Civil and Environmental Engineering
Vol. 17, Issue 2, 450-455, DOI: 10.2478/cee-2021-0047

FORMULATION OF A BASIC CONSTITUTIVE MODEL FOR FINE - GRAINED SOILS USING THE HYPOPLASTIC FRAMEWORK

Filip GAGO[1,*], Alessandro VALLETTA[2], Juraj MUŽÍK[1]

[1] Department of Geotechnics, Faculty of Civil Engineering, University of Žilina, Univerzitná 8215/1, 010 26 Žilina, Slovakia.
[2] Department of Engineering and Architecture, University of Parma - Parco Area delle Scienze 181/a, 43124 Parma, Italy.
[*] corresponding author: filip.gago@uniza.sk

Abstract

A hypoplastic approach to constitutive modelling was developed by Kolymbas 1996 considering a non-linear tensor function in the form of strain and stress rate. However, the implicit formulation of the hypoplastic model with indirect material parameters severely limits its applicability to real-world geotechnical problems. In many cases, the numerical analysis of geotechnical problems relies on simple elastoplastic constitutive models that cannot model a wide range of soil response aspects. One promising paradigm of constitutive modelling in geotechnics is hypoplasticity, but many of the hypoplastic models belong to advanced models. In the article, we present the simple hypoplastic model as an alternative to the widely used Mohr-Coulomb elastoplastic model.

Keywords:

Constitutive modelling;
Hypoplasticity;
Soil stiffness;
Finite element method;
Foundation settlement.

1 Introduction

We can characterize the strength of soils and other granular materials in terms of the friction angle *φ* and cohesion *c*. These parameters define a failure criterion and provide a sufficient description of the material properties relevant to stability problems (e.g. bearing capacity of foundations) that are mostly handled using the standard Mohr-Coulomb model.

However, this approach is inadequate for deformation problems. In that case, we need a constitutive law that relates the stresses and the strains. For soils, this relation may not be the available set of equations of elasticity theory, as soil deformations are markedly irreversible. Nevertheless, the following general properties characterize the inelastic behaviour of soils [1]:

- The instantaneous stiffness changes drastically upon deformation reversal. In other words, there are very different stiffnesses at loading and unloading.
- The instantaneous stiffness depends on the stress level: In a first approximation, it increases linearly with stress level. In other words, the soil becomes stiffer with proportionally increasing stress.
- There are limit stress states characterized by vanishing stiffnesses under some specified deformations. The limit state is connected with the so-called peak obtained from triaxial tests.
- The stress increases with the logarithm of the strain rate.

The article presents the implementation of the hypoplastic constitutive model as an alternative to the standard Mohr-Coulomb model used in geotechnical numerical analysis. The practical use of numerical models in geotechnical engineering is strongly affected by the material parameters available. That is why we adopted the hypoplastic implementation without considering critical and asymptotic states that characterize the response of advanced hypoplastic models for fine-grained soils. The hypoplastic model in the "elastic" stress space before yielding represents a more advantageous model than the standard Hardening-Soil (HS) model. The HS model requires the determination of three stiffnesses (E_0, E_{50}, and E_{ur}) for stress path modelling. The hypoplastic model requires only one stiffness parameter, and in other aspects, the material behaviour is primarily captured by the tensor member generator itself. This assumption also applies to behaviour material during unloading, and this aspect is modelled by the simple concept of absolute value representing the relevant stiffness reduction or increase under unloading or loading conditions [2].

It is a widespread opinion that the natural generalization of elasticity theory is the so-called theory of elastoplasticity. This is why most constitutive laws so far proposed for soils belong to the family of elastoplastic laws. Such laws consist of a set of linear relations connecting the increments of stress and strain. According to the direction of stress (or strain) increment, the appropriate linear relation must be chosen. Mathematically, this choice is made by switch functions. Numerically, the appropriate linear relation can only be found by iterations. Elastoplastic equations have advanced to a considerable degree of sophistication, and many of them have been reported to be successful in describing many aspects of soil behaviour [1]. However, the complex structure and many auxiliary notions of elastoplastic formulations, such as yield surface, plastic potential, and hardening/softening behaviour, hinder a direct insight into the modelled soil behaviour. It should be kept in mind that the usual elastoplastic formulations are not the only possible approach to the inelastic behaviour of solids. In this paper, a constitutive equation is proposed that departs entirely from the theory of elastoplasticity.

