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The distribution of the local time for "pseudoprocesses" and its connection with fractional diffusion equations

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Abstract

We prove that the pseudoprocesses governed by heat-type equations of order $n \geqslant 2$ have a local time in zero (denoted by $L_0^n(t)$) whose distribution coincides with the folded fundamental solution of a fractional diffusion equation of order $2(n-1)/n$, $n \geqslant 2$.

The distribution of $L_0^n(t)$ is also expressed in terms of stable laws of order $n/(n-1)$ and their form is analyzed. Furthermore, it is proved that the distribution of $L_0^n(t)$ is connected with a wave equation as $n \to \infty$.

The distribution of the local time in zero for the pseudoprocess related to the Myiamoto's equation is also derived and examined together with the corresponding telegraph-type fractional equation.

Keywords: Heat-type equation; Fractional diffusion equations; Local time; Feynman–Kac functional; Wright functions; Stable laws; Vandermonde determinant; Mittag–Leffler functions

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1. Introduction

It is well-known that, for the standard Brownian motion B, the local time in zero, which is defined as

$$L_0(t) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{[-\varepsilon,\varepsilon]}(B(s))\, \mathrm{d}s \tag{1.1}$$

has a half-normal distribution coinciding with that of the maximum, that is

$$\Pr\{L_0(t) \in \mathrm{d}s\} = \Pr\left\{ \max_{0 \leqslant z \leqslant t} B(z) \in \mathrm{d}s \right\} = \frac{2\mathrm{d}s}{\sqrt{2\pi t}} \mathrm{e}^{-s^2/2t}, \quad s > 0, \quad t > 0. \tag{1.2}$$

We analyze here the distribution of the local time of pseudoprocesses related to higher-order heat-type equations of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = c_n \dfrac{\partial^n u}{\partial x^n}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \delta(x), \end{cases} \tag{1.3}$$

where $c_n = (-1)^{p+1}$ for $n = 2p$, $p \in \mathbb{N}$, while $c_n = \pm 1$ for $n = 2p + 1$.

The pseudoprocesses connected with equations of the form (1.3) were introduced in the 1960s and have been studied since then by many authors such as Krylov [12], Daletsky and Fomin [7], Ladokhin [14,15] Miyamoto [18] and Daletsky [6].

The fundamental solution u_n to (1.3) is used to construct a signed measure Q_n in the following way. We consider a space of bounded functions (the sample paths of the pseudoprocess) $\mathscr{X} = \{x : t \in [0, \infty) \to x(t)\}$ and a decomposition of the set $[0, t]$ by means of the time points $0 = t_0 < \cdots < t_n = t$.

We define the cylinders

$$C = \{x : a_1 \leqslant x(t_1) \leqslant b_1, \ldots, \leqslant a_n \leqslant x(t_n) \leqslant b_n\}, \tag{1.4}$$

where a_j, b_j are real numbers and

$$Q_n(C) = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{j=1}^n u_n(x_j - x_{j-1}; t_j - t_{j-1})\, \mathrm{d}x_j, \tag{1.5}$$

where we put $x_j = x(t_j)$ (see [12,19–21]).

For fixed $t_1, \ldots, t_n$, Q_n is a finite σ-additive measure on the Borel field generated by the cylinders C and has finite total variation ρ_n:

$$\begin{aligned} \rho_n &= \int_{-\infty}^{+\infty} \mathrm{d}x_1 \cdots \int_{-\infty}^{+\infty} \mathrm{d}x_n \prod_{j=1}^n |u_n(x_j - x_{j-1}; t_j - t_{j-1})| \\ &= \left[\int_{-\infty}^{+\infty} |u_n(x,t)|\, \mathrm{d}x \right]^n > 1, \end{aligned}$$

because $\int_{-\infty}^{+\infty} u_n(x,t)\, \mathrm{d}x = 1$ and u_n is sign varying.

The extension of Q_n is outlined with some details in [7,18], Nishioka [19] for $n = 4$ and follows easily for the general case, in the same way. The mean value with respect to this signed measure is also defined and discussed in [19,21].

The fundamental solution to (1.3) can be written as

$$u_n(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{e}^{c_n(-\mathrm{i}\alpha)^n t}\,\mathrm{d}\alpha \tag{1.6}$$

and becomes, for n even, $c_n = (-1)^{\frac{n}{2}+1}$

$$u_n(x,t) = \frac{1}{\pi}\int_0^{+\infty} \cos(\alpha x)\mathrm{e}^{-\alpha^n t}\,\mathrm{d}\alpha \tag{1.7}$$

while, for n odd, is equal to

$$u_n(x,t) = \begin{cases} \dfrac{1}{\pi}\displaystyle\int_0^{+\infty} \cos(\alpha x + (-1)^{\frac{n+1}{2}}\alpha^n t)\,\mathrm{d}\alpha, & c_n = -1, \\ \dfrac{1}{\pi}\displaystyle\int_0^{+\infty} \cos(\alpha x + (-1)^{\frac{n-1}{2}}\alpha^n t)\,\mathrm{d}\alpha, & c_n = 1. \end{cases} \tag{1.8}$$

Formula (1.6) displays a property of autosimilarity which can be expressed as follows:

$$u_n(x,t) = \frac{1}{t^{1/n}} u_n\left(\frac{x}{t^{1/n}}, 1\right), \quad x \in \mathbb{R}, \ t > 0.$$

The study of functions (1.6) goes back (for some special cases) to Bernstein [3], Hardy and Littlewood [10], Pólya [25], Lévy [16] (see [17,5]). The analysis of the asymptotic behavior of $u_n(x,t)$, for $x \to \pm\infty$, is carried out by Li and Wong [17] (by means of the steepest descent method) and by Lachal [13] and shows that they have an infinite number of zeros and display an oscillatory behavior.

From (1.7) we can conclude that $u_n(x,t)$ is symmetric for n even, while, for n odd, we get

$$u_n^+(x,t) = u_n^-(-x,t), \tag{1.9}$$

where we indicate by u_n^+ and u_n^- the fundamental solutions for $c_n = 1$ and -1, respectively.

For $x = 0$ all the functions u_n are positive-valued (see formula (10) of Lachal [13]). We also note that the integrals $\int_0^{+\infty} u_n(x,t)\,\mathrm{d}x$ do not depend on t (see [13, formula (11)]) and they are equal to

$$\int_0^{+\infty} u_n(x,t)\,\mathrm{d}x = \begin{cases} \dfrac{1}{2} & n \text{ even}, \\ \dfrac{1}{2}\left(1 - \dfrac{1}{n}\right) & n \text{ odd}, c_n = (-1)^{\frac{n-1}{2}}, \\ \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right) & n \text{ odd}, c_n = (-1)^{\frac{n+1}{2}}. \end{cases} \tag{1.10}$$

From (1.6) we have also that

$$\int_{-\infty}^{+\infty} u_n(x,t)\,\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{d}\alpha \mathrm{e}^{c_n(-\mathrm{i}\alpha)^n t}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x = \int_{-\infty}^{+\infty} \delta(\alpha)\mathrm{e}^{c_n(-\mathrm{i}\alpha)^n t}\,\mathrm{d}\alpha = 1. \tag{1.11}$$

For the special case $n = 3$, the function $u_3(x,t)$ can be expressed by means of the Airy functions.

For the investigation of functionals of pseudoprocesses a key role is played by the Feynman–Kac functional (see the authoritative analysis in [4]), which has been generalized to the case of pseudoprocesses by Daletsky and Fomin [7] and by Krylov [12], who applied it to the derivation of the arc-sine law for pseudoprocesses of even order.

For the convenience of the reader we state this result here:

Theorem 1.1. *Let V be a piecewise continuous function. The initial-value problem*

$$\begin{cases} \dfrac{\partial w}{\partial t} = c_n \dfrac{\partial^n w}{\partial x^n} - Vw, \\ w(x,0) = 1 \end{cases} \tag{1.12}$$

has solution

$$w(x,t) = E_x\left\{\mathrm{e}^{-\int_0^t V(X(s))\,\mathrm{d}s}\right\}, \tag{1.13}$$

where

$$E_x\left\{\mathrm{e}^{-\int_0^t V(X(s))\,\mathrm{d}s}\right\} = \lim_{n\to\infty} \underbrace{\int \cdots \int}_{\mathbb{R}^n} \mathrm{e}^{-\sum_{j=1}^n V(x_{j-1})(t_j - t_{j-1})} \prod_{j=1}^n u(x_j - x_{j-1}; t_j - t_{j-1})\,\mathrm{d}x_j, \tag{1.14}$$

provided that the limit exists.

The previous theorem transforms the research on the distribution of the functionals of pseudoprocesses into an analytical problem (namely the solution of the Cauchy problem (1.12)).

For $V(x) = \beta 1_{[0,\infty)}(x)$, from Theorem 1.1, it is possible to obtain the Laplace transform of the sojourn time on the positive half-line. This has been done by Krylov [12], in the even order case, and, in the odd case, by Orsingher [22], Hochberg and Orsingher [11] and Lachal [13].

