^2}$, and

$$\frac{1}{2}|z_1-z_2|^2\Big|_{z_2=-z_1} = \left(|z_1|^2+|z_2|^2 - 2|z_1+z_2|\right)\Big|_{z_2=-z_1} = 2|z_1|^2\,,$$

therefore (3.28) holds. □

Let $B_{\mathbb{D},2}(a;c) = \{z \in \mathbb{D} : b_{\mathbb{D},2}(a,z) < c\}$.

Theorem 3.29. *Let a and r be numbers satisfying $b_{\mathbb{D},2}(a,a+r)=c$ and $0<a<a+r<1$. Then*

$$\{|z-a|<r\}\subset B_{\mathbb{D},2}(a;c)\subset\{|z|<a+r\}.$$

Proof. We will prove that the inequalities $b_{\mathbb{D},2}(a,a+re^{i\theta})\le b_{\mathbb{D},2}(a,a+r)\le b_{\mathbb{D},2}(a,(a+r)e^{i\theta})$ hold for all $\theta\in\mathbb{R}$.

Observe that $b_{\mathbb{D},2}(w,z)=\frac{|w-z|}{\sqrt{2+|w|^2+|z|^2-2|w+z|}}$ holds for $w,z\in\mathbb{D}$, by Theorem 3.27.

At first, we will show $\big(b_{\mathbb{D},2}(a,a+r)\big)^2\le\big(b_{\mathbb{D},2}(a,(a+r)e^{i\theta})\big)^2$. Let

$$u(\theta)=\left|a-(a+r)e^{i\theta}\right|^2\big(2+a^2+(a+r)^2-2(2a+r)\big)-r^2\big(2+a^2+(a+r)^2-2|a+(a+r)e^{i\theta}|\big).$$

Then, u can also be written as

$$\begin{aligned}u(\theta)=&2r^2\sqrt{2(ar+a^2)\cos\theta+(r^2+2ar+2a^2)}\\&+2((a-1)r^3+(3a^2-4a)r^2+(4a^3-6a^2+2a)r+2a^4-4a^3+2a^2)\\&-2a(r+a)((1-r-a)^2+(a-1)^2)\cos\theta.\end{aligned}$$

Set $t=\cos\theta$ and $u(\theta)=\tilde{u}(t)$. Here we need to show $\tilde{u}(t)\ge 0$ holds for $-1\le t\le 1$.

The function $\tilde{u}(t)$ has the unique critical point t_0 and attains the maximum at the point. Moreover, we have $\tilde{u}(1)=0$ and $\tilde{u}(-1)=4a(1-r-a)\big(r(2-2a-r)+2a(1-a)\big)>0$. Therefore, $b_{\mathbb{D},2}(a,a+r)\le b_{\mathbb{D},2}(a,(a+r)e^{i\theta})$ holds for for all $\theta\in\mathbb{R}$.

The inequality

$$b_{\mathbb{D},2}(a,a+re^{i\theta})\le b_{\mathbb{D},2}(a,a+r)\,,$$

which holds by the proof of Theorem 3.34, completes the proof. □

It follows from (2.2) that the closures of $s_{\mathbb{D}}$-disks centered at some point $z_0\in\mathbb{D}$ are compact subsets of $\mathbb{D}$. Looking at Figure 1 we notice a topological difference: the $b_{\mathbb{D},2}$-disks centered at some point $(a,0)$, $a\in(-1,1)$, with radius 1 touch the boundary $\partial\mathbb{D}$ at the points $(\pm1,0)$. Moreover, it follows from (3.28) of Theorem 3.27 that $b_{\mathbb{D},2}$-disk $B_{\mathbb{D},2}(a;1)$ forms the elliptic disk $\left\{x+iy\,:\,x^2+\frac{y^2}{1-a^2}\le 1\right\}$.

Theorem 3.30. *Let a and r be numbers satisfying $b_{\mathbb{D},2}(a,a+r)=c$ and $0<a<a+r<1$. Then*

$$B_{\mathbb{D},2}(a;c)\subset\{|z-a|<R\}\cap\mathbb{D},$$

where R is the number satisfying $b_{\mathbb{D},2}(a,a-R)=c$ and $-1<a-R<a$.

Proof. We will show that $b_{\mathbb{D},2}(a,a+r)\le b_{\mathbb{D},2}(a,a-Re^{i\theta})$ holds for all $\theta\in\mathbb{R}$.

As the value R satisfies $b_{\mathbb{D},2}(a,a+r)=b_{\mathbb{D},2}(a,a-R)$, the equality

$$\frac{r}{\sqrt{2+a^2+(a+r)^2-2(2a+r)}}=\frac{R}{\sqrt{2+a^2+(a-R)^2-2|2a-R|}}$$

follows from Theorem 3.27. Squaring the both sides,

$$r^2\big(2+a^2+(a-R)^2-2|2a-R|\big)=R^2\big(2+a^2+(a+r)^2-2(2a+r)\big).\tag{3.31}$$

Solving the equation (3.31) for R, we have

$$R = \begin{cases} \frac{r(1-a)}{1-a-r} & \text{if } 2a - R \geq 0 \text{ (i.e. } 2a(1-a) - r(1+a) \geq 0), \\ \frac{r(1+a)}{1-a} & \text{if } 2a - R < 0 \text{ (i.e. } 2a(1-a) - r(1+a) < 0). \end{cases}$$

Here we consider the function

$$v(\theta) = |a + Re^{i\theta}|^2 - 2|2a + Re^{i\theta}| = a^2 + 2aR\cos\theta + R^2 - 2\sqrt{4a^2 + 4aR\cos\theta + R^2}.$$

Set $t = \cos\theta$ and $v(\theta) = \widetilde{v}(t)$. Then, $\widetilde{v}(t)$ is convex downward in $-1 \leq t \leq 1$ since $\widetilde{v}$ has the unique critical point and attains the minimum at the point.

At first, we will show that $\big(b_{\mathbb{D},2}(a, a+r)\big)^2 \leq \big(b_{\mathbb{D},2}(a, a+Re^{i\theta})\big)^2$ holds for $R = \frac{r(1-a)}{1-a-r}$ and $2a - R > 0$. Let $\widetilde{u}_1(t) = R^2\big(2 + a^2 + (a+r)^2 - 2(2a+r)\big) - r^2\big(2 + a^2 + \widetilde{v}(t)\big)$. Then, $\widetilde{u}_1$ is concave in $-1 \leq t \leq 1$, and satisfies $\tilde{u}_1(1) = \frac{4r^3(1-a)^2}{1-r-a} > 0$ and $\tilde{u}_1(-1) = 0$. Therefore, $\tilde{u}_1(t) \geq 0$ holds for $-1 \leq t \leq 1$ and the assertion is obtained for this case.

Next, similarly, for $R = \frac{r(1+a)}{1-a}$ and $2a - R < 0$, we have $\tilde{u}_1(1) = \frac{4ar^2}{1-a}\big((1-a-r)(1+a) + (1-a)^2\big) > 0$ and $\tilde{u}_1(-1) = 0$. Therefore, $\big(b_{\mathbb{D},2}(a, a+r)\big)^2 \leq \big(b_{\mathbb{D},2}(a, a+Re^{i\theta})\big)^2$ also holds for this case.

From the above arguments the assertion of the theorem is obtained. □

Remark 3.32. *The disk $D(0, a+r) = \{|z| < a+r\}$ in Theorem 3.29 always satisfies $D(0, a+r) \subset \mathbb{D}$, but the disk $D(a, R) = \{|z-a| < R\}$ in Theorem 3.30 may intersect the unit circle. So, there is no inclusion relation between these two disks (see Figure 2).*

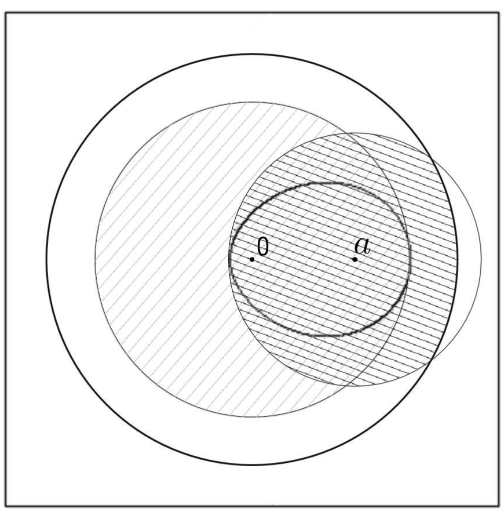

FIGURE 2. The oval in the figure is the boundary of $B_{\mathbb{D},2}(a; 0.5)$ with $a = 0.5$. The disk with center the origin indicates the upper bound in Theorem 3.29. The shaded region corresponds to Theorem 3.30.

3.33. Inequalities of Barrlund's metric for $p \in (1, \infty)$ **.**

Let $B_{\mathbb{D},p}(a; c) = \{z \in \mathbb{D} : b_{\mathbb{D},p}(a, z) < c\}$.