2 Constitutive equation formulation using the hypoplastic framework

In this paper, the symbolic tensor notation of [1] is followed. Herein, the several tensors have the following representation in components: σ is the Cauchy stress,

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, \tag{1}$$

$\dot{\varepsilon}$ and w are the stretching and spin tensors, respectively,

$$\dot{\varepsilon} = \frac{1}{2}(\nabla v + \nabla v^T);\ w = \frac{1}{2}(\nabla v - \nabla v^T), \tag{2}$$

$$\dot{\varepsilon} = \begin{pmatrix} \dot{\varepsilon}_{11} & \dot{\varepsilon}_{12} & \dot{\varepsilon}_{13} \\ \dot{\varepsilon}_{21} & \dot{\varepsilon}_{22} & \dot{\varepsilon}_{23} \\ \dot{\varepsilon}_{31} & \dot{\varepsilon}_{32} & \dot{\varepsilon}_{33} \end{pmatrix};\ w = \begin{pmatrix} 0 & w_{12} & w_{13} \\ -w_{12} & 0 & w_{23} \\ -w_{13} & -w_{23} & 0 \end{pmatrix}, \tag{3}$$

where tensor w represents the rotation of the element and $\dot{\varepsilon}$ represents the rate of deformation. Trace of tensor X is the first invariant of X

$$tr(X) = x_{11} + x_{22} + x_{33}. \tag{4}$$

The constitutive equation represents the co-rotated stress rate $\dot{\boldsymbol{\sigma}}$ as a single tensor valued function $\dot{\sigma} = h(\sigma, \dot{\varepsilon})$ of the stress σ and the stretching tensor $\dot{\varepsilon}$. Apart from those quantities, only some material constants C_i appear in this equation. Note that the somewhat artificial distinction between elastic and plastic strains is not used here. Due to the formulation in rates, the constitutive equation is equally well applicable to problems with small and large deformations, provided that in the course of a numeric integration, the spatial coordinates of the material points are updated at every time step.

Although the final form of the constitutive equation is simple enough, a systematic, step-by-step presentation will be given in the following in the hope that this will facilitate comprehension. Let us start with the linear elastic constitutive equation, as this can be expressed in rates:

$$\dot{\sigma} = 2\mu\dot{\varepsilon} + \lambda tr(\dot{\varepsilon})1. \tag{5}$$

The constants μ and λ (so-called Lame constants) represent the stiffness of the material. With soils, the stiffness is not constant, but it depends on both σ and $\dot{\varepsilon}$. In particular, instantaneous stiffnesses increase with increasing stress levels for constant stress and strain ratios. To model this fact as simple as possible, one can replace strain rate $\dot{\varepsilon}$ in equation (5) by the symmetric tensor $(\sigma\dot{\varepsilon} + \dot{\varepsilon}\sigma)/2$. Renaming the Lame constants μ and λ to more general C_1 and C_2, one can rewrite equation (1) as:

$$\dot{\sigma} = C_1 \frac{(\sigma\dot{\varepsilon} + \dot{\varepsilon}\sigma)}{2} + C_2 tr(\sigma\dot{\varepsilon})1. \tag{6}$$

The above equation is a hypoelastic one and exhibits stress-dependent stiffnesses. Since (6) is homogeneous to the first degree in *σ,* instantaneous stiffness increases linearly with the stress level. Equation (6) is also linear in $\dot{\varepsilon}$. That means that it is rate-independent (i.e. the material behaviour is invariant to time scale transformations) [3]. Nevertheless, it also means that the stiffnesses are not changed if we replace $\dot{\varepsilon}$ by $-\dot{\varepsilon}$. Thus, equal stiffnesses are predicted, e.g. for loading and unloading, which does not correspond to reality. To meet this shortcoming, we added a further term that is still homogeneous of the first degree in $\dot{\varepsilon}$ (to preserve rate independence) but non-linear in $\dot{\varepsilon}$. The equation now reads,

$$\dot{\sigma} = C_1 \frac{(\sigma\dot{\varepsilon} + \dot{\varepsilon}\sigma)}{2} + C_2 tr(\sigma\dot{\varepsilon})1 + C_3\sigma\sqrt{tr(\dot{\varepsilon}^2)} \ . \tag{7}$$

To describe the soil behaviour realistically, one needs to extend the equation (7) by further term, which is similar to the third one:

$$\dot{\sigma} = C_1 \frac{(\sigma\dot{\varepsilon} + \dot{\varepsilon}\sigma)}{2} + C_2 tr(\sigma\dot{\varepsilon})1 + C_3\sigma\sqrt{tr(\dot{\varepsilon}^2)} + C_4 \frac{\sigma^2}{tr(\sigma)}\sqrt{tr(\dot{\varepsilon}^2)} \ . \tag{8}$$

The obtained equation (8) is capable of describing many aspects of sand behaviour. For example, the familiar stress-strain curves for triaxial, oedometric, and simple shear loading and unloading can be easily obtained. Other properties such as limit condition and dilatancy are also included.