For $V(x) = \beta 1_{[a,\infty)}(x)$, $a > 0$, the distribution of the maximum of pseudoprocesses has been obtained in [1,2], for $n = 3, 4$, and in [13], for any order $n > 2$.

Other functionals, such as the hitting time and place of a half-line have been studied by Nishioka [20], for the case $n = 4$.

Our paper is devoted to the study of the distribution of the local time in zero

$$L_0^n(t) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t 1_{[-\varepsilon,\varepsilon]}(X(s))\,\mathrm{d}s, \tag{1.15}$$

for the pseudoprocess $X(t), t > 0$. This analysis can be carried out by applying Theorem 1.1 to the case where

$$V(x) = \begin{cases} \dfrac{\beta}{2\varepsilon}, & x \in [-\varepsilon, \varepsilon], \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

Our first result is the following double Laplace transform:

$$\int_0^\infty e^{-\lambda t} E e^{-\beta L_0^n(t)} \, dt = \frac{1}{\sqrt[n]{\lambda}} \frac{1}{\sqrt[n]{\lambda^{n-1}} + \frac{\beta}{\delta_n \sin \frac{\pi}{\delta_n}}}, \quad \lambda, \beta > 0, \tag{1.17}$$

where

$$\delta_n = \begin{cases} n & \text{for } n \text{ even,} \\ 2n & \text{for } n \text{ odd.} \end{cases} \tag{1.18}$$

We also prove that the local time $L_0^n(t)$ possesses a genuine probability distribution for all n which is connected with the solution of the following fractional diffusion equation:

$$\begin{cases} \dfrac{\partial^{2(n-1)/n} u}{\partial t^{2(n-1)/n}} = \dfrac{1}{\left(\delta_n \sin \frac{\pi}{\delta_n}\right)^2} \dfrac{\partial^2 u}{\partial x^2}, \\ u(x, 0) = \delta(x), \quad x \in \mathbb{R}, \quad t > 0. \\ u_t(x, 0) = 0. \end{cases} \tag{1.19}$$

The connection between the distribution of $L_0^n(t)$ and Eq. (1.19) is given by the relationship

$$\Pr\{L_0^n(t) \in ds\} = 2\, ds\, u_{2(n-1)/n}(s, t) \quad \text{for } s > 0, \tag{1.20}$$

where $u_{2(n-1)/n}(s, t) 1_{[0,\infty)}(s)$ is the folded solution of the fractional equation (1.19).

For $n = 2$ formula (1.20) coincides, up to a constant, with the well-known relationship of Brownian motion (1.2): indeed, in this case, (1.19) becomes the heat equation with diffusion coefficient $1/2$.

Formula (1.20) permits us to derive the form of the distribution of $L_0^n(t)$ from that of $u_{2(n-1)/n}$ (which is studied in [9]), via its representation in terms of stable laws. The distribution of $L_0^n(t)$ has a positive maximum, the position of which is given by $x = k_{2(n-1)/n} t^{n/(n-1)}$ and $k_{2(n-1)/n}$ is a constant depending on the degree of the fractional equation (1.19). This implies that, for pseudoprocesses, this law has a unimodal structure and the most likely values of $L_0^n(t)$ are, in general, not located at the origin and move rightward as $n \to \infty$. Only in the Brownian case ($n = 2$) the distribution decreases for all $x > 0$.

At the limit, as $n \to \infty$, Eq. (1.19) becomes the wave equation with propagation parameter equal to $1/\pi$. This means that, in this case, the distribution of $L_0^n(t)$ tends to degenerate around $x = t/\pi$.

The last section of the paper is devoted to the study of the distribution of the local time in zero, denoted as $L_0^M(t)$, of a pseudoprocess related to the Myiamoto equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + 2k\frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \ t>0, \ k \in \mathbb{R},$$
$$u(x,0) = \delta(x). \tag{1.21}$$

We obtain the double Laplace transform

$$W(\beta, \lambda, 0) = \frac{2\sqrt{2}\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}(2\sqrt{2}\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}+\beta)}, \quad \lambda, \beta>0 \tag{1.22}$$

and we are able to show that the distribution of $L_0^M(t)$ can be constructed from the fundamental solution to the fractional telegraph-type equation:

$$\begin{cases} \dfrac{\partial^{3/2} u}{\partial t^{3/2}} + k\dfrac{\partial u}{\partial t} = \dfrac{1}{2^3}\dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \delta(x), \quad x \in \mathbb{R}, \ t>0, \ k \in \mathbb{R}, \\ u_t(x,0) = 0. \end{cases} \tag{1.23}$$

From our analysis we obtain that

$$\int_0^\infty e^{-\lambda t} \Pr\{L_0^M(t) \in ds\}\, dt = 2\sqrt{2}\frac{\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}} e^{-2\sqrt{2}s\sqrt{\sqrt{\lambda}+k}\sqrt{\lambda}}\, ds, \quad s>0 \tag{1.24}$$

but extracting from (1.24) the distribution of $L_0^M(t)$ seems prohibitively complicated.

2. About the double Laplace transform of the local time

In this section we derive the general expression for the double Laplace transform of the local time in zero for the pseudoprocesses $X(t), t>0$

$$W_n(\beta, \lambda, 0) = \int_0^\infty e^{-\lambda t} E(e^{-\beta L_0^n(t)} | X(0) = 0)\, dt, \quad n \geqslant 2, \ \beta, \lambda > 0. \tag{2.1}$$

By choosing $V(x)$ as specified in (1.16) the Feynman–Kac functional (1.13) becomes

$$w_n^\varepsilon(\beta, t, x) = E\left(e^{-\frac{\beta}{2\varepsilon}\int_0^t 1_{[-\varepsilon,\varepsilon]}(X(s))\, ds} \middle| X(0) = x \right) \tag{2.2}$$

and its limit for $\varepsilon \to 0$ represents the Laplace transform of $L_0^n(t)$.

By applying Theorem 1.1 function (2.2) can be obtained by solving the Cauchy problem

$$\begin{cases} \dfrac{\partial w}{\partial t} = c_n \dfrac{\partial^n w}{\partial x^n} - \dfrac{\beta}{2\varepsilon} 1_{[-\varepsilon,\varepsilon]}(x) w, \\ w(x,0) = 1. \end{cases} \tag{2.3}$$

The technique used to solve (2.3) is based on the Laplace transform

$$W_n(\beta,\lambda,x)=\int_0^\infty \mathrm{e}^{-\lambda t}w_n(\beta,t,x)\,\mathrm{d}t=\int_0^\infty \mathrm{e}^{-\lambda t}E(\mathrm{e}^{-\beta L_0^n(t)}|X(0)=x)\mathrm{d}t, \tag{2.4}$$

where $w_n(\beta,t,x)=\lim_{\varepsilon\to 0}w_n^\varepsilon(\beta,t,x)$.

By taking the Laplace transform of (2.3) we get

$$-1+\lambda W_n^\varepsilon = c_n\frac{\partial^n W_n^\varepsilon}{\partial x^n}-\frac{\beta}{2\varepsilon}1_{[-\varepsilon,\varepsilon]}(x)W_n^\varepsilon \tag{2.5}$$

with the initial condition in (2.3) yielding the non-homogeneous term.

Then, by integrating (2.5) in $[-\varepsilon,\varepsilon]$ and letting $\varepsilon\to 0$, we prove that the derivative of order $n-1$ must have a discontinuity in $x=0$, so that we have the condition

$$c_n\left\{\frac{\partial^{n-1}W_n}{\partial x^{n-1}}\bigg|^{x=0^+}-\frac{\partial^{n-1}W_n}{\partial x^{n-1}}\bigg|^{x=0^-}\right\}=\beta W_n(\beta,\lambda,0). \tag{2.6}$$

Our problem consists therefore in solving the nth-order linear equation

$$c_n\frac{\partial^n W_n}{\partial x^n}=\lambda W_n-1,\quad \text{for } x\neq 0 \tag{2.7}$$

with the matching conditions

$$\begin{aligned}&\frac{\partial^j W_n}{\partial x^j}\bigg|^{x=0^+}-\frac{\partial^j W_n}{\partial x^j}\bigg|^{x=0^-}=0,\quad j=0,1,\ldots,n-2,\\ &c_n\left\{\frac{\partial^{n-1}W_n}{\partial x^{n-1}}\bigg|^{x=0^+}-\frac{\partial^{n-1}W_n}{\partial x^{n-1}}\bigg|^{x=0^-}\right\}=\beta W_n(\beta,\lambda,0).\end{aligned} \tag{2.8}$$

We state the following result:

Theorem 2.1. *The double Laplace transform* $W_n(\beta,\lambda,0)$ *has the following explicit form*:

$$W_n(\beta,\lambda,0)=\begin{cases}\dfrac{1}{\sqrt[n]{\lambda}}\dfrac{1}{\sqrt[n]{\lambda^{n-1}}+\frac{\beta}{2n\sin\frac{\pi}{2n}}} & \text{for } n \text{ odd},\\[2ex] \dfrac{1}{\sqrt[n]{\lambda}}\dfrac{1}{\sqrt[n]{\lambda^{n-1}}+\frac{\beta}{n\sin\frac{\pi}{n}}} & \text{for } n \text{ even},\end{cases} \tag{2.9}$$

for $\lambda,\beta>0$ *and* $n\geqslant 2$.