Theorem 3.34. *The following holds for* $p > 1 > a > 0$,

$$\{|z-a| < r\} \subset B_{\mathbb{D},p}(a;c),$$

where r *is a number satisfying* $b_{\mathbb{D},p}(a, a+r) = c$ *and* $0 < a < a+r < 1$.

Proof. We will show the inequality $b_{\mathbb{D},p}(a, a+re^{i\theta}) \le b_{\mathbb{D},p}(a, a+r)$, that is, we will show that

$$\inf_{z\in\partial\mathbb{D}} \big(|a-z|^p + |a+r-z|^p\big) \le \inf_{w\in\partial\mathbb{D}} \big(|a-w|^p + |a+re^{i\theta}-w|^p\big) \tag{3.35}$$

holds for all $\theta \in \mathbb{R}$.

The function $|a-z|^p + |a+r-z|^p$ on the left hand side of (3.35) attains its minimum at $z = 1$ because $0 \le a < a+r \le 1$. Therefore, we see that

$$\inf_{z\in\partial\mathbb{D}} \big(|a-z|^p + |a+r-z|^p\big) = (1-a)^p + \big(1-(a+r)\big)^p. \tag{3.36}$$

Since the distance between the point $a+re^{i\theta}$ and the unit circle is $d_{\mathbb{D}}(a+re^{i\theta}) = 1-|a+re^{i\theta}|$, we have

$$\begin{aligned}\inf_{w\in\partial\mathbb{D}} \big(|a-w|^p + |a+re^{i\theta}-w|^p\big) &\ge \inf_{u\in\partial\mathbb{D}} |a-u|^p + \inf_{v\in\partial\mathbb{D}} |a+re^{i\theta}-v|^p \\ &= (1-a)^p + \big(1-|a+re^{i\theta}|\big)^p.\end{aligned}$$

Here, $\big(1-|a+re^{i\theta}|\big)^p \ge \big(1-(a+r)\big)^p$ holds as $|a+re^{i\theta}| \le a+r$ ($\forall\theta\in\mathbb{R}$). Hence, we have

$$\inf_{w\in\partial\mathbb{D}} \big(|a-w|^p + |a+re^{i\theta}-w|^p\big) \ge (1-a)^p + \big(1-(a+r)\big)^p = \inf_{z\in\partial\mathbb{D}} \big(|a-z|^p + |a+r-z|^p\big),$$

and the assertion is obtained. □

Lemma 3.37. *For* $z_1, z_2 \in \mathbb{D}\setminus\{0\}$, $z_1 \ne z_2$, *and* $p \ge 1$ *we have* $b_{\mathbb{D},p}(z_1,z_2) < b_{\mathbb{C}\setminus\overline{\mathbb{D}},p}\Big(\dfrac{1}{z_1}, \dfrac{1}{z_2}\Big)$. *In particular,* $s_{\mathbb{D}}(z_1,z_2) < s_{\mathbb{C}\setminus\overline{\mathbb{D}}}\Big(\dfrac{1}{z_1}, \dfrac{1}{z_2}\Big)$, *also holds (the case of* $p=1$*).*

Proof. At first, we observe that

$$\begin{aligned} b_{\mathbb{C}\setminus\overline{\mathbb{D}},p}\Big(\frac{1}{z_1}, \frac{1}{z_2}\Big) &= \sup_{w\in\partial\mathbb{D}} \frac{\big|\frac{1}{z_1} - \frac{1}{z_2}\big|}{\sqrt[p]{\big|\frac{1}{z_1} - w\big|^p + \big|w - \frac{1}{z_2}\big|^p}} \\ &= \sup_{w\in\partial\mathbb{D}} \frac{|z_1-z_2|}{\sqrt[p]{|z_2|^p|1-wz_1|^p + |z_1|^p|1-wz_2|^p}}. \end{aligned}$$

Suppose that the functions

$$w \mapsto \sqrt[p]{|z_1-w|^p + |w-z_2|^p} \qquad \text{and} \qquad w \mapsto \sqrt[p]{|z_2|^p|1-wz_1|^p + |z_1|^p|1-wz_2|^p}$$

defined on $\partial\mathbb{D}$ attain their minima at $u \in \partial\mathbb{D}$ and $v \in \partial\mathbb{D}$, respectively.

Therefore, we have $b_{\mathbb{D},p}(z_1,z_2) = \dfrac{|z_1-z_2|}{\sqrt[p]{|z_1-u|^p + |u-z_2|^p}}$ and

$$b_{\mathbb{C}\setminus\overline{\mathbb{D}},p}\Big(\frac{1}{z_1}, \frac{1}{z_2}\Big) = \frac{|z_1-z_2|}{\sqrt[p]{|z_2|^p|1-vz_1|^p + |z_1|^p|1-vz_2|^p}}.$$

Then, for $z_1, z_2 \in \mathbb{D}$, we have

$$\begin{aligned}|z_2|^p|1-vz_1|^p+|z_1|^p|1-vz_2|^p &\le |z_2|^p|1-\overline{u}z_1|^p+|z_1|^p|1-\overline{u}z_2|^p\\ &=|z_2|^p|u-z_1|^p+|z_1|^p|u-z_2|^p<|u-z_1|^p+|u-z_2|^p\,.\end{aligned}$$

The first inequality holds from the assumption that the denominator attains minima at v, and the second equality holds from $u\overline{u}=1$. Hence,

$$\frac{|z_1-z_2|}{\sqrt[p]{|u-z_1|^p+|u-z_2|^p}}<\frac{|z_1-z_2|}{\sqrt[p]{|z_2|^p|1-vz_1|^p+|z_1|^p|1-vz_2|^p}}$$

holds, and the assertion is obtained. □

We give next a lower bound for $b_{\mathbb{H},p},\ p\ge 1$.

Theorem 3.38. *For $z_1, z_2 \in \mathbb{H}$ and $p \ge 1$ let*

$$T_p(z_1,z_2)=\frac{|z_1-z_2|}{|z_1-\overline{z}_2|\sqrt[p]{\alpha^p+(1-\alpha)^p}}\,,\quad \alpha=\frac{\mathrm{Im}(z_1)}{\mathrm{Im}(z_1)+\mathrm{Im}(z_2)}\,.$$

Then

$$b_{\mathbb{H},p}(z_1,z_2)\ge T_p(z_1,z_2)\ge\frac{|z_1-z_2|}{|z_1-\overline{z}_2|}=s_{\mathbb{H}}(z_1,z_2)\,. \tag{3.39}$$

In particular, $b_{\mathbb{H},1}(z_1,z_2)=T_1(z_1,z_2)=s_{\mathbb{H}}(z_1,z_2)$. For $p>1$ the first inequality (3.39) *holds as an equality if and only if* $\mathrm{Re}\,(z_1)=\mathrm{Re}\,(z_2)$ *or* $\mathrm{Im}\,(z_1)=\mathrm{Im}\,(z_2)$.

Proof. Fix $z_1, z_2 \in \mathbb{H}$ and let $\{w\}=[z_1,\overline{z}_2]\cap\mathbb{R}$. By geometry $\dfrac{|z_1-w|}{|z_1-\overline{z}_2|}=\alpha$ and hence $|z_1-w|=\alpha|z_1-\overline{z}_2|$. By the definition,

$$b_{\mathbb{H},p}(z_1,z_2)\ge\frac{|z_1-z_2|}{\sqrt[p]{|z_1-w|^p+|z_2-w|^p}}=\frac{|z_1-z_2|}{|z_1-\overline{z}_2|\sqrt[p]{\alpha^p+(1-\alpha)^p}}\,.$$

Now we consider the equality cases.