Before any particular numerical calculation, the material constants C_1, C_2, C_3, C_4 must be determined. The material constants of the hypoplastic equation have relations with natural material characteristics, but this aspect represents a severe drawback of hypoplastic behaviour. Therefore, the process of hypoplastic parameters determination needs not only scalar values of soil characteristics but the course of soil tests [4]. From triaxial and oedometric tests, the empirical relation for constants C_1, C_2, C_3, C_4 can be derived, assuming the fine-grained soil.

$$a = \frac{1+\sin\varphi}{1+\sin\varphi}; \quad b = \frac{a-1}{a+2}[(1-a)(2a+1)+3a] \ . \tag{9}$$

Moreover, one can obtain the following relations.

$$C_1 = \frac{E_0}{\sigma}; C_2 = C_1 \frac{a}{b}\left(1.5 - 3\left(\frac{1+\frac{a}{2}}{a+2}\right)\right)$$
$$C_3 = \frac{-C_1}{b}\left[(1-a)(a+0.5) + 3a\frac{1+\frac{a}{2}}{a+2}\right]; C_4 = 3(C_2 - C_3) \ . \tag{10}$$

The C_1 can be obtained by fitting the constitutive law to oedometric compression tests. Next, the model's stiffness is determined using the oedometric E_{oed} modulus as parameter E_0 with corresponding reference pressure as *σ*. Next, the strength of soil material using parameters C_i is represented by shear resistance angle only. Finally, cohesion *c* is incorporated by translating the stress point in the direction of the hydrostatic line [5].

$$\sigma = \begin{pmatrix} \sigma_{11} - c & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - c & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - c \end{pmatrix} . \tag{11}$$

The constitutive law can now be implemented in any numerical calculation. So-called element tests (i.e. tests with homogeneous deformation of the samples) can be immediately simulated without any advanced numerical methods such as FEM or finite differences: Any loading (and/or unloading) process starting from a known initial stress state can be numerically simulated by stepwise or rather multistep (in case of rate dependent version of constitutive model) integration [6] of the constitutive equation (8). If all boundary conditions are of the kinematic type (e.g. oedometer test), the $\dot{\varepsilon}$-tensor is wholly known, and the corresponding stress rate $\dot{\sigma}$ can be directly obtained from equation (8). If some of the boundary conditions are of the static type (e.g. with the triaxial test, where constant lateral stress is imposed on the sample), then at each integration step, the stretching tensor $\dot{\varepsilon}$ must be determined in such a way that the imposed stress condition is fulfilled. To do this, an algebraic equation or a system of algebraic equations must be numerically solved.

3 Numerical simulation of shallow foundation settlement

As a numerical example, we present the simple concrete shallow foundation on fine soil, Fig. 1. Surface settlement under the foundation depends on the stiffness of underlying soil and, in the case of the numerical model, on the ability of the constitutive model to resolve the stiffness aspects of the material [7].

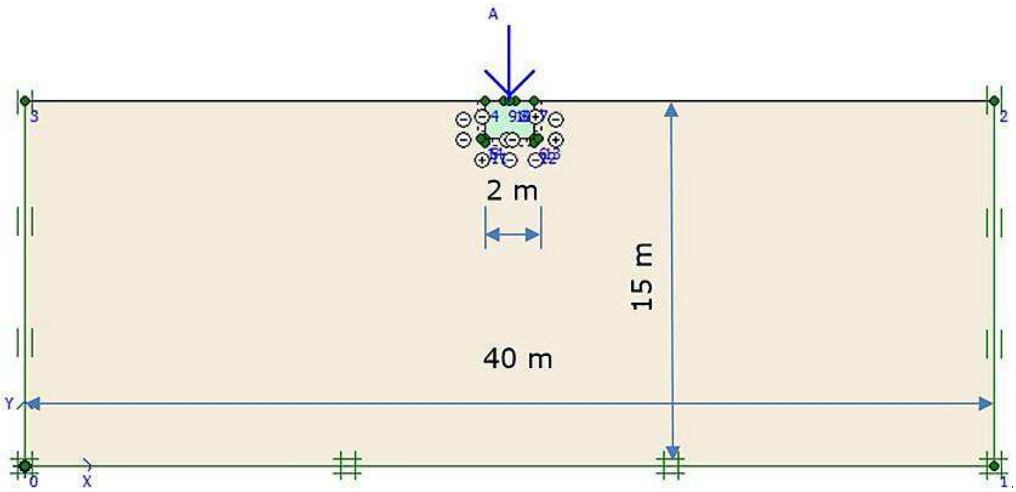

Fig. 1: The shallow foundation settlement simulation situation with dimensions in Plaxis.