Proof. The general bounded solution to Eq. (2.7) reads

$$W_n(\beta,\lambda,x)=\begin{cases}\sum_{k\in I}b_k\mathrm{e}^{\theta_k\sqrt[n]{\lambda}x}+\frac{1}{\lambda}, & x>0,\\ \sum_{k\in J}d_k\mathrm{e}^{\theta_k\sqrt[n]{\lambda}x}+\frac{1}{\lambda}, & x<0,\end{cases} \tag{2.10}$$

where θ_k, $k=0,1,\dots,n-1$ are the roots of c_n, $I=\{k\in(0,1,\dots,n-1):\mathbb{R}\theta_k<0\}$ and $J=\{k\in(0,1,\dots,n-1):\mathbb{R}\theta_k>0\}$.

The boundedness of the solutions explains why, for $x>0$, the first sum is restricted to I, while, for $x<0$, the second one applies only to J.

By taking into account the matching conditions (2.8) we get the linear system

$$\begin{cases} \sum_{k\in I}\theta_k^l b_k - \sum_{k\in J}\theta_k^l d_k = 0 \quad \text{for } l=0,1,\dots,n-2, \\ \sum_{k\in I}\theta_k^{n-1} b_k - \sum_{k\in J}\theta_k^{n-1} d_k = \dfrac{\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}}. \end{cases} \tag{2.11}$$

System (2.11) can be rewritten in a Vandermonde form as follows: let

$$x_k = \begin{cases} b_k & \text{for } k\in I, \\ -d_k & \text{for } k\in J, \end{cases}$$

then (2.11) can be rewritten as

$$\sum_{k=0}^{n-1}\theta_k^l x_k = \begin{cases} 0, & l=0,1,\dots,n-2, \\ \dfrac{\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}}, & l=n-1 \end{cases} \tag{2.12}$$

or, alternatively, as

$$\begin{pmatrix} 1 & 1 & .. & 1 & .. & 1 \\ \theta_0 & \theta_1 & .. & \theta_k & .. & \theta_{n-1} \\ \theta_0^2 & \theta_1^2 & .. & \theta_k^2 & .. & \theta_{n-1}^2 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ \theta_0^{n-1} & \theta_1^{n-1} & .. & \theta_k^{n-1} & .. & \theta_{n-1}^{n-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ .. \\ .. \\ .. \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ .. \\ .. \\ 0 \\ \dfrac{\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}} \end{pmatrix}. \tag{2.13}$$

A crucial point in solving (2.13) is the use of the Vandermonde determinant. We first note that the determinant of the matrix appearing in (2.13) can be written as

$$\prod_{j=1}^{n-1}(\theta_j-\theta_0)\prod_{j=2}^{n-1}(\theta_j-\theta_1)\dots\prod_{j=n-1}^{n-1}(\theta_j-\theta_{n-2}) = \prod_{r=0}^{n-2}\prod_{j=r+1}^{n-1}(\theta_j-\theta_r) = \prod_{j=1}^{n-1}\prod_{r=0}^{j-1}(\theta_j-\theta_r). \tag{2.14}$$

In order to evaluate x_k we must replace the $(k+1)$th column of the matrix in (2.13) with the right-hand side vector of the system thus obtaining a Vandermonde submatrix of rank $n-1$ whose value repeats the form (2.14) with the only additional condition that $j,r\neq k$.

Thus

$$\begin{aligned} x_k &= \frac{\frac{(-1)^{n+k+1}}{c_n}\frac{\beta W_n(\beta,\lambda,0)}{\sqrt[n]{\lambda^{n-1}}}\prod_{j=1}^{n-1}\prod_{\substack{r=0\\ j,r\neq k}}^{j-1}(\theta_j-\theta_r)}{\prod_{j=1}^{n-1}\prod_{r=0}^{j-1}(\theta_j-\theta_r)} \\ &= \frac{\frac{(-1)^{n+k+1}}{c_n}\frac{\beta W_n(\beta,\lambda,0)}{\sqrt[n]{\lambda^{n-1}}}\prod_{j=1}^{k-1}\prod_{r=0}^{j-1}(\theta_j-\theta_r)\prod_{j=k+1}^{n-1}\prod_{\substack{r=0\\ r\neq k}}^{j-1}(\theta_j-\theta_r)}{\prod_{j=1}^{k-1}\prod_{r=0}^{j-1}(\theta_j-\theta_r)\prod_{r=0}^{k-1}(\theta_k-\theta_r)\prod_{j=k+1}^{n-1}\prod_{r=0}^{j-1}(\theta_j-\theta_r)} \\ &= \frac{(-1)^{n+k+1}}{c_n}\frac{\beta W_n(\beta,\lambda,0)}{\sqrt[n]{\lambda^{n-1}}}\frac{1}{\prod_{r=0}^{k-1}(\theta_k-\theta_r)\prod_{j=k+1}^{n-1}(\theta_j-\theta_k)} \\ &= \frac{(-1)^{n+k+1}\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}}\frac{1}{(-1)^k\prod_{\substack{r=0\\ r\neq k}}^{n-1}(\theta_r-\theta_k)} \\ &= \frac{(-1)^{n+1}\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}\prod_{\substack{r=0\\ r\neq k}}^{n-1}(\theta_r-\theta_k)}. \end{aligned} \tag{2.15}$$

We are interested in solution (2.10), for $x=0$, which reads

$$\begin{aligned} W_n(\beta,\lambda,0) &= \frac{1}{\lambda}+\sum_{k\in I}x_k = \frac{1}{\lambda}+\sum_{k\in I}b_k \\ &= \frac{1}{\lambda}+\frac{(-1)^{n+1}\beta W_n(\beta,\lambda,0)}{c_n\sqrt[n]{\lambda^{n-1}}}\sum_{k\in I}\frac{1}{\prod_{\substack{r=0\\ r\neq k}}^{n-1}(\theta_r-\theta_k)}. \end{aligned} \tag{2.16}$$

From (2.16) we can extract the following expression for the double Laplace transform $W_n(\beta,\lambda,0)$ as follows:

$$W_n(\beta,\lambda,0) = \frac{\frac{1}{\lambda}}{1+\frac{(-1)^n\beta}{c_n\sqrt[n]{\lambda^{n-1}}}\sum_{k\in I}\left(\prod_{\substack{r=0\\ r\neq k}}^{n-1}(\theta_r-\theta_k)\right)^{-1}}\ . \tag{2.17}$$

Our next task is the evaluation of the product in (2.17). Since

$$x^n-c_n=\prod_{r=0}^{n-1}(x-\theta_r)=\prod_{\substack{r=0\\ r\neq k}}^{n-1}(x-\theta_r)(x-\theta_k),$$

where $\theta_j, j=0,\dots,n-1$ are the roots of c_n, we have that

$$\begin{aligned} \prod_{\substack{r=0\\ r\neq k}}^{n-1}(\theta_k-\theta_r) &= \lim_{x\to\theta_k}\frac{x^n-c_n}{x-\theta_k}=n\theta_k^{n-1}=\begin{cases} n\mathrm{e}^{\mathrm{i}\frac{2k\pi}{n}(n-1)} & \text{for } c_n=1,\\ n\mathrm{e}^{\mathrm{i}\frac{(2k+1)\pi}{n}(n-1)} & \text{for } c_n=-1,\end{cases} \\ &= \begin{cases} n\mathrm{e}^{-\frac{2k\pi\mathrm{i}}{n}} & \text{for } c_n=1,\\ -n\mathrm{e}^{-\frac{(2k+1)\pi\mathrm{i}}{n}} & \text{for } c_n=-1.\end{cases} \end{aligned} \tag{2.18}$$

In conclusion we have the following expression for the product in (2.17):

$$\prod_{\substack{r=0\\ r\neq k}}^{n-1}\frac{1}{(\theta_r-\theta_k)}=\begin{cases}\dfrac{(-1)^{n-1}}{n}e^{\frac{2k\pi i}{n}} & \text{for } c_n=1,\\[2mm] \dfrac{(-1)^{n}}{n}e^{\frac{(2k+1)\pi i}{n}} & \text{for } c_n=-1.\end{cases} \tag{2.19}$$

In order to perform the sum $\sum_{k\in I}e^{\frac{2k\pi i}{n}}$ we need the following elementary formula:

$$\begin{aligned}\sum_{k=a}^{b}e^{\frac{2k\pi i}{n}}&=\frac{1-e^{\frac{2\pi i}{n}(b+1)}}{1-e^{\frac{2\pi i}{n}}}-\frac{1-e^{\frac{2\pi i}{n}a}}{1-e^{\frac{2\pi i}{n}}}=\frac{e^{\frac{\pi(a+b)i}{n}}}{e^{\frac{\pi i}{n}}}\frac{e^{\frac{\pi i}{n}(a-b)}-e^{\frac{\pi i}{n}(2+b-a)}}{e^{-\frac{\pi i}{n}}-e^{\frac{\pi i}{n}}}\\ &=e^{\frac{\pi(a+b)i}{n}}\frac{e^{\frac{\pi i}{n}(b-a+1)}-e^{-\frac{\pi i}{n}(b-a+1)}}{e^{\frac{\pi i}{n}}-e^{-\frac{\pi i}{n}}}=\frac{\sin\frac{(b-a+1)\pi}{n}}{\sin\frac{\pi}{n}}e^{\frac{\pi(a+b)i}{n}}\end{aligned} \tag{2.20}$$

(see also [13, formula (25)]).