Fix $p>1$. The equality $b_{\mathbb{H},p}(z_1,z_2)=T_p(z_1,z_2)$ is equivalent to

$$\frac{|z_1-z_2|}{\sqrt[p]{|z_1-w|^p+|z_2-w|^p}}=\frac{|z_1-z_2|}{\min\limits_{z\in\partial\mathbb{H}}\sqrt[p]{|z_1-z|^p+|z_2-z|^p}}.$$

Assume that $z_1\ne z_2$. Then the above equality holds if and only if

$$|z_1-z|^p+|z_2-z|^p\ge|z_1-w|^p+|z_2-w|^p\ \text{ for every } z\in\partial\mathbb{H}\,. \tag{3.40}$$

<u>*Sufficiency*</u> By Hölder's inequality, $|z_1-z|^p+|z_2-z|^p\ge 2^{1-p}\left(|z_1-z|+|z_2-z|\right)^p$. By the definition of w, we have

$$\min_{\zeta\in\partial\mathbb{H}}\left(|z_1-\zeta|+|z_2-\zeta|\right)=|z_1-w|+|z_2-w|\,,$$

hence

$$|z_1-z|^p+|z_2-z|^p\ge 2^{1-p}\left(|z_1-w|+|z_2-w|\right)^p\ \text{ for every } z\in\partial\mathbb{H}\,.$$

Case 1. Assume that $\mathrm{Im}\,(z_1)=\mathrm{Im}\,(z_2)$. Then $\alpha=\frac{1}{2}$ and $|z_1-w|=|z_2-w|=\frac{1}{2}\,|z_1-\overline{z_2}|$, therefore

$$2^{1-p}\left(|z_1-w|+|z_2-w|\right)^p=2\,|z_1-w|^p=|z_1-w|^p+|z_2-w|^p\,.$$

It follows that (3.40) holds.
Case 2. Assume that Re $(z_1) =$ Re (z_2). Then $w =$ Re $(z_1) =$ Re (z_2). For every $z \in \partial\mathbb{H}$ we have

$$|z_k - z| = \sqrt{\mathrm{Re}^2(z_k - z) + \mathrm{Im}^2(z_k)} \geq |\mathrm{Im}(z_k)| = |z_k - w|$$

for $k = 1, 2$, therefore, (3.40) holds.
Necessity Denote Re $(z_k) = x_k$ for $k = 1, 2$. Then $w = (1-\alpha)x_1 + \alpha x_2$.

Let $f(t) = |z_1 - t|^p + |z_2 - t|^p$, $t \in \mathbb{R}$. Since $t = w$ is a minimum point, it follows that $f'(w) = 0$.

But $f'(t) = p\big(|z_1 - t|^{p-2}(t - x_1) + |z_2 - t|^{p-2}(t - x_2)\big)$, $t \in \mathbb{R}$. Then

$$\begin{aligned} f'(w) &= p\big(|z_1 - w|^{p-2}(w - x_1) + |z_2 - w|^{p-2}(w - x_2)\big) \\ &= p\,|z_1 - \overline{z_2}|^{p-2}(x_2 - x_1)\big(\alpha^{p-1} - (1-\alpha)^{p-1}\big). \end{aligned}$$

We see that $f'(w) = 0$ if and only if Re $(z_1) =$ Re (z_2) or $\alpha = \frac{1}{2}$ (i.e. Im $(z_1) =$ Im (z_2)). □

Remark 3.41. According to numerical tests, we have the following particular values

$$T_2(1 + i6, -2 + i3) = 3/5\,, \quad T_2(-4 + i4, 4 + i12) = 4/5\,,$$
$$T_p(-t + it, 1 + i) = 1 \quad \text{for all } p \geq 1\,, t > 0\,.$$

Theorem 3.42. *For $z_1, z_2 \in \mathbb{H}$ and $p \geq 1$ let*

$$U_p(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt[p]{\alpha^p + \beta^p}}\,, \quad \alpha = \sqrt{\mathrm{Im}(z_1)^2 + c^2}\,, \quad \beta = \sqrt{\mathrm{Im}(z_2)^2 + c^2}\,,$$
$$c = |\mathrm{Re}(z_1 - z_2)|/2.$$

Then

$$b_{\mathbb{H},p}(z_1, z_2) \geq U_p(z_1, z_2)\,. \tag{3.43}$$

Proof. Fix $z_1, z_2 \in \mathbb{H}$ and let $u = \mathrm{Re}(z_1 + z_2)/2$. The Pythagorean theorem yields

$$|z_1 - u| = \alpha\,, |z_2 - u| = \beta\,,$$

and hence by the definition of the Barrlund metric the claim follows. □

We will compare below the above lower bounds T_p and U_p for the Barrlund metric.

Lemma 3.44. *For $z_1, z_2 \in \mathbb{H}$ let*

$$m = \frac{1}{2}(\mathrm{Re}\,(z_1) + \mathrm{Re}\,(z_2))\,, \quad \alpha = \frac{\mathrm{Im}\,(z_1)}{\mathrm{Im}\,(z_1) + \mathrm{Im}\,(z_2)}\,, \quad w = (1-\alpha)\mathrm{Re}\,(z_1) + \alpha\mathrm{Re}\,(z_2)\,,$$

$$U_p\,(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt[p]{|m - z_1|^p + |m - z_2|^p}}\,, \quad T_p\,(z_1, z_2) := \frac{|z_1 - z_2|}{\sqrt[p]{|w - z_1|^p + |w - z_2|^p}}\,.$$

If $p \geq 2$, then

$$U_p\,(z_1, z_2) \geq T_p\,(z_1, z_2)\,.$$

Proof. We will use Lemma 3.16. Let $S_p(t) = |t - z_1|^p + |t - z_2|^p$, $t \in \mathbb{R}$. We proved that the derivative S_p' is increasing on $\mathbb{R}$ and has a zero t_0, which is the unique minimum point of S_p, since S_p is decreasing on $(-\infty, t_0]$ and increasing on $[t_0, \infty)$.

With our notations,

$$U_p(z_1,z_2) - T_p(z_1,z_2) = \frac{|z_1-z_2|}{(S_p(m)S_p(w))^{1/p}}\left((S_p(w))^{1/p} - (S_p(m))^{1/p}\right). \tag{3.45}$$

If $p=2$, we proved that $S_p(m) \le S_p(t)$ for every $t \in \mathbb{R}$, in particular $S_p(m) \le S_p(w)$, hence $U_p(z_1,z_2) \ge T_p(z_1,z_2)$.

Assume now that $p > 2$.

We have to compare m, w and t_0.

$$m - w = \left(\alpha - \frac{1}{2}\right)\mathrm{Re}\,(z_1-z_2) = \frac{1}{2\mathrm{Im}\,(z_1+z_2)}\mathrm{Re}\,(z_1-z_2)\,\mathrm{Im}\,(z_1-z_2).$$

If $\mathrm{Im}\,(z_1) = \mathrm{Im}\,(z_2)$ or $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$, then $m = w$ and $U_p(z_1,z_2) = T_p(z_1,z_2)$ for every $p \ge 2$ and the claim follows.

Now assume that $\mathrm{Re}\,(z_1) \neq \mathrm{Re}\,(z_2)$ and $\mathrm{Im}\,(z_1) \neq \mathrm{Im}\,(z_2)$.

Let $g_p(\lambda) = S_p'((1-\lambda)\mathrm{Re}\,(z_1) + \lambda\mathrm{Re}\,(z_2))$, $\lambda \in [0,1]$. We have

$$\begin{aligned} g_p(\lambda) =& p\mathrm{Re}\,(z_2-z_1) \\ &\times \left[\lambda\left|\lambda\mathrm{Re}\,(z_2-z_1) - i\mathrm{Im}\,(z_1)\right|^{p-2} - (1-\lambda)\left|(1-\lambda)\,\mathrm{Re}\,(z_2-z_1) + i\mathrm{Im}\,(z_2)\right|^{p-2}\right]. \end{aligned}$$

Then

$$g_p\left(\frac{1}{2}\right) = \frac{p}{2}\mathrm{Re}\,(z_2-z_1)\left[\left|\frac{1}{2}\mathrm{Re}\,(z_2-z_1) - i\mathrm{Im}\,(z_1)\right|^{p-2} - \left|\frac{1}{2}\mathrm{Re}\,(z_2-z_1) + i\mathrm{Im}\,(z_2)\right|^{p-2}\right].$$

Then

$$\mathrm{Re}\,(z_1-z_2)\,\mathrm{Im}\,(z_1-z_2) g_p\left(\frac{1}{2}\right) < 0,$$

since $p > 2$.

Case 1. $\mathrm{Re}\,(z_1-z_2)\,\mathrm{Im}\,(z_1-z_2) > 0$.

We have $w < m$. On the other hand, $g_p\left(\frac{1}{2}\right) < 0$, hence $m < t_0$. Since $w < m < t_0$ and S_p is decreasing on $(-\infty, t_0]$, we have $S_p(w) \ge S_p(m)$.

Case 2. $\mathrm{Re}\,(z_1-z_2)\,\mathrm{Im}\,(z_1-z_2) < 0$.

Now $w > m$ and $g_p\left(\frac{1}{2}\right) > 0$, hence $m > t_0$. Since $w > m > t_0$ and S_p is increasing on $[t_0, \infty)$, we have $S_p(w) \ge S_p(m)$. In both cases inequality (3.45) shows that $U_p(z_1,z_2) - T_p(z_1,z_2) \ge 0$. □

3.46. Barrlund's metric for $p = \infty$.

Let $G \subset \mathbb{R}^n$ be a proper subdomain. Let

$$b_{G,\infty}(z_1,z_2) = \sup_{w \in \partial G} \frac{|z_1 - z_1|}{\max\{|z_1 - w|, |z_2 - w|\}}.$$

For $G = \mathbb{R}^n \setminus \{0\}$, D. Day [6] proved that $b_{G,\infty}$ is a metric.