The soil and concrete properties are shown in Table 1. The shallow concrete foundation is constructed at a depth of 1.5 m. The dimensions and load boundary conditions are described in detail in Fig. 2.

Table 1: Parameters of material models used in the settlement simulation.

Soil type		**F6/CI**	**Concrete**
Constitutive model		**Mohr-Coulomb**	**Linear elastic**
Volumetric weight (saturated)	[kN/m^3]	20.0	25.0
Oedometric modulus - E_{oed}	[kN/m^2]	6 000	30 000 000
Poisson´s ratio	[-]	0.35	0.15
Cohesion *c*	[kN/m^2]	27.0	
Internal friction angle φ	[°]	23.7	
Constitutive model		**Hypoplastic**	
C_1	[-]	60.0	
C_2	[-]	0.00	
C_3	[-]	-67.56	
C_4	[-]	202.70	

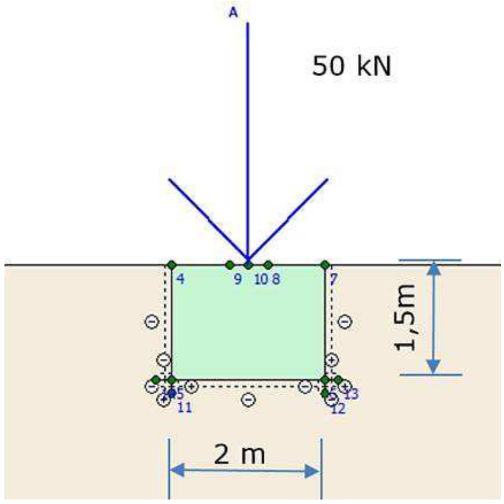

Fig. 2: The load boundary conditions and shallow foundation detail.

A wide range of civil engineering problems is analyzed using numerical methods such as FEM [8, 9], as they can a burdensome and, in most cases, financially unfeasible experimental methods. In order to demonstrate advantages of basic model based on hypoplastic formulation, another study and verification should be introduced egg. calibration with structural and geotechnical monitoring [10-12]. However, one of the most significant advantages of numerical analysis is that it allows the isolation of the governing parameters. The simulation performed using commercial FEM software Plaxis used the model with concrete defined by simple linear elastic model and soil as Mohr-Coulomb elastoplastic material. The other simulation was performed using in-house FEM code with hypoplastic constitutive model implemented using theoretical assumption described in this article, with material parameters in Table 1.