The set I can be given explicitly in all cases examined as follows.

We start by considering the odd-order case, that is $n=2p+1$. For $c_n=1$

$$I=\begin{cases}\left\{\frac{p}{2}+1,\dots,\frac{3p}{2}\right\} & \text{for } p \text{ even},\\[2mm] \left\{\frac{p+1}{2},\dots,\frac{3p+1}{2}\right\} & \text{for } p \text{ odd},\end{cases} \tag{2.21}$$

while for $c_n=-1$

$$I=\begin{cases}\left\{\frac{p}{2},\dots,\frac{3p}{2}\right\} & \text{for } p \text{ even},\\[2mm] \left\{\frac{p+1}{2},\dots,\frac{3p-1}{2}\right\} & \text{for } p \text{ odd}.\end{cases} \tag{2.22}$$

In order to verify the previous expressions we note that, in general, the roots $\theta_k=e^{\frac{2k\pi i}{n}}$ of 1 and the roots $\theta_k=e^{\frac{(2k+1)\pi i}{n}}$ of -1 have negative real parts for $k\in U=(\frac{n}{4},\frac{3n}{4})$ and $k\in V=(\frac{n-2}{4},\frac{3n-2}{4})$, respectively.

These intervals become, for $n=2p+1$, $U=(\frac{2p+1}{4},\frac{6p+3}{4})$ and $V=(\frac{2p-1}{4},\frac{6p+1}{4})$. We need then to distinguish between the cases of odd and even p.

If $p=2m$, clearly $U=(m+\frac{1}{4},3m+\frac{3}{4})$ so that, for $c_n=1$ we have that $k\in[m+1,3m]=[\frac{p}{2}+1,\frac{3p}{2}]$, while $V=(m-\frac{1}{4},3m+\frac{1}{4})$ and thus, for $c_n=-1$, we get $k\in[\frac{p}{2},\frac{3p}{2}]$.

If $p=2m+1$, we see that $U=(m+\frac{1}{4},3m+\frac{9}{4})$ so that, for $c_n=1$ it results that $k\in[m+1,3m+2]=[\frac{p+1}{2},\frac{3p+1}{2}]$. Finally $V=(m+\frac{1}{4},3m+\frac{7}{4})$ and thus, for $c_n=-1$, we get $k\in[m+1,3m+1]=[\frac{p+1}{2},\frac{3p-1}{2}]$.

We now show that, for $n=2p+1$,

$$\sum_{k\in I}e^{\frac{2k\pi i}{n}}=-\frac{1}{2\sin\frac{\pi}{2n}} \tag{2.23}$$

in the four cases $c_n=1$ (for $p=2m$ and $p=2m+1$) and $c_n=-1$ (again for $p=2m$ and $p=2m+1$).

Consider first the cases where $c_n = 1$: by applying formula (2.20) we get that

$$\sum_{k\in I} e^{\frac{2k\pi i}{n}} = \begin{cases} \sum_{k=\frac{p}{2}+1}^{\frac{3p}{2}} e^{\frac{2k\pi i}{n}} = \frac{e^{i\pi}\sin\left(\frac{n-1}{2n}\right)\pi}{\sin\frac{\pi}{n}} = \frac{e^{i\pi}\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{\pi}{n}}, & p = 2m, \\ \sum_{k=\frac{p+1}{2}}^{\frac{3p+1}{2}} e^{\frac{2k\pi i}{n}} = \frac{e^{i\pi}\sin\left(\frac{n+1}{2n}\right)\pi}{\sin\frac{\pi}{n}} = \frac{e^{i\pi}\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{\pi}{n}}, & p = 2m+1. \end{cases}$$

For the case $c_n = -1$ we have, analogously

$$\sum_{k\in I} e^{\frac{(2k+1)\pi i}{n}} = \begin{cases} \sum_{k=\frac{p}{2}}^{\frac{3p}{2}} e^{\frac{(2k+1)\pi i}{n}} = \frac{e^{i\pi}\sin\left(\frac{n+1}{2n}\right)\pi}{\sin\frac{\pi}{n}} = \frac{e^{i\pi}\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{\pi}{n}}, & p = 2m, \\ \sum_{k=\frac{p+1}{2}}^{\frac{3p-1}{2}} e^{\frac{(2k+1)\pi i}{n}} = \frac{e^{i\pi}\sin\left(\frac{n-1}{2n}\right)\pi}{\sin\frac{\pi}{n}} = \frac{e^{i\pi}\cos\left(\frac{\pi}{2n}\right)}{\sin\frac{\pi}{n}}, & p = 2m+1 \end{cases}$$

and thus result (2.23) is confirmed.

In the even-order case, that is $n = 2p$, $c_n = (-1)^{p+1}$ we have two cases, namely $p = 2m+1$ and $p = 2m$.

For $p = 2m+1$ the roots $\theta_k = e^{\frac{2k\pi i}{n}}$ of 1, have negative real parts for k belonging to the set

$$I = \left\{\frac{p+1}{2}, \dots, \frac{3p-1}{2}\right\} \tag{2.24}$$

and thus

$$\sum_{k\in I} e^{\frac{2k\pi i}{n}} = \sum_{k=\frac{p+1}{2}}^{\frac{3p-1}{2}} e^{\frac{2k\pi i}{n}} = \frac{e^{i\pi}\sin\frac{\pi}{2}}{\sin\frac{\pi}{n}} = -\frac{1}{\sin\frac{\pi}{n}}. \tag{2.25}$$

For $p = 2m$ the roots $\theta_k = e^{\frac{(2k+1)\pi i}{n}}$ of -1, have negative real parts for k belonging to the set

$$I = \left\{\frac{p}{2}, \dots, \frac{3p}{2} - 1\right\} \tag{2.26}$$

and we have again, by applying formula (2.20) that

$$\sum_{k\in I} e^{\frac{(2k+1)\pi i}{n}} = \sum_{k=\frac{p}{2}}^{\frac{3p}{2}-1} e^{\frac{(2k+1)\pi i}{n}} = -\frac{1}{\sin\frac{\pi}{n}}. \tag{2.27}$$

In view of (2.23), (2.25) and (2.27) we obtain the claimed result (2.9). □

Remark 2.1. If $n = 2$, $c_2 = \frac{1}{2}$ we obtain, for Brownian motion B, that

$$\int_0^\infty e^{-\lambda t} E\left(e^{-\beta L_0(t)} \middle| B(0) = 0\right) dt$$

$$= \frac{\sqrt{2}}{\sqrt{\lambda}\left(\sqrt{2\lambda} + \beta\right)}$$

$$= \int_0^\infty e^{-\lambda t}\, dt \int_0^\infty e^{-\beta s}\, ds \int_s^\infty 2y \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t^3}}\, dy. \tag{2.28}$$

From (2.28) it is straightforward that, for $s > 0$, we have

$$\Pr\{L_0(t) \in ds\} = ds \int_s^\infty 2y \frac{e^{-\frac{y^2}{2t}}}{\sqrt{2\pi t^3}}\, dy = 2\frac{e^{-\frac{s^2}{2t}}\, ds}{\sqrt{2\pi t}} = \Pr\left\{\max_{0 \leqslant z \leqslant t} B(z) \in ds\right\}. \tag{2.29}$$

From the last expression we can conclude that the law of $L_0(t)$ is obtained by folding the transition function of Brownian motion (the half-normal) around the origin. Furthermore (2.29) shows the important coincidence of the distribution of the local time with that of the maximum of B.

For a non-standard Brownian motion (with $n = 2$ and $c_2 = 1$), instead of (2.28), we can derive from (2.9), the following different result:

$$W_2(\beta, \lambda, 0) = \frac{2}{\sqrt{\lambda}\left(2\sqrt{\lambda} + \beta\right)}$$

$$= \int_0^\infty e^{-\lambda t}\, dt \int_0^\infty e^{-\beta s}\, ds \int_s^\infty 2^2\sqrt{2}y \frac{e^{-\frac{(\sqrt{2}y)^2}{2t}}}{\sqrt{2\pi t^3}}\, dy. \tag{2.30}$$

From the previous expression it is transparent that

$$\Pr\{L_0^2(t) \in ds\} = 2\frac{e^{-\frac{s^2}{t}}\, ds}{\sqrt{\pi t}}, \quad s > 0, \quad t > 0. \tag{2.31}$$

Distribution (2.31) is obtained by folding the normal law with variance parameter 1/2.