Note that $\max\{|z_1-w|,|z_2-w|\} = \lim_{p\to\infty} \sqrt[p]{|z_1-w|^p + |z_2-w|^p}$. It follows that

$$b_{G,p}(z_1,z_2) \le b_{G,\infty}(z_1,z_2) \le 2^{\frac{1}{p}} b_{G,p}(z_1,z_2)$$

for all $z_1, z_2 \in G$ and $1 \le p < \infty$.

Recall that the power p ellipse E_p is written as $|z-z_1|^p+|z-z_2|^p=r^p$. We have the following result for the shape of the power ∞ ellipse.

Lemma 3.47. *The power ∞ ellipse is given by*

$$E_\infty : \partial\{|z-z_1|<r \text{ and } |z-z_2|<r\}.$$

Proof. The assertion holds from $\lim_{p\to\infty}\sqrt[p]{|z-z_1|^p+|z-z_2|^p}=\max\{|z-z_1|,|z-z_2|\}$. □

3.46.1. **The domain $G=\mathbb{H}$.**

Theorem 3.48. *For $z_1,z_2\in\mathbb{H}$*

$$b_{\mathbb{H},\infty}(z_1,z_2)=\begin{cases}\dfrac{2|\mathrm{Re}\,(z_1-z_2)|}{|z_1-\overline{z_2}|} & \text{if } \min\{\mathrm{Re}\,(z_1),\mathrm{Re}\,(z_2)\}<\tilde{z}<\max\{\mathrm{Re}\,(z_1),\mathrm{Re}\,(z_2)\},\\ \dfrac{|z_1-z_2|}{\max\{\mathrm{Im}\,(z_1),\ \mathrm{Im}\,(z_2)\}} & \text{otherwise.}\end{cases}$$

where $\tilde{z}=\frac{\overline{z_1}z_1-\overline{z_2}z_2}{(z_1-z_2)+(\overline{z_1}-\overline{z_2})}$ *if* $\mathrm{Re}\,(z_1)\neq\mathrm{Re}\,(z_2)$.

Proof. Assume first that $\mathrm{Re}\,(z_1)\neq\mathrm{Re}\,(z_2)$. Let $\tilde{z}$ be the intersection point of the real axis and the perpendicular bisector ℓ of the segment $[z_1,z_2]$. The line ℓ and $\tilde{z}$ are given by

$$\ell:\ (\overline{z_1}-\overline{z_2})z+(z_1-z_2)\overline{z}=\overline{z_1}z_1-\overline{z_2}z_2 \quad\text{and}\quad \tilde{z}=\frac{\overline{z_1}z_1-\overline{z_2}z_2}{(z_1-z_2)+(\overline{z_1}-\overline{z_2})}.$$

Then, we need to consider the following two cases.

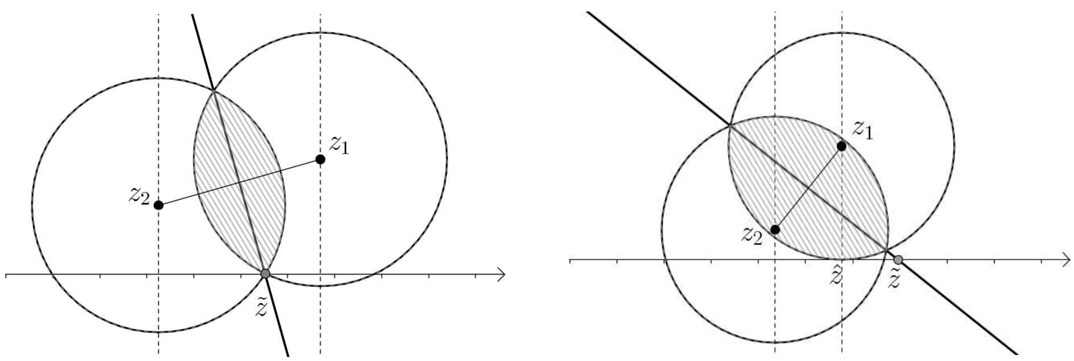

FIGURE 3. The left and right figures indicate the case (1) and (2) respectively.

(1) $\min\{\mathrm{Re}\,(z_1),\ \mathrm{Re}\,(z_2)\}\leq\tilde{z}\leq\max\{\mathrm{Re}\,(z_1),\ \mathrm{Re}\,(z_2)\}$

The limit $\lim_{p\to\infty}\sqrt[p]{|z_1-z|^p+|z-z_2|^p}=\max\{|z_1-z|,|z_2-z|\}$ attains the minimum at $z=\tilde{z}$ and its minimum is

$$|z_1-\tilde{z}|=\left|\frac{(z_1-z_2)(z_1-\overline{z_2})}{2\mathrm{Re}\,(z_1-z_2)}\right|.$$

Therefore in this case,

$$b_{\mathbb{H},\infty}(z_1,z_2)=\frac{2|\mathrm{Re}\,(z_1-z_2)|}{|z_1-\overline{z_2}|}.$$

(2) $\tilde{z} \le \min\{\mathrm{Re}\,(z_1),\, \mathrm{Re}\,(z_2)\}$ or $\max\{\mathrm{Re}\,(z_1),\, \mathrm{Re}\,(z_2)\} \le \tilde{z}$
In this case,
$$\max\{|z_1 - z|\,, |z_2 - z|\}$$
attains the minimum at the finite endpoint of the interval (for example, point $\hat{z}$ on the Figure 3) where $\tilde{z}$ belongs and the minimum is $\max\{\mathrm{Im}\,(z_1),\ \mathrm{Im}\,(z_2)\}$. Then
$$b_{\mathbb{H},\infty}(z_1, z_2) = \frac{|z_1 - z_2|}{\max\{\mathrm{Im}\,(z_1),\ \mathrm{Im}\,(z_2)\}}.$$
If $\mathrm{Re}\,(z_1) = \mathrm{Re}\,(z_2)$, then the above formula also holds. □

An upper bound of $b_{\mathbb{H},p}(z_1, z_2)$ is given as follows.

Proposition 3.49. *For $z_1, z_2 \in \mathbb{H}$*
$$b_{\mathbb{H},p}(z_1, z_2) \le \frac{|z_1 - z_2|}{\max\{\mathrm{Im}\,(z_1), \mathrm{Im}\,(z_2)\}}.$$

Proof. From Theorem 3.9, (3.10), the inequality
$$s_{\mathbb{H}}(z_1, z_2) \le b_{\mathbb{H},p}(z_1, z_2) \le b_{\mathbb{H},\infty}(z_1, z_2)$$
holds. Also, from the proof of the above lemma the inequality
$$|\max\{\mathrm{Im}\,(z_1), \mathrm{Im}\,(z_2)\}| \le |z_k - \tilde{z}|$$
$(k = 1, 2)$ holds. Therefore, we have
$$\frac{2|\mathrm{Re}\,(z_1 - z_2)|}{|z_1 - \overline{z_2}|} \le \frac{|z_1 - z_2|}{\max\{\mathrm{Im}\,(z_1), \mathrm{Im}\,(z_2)\}},$$
and the assertion is obtained. □

3.46.2. **The domain $G = \mathbb{D}$.**

Lemma 3.50. *Suppose $z_1, z_2 \in \mathbb{D}$ satisfy $r = |z_1| \le |z_2|$. Set $z_1 = re^{i\theta}$.*
Then, the following (1), (2) and (3) are equivalent to each other.
(1) *$b_{\mathbb{D},\infty}(z_1, z_2)$ attains its supremum at $u = \frac{z_1}{|z_1|} = e^{i\theta}$.*
(2) *$z_2 \in \left\{|z - e^{i\theta}| \le 1 - r\right\} \cap \mathbb{D}$.*
(3) *the power ∞ ellipse $\lim_{p\to\infty} \sqrt[p]{|z - z_1|^p + |z - z_2|^p} = 1 - r$ tangents to the unit circle.*

Proof. (1) ⇔ (3) The power ∞ ellipse in (3) is written as
$$\partial\{|z - z_1| \le 1 - r \text{ and } |z - z_2| \le 1 - r\}.$$
The circle $|z - z_1| = 1 - r$ is inscribed in the unit circle, and the point $\frac{z_1}{|z_1|} = e^{i\theta}$ is the point of tangency of these two circles. In this case, if power ∞ ellipse with foci z_1 and z_2 tangent to the unit circle at a point in its "arc", the point of tangency is also given by $u = e^{i\theta}$ (see the left figure in Figure 4). Clearly, the converse also holds.
(1) ⇒ (2) From the above argument, the following is also obtained: if the unit circle tangent to a power ∞ ellipse at a point in "arc", $b_{\mathbb{D},\infty}(z_1, z_2)$ attains its supremum at the tangent point $u = \frac{z_1}{|z_1|}$.