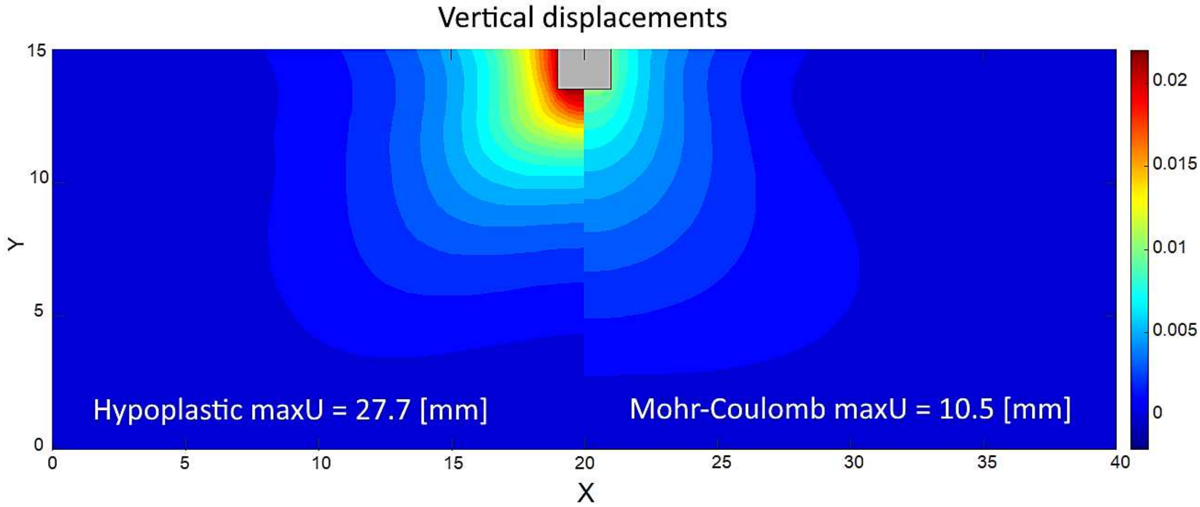

Fig. 3: Vertical displacements contours obtained using Hypoplastic (left) and Mohr-Coulomb (right) model.

The results in the context of displacements (Fig. 3) showed the difference in the settlement obtained using the Hypoplastic and standard Mohr-Coulomb constitutive model. As shown in the Fig. 3, the settlement troughs obtained with the hypoplastic U_{max} = 27.7 mm model agree well with the assumption of stiffness development for a stress accumulation during the loading. To capture such behaviour using a simple elastoplastic model, one needs to assess the stress range in the problem domain.

4 Conclusions

The results showed the qualitative drawbacks of simple elastoplastic models, such as Mohr-Coulomb, in the "elastic" domain, wherein in the case of simple models, an engineer needs to assume stress conditions to define the model stiffness parameters. From a structural design perspective, the logarithmic dependency of stiffness in soils can easily be incorporated into an appropriate numerical model to optimize the foundation design.

References

[1] KOLYMBAS, D.: An outline of hypoplasticity. Arch. Appl. Mech., 61, 1996, pp. 143–151.
[2] BARDET, J. P.: Lode dependences for isotropic pressure-sensitive elastoplastic materials. J. Appl. Mech., ASME 57, 1990, pp. 498–506.
[3] WANG, X. – WU, W.: An Updated Hypoplastic Constitutive Model, Its Implementation and Application. In: Bifurcations, Instabilities, and Degradations in Geomaterials. Springer Series in Geomechanics and Geoengineering, Vol. 0, 2011.
[4] BAUER, E.: Calibration of a comprehensive constitutive model for granular materials. Soils and Foundations, 36, 1995, pp. 13–26.
[5] WU, W. - BAUER, E.: A simple hypoplastic constitutive model for sand. Int. J. Numer. Anal. Methods Geomech., 18, 1994, pp. 833–862.
[6] VALAŠKOVÁ, V. - VLČEK, J.: Analysis of the Tyre Loads of the Truck Tatra 815 in Interaction with the Ground. Civil and Environmental Engineering, Vol. 12, Iss. 2, 2016, pp. 78-83, https://doi.org/10.1515/cee-2016-0011.
[7] IŽVOLT, L. - DOBEŠ, P.: Analysis of Measuring the Deformation Resistance of the Subgrade Surface of a Modernised Line Považská Teplá – Žilina. Civil and Environmental Engineering, Vol. 16, Iss. 1, 2020, pp. 210-218, https://doi.org/10.2478/cee-2020-0020.
[8] PROKOP, J. - VIČAN, J. - JOŠT, J.: Numerical Analysis of the Beam-Column Resistance Compared to Methods by European Standards. Appl. Sci. 11, 2021, 3269.
[9] FARBÁK, M. - JOŠT, J. - HLINKA, R. - ROSMANIT, M.: Numerical Analysis of the Load-Displacement Behaviour of Cast-in-Place Progressive Anchorage in Reinforced Concrete Members. Appl. Sci. 11, 2021, 2343.
[10] SOBALA, D. – RYBAK, J.: Role to Be Played by Independent Geotechnical Supervision in the Foundation for Bridge Construction. 2017 IOP Conf. Ser.: Mater. Sci. Eng. 245 022073.
[11] DRUSA M. - VLČEK J.: Importance of Results Obtained from Geotechnical Monitoring for Evaluation of Reinforced Soil Structure – Case Study. Journal of Applied Engineering Sciences, Vol. 6, Iss. 1, 2016.
[12] PARKASIEWICZ, B. – KADELA, M. – BETKOWSKI, P. – SIENKO, R. - BEDNARSKI, L.: Application of Structure Monitoring Systems to the Assessment of the Behaviour of Bridges in Mining Areas. 2017 IOP Conf. Ser.: Mater. Sci. Eng. 245 032018.