Note that for the standard Brownian motion (that is for $c_2 = \frac{1}{2}$) the Gaussian distribution, by means of which the law of $L_0(t)$ is constructed, coincides with the transition function of the Brownian motion and both satisfy the heat equation

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \tag{2.32}$$

with $\sigma^2 = 1$. On the contrary, in the non-standard case, for $c_2 = 1$, the transition function of the Brownian motion satisfies Eq. (2.32) with $\sigma^2 = 2$, while the law of $L_0^2(t)$ is a solution of (2.32) with $\sigma^2 = 1/2$.

In general, for any $n > 2$, the equation satisfied by the transition function of the pseudoprocesses is a higher-order heat equation, while the law of $L_0^n(t)$ is obtained by folding around the origin the solution of a time-fractional equation of order $2(n-1)/n$. To give a deeper insight on this point we examine the case $n=3$ in the next remark.

Remark 2.2. Another special case which merits some attention is that related to $n=3$ for which the double Laplace transform (2.9) becomes

$$W_3(\beta,\lambda,0) = \frac{3}{\sqrt[3]{\lambda}\left(3\sqrt[3]{\lambda^2}+\beta\right)} = 3^2\sqrt[3]{\lambda}\int_0^\infty e^{-\beta x}\,dx\int_x^\infty e^{-3y\sqrt[3]{\lambda^2}}\,dy$$
$$= \int_0^\infty e^{-\lambda t} E_{\frac{2}{3},1}\left(-\frac{\beta t^{\frac{2}{3}}}{3}\right)dt, \tag{2.33}$$

where

$$E_{\nu,\eta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\nu k+\eta)}$$

is the Mittag–Leffler function.

The first relationship of (2.33) shows that

$$\int_0^\infty e^{-\lambda t}\Pr\{L_0^3(t)\in ds\}\,dt = ds\,3^2\sqrt[3]{\lambda}\int_s^\infty e^{-3y\sqrt[3]{\lambda^2}}\,dy = \frac{3ds}{\sqrt[3]{\lambda}}e^{-3s\sqrt[3]{\lambda^2}}, \quad s>0. \tag{2.34}$$

Now the Laplace transform of the solution $u_{4/3}(x,t)$ of the initial value problem

$$\begin{cases} \dfrac{\partial^{\frac{4}{3}}u}{\partial t^{\frac{4}{3}}} = \dfrac{1}{3^2}\dfrac{\partial^2 u}{\partial x^2}, \quad x\in\mathbb{R},\ t>0,\\ u(x,0)=\delta(x),\\ u_t(x,0)=0 \end{cases} \tag{2.35}$$

is

$$\int_0^\infty e^{-\lambda t}u_{4/3}(x,t)\,dt = \frac{3}{2}\frac{1}{\sqrt[3]{\lambda}}e^{-3|x|\sqrt[3]{\lambda^2}}, \tag{2.36}$$

as can be derived from formula (3.3) of Orsingher and Beghin [23].

By comparing (2.34) and (2.36) we have that

$$\int_0^\infty e^{-\lambda t}\Pr\left\{L_0^3(t)\in ds\right\}dt = 2ds\int_0^\infty e^{-\lambda t}u_{4/3}(s,t)\,dt \tag{2.37}$$

so that the distribution of $L_0^3(t)$ is obtained by folding the solution of the fractional equation (2.35).

As we shall see in the next section, this is a general result for the law of the local time of pseudoprocesses. Unlike the case of standard Brownian motion, the transition function of pseudoprocesses and the distribution of local time are related to different equations: the former is the fundamental solution to the higher-order heat equation (1.3), while the latter resolves the fractional equation (2.35).

The distribution of $L_0^3(t)$ can be given different representations. One emerges from the above discussion: by using the expression of the solution to the fractional equation (2.35) given in formula (3.5) of Orsingher and Beghin [23], we can write

$$\Pr\left\{L_0^3(t) \in \mathrm{d}s\right\} = \frac{3\mathrm{d}s}{\Gamma\left(\frac{1}{3}\right)} \int_0^t (t-w)^{-\frac{2}{3}} \overline{p}_{\frac{2}{3}}(s,w)\,\mathrm{d}w \quad s>0, \tag{2.38}$$

where $\overline{p}_{\frac{2}{3}}(s;w)$ is the law of the stable r.v. $X(\sigma,1,0)$ with skewness parameter $\beta = 1$ and scale parameter $\sigma = (\frac{3}{2}s)^{\frac{3}{2}}$ (see [27, p. 15], for the previous notation), which possesses Laplace transform

$$\int_0^\infty \mathrm{e}^{-\lambda t} \overline{p}_{\frac{2}{3}}(s;t)\,\mathrm{d}t = \mathrm{e}^{-3s\sqrt[3]{\lambda^2}}, \quad s>0. \tag{2.39}$$

Another representation can be given in terms of Wright functions or as integrals on suitable Hankel contours. However the most important and meaningful one can be derived from Fujita [9]: this shows that the solution to (2.35) is symmetric and possesses two maxima at $x = \pm k_{4/3} t^{\frac{3}{2}}$, where $k_{4/3}$ is a constant. The distribution of $L_0^3(t)$, which is obtained by folding $u_{4/3}(s,t)$ around the origin, has consequently a Gamma-like form and differs substantially from that of the local time of Brownian motion.

We finally note that in the third-order case (and in general for any odd order) the distribution of $L_0^3(t)$ does not depend on the sign of the equation governing the pseudoprocess: as can be seen from (2.9) the law of the local time is exactly the same in the two cases $c_n = 1$ and $c_n = -1$. This is a peculiar feature of the functional considered here, which does not hold, for example, for sojourn time distribution (see [11]) and for the maximal law (see [22]). An intuitive explanation of this coincidence can be the following: the interval concerned by the functional $L_0^n(t)$ is symmetric around zero and, as shown by formula (1.9), the transition function $u_n^-(x,t)$ is obtained by reflecting around the origin $u_n^+(x,t)$.

3. Fractional diffusion equations related to the distribution of the local time of pseudoprocesses

In this section we show that the distribution of $L_0^n(t)$ is obtained by folding the fundamental solution of a fractional diffusion equation of order $2(n-1)/n$. This is the main difference between the case of Brownian motion and that of pseudoprocesses. Another important difference is that in the latter case no connection between the law of the local time and that of the maximum emerges. In our view this is due to the fact that the maximum of pseudoprocesses does not obey the reflection principle and, in general, the maximum is not distributed as a genuine random variable (consult on this point Beghin et al. [2] and Lachal [13]).

We consider here the fractional equation

$$\begin{cases} \dfrac{\partial^{2(n-1)/n}u}{\partial t^{2(n-1)/n}} = \dfrac{1}{\left(\delta_n \sin\frac{\pi}{\delta_n}\right)^2}\dfrac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \delta(x), \quad x \in \mathbb{R}, \ t>0, \ n\geqslant 2, \\ u_t(x,0) = 0, \end{cases} \tag{3.1}$$

where

$$\delta_n = \begin{cases} n & \text{for } n \text{ even}, \\ 2n & \text{for } n \text{ odd}. \end{cases}$$

Eq. (3.1) is a sort of interpolation between the heat equation ($n = 2$) and the wave equation, which is obtained as $n \to \infty$. It is a time-fractional equation and the fractional derivative must be understood in the sense of Dzherbashyan–Caputo, that is as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t \dfrac{\partial^m u(x,s)}{\partial s^m}\dfrac{\mathrm{d}s}{(t-s)^{1+\alpha-m}}, & \text{for } m-1<\alpha<m, \\ \dfrac{\mathrm{d}^m}{\mathrm{d}t^m}f(t), \quad \text{for } \alpha = m, \end{cases} \tag{3.2}$$

where $m-1 = \lfloor\alpha\rfloor$ ($\lfloor\alpha\rfloor$ denoting the integer part of the real number α) (for general reference on fractional calculus, see [26]). In our case it is $\alpha = 2(n-1)/n$ and therefore $m = 2$.

Since the degree of (3.1) is a number inside the interval $[1,2]$, two initial conditions are needed.

Equations like (3.1) have been studied by numerous authors (see, for example, [28,29]) and, for degrees $\alpha\leqslant 2$, are called fractional diffusion equations (see [24] for a general presentation of this topic). For $1\leqslant\alpha\leqslant 2$, fractional diffusion equations have been studied by Fujita [9].