Here we consider the case when the unit circle intersects with a power ∞ ellipse at one of the vertices. Let D be the set consisting of the points z_2 in which $b_{\mathbb{D},\infty}$ attains its supremum

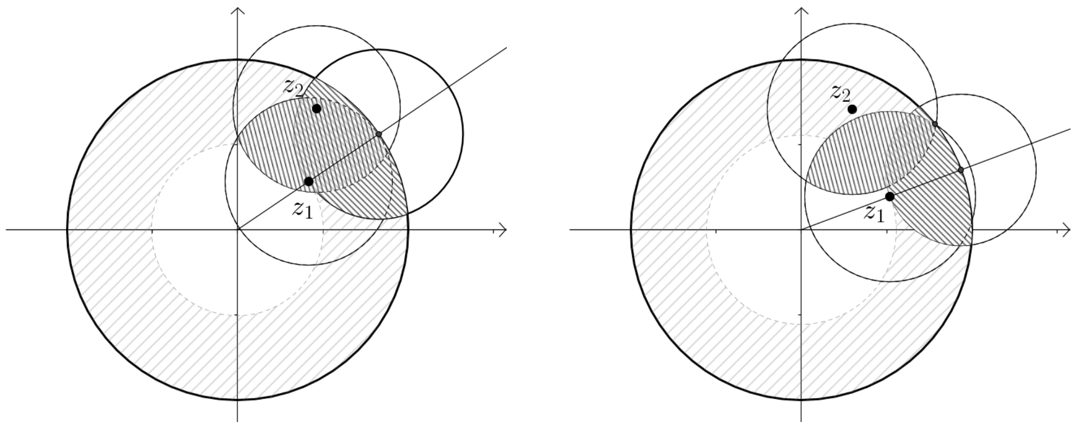

FIGURE 4. The power ∞ ellipse and the set $\left\{|z-e^{i\theta}|\le 1-r\right\}\cap\mathbb{D}$.

at a vertex of corresponding power ∞ ellipse. Then, for each boundary point $z_2 \in \partial D$, $b_{\mathbb{D},\infty}(z_1,z_2)$ attains the supremum at the vertex $u=e^{i\varphi}$ of power ∞ ellipse.

Now, let ℓ be the line passing through $e^{i\theta}$ and $e^{i\varphi}$, and z^* the reflection point of z_1 with respect to the line ℓ. Then, we have

$$\ell \ : \ z+e^{i\theta}e^{i\varphi}\overline{z}=e^{i\theta}+e^{i\varphi}\,, \quad \text{and} \quad z^*=e^{i\theta}+(1-r)e^{i\varphi}\,.$$

The trace of z^* forms the circle

$$|z-e^{i\theta}|=1-r\,, \tag{3.51}$$

as the point $e^{i\varphi}$ ranges over the unit circle. Clearly, if we choose the point z_2 in the inside of the disk (3.51), the unit circle tangents to a power ∞ ellipse with tangency a point in "arc".

(2) $\Rightarrow$ (3) From the above argument, it is clear that if z_2 is in the disk $|z-e^{i\theta}|\le 1-r$ (and $z_2\in\mathbb{D}$), the power ∞ ellipse with foci z_1,z_2 is inscribed in the unit circle and the tangent point is a point in "arc" part of the power ∞ ellipse. As the distance from z_1 to the unit circle is $1-r$, the power ∞ ellipse is written by $\lim_{p\to\infty}\sqrt[p]{|z-z_1|^p+|z-z_2|^p}=1-r$. $\square$

Theorem 3.52. *Let $z_1,z_2\in\mathbb{D}\setminus\{0\}$ be distinct points. Then*

$$b_{\mathbb{D},\infty}(z_1,z_2)=\begin{cases}\dfrac{|z_1-z_2|}{1-\min\left\{|z_1|,|z_2|\right\}} & \textit{if } |z_1|\le 1-\left|z_2-\frac{z_1}{|z_1|}\right| \textit{ or } |z_2|\le 1-\left|z_1-\frac{z_2}{|z_2|}\right|,\\[2ex] \dfrac{|z_1-z_2|}{\min\left\{|z'-z_1|,|z''-z_1|\right\}} & \textit{otherwise.}\end{cases}$$

Here z' and z'' are the intersections of the perpendicular bisector of the segment $[z_1,z_2]$ with the the unit circle $\partial\mathbb{D}$, and are given by

$$\{z',z''\}=\left\{\frac{z_1-z_2}{|z_1-z_2|}\left(\frac{|z_1|^2-|z_2|^2}{2\,|z_1-z_2|}\pm i\sqrt{1-\left(\frac{|z_1|^2-|z_2|^2}{2\,|z_1-z_2|}\right)^2}\right)\right\}. \tag{3.53}$$

Proof. Let $z_1, z_2 \in \mathbb{D}$. Denote $M(z) := \max\{|z-z_1|, |z-z_2|\}$, $z \in \mathbb{C}$ and $m := \min\limits_{z\in\partial\mathbb{D}} M(z)$. Then

$$b_{\mathbb{D},\infty}(z_1, z_2) = \frac{|z_1 - z_2|}{m}.$$

If $z_1 = z_2$, then $m = 1 - |z_1|$ and $b_{\mathbb{D},\infty}(z_1, z_2) = 0$. If $z_1 = 0 \neq z_2$ or $z_2 = 0 \neq z_1$, then $m = 1$. In the following we assume that $z_1, z_2 \in \mathbb{D}\setminus\{0\}$ are distinct.

The perpendicular bisector $\mathcal{L}$ of the segment $[z_1, z_2]$ has the equation $\mathcal{L} : L(z) = 0$, where

$$L(z) = (\overline{z_1} - \overline{z_2})\, z + (z_1 - z_2)\, \overline{z} - \left(|z_1|^2 - |z_2|^2\right).$$

The closed half-planes determined by $\mathcal{L}$ are $H_1 = \{z \in \mathbb{C} : L(z) \geq 0\}$ and $H_2 = \{z \in \mathbb{C} : L(z) \leq 0\}$. Since $L(z_1) = |z_1 - z_2|^2 > 0$ and $L(z_2) = -L(z_1) < 0$, we have $z_k \in H_k \setminus \mathcal{L}$ for $k = 1, 2$. Note that $L(0) = |z_2|^2 - |z_1|^2$ and

$$M(z) = \begin{cases} |z - z_2| & \text{if } z \in H_1, \\ |z - z_1| & \text{if } z \in H_2. \end{cases}$$

Then $m = \min\{m_1, m_2\}$, where $m_1 := \min\limits_{z\in\partial\mathbb{D}\cap H_2} |z - z_1|$ and $m_2 := \min\limits_{z\in\partial\mathbb{D}\cap H_1} |z - z_2|$.

The minimum in the definition of m_1 is attained at $z = \frac{z_1}{|z_1|}$ if $\frac{z_1}{|z_1|} \in H_2$, respectively at some $z \in \{z', z''\}$ if $\frac{z_1}{|z_1|} \in H_1$. Then $m_1 = 1 - |z_1|$ if $\frac{z_1}{|z_1|} \in H_2$ and $m_1 = \min\{|z' - z_1|, |z'' - z_1|\}$ if $\frac{z_1}{|z_1|} \in H_1$.

Denote $m_3 := 1 - \min\{|z_1|, |z_2|\}$ and $m_4 := \min\{|z' - z_1|, |z'' - z_1|\} = \min\{|z' - z_2|, |z'' - z_2|\}$. Note that $m_4 \geq m_3$.

We will assume that $|z_1| \leq |z_2|$, equivalently, $0 \in H_1$. The case $|z_2| \leq |z_1|$ is similar.

$0 \in H_1$ yields $\frac{z_2}{|z_2|} \in H_2$, otherwise by the convexity of H_1 we get $z_2 \in H_1$, which is false. So, $0 \in H_1$ implies $m_2 = m_4$.

If $0 \in H_1$ and $\frac{z_1}{|z_1|} \in H_1$, then $m_1 = m_4$, hence $m = m_4$. If $0 \in H_1$ and $\frac{z_1}{|z_1|} \in H_2$, then $m_1 = 1 - |z_1| = m_3 \leq m_4$, hence $m = m_3$.

We obtain

$$m = \begin{cases} m_4 & \text{if } (0 \in H_1 \text{ and } \frac{z_1}{|z_1|} \in H_1) \text{ or } (0 \in H_2 \text{ and } \frac{z_2}{|z_2|} \in H_2), \\ m_3 & \text{if } (0 \in H_1 \text{ and } \frac{z_1}{|z_1|} \in H_2) \text{ or } (0 \in H_2 \text{ and } \frac{z_2}{|z_2|} \in H_1). \end{cases}$$

In particular, there are the following special cases. If $0 \in H_1 \cap H_2$ (i.e. $|z_1| = |z_2|$), then $\frac{z_1}{|z_1|} \in H_1$ and $\frac{z_2}{|z_2|} \in H_2$, hence $m = m_4$. If $\frac{z_1}{|z_1|}, \frac{z_2}{|z_2|} \in H_1 \cap H_2$, then $m = m_3 = m_4$.

Since $L\big(\frac{z_1}{|z_1|}\big) = \big|z_2 - \frac{z_1}{|z_1|}\big|^2 - (1 - |z_1|)^2$, we have $\frac{z_1}{|z_1|} \in H_2$ if and only if

$$E(z_1, z_2) := \left|z_2 - \frac{z_1}{|z_1|}\right| - (1 - |z_1|) \leq 0,$$

i.e. z_2 belongs to the closed disk bounded by the circle $\mathcal{C}_1$ centered at $\frac{z_1}{|z_1|}$, passing through z_1.

Note that $\big|z_2 - \frac{z_1}{|z_1|}\big| \geq 1 - |z_2|$ and $\big|z_1 - \frac{z_2}{|z_2|}\big| \geq 1 - |z_1|$ whenever $z_1 \neq 0 \neq z_2$, by the triangle inequality.

The formulas for m and the above analytical characterizations of $0 \in H_j$ and of $\frac{z_k}{|z_k|} \in H_j$ for $j, k \in \{1, 2\}$ imply the claim.

Moreover, z', z'' are the roots of the quadratic equation

$$(\overline{z_1}-\overline{z_2})z^2-\left(|z_1|^2-|z_2|^2\right)z+(z_1-z_2)=0,$$

as $z\in\{z',z''\}$ implies $L\left(\frac{1}{z}\right)=L(\overline{z})=\overline{L(z)}=0$. □

Remark 3.54. The formula (3.53) is invariant to rotations around the origin.

It follows that $\min\{|z'-z_1|,|z''-z_1|\}=|z^*-z_1|$, with

$$z^*=\frac{z_1-z_2}{|z_1-z_2|}\left(\frac{|z_1|^2-|z_2|^2}{2|z_1-z_2|}+i\cdot\text{signum}\big(\text{Im }(\overline{z_1}z_2)\big)\sqrt{1-\left(\frac{|z_1|^2-|z_2|^2}{2|z_1-z_2|}\right)^2}\right),$$

where we assume $\text{Im }(\overline{z_1}z_2)\neq 0$.

If $\text{Im }(\overline{z_1}z_2)=0$, i.e. $0,\ z_1,\ z_2$ are collinear, then $|z'-z_1|=|z''-z_1|$ and we can choose any $z^*\in\{z',z''\}$.

4. Barrlund's metric and quasiconformal maps

In this section we will study how Barrlund's metric behaves under quasiconformal mappings. We first consider the case of Möbius transformations.

The main property of the hyperbolic metric is its invariance under the Möbius self-mapping $T_a:\mathbb{D}\to\mathbb{D}\,,z\mapsto\frac{z-a}{1-\overline{a}z}\,,|a|<1$, of the unit disk:

$$\rho_{\mathbb{D}}(T_a(z_1),T_a(z_2))=\rho_{\mathbb{D}}(z_1,z_2)$$

for all $z_1,z_2,a\in\mathbb{D}$. In other words, the mapping T_a is an isometry. Now making use of (2.2), Theorem 3.9, and the properties of the triangular ratio metric, we can prove that T_a is a Lipschitz mapping with respect to the Barrlund metric. The proof is based on [11, Theorem 4.8] and the same proof would also give similar results for Möbius transformations between half planes.

Theorem 4.1. *Let $p\geq 1$. For $a,\ z_1,\ z_2\in\mathbb{D}$ we have*

$$b_{\mathbb{D},p}(T_a(z_1),T_a(z_2))\leq 2^{2-\frac{1}{p}}\frac{b_{\mathbb{D},p}(z_1,z_2)}{1+b_{\mathbb{D},p}(z_1,z_2)^2}.$$

Proof. By [11, Theorem 4.8] $s_{\mathbb{D}}(T_a(z_1),T_a(z_2))\leq 2\frac{s_{\mathbb{D}}(z_1,z_2)}{1+s_{\mathbb{D}}(z_1,z_2)^2}$ and by Theorem 3.9 $s_{\mathbb{D}}\leq b_{\mathbb{D},p}\leq 2^{1-\frac{1}{p}}s_{\mathbb{D}}$ on $\mathbb{D}$. The claim follows using the fact that $t\mapsto\frac{t}{1+t^2}$ is increasing on $[0,1]$. □

We give a generalization of [5, Theorem 3.31] for $n=2$, which can be extended to the case $n\geq 2$.

Theorem 4.2. *Let $1\leq p\leq\infty$ and $a\in\mathbb{D}$. Then $T_a:(\mathbb{D},b_{\mathbb{D},p})\to(\mathbb{D},b_{\mathbb{D},p})$ is L-bilipschitz with $L=\frac{1+|a|}{1-|a|}$.*

Proof. For every $u,v\in\mathbb{D}$,

$$T_a(u)-T_a(v)=b\frac{u-v}{(u-a^*)(v-a^*)},$$

where $a^*=a/|a|^2$ and $b=(1-|a|^2)/\overline{a}^2$.

Let $z_1, z_2 \in \mathbb{D}$ be distinct points. We prove that

$$\frac{1-|a|}{1+|a|} b_{\mathbb{D},p}(z_1,z_2) \le b_{\mathbb{D},p}(T_a(z_1),T_a(z_2)) \le \frac{1+|a|}{1-|a|} b_{\mathbb{D},p}(z_1,z_2). \tag{4.3}$$

If $1 \le p < \infty$, for every $w \in \partial\mathbb{D}$

$$\begin{aligned} Q_p(z_1,z_2,w) &:= \left(\frac{|T_a(z_1)-T_a(z_2)|}{\sqrt[p]{|T_a(z_1)-T_a(w)|^p + |T_a(z_2)-T_a(w)|^p}}\right) \Big/ \left(\frac{|z_1-z_2|}{\sqrt[p]{|z_1-w|^p+|z_2-w|^p}}\right) \\ &= \left(\frac{|z_1-w|^p+|z_2-w|^p}{c^p|z_1-w|^p + d^p|z_2-w|^p}\right)^{1/p}, \end{aligned}$$

where $c := |z_2 - a^*| \,/\, |w-a^*|$ and $d := |z_1-a^*| \,/\, |w-a^*|$.

Since $|w-a^*| \le 1+|a|^{-1}$ and $|z_1-a^*|, |z_2-a^*| \ge |a|^{-1}-1$, we have $c, d \ge (1-|a|)/(1+|a|)$. Therefore, $Q_p(z_1,z_2,w) \le \frac{1+|a|}{1-|a|} =: L$, hence

$$\begin{aligned} \frac{|T_a(z_1)-T_a(z_2)|}{\sqrt[p]{|T_a(z_1)-T_a(w)|^p + |T_a(z_2)-T_a(w)|^p}} &\le L\frac{|z_1-z_2|}{\sqrt[p]{|z_1-w|^p+|z_2-w|^p}} \\ &\le L b_{\mathbb{D},p}(z_1,z_2). \end{aligned}$$

As $T_a(\partial\mathbb{D}) = \partial\mathbb{D}$, taking supremum over all $w \in \partial\mathbb{D}$ yields

$$b_{\mathbb{D},p}(T_a(z_1),T_a(z_2)) \le \frac{1+|a|}{1-|a|} b_{\mathbb{D},p}(z_1,z_2).$$

Having $T_a^{-1} = T_{-a}$, it follows similarly that $b_{\mathbb{D},p}(z_1,z_2) \le \frac{1+|a|}{1-|a|} b_{\mathbb{D},p}(T_a(z_1),T_a(z_2))$. Then (4.3) holds.

If $p = \infty$, for every $w \in \partial\mathbb{D}$

$$\begin{aligned} R(z_1,z_2,w) &:= \left(\frac{|T_a(z_1)-T_a(z_2)|}{\max\{|T_a(z_1)-T_a(w)|, |T_a(z_2)-T_a(w)|\}}\right) \Big/ \left(\frac{|z_1-z_2|}{\max\{|z_1-w|,|z_2-w|\}}\right) \\ &= \frac{\max\{|z_1-w|,|z_2-w|\}}{\max\{c|z_1-w|, d|z_2-w|\}}, \end{aligned}$$

with c, d as above. Then

$$\begin{aligned} \frac{|T_a(z_1)-T_a(z_2)|}{\max\{|T_a(z_1)-T_a(w)|, |T_a(z_2)-T_a(w)|\}} &\le L\frac{|z_1-z_2|}{\max\{|z_1-w|,|z_2-w|\}} \\ &\le L b_{\mathbb{D},\infty}(z_1,z_2), \end{aligned}$$

hence $b_{\mathbb{D},\infty}(T_a(z_1),T_a(z_2)) \le \frac{1+|a|}{1-|a|} b_{\mathbb{D},\infty}(z_1,z_2)$. As above, it follows that (4.3) also holds for $p = \infty$. □

Conjecture 4.4. *By the above results we see that there exists for $p \in [1,\infty]$, $a \in \mathbb{D}$, the least constant $R(p,a)$ such that for all $z_1, z_2 \in \mathbb{D}$*

$$b_{\mathbb{D},p}(T_a(z_1),T_a(z_2)) \le R(p,a) b_{\mathbb{D},p}(z_1,z_2)\,.$$

On the basis of computer experiments we expect that the following inequality holds for $p = 1, 2$

$$R(p,a) \le 1 + |a|\,.$$

In the case $p = 1$ Conjecture 4.4 was formulated in [5] and it was shown in [5, Thm 1.5] that $R(1,a) \ge 1 + |a|$. We now extend this last inequality for all p .