By formula (3.3) of Orsingher and Beghin [23], we have that the Laplace transform of the solution $u_{2(n-1)/n}$ to (3.1) is

$$\int_0^\infty \mathrm{e}^{-\lambda t}u_{2(n-1)/n}(x,t)\,\mathrm{d}t = \frac{1}{2}\delta_n \sin\frac{\pi}{\delta_n}\frac{1}{\lambda^{1/n}}\mathrm{e}^{-\delta_n \sin\frac{\pi}{\delta_n}|x|\lambda^{\frac{n-1}{n}}}, \quad x\in\mathbb{R}, \ n\geqslant 2. \tag{3.3}$$

We now define the folded solution $\overline{u}_{2(n-1)/n}(x,t)$ as follows:

$$\overline{u}_{2(n-1)/n}(x,t) = \begin{cases} 2u_{2(n-1)/n}(x,t), & x\geqslant 0, \\ 0, & x<0. \end{cases} \tag{3.4}$$

Clearly the Laplace transform of $\overline{u}_{2(n-1)/n}$ is

$$\int_0^\infty \mathrm{e}^{-\lambda t}\overline{u}_{2(n-1)/n}(x,t)\,\mathrm{d}t = \delta_n \sin\frac{\pi}{\delta_n}\frac{1}{\lambda^{1/n}}\mathrm{e}^{-x\delta_n \sin\frac{\pi}{\delta_n}\lambda^{\frac{n-1}{n}}}, \quad x\geqslant 0, \ n\geqslant 2. \tag{3.5}$$

It is now a very simple matter to check that the Laplace transform of (3.5) with respect to x gives (2.9).

Theorem 3.1. *The explicit solution to the Cauchy problem* (3.1) *reads*

$$u_{2(n-1)/n}(x,t) = \frac{t^{-\frac{n-1}{n}}}{2}\delta_n \sin\frac{\pi}{\delta_n} W_{-\frac{n-1}{n},\frac{1}{n}}\left(-|x|t^{-\frac{n-1}{n}}\delta_n \sin\frac{\pi}{\delta_n}\right)$$
$$= \frac{t^{-\frac{n-1}{n}}}{2}\delta_n \sin\frac{\pi}{\delta_n}\sum_{k=0}^{\infty}\frac{(-1)^k\left(|x|t^{-\frac{n-1}{n}}\delta_n \sin\frac{\pi}{\delta_n}\right)^k}{k!\Gamma\left(-\frac{n-1}{n}k+\frac{1}{n}\right)}, \quad x\in\mathbb{R},\ t>0, \tag{3.6}$$

where

$$W_{v,\eta}(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!\Gamma(vk+\eta)}$$

is the Wright function.

Proof. The Fourier–Laplace transform of the solution to (3.1) is

$$\int_0^{\infty} e^{-\lambda t}\,dt\int_{-\infty}^{\infty} e^{i\beta x}u_{2(n-1)/n}(x,t)\,dx = \frac{1}{\lambda^{\frac{2}{n}-1}}\frac{1}{\lambda^{\frac{2(n-1)}{n}}+\left(\delta_n\sin\frac{\pi}{\delta_n}\right)^{-2}\beta^2} \tag{3.7}$$

and, by inverting the Laplace transform, we get that

$$\int_{-\infty}^{\infty} e^{i\beta x}u_{2(n-1)/n}(x,t)\,dx = E_{\frac{2(n-1)}{n},1}\left(-t^{\frac{2(n-1)}{n}}\left(\delta_n\sin\frac{\pi}{\delta_n}\right)^{-2}\beta^2\right). \tag{3.8}$$

By considering that

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{H_a} e^w w^{-z}\,dz, \tag{3.9}$$

where H_a is the Hankel contour, it is very easy to obtain that

$$u_{2(n-1)/n}(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x}E_{\frac{2(n-1)}{n},1}\left(-t^{\frac{2(n-1)}{n}}\left(\delta_n\sin\frac{\pi}{\delta_n}\right)^{-2}\beta^2\right)d\beta$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\beta x}\frac{d\beta}{2\pi i}\int_{H_a} e^w\frac{w^{\frac{2(n-1)}{n}-1}\,dw}{w^{\frac{2(n-1)}{n}}+t^{\frac{2(n-1)}{n}}\left(\delta_n\sin\frac{\pi}{\delta_n}\right)^{-2}\beta^2}$$
$$= \frac{\delta_n\sin\frac{\pi}{\delta_n}}{2}t^{-\frac{n-1}{n}}\frac{1}{2\pi i}\int_{H_a}\frac{e^{w-|x|t^{-\frac{n-1}{n}}\left(\delta_n\sin\frac{\pi}{\delta_n}\right)w^{\frac{n-1}{n}}}\,dw}{w^{\frac{1}{n}}}. \tag{3.10}$$

Since the Wright function, in view of (3.9), can also be written as

$$W_{v,\eta}(y) = \frac{1}{2\pi i}\int_{H_a}\frac{e^{w+yw^{-v}}}{w^{\eta}}\,dw,$$

we obtain (for $v=(1-n)/n$, $\eta=1/n$, and $y=-|x|t^{-\frac{n-1}{n}}(\delta_n\sin\frac{\pi}{\delta_n})$) the first relationship in (3.6). □

Remark 3.1. Let us denote by $p_\alpha(x;\gamma,\lambda)$ the stable density

$$p_\alpha(x;\gamma,\lambda) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{-\mathrm{i}\beta x - \lambda|\beta|^\alpha \mathrm{e}^{-\mathrm{i}\frac{\pi\gamma}{2}\operatorname{sign}\beta}\right\} \mathrm{d}\beta. \tag{3.11}$$

From formula (6.9), p. 583 of Feller [8] we know that a stable density $p_\alpha(x;\gamma,\lambda)$, of order $\alpha \in (1,2)$ and with parameter $\lambda = 1$, has the following series representation for $x > 0$:

$$p_\alpha(x;\gamma,1) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-x)^{k-1}\frac{\Gamma\left(1+\frac{k}{\alpha}\right)}{k!}\sin\frac{k\pi(\gamma+\alpha)}{2\alpha}, \tag{3.12}$$

which can be extended on the negative half-line by means of the relationship

$$p_\alpha(x;\gamma,1) = p_\alpha(-x;-\gamma,1).$$

We note that the derivation of expression (3.12) from formula (3.11) is carried out by integrating the complex-valued function $g(\beta) = \exp\{-\mathrm{i}\beta x - |\beta|^\alpha \mathrm{e}^{-\mathrm{i}\frac{\pi\gamma}{2}}\}$, $\beta \in \mathbb{C}$, on the contour made up by the circular sector of radii (r, R) and of amplitude $\pi\gamma/2\alpha$ (see Fig. 1) and then applying the Cauchy theorem and letting $r \to 0$, $R \to \infty$.

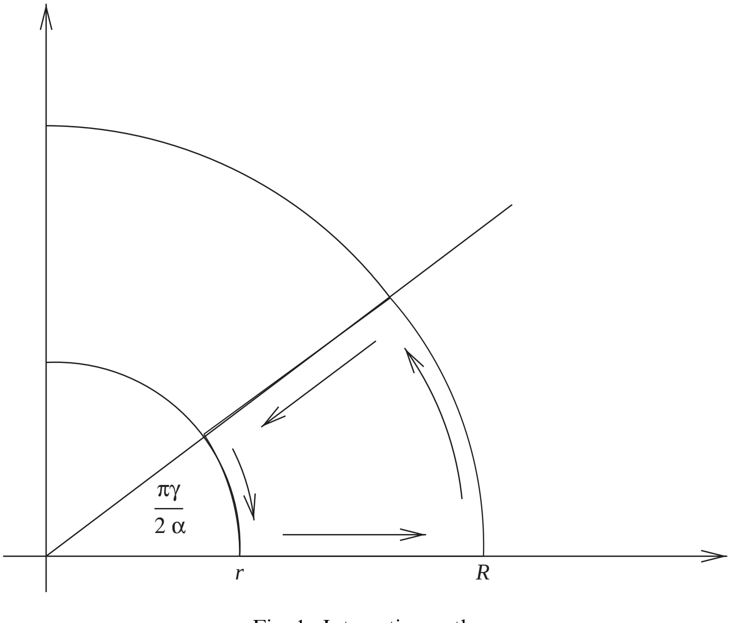

Fig. 1. Integration path.

Formula (3.12) in the special case $\gamma = 2 - \alpha$ reduces to

$$p_\alpha(x;\gamma,1) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-x)^{k-1}\frac{\Gamma\left(1+\frac{k}{\alpha}\right)}{k!}\sin\frac{k\pi}{\alpha} = \sum_{k=1}^{\infty}(-x)^{k-1}\frac{\Gamma\left(1+\frac{k}{\alpha}\right)}{\Gamma\left(\frac{k}{\alpha}\right)\Gamma\left(1-\frac{k}{\alpha}\right)}\frac{1}{k!}$$
$$= \frac{1}{\alpha}\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\Gamma\left(1-\frac{1}{\alpha}-\frac{k}{\alpha}\right)} = \frac{1}{\alpha}W_{-\frac{1}{\alpha},1-\frac{1}{\alpha}}(-x), \tag{3.13}$$

for any $x \in \mathbb{R}$.