Theorem 4.5. *For all* $1 \le p \le \infty$ *and* $a \in \mathbb{D}$ $R(p,a) \ge 1 + |a|$.

Proof. We may assume $a \ne 0$, as $R(p,0) = 1$. Denote $\alpha = \arg(-a)$. Then $T_a(re^{i\alpha}) = \frac{r+|a|}{1+r|a|}e^{i\alpha}$ for all $r \in [0,1)$.

Let $0 \le r < s < 1$. For all $t \in \mathbb{R}$,

$$b_{\mathbb{D},p}(re^{it}, se^{it}) = \frac{s-r}{\sqrt[p]{(1-r)^p + (1-s)^p}}$$

and $b_{\mathbb{D},\infty}(re^{it}, se^{it}) = \frac{s-r}{1-r}$.

Note that $0 < e^{-i\alpha}T_a(re^{i\alpha}) < e^{-i\alpha}T_a(se^{i\alpha}) < 1$.

Assume that $1 \le p < \infty$. Then

$$\begin{aligned} b_{\mathbb{D},p}(T_a(re^{i\alpha}), T_a(se^{i\alpha})) &= \frac{\frac{s+|a|}{1+s|a|} - \frac{r+|a|}{1+r|a|}}{\sqrt[p]{(1-\frac{r+|a|}{1+r|a|})^p + (1-\frac{s+|a|}{1+s|a|})^p}} \\ &= \frac{(1+|a|)(s-r)}{\sqrt[p]{(1+s|a|)^p(1-r)^p + (1+r|a|)^p(1-s)^p}}. \end{aligned}$$

Therefore,

$$\begin{aligned} R(p,a) &\ge \frac{b_{\mathbb{D},p}(T_a(re^{i\alpha}), T_a(se^{i\alpha}))}{b_{\mathbb{D},p}(re^{it}, se^{it})} \\ &= (1+|a|)\sqrt[p]{\frac{(1-r)^p + (1-s)^p}{(1+s|a|)^p(1-r)^p + (1+r|a|)^p(1-s)^p}}. \end{aligned}$$

Similarly, $b_{\mathbb{D},\infty}(T_a(re^{i\alpha}), T_a(se^{i\alpha})) = \frac{\frac{s+|a|}{1+s|a|} - \frac{r+|a|}{1+r|a|}}{1-\frac{r+|a|}{1+r|a|}} = \frac{(1+|a|)(s-r)}{(1+s|a|)(1-r)}$, hence

$$R(\infty,a) \ge \frac{b_{\mathbb{D},\infty}(T_a(re^{i\alpha}), T_a(se^{i\alpha}))}{b_{\mathbb{D},\infty}(re^{it}, se^{it})} = \frac{1+|a|}{1+s|a|}.$$

As $s \to 0$, it follows that $r \to 0$ and $R(p,a) \ge 1 + |a|$ for $1 \le p \le \infty$. □

By [5, Corollary 3.30] and Theorem 3.9 (extended to include the case $p = \infty$), we obtain

Proposition 4.6. *Let* $f : G \to \Omega$ *be a Möbius transformation onto* Ω, *where* $G, \Omega \in \{\mathbb{D}, \mathbb{H}\}$ *and let* $1 \le p \le \infty$. *Then* $f : (G, b_{G,p}) \to (\Omega, b_{\Omega,p})$ *is L-Lipschitz with* $L = 2^{2-1/p}$ *if* $G = \mathbb{D}$, *respectively* $L = 2^{1-1/p}$ *if* $G = \mathbb{H}$.

We also recall some notation about special functions and the fundamental distortion result of quasiregular maps, a variant of the Schwarz lemma for these maps. For $r \in (0,1)$ and $K > 0$, we define the distortion function

$$\varphi_K(r) = \mu^{-1}(\mu(r)/K),$$

where $\mu(r)$ is the modulus of the planar Grötzsch ring, see [1, pp. 92-94], [23, Exercise 5.61].

Lemma 4.7. [23, Theorem 11.2] *Let $f : D \to G$, $D, G \in \{\mathbb{B}^n, \mathbb{H}^n\}$ be a non-constant K-quasiregular mapping with $fD \subset G$. Then for all z_1, $z_2 \in D$,*

$$\tanh \tfrac{1}{2}\rho_G\left(f(z_1), f(z_2)\right) \le \varphi_K\left(\tanh \tfrac{1}{2}\rho_D(z_1, z_2)\right) \le 4^{1-1/K}\left(\tanh \tfrac{1}{2}\rho_D(z_1, z_2)\right)^{1/K} .$$

4.8. **Proof of Theorem 1.7.** By Theorem 3.9 and Lemma 4.7

$$\begin{aligned} b_{\mathbb{H},p}(f(z_1), f(z_2)) &\le 2^{1-1/p}\, s_{\mathbb{H}}(f(z_1), f(z_2)) = 2^{1-1/p} \tanh \frac{\rho_{\mathbb{H}}(f(z_1), f(z_2))}{2} \\ &\le 4^{1-1/K}\, 2^{1-1/p} \left(\tanh \frac{\rho_{\mathbb{H}}(z_1, z_2)}{2}\right)^{1/K} \\ &= 4^{1-1/K}\, 2^{1-1/p}\, (s_{\mathbb{H}}(z_1, z_2))^{1/K} \le 4^{1-1/K}\, 2^{1-1/p}\, b_{\mathbb{H},p}(z_1, z_2)^{1/K} . \quad \square \end{aligned}$$

Remark 4.9. Theorem 1.7 is sharp in the following sense. If $p = 1$, then the conclusion is

$$s_{\mathbb{H}}(f(z_1), f(z_2)) \le 4^{1-1/K} s_{\mathbb{H}}(z_1, z_2)^{1/K}$$

and the constant $4^{1-1/K}$ cannot be replaced by any number $c < 1$. Moreover, if $p = 2, K = 1$, the result says that

$$b_{\mathbb{H},2}(f(z_1), f(z_2)) \le \sqrt{2} b_{\mathbb{H},2}(z_1, z_2)^{1/K} .$$

The constant $\sqrt{2}$ is sharp, because by numerical experiments this constant is attained if $h(x) = x/|x|^2$, which maps $\mathbb{H}$ onto itself, and $z_1 = ic, z_2 = 2 + it$ where $c > 0$ and $t > 0$ are close to zero.

We generalize [11, Theorem 4.4], using also some ideas from [13, Proposition 2.2].

Theorem 4.10. *Let G, $D \subsetneq \mathbb{R}^n$ be domains and $1 \le p < \infty$. Let $f : G \to D$ be a surjective mapping satisfying the L-bilipschitz condition with respect to the p-Barrlund metric, for some $L \ge 1$, i.e.*

$$b_{G,p}(z_1, z_2)/L \le b_{D,p}(f(z_1), f(z_2)) \le L b_{G,p}(z_1, z_1) \tag{4.11}$$

for all $z_1, z_2 \in G$. Then f is a quasiconformal homeomorphism (either sense-preserving or sense-reversing), with the linear dilatation bounded from above by $4^{1-\frac{1}{p}} L^2$.

Proof. The first inequality in (4.11) shows that f is injective, hence f is bijective. We will prove that f is continuous. Since the inverse f^{-1} also satisfies the L-bilipschitz condition with respect to the p-Barrlund metric, it will follow that f^{-1} is continuous, therefore f is a homeomorphism.

Let $z_1, z_2 \in G$.

It is easy to see that

$$b_{G,p}(z_1, z_2) \le \frac{|z_1 - z_2|}{(d_G(z_1)^p + d_G(z_2)^p)^{1/p}}, \tag{4.12}$$

hence, for all $z_1, z_2 \in G$,

$$|z_1 - z_2| \ge (d_G(z_1)^p + d_G(z_2)^p)^{1/p}\, b_{G,p}(z_1, z_2).$$

Now let $w \in \partial G$ with $d_G(z_1) = |z_1 - w|$. Then

$$b_{G,p}(z_1, z_2) \ge s_G(z_1, z_2) \ge \frac{|z_1 - z_2|}{|z_1 - w| + |w - z_2|}.$$

But $|w-z_2| \le |z_1-w|+|z_1-z_2|$, hence $b_{G,p}(z_1,z_2) \ge \frac{|z_1-z_2|}{2d_G(z_1)+|z_1-z_2|}$. By symmetry, we get as in [11] the stronger inequality

$$b_{G,p}(z_1,z_2) \ge \frac{|z_1-z_2|}{|z_1-z_2|+2\min\{d_G(z_1),d_G(z_2)\}}. \tag{4.13}$$

If $0<b_{G,p}(z_1,z_2)<1$ this implies

$$|z_1-z_2| \le \frac{2\min\{d_G(z_1),d_G(z_2)\}}{\frac{1}{b_{G,p}(z_1,z_2)}-1}.$$

Fix $z\in G$. For every $u\in G\setminus\{z\}$ we have $f(u)\ne f(z)$ and using inequalities corresponding to (4.13) and (4.12), respectively, we get

$$\begin{aligned} 1+\frac{2d_D(f(z))}{|f(u)-f(z)|} &\ge 1+\frac{2\min\{d_D(f(u)),d_D(f(z))\}}{|f(u)-f(z)|} \ge \frac{1}{b_{D,p}(f(u),f(z))} \\ &\ge \frac{1}{Lb_{G,p}(u,z)} \ge \frac{1}{L}\cdot\frac{(d_G(u)^p+d_G(z)^p)^{1/p}}{|u-z|} \ge \frac{1}{L}\frac{d_G(z)}{|u-z|}. \end{aligned} \tag{4.14}$$

If $0<|u-z|<\frac{1}{L}d_G(z)$ it follows that $0<b_{D,p}(f(u),f(z))<1$ and

$$|f(u)-f(z)| \le 2Ld_G(z)\frac{|u-z|}{d_G(z)-L|u-z|}.$$

We conclude that f is continuous at the arbitrary point $z\in G$.

The linear dilatation of the homeomorphism f at $z\in G$ is defined by

$$H_f(z) := \limsup_{r\to 0}\frac{L_f(z,r)}{l_f(z,r)},$$

where $L_f(z,r) := \sup\{|f(z_1)-f(z)| : |z_1-z|=r\}$ and $l_f(z,r) := \inf\{|f(z_1)-f(z)| : |z_1-z|=r\}$.

If $u\in G$ with $0<|u-z|<\frac{1}{L}d_G(z)$, revisiting inequalities (4.14) we get

$$|f(u)-f(z)| \le \frac{2\min\{d_D(f(u)),d_D(f(z))\}}{\frac{1}{L}\cdot\frac{(d_G(u)^p+d_G(z)^p)^{1/p}}{|u-z|}-1}.$$

On the other hand, for every $v\in G$,

$$\begin{aligned} |f(v)-f(z)| &\ge \Big(d_D(f(v))^p+d_D(f(z))^p\Big)^{1/p} b_{D,p}(f(v),f(z)) \\ &\ge \frac{1}{L}\Big(d_D(f(v))^p d_D(f(z))^p\Big)^{1/p} b_{G,p}(v,z) \\ &\ge \frac{1}{L}\Big(d_D(f(v))^p d_D(f(z))^p\Big)^{1/p}\frac{|v-z|}{|v-z|+2\min\{d_G(v),d_G(z)\}}. \end{aligned}$$

For every ε with $0<\varepsilon<d_D(f(z))$ consider $\delta(\varepsilon,z)>0$ such that $|f(z_1)-f(z_2)|<\varepsilon$ for every $z_1\in G$ with $|z_1-z|<\delta(\varepsilon,z)$.

Let $0 < r < \min\left\{\frac{1}{L}d_G(z), \delta\left(\varepsilon, z\right)\right\}$. Assuming that $|u-z| = |v-z| = r$ we obtain from the above inequalities

$$\frac{|f(u)-f(z)|}{|f(v)-f(z)|} \leq L^2 \frac{2\min\left\{d_D\left(f(u)\right), d_D(f(z))\right\}}{\left(d_D(f(v))^p d_D(f(z))^p\right)^{1/p}} \frac{2\min\left\{d_G\left(v\right), d_G(z)\right\} + r}{\left(d_G(u)^p + d_G(z)^p\right)^{1/p} - Lr}.$$

Then

$$\frac{L_f(z,r)}{l_f(z,r)} \leq L^2 \frac{2d_D(f(z))}{\left((d_D(f(z))-\varepsilon)^p d_D(f(z))^p\right)^{1/p}} \frac{2d_G(z)+r}{\left((d_G(z)-r)^p + d_G(z)^p\right)^{1/p} - Lr}.$$

As r tends to zero, we conclude that

$$H_f(z) \leq L^2 \frac{2^{2-\frac{1}{p}} d_D(f(z))}{\left((d_D(f(z))-\varepsilon)^p d_D(f(z))^p\right)^{1/p}},$$

hence letting $\varepsilon \to 0$ it follows that $H_f(z) \leq 4^{1-\frac{1}{p}} L^2$. □

As expected, the above result has a counterpart in the case $p = \infty$.

Theorem 4.15. *Let G, $D \subsetneq \mathbb{R}^n$ be domains and let $f : G \to D$ be a surjective mapping satisfying the L-bilipschitz condition with respect to the ∞-Barrlund metric, for some $L \geq 1$, i.e.*

$$b_{G,\infty}(z_1,z_2)/L \leq b_{D,\infty}(f(z_1), f(z_2)) \leq L b_{G,\infty}(z_1,z_2) \tag{4.16}$$

for all $z_1, z_2 \in G$. Then f is a quasiconformal homeomorphism (either sense-preserving or sense-reversing), with the linear dilatation bounded from above by $4L^2$.

Proof. Clearly, f is a bijection. For every $z_1, z_2 \in G$,

$$\frac{|z_1-z_2|}{|z_1-z_2| + 2\min\left\{d_G\left(z_1\right), d_G(z_2)\right\}} \leq b_{G,\infty}(z_1,z_2) \leq \frac{|z_1-z_2|}{\max\left\{d_G\left(z_1\right), d_G(z_2)\right\}}.$$

If $0 < b_{G,\infty}(z_1,z_2) < 1$ then

$$|z_1-z_2| \leq \frac{2\min\left\{d_G\left(z_1\right), d_G(z_2)\right\}}{\frac{1}{b_{G,\infty}(z_1,z_2)} - 1}.$$

Fix $z \in G$. For every $u \in G \setminus \{z\}$ we have $f(u) \neq f(z)$ and

$$\begin{aligned} 1 + \frac{2d_D(f(z))}{|f(u)-f(z)|} &\geq 1 + \frac{2\min\left\{d_D\left(f(u)\right), d_D(f(z))\right\}}{|f(u)-f(z)|} \geq \frac{1}{b_{D,\infty}(f(u), f(z))} \\ &\geq \frac{1}{L b_{G,\infty}(u,z)} \geq \frac{1}{L}\frac{\max\left\{d_G\left(u\right), d_G(z)\right\}}{|u-z|} \geq \frac{1}{L}\frac{d_G(z)}{|u-z|}. \end{aligned}$$

As in the proof of Theorem 4.10, the continuity of f follows. Moreover, f^{-1} is continuous on D. If $0 < |u-z| < \frac{1}{L}d_G(z)$ it follows that $0 < b_{D,\infty}(f(u), f(z)) < 1$ and

$$|f(u)-f(z)| \leq \frac{2\min\left\{d_{fG}\left(f(u)\right), d_{fG}(f(z))\right\}}{\frac{1}{L}\cdot\frac{\max\{d_G(u), d_G(z)\}}{|u-z|} - 1}.$$

For every $v \in G$,

$$\begin{aligned} |f(v)-f(z)| &\geq \max\{d_D(f(v)), d_D(f(z))\}\, b_{D,\infty}(f(v), f(z)) \\ &\geq \frac{1}{L}\max\{d_D(f(v)), d_D(f(z))\}\, b_{G,\infty}(v,z) \\ &\geq \frac{1}{L}\max\{d_D(f(v)), d_D(f(z))\}\, \frac{|v-z|}{|v-z|+2\min\{d_G(v), d_G(z)\}}. \end{aligned}$$

If $0 < r < \frac{1}{L} d_G(z)$ and $|u-z| = |v-z| = r$, the latter inequalities yield

$$\frac{|f(u)-f(z)|}{|f(v)-f(z)|} \leq L^2 \frac{2\min\{d_D(f(u)), d_D(f(z))\}}{\max\{d_D(f(v)), d_D(f(z))\}} \frac{2\min\{d_G(v), d_G(z)\}+r}{\max\{d_G(u), d_G(z)\}-Lr}.$$

Then

$$\frac{L_f(z,r)}{l_f(z,r)} \leq 2L^2 \frac{2d_G(z)+r}{d_G(z)-Lr},$$

hence $H_f(z) \leq 4L^2$. □

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