By comparing (3.6) with (3.13) we get that

$$u_{2(n-1)/n}(x,t) = \frac{1}{2}\frac{\delta_n \sin\frac{\pi}{\delta_n}}{\frac{n-1}{n}}t^{-\frac{n-1}{n}}p_{\frac{n}{n-1}}\left(|x|t^{-\frac{n-1}{n}}\delta_n\sin\frac{\pi}{\delta_n};\frac{n-2}{n-1},1\right)$$
$$= \frac{1}{2}\frac{\delta_n \sin\frac{\pi}{\delta_n}}{\frac{n-1}{n}}p_{\frac{n}{n-1}}\left(|x|\delta_n\sin\frac{\pi}{\delta_n};\frac{n-2}{n-1},t\right), \tag{3.14}$$

where, in the last step, we have used the following well-known property of the stable law

$$p_\alpha(x;\gamma,\lambda) = \lambda^{-\frac{1}{\alpha}}p_\alpha\left(x\lambda^{-\frac{1}{\alpha}};\gamma,1\right).$$

We must note that the function $p_{\frac{n}{n-1}}$ above is obtained from the positive branch of the stable law (3.13) by symmetry.

Now, from formula (3.14), in view of (3.11), as $n \to \infty$, we obtain, for $x>0$, that

$$\lim_{n\to\infty} u_{2(n-1)/n}(x,t) = \frac{\pi}{2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\{-\mathrm{i}\beta x\pi + \mathrm{i}t|\beta|\operatorname{sign}\beta\}\,\mathrm{d}\beta$$
$$= \frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left\{-\mathrm{i}\beta x + \frac{\mathrm{i}t}{\pi}\beta\right\}\mathrm{d}\beta = \frac{1}{2}\delta\left(x-\frac{t}{\pi}\right). \tag{3.15}$$

Analogously, for $x<0$, we obtain that

$$\lim_{n\to\infty} u_{2(n-1)/n}(x,t) = \frac{1}{2}\delta\left(x+\frac{t}{\pi}\right). \tag{3.16}$$

On the other hand, Eq. (3.1) tends, as $n \to \infty$, to the wave equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{1}{\pi^2}\frac{\partial^2 u}{\partial x^2}, \\ u(x,0) = \delta(x), \\ u_t(x,0) = 0 \end{cases}$$

whose solution is $u(x,t) = \frac{1}{2}[\delta(x-\frac{t}{\pi}) + \delta(x+\frac{t}{\pi})]$.

Remark 3.2. The main merit of representation (3.14) is that it permits us to give a picture of the distribution of $L_0^n(t)$. The function $u_{2(n-1)/n}(x,t)$, for $x \in \mathbb{R}$ and for a fixed t, has the form depicted in Fig. 2A (see also [9]).

It possesses two maxima at $x = \pm k_{2(n-1)/n}t^{\frac{n}{n-1}}$, where $k_{2(n-1)/n}$ is a constant depending on the degree of the fractional equation, and a minimum in

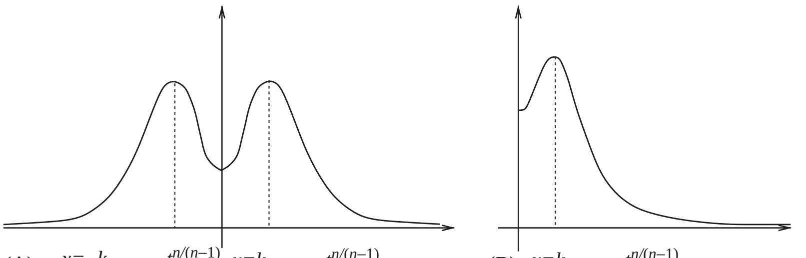

Fig. 2. (A,B) The fundamental solution of (3.1) and the distribution of $L_0^n(t)$.

zero where

$$u_{2(n-1)/n}(0,t) = \frac{1}{2}\delta_n \sin\frac{\pi}{\delta_n}\frac{t^{-\frac{n-1}{n}}}{\Gamma\left(\frac{1}{n}\right)}.$$

Fig. 2A clearly shows that the fundamental solution to the fractional equation of order $\frac{2(n-1)}{n} \in (1,2)$ is defined and continuous for any $x \in \mathbb{R}$ (as happens for $n = 2$) and lets the two-peak structure of the solution of the wave equation (obtained for $n \to \infty$) emerge.

Therefore, the distribution of $L_0^n(t)$ must have the form depicted in Fig. 2B, with a variance decreasing as $n \to \infty$. This last feature of the distribution can be derived by using, for the stable density, the alternative canonical form (for $\alpha \neq 1$)

$$\overline{p}_\alpha(x;\beta,\sigma) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{-\mathrm{i}ux - \sigma|u|^\alpha\left(1 + iu\tan\left(\frac{\pi\alpha}{2}\right)\operatorname{sign} u\right)\right\}\mathrm{d}u. \tag{3.17}$$

Thus, in our case, the solution $u_{2(n-1)/n}$ to the fractional equation can be expressed, in terms of the stable law $\overline{p}_{\frac{n}{n-1}}(x;\beta,\sigma)$ with

$$\sigma = \lambda\cos\left(\frac{\pi\gamma}{2}\right) = t\cos\left(\frac{\pi}{2}\frac{n-2}{n-1}\right),$$
$$\beta = -\frac{\tan\frac{\pi\gamma}{2}}{\tan\frac{\pi\alpha}{2}} = 1,$$

as

$$u_{2(n-1)/n}(x,t) = \frac{1}{2}\frac{\delta_n \sin\frac{\pi}{\delta_n}}{\frac{n-1}{n}}\overline{p}_{\frac{n}{n-1}}\left(|x|\delta_n\sin\frac{\pi}{\delta_n}; 1, t\cos\left(\frac{\pi}{2}\frac{n-2}{n-1}\right)\right).$$

The analysis of this section permits us to state the following result, which gives the explicit distribution of the local time $L_0^n(t)$, for any $n \geqslant 2$, in the form of a Riemann–Liouville integral.

Theorem 3.2. *The distribution of the local time $L_0^n(t)$ for any $n \geqslant 2$ is given by*

$$\Pr\left\{L_0^n(t) \in \mathrm{d}x\right\} = \overline{u}_{2(n-1)/n}(x,t)\,\mathrm{d}x = \delta_n \sin\frac{\pi}{\delta_n}\frac{\mathrm{d}x}{\Gamma\left(\frac{1}{n}\right)}\int_0^t \overline{p}_{\frac{n-1}{n}}(z;x)\frac{\mathrm{d}z}{\sqrt[n]{(t-z)^{n-1}}}, \quad x \geqslant 0, \tag{3.18}$$

where $\overline{p}_{\frac{n-1}{n}}(t;x)$ *denotes the stable density of order* $\frac{n-1}{n}$, $\beta = 1$ *and* $\sigma = (x\delta_n \sin\frac{\pi}{\delta_n}\sin\frac{\pi}{2n})^{n/(n-1)}$, *with Laplace transform*

$$\int_0^\infty \mathrm{e}^{-\lambda t}\overline{p}_{\frac{n-1}{n}}(t;x)\,\mathrm{d}t = \mathrm{e}^{-x\delta_n \sin\frac{\pi}{\delta_n}\lambda^{\frac{n-1}{n}}}, \quad x \geqslant 0. \tag{3.19}$$

Proof. By inverting the Laplace transform (3.5) we obtain $\overline{u}_{2(n-1)/n}(x,t)$ as the convolution of the inverse of (3.19) with the inverse of $\lambda^{-\frac{1}{n}}$.

We can check that the expression of the local time distribution given in (3.18) is non-negative and integrates to one.

In effect we have that

$$\begin{aligned}\int_0^\infty \Pr\{L_0^n(t) \in \mathrm{d}s\} &= \delta_n \sin\frac{\pi}{\delta_n}\frac{1}{\Gamma\left(\frac{1}{n}\right)}\int_0^\infty \mathrm{d}s\int_0^t \overline{p}_{\frac{n-1}{n}}(z;s)\frac{\mathrm{d}z}{\sqrt[n]{(t-z)^{n-1}}}\\ &= \delta_n \sin\frac{\pi}{\delta_n}\frac{1}{\Gamma\left(\frac{1}{n}\right)}\int_0^t \mathrm{d}z\int_0^\infty \mathrm{d}s\,\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{\lambda z}\frac{\mathrm{e}^{-s\delta_n \sin\frac{\pi}{\delta_n}\lambda^{\frac{n-1}{n}}}}{\sqrt[n]{(t-z)^{n-1}}}\,\mathrm{d}\lambda\\ &= \frac{1}{\Gamma\left(\frac{1}{n}\right)}\int_0^t \frac{\mathrm{d}z}{\sqrt[n]{(t-z)^{n-1}}}\frac{1}{2\pi\mathrm{i}}\int_{\gamma-\mathrm{i}\infty}^{\gamma+\mathrm{i}\infty}\mathrm{e}^{\lambda z}\lambda^{\frac{1}{n}-1}\,\mathrm{d}\lambda\\ &= \frac{1}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}\int_0^t \frac{\mathrm{d}z}{\sqrt[n]{(t-z)^{n-1}z}} = \frac{B\left(\frac{1}{n},1-\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)} = 1.\end{aligned}$$

This result can also be obtained directly from (2.9) by putting $\beta = 0$.

It is easy to check that, for $n = 3$, distribution (3.18) coincides with formula (2.38).

4. The local time for pseudoprocesses governed by Myiamoto's equation

The pseudoprocess connected with Myiamoto's equation, that is

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} + 2k\frac{\partial^2 u}{\partial x^2} \tag{4.1}$$

has been examined in a paper by Myiamoto [18] and the distribution of the related maximum is obtained in [2].

The technique necessary to derive the distribution of the local time in zero $L_0^M(t)$ for the pseudoprocess governed by (4.1) can be easily adapted from the above analysis.

In this case we need to solve the following equation:

$$-1+\lambda W=-\frac{\partial^4 W}{\partial x^4}+2k\frac{\partial^2 W}{\partial x^2}, \quad x\neq 0 \tag{4.2}$$

subject to the matching conditions

$$\begin{aligned} &\left.\frac{\partial^j W}{\partial x^j}\right|^{x=0^+}-\left.\frac{\partial^j W}{\partial x^j}\right|^{x=0^-}=0, \quad j=0,1,2,\\ &-\left.\frac{\partial^3 W}{\partial x^3}\right|^{x=0^+}+\left.\frac{\partial^3 W}{\partial x^3}\right|^{x=0^-}=\beta W(\beta,\lambda,0). \end{aligned} \tag{4.3}$$

The general bounded solution to (4.2) reads

$$W(\beta,\lambda,x)=\begin{cases} Ae^{\varphi_1 x}+Be^{\varphi_2 x}+\frac{1}{\lambda}, & x>0,\\ Ce^{\varphi_0 x}+De^{\varphi_3 x}+\frac{1}{\lambda}, & x<0, \end{cases} \tag{4.4}$$

where

$$\varphi_0=\sqrt{\frac{k+\sqrt{\lambda}}{2}}+i\sqrt{\frac{-k+\sqrt{\lambda}}{2}}, \quad \varphi_1=-\sqrt{\frac{k+\sqrt{\lambda}}{2}}+i\sqrt{\frac{-k+\sqrt{\lambda}}{2}},$$
$$\varphi_2=-\sqrt{\frac{k+\sqrt{\lambda}}{2}}-i\sqrt{\frac{-k+\sqrt{\lambda}}{2}}, \quad \varphi_3=\sqrt{\frac{k+\sqrt{\lambda}}{2}}-i\sqrt{\frac{-k+\sqrt{\lambda}}{2}}.$$

By solving the linear Vandermonde system emerging from (4.4) and based on conditions (4.3), we get that

$$\begin{aligned} A&=\frac{\beta W(\beta,\lambda,0)}{(\varphi_3-\varphi_2)(\varphi_0-\varphi_2)(\varphi_1-\varphi_2)},\\ B&=-\frac{\beta W(\beta,\lambda,0)}{(\varphi_3-\varphi_1)(\varphi_0-\varphi_1)(\varphi_1-\varphi_2)} \end{aligned} \tag{4.5}$$

and thus

$$\begin{aligned} W(\beta,\lambda,0)&=A+B+\frac{1}{\lambda}\\ &=\frac{\beta W(\beta,\lambda,0)}{(\varphi_1-\varphi_2)}\left[\frac{1}{(\varphi_3-\varphi_2)(\varphi_0-\varphi_2)}-\frac{1}{(\varphi_3-\varphi_1)(\varphi_0-\varphi_1)}\right]+\frac{1}{\lambda}\\ &=\frac{\beta W(\beta,\lambda,0)}{2i\sqrt{\frac{-k+\sqrt{\lambda}}{2}}}\left[\frac{1}{4\sqrt{\frac{k+\sqrt{\lambda}}{2}}\left(\sqrt{\frac{k+\sqrt{\lambda}}{2}}+i\sqrt{\frac{-k+\sqrt{\lambda}}{2}}\right)}\right. \end{aligned}$$

$$
\left. - \frac{1}{4\sqrt{\frac{k+\sqrt{\lambda}}{2}}\left(\sqrt{\frac{k+\sqrt{\lambda}}{2}} - \mathrm{i}\sqrt{\frac{-k+\sqrt{\lambda}}{2}}\right)}\right] + \frac{1}{\lambda}
$$

$$
= \frac{\beta W(\beta,\lambda,0)}{8\mathrm{i}\sqrt{\frac{-k+\sqrt{\lambda}}{2}}\sqrt{\frac{k+\sqrt{\lambda}}{2}}} \frac{(-2\mathrm{i})\sqrt{\frac{-k+\sqrt{\lambda}}{2}}}{\frac{k+\sqrt{\lambda}}{2} + \frac{-k+\sqrt{\lambda}}{2}} + \frac{1}{\lambda} = -\frac{\beta W(\beta,\lambda,0)}{2\sqrt{2}\sqrt{\sqrt{\lambda}+k}\sqrt{\lambda}} + \frac{1}{\lambda}. \tag{4.6}
$$

From (4.6) we have that

$$
W(\beta,\lambda,0) = \frac{2\sqrt{2}\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}'\left(2\sqrt{2}\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}+\beta\right)} \tag{4.7}
$$

which coincides, for $k=0$, with formula (2.9) in the particular case $n=4$.

In order to invert the double Laplace transform, formula (4.7) can also be rewritten as

$$
\begin{aligned}
W(\beta,\lambda,0) &= \frac{\left(2\sqrt{2}\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}\right)^2}{\lambda} \int_0^\infty \mathrm{e}^{-\beta x} \int_x^\infty \mathrm{e}^{-2\sqrt{2}y\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}}\,\mathrm{d}y\,\mathrm{d}x \\
&= \frac{2\sqrt{2}\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}} \int_0^\infty \mathrm{e}^{-\beta x - 2\sqrt{2}x\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}}\,\mathrm{d}x.
\end{aligned} \tag{4.8}
$$

We consider then the fractional telegraph-type equation

$$
\begin{cases}
\dfrac{\partial^{3/2}u}{\partial t^{3/2}} + k\dfrac{\partial u}{\partial t} = \dfrac{1}{2^3}\dfrac{\partial^2 u}{\partial x^2}, \\
u(x,0) = \delta(x), \quad x \in \mathbb{R},\ t>0,\ k \in \mathbb{R}, \\
u_t(x,0) = 0.
\end{cases} \tag{4.9}
$$

By means of arguments similar to those of Orsingher and Beghin [23] it is easy to check that the Laplace–Fourier transform of the solution $u_M(x,t)$ to (4.9) is equal to

$$
\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}t \int_{-\infty}^\infty \mathrm{e}^{-\mathrm{i}\beta x} u_M(x,t)\,\mathrm{d}x = \frac{\sqrt{\lambda}+k}{\sqrt{\lambda^3} + k\lambda + \frac{\beta^2}{2^3}} \tag{4.10}
$$

so that

$$
\int_0^\infty \mathrm{e}^{-\lambda t} u_M(x,t)\,\mathrm{d}t = \frac{2\sqrt{2}}{2}\frac{\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}} \mathrm{e}^{-2\sqrt{2}|x|\sqrt{\lambda}\sqrt{\sqrt{\lambda}+k}}. \tag{4.11}
$$

If we define now

$$
\overline{u}_M(x,t) = \begin{cases} 2u_M(x,t), & x \geqslant 0, \\ 0, & x<0, \end{cases}
$$

we have that

$$\int_0^\infty e^{-\lambda t}\overline{u}_M(x,t)\,dt = 2\sqrt{2}\frac{\sqrt{\sqrt{\lambda}+k}}{\sqrt{\lambda}}e^{-2\sqrt{2}x\sqrt{\sqrt{\lambda}+k}\sqrt{\lambda}}, \quad x \geqslant 0. \tag{4.12}$$

By the uniqueness of the Laplace transform we conclude that

$$\Pr\left\{L_0^M(t) \in ds\right\} = ds\overline{u}_M(s,t), \quad s \geqslant 0.$$

Note that, in the special case $k = 0$, the Myiamoto equation becomes the heat-type equation of order $n = 4$, while the associated fractional equation (4.9) coincides with (3.1) for $n = 4$. Therefore the local time has, in this special case, a distribution which can be obtained from (3.14) by putting $n = 4$, that is

$$u_{\frac{3}{2}}(x,t) = \frac{4\sqrt{2}}{3}p_{\frac{4}{3}}\left(2\sqrt{2}|x|; \frac{2}{3}, t\right). \tag{4.13}$$

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