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RESULTS ON PARABOLIC EQUATIONS RELATED TO SOME CAFFARELLI–KOHN–NIRENBERG INEQUALITIES*

ANDREA DALL'AGLIO[†], DANIELA GIACHETTI[†], AND IRENEO PERAL[‡]

Abstract. In this paper problem

$$
\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \lambda \dfrac{u^{p-2}u}{|x|^{p(\gamma+1)}} \quad \text{in } \Omega\times(0,\infty),\ 0\in\Omega, \\ u(x,t)=0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=\psi(x)\geq 0 \end{cases} \tag{0.1}
$$

is studied when $1 < p < N$, $-\infty < (\gamma+1) < \frac{N}{p}$, and under hypotheses on the initial data.

Key words. nonlinear parabolic equations, p-laplacian, existence, behavior of solutions, critical problems, Caffarelli–Kohn–Nirenberg inequalities

AMS subject classifications. 35K25, 35K55, 35K57, 35K65, 46E30, 46E35

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1. Introduction. The results by Baras and Goldstein in [7] concerning a blow-up for the solution to the heat equation with a critical potential of the type

$$
\begin{cases} u_t - \Delta u = \lambda \dfrac{u}{|x|^2} \quad \text{in } \Omega\times(0,\infty),\ 0\in\Omega, \\ u(x,t)=0 \quad \text{on } \partial\Omega\times(0,\infty), \\ u(x,0)=\psi(x)\geq 0 \end{cases} \tag{1.1}
$$

have attracted in recent years the interest of research on some related problems. Roughly speaking, apparently, the main ingredient of the problem studied by Baras and Goldstein is a classical Hardy inequality,

$$
\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^2}\,dx \leq C_N \int_{\mathbb{R}^n} |\nabla u|^2\,dx, \tag{1.2}
$$

where $C_N = (\frac{2}{N-2})^2$ is the optimal constant that is not achieved in the Sobolev space $\mathcal{D}^{1,2}(\mathbb{R}^n)$. For problem (1.1) Baras and Goldstein have proved that if $\lambda \leq C_N^{-1}$, then there exists a global solution if the initial datum is in a convenient class, while if $\lambda > C_N^{-1}$, there is no solution in the sense that if we consider the solutions u_n of the problems with truncated potential $W_n(x) = \min\{n, |x|^{-2}\}$, then

$$
\lim_{n\to\infty} u_n(x,t) = +\infty \quad \text{for all } (x,t)\in\Omega\times\mathbb{R}^+.
$$

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[†]Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università di Roma La Sapienza, Via A. Scarpa 16 I-00161, Rome, Italy (aglio@dmmm.uniroma1.it, giachett@dmmm.uniroma1.it).

[‡]Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain (ireneo.peral@uam.es). This author was partially supported by Project BFM2001-0183, MCYT (Spain).

We will call this behavior *spectral instantaneous complete blow-up*. On the other hand, we have the following extension of Hardy's inequality:

$$\int_{\mathbb{R}^n} \frac{|u|^p}{|x|^{(\gamma+1)p}}\, dx \le C_{n,p,\gamma} \int_{\mathbb{R}^n} \frac{|\nabla u|^p}{|x|^{\gamma p}}\, dx, \quad -\infty < \gamma < \frac{N-p}{p}. \tag{1.3}$$

This is a particular limit case of the following Caffarelli–Kohn–Nirenberg inequalities which are proven in [13] (see also [14], [4], and [11]).

PROPOSITION 1.1. *Assume that $1 < p < N$. Then there exists a positive constant $C_{N,p,\gamma,q}$ such that, for every $u \in C_0^\infty(\mathbb{R}^N)$,*

$$\left(\int_{\mathbb{R}^n} \frac{|u|^q}{|x|^{\delta q}}\, dx\right)^{p/q} \le C_{N,p,\gamma,q} \int_{\mathbb{R}^n} \frac{|\nabla u|^p}{|x|^{\gamma p}}\, dx, \tag{1.4}$$

where p, q, γ, δ are related by

$$\frac{1}{q} - \frac{\delta}{N} = \frac{1}{p} - \frac{\gamma+1}{N}, \quad \gamma \le \delta \le \gamma + 1, \tag{1.5}$$

and $\delta q < N$, $\gamma p < N$.

Remark 1.2.

(i) Inequality (1.3) holds a fortiori in every open set Ω.

(ii) One can take

$$C_{n,p,\gamma} = \left(\frac{p}{N - p(\gamma+1)}\right)^p \tag{1.6}$$

in (1.3). This choice of $C_{n,p,\gamma}$ is optimal in every open set Ω containing 0. (The arguments are similar to those in [19] for $\gamma = 0$.)

(iii) If $0 \in \Omega$, the optimal constant is never attained in (1.3).

Remark 1.3. The other limit case for inequality (1.4) is for $\delta = \gamma$, and then one obtains a weighted Sobolev inequality

$$\left(\int_{\mathbb{R}^n} \frac{|u|^{p^*}}{|x|^{\gamma p^*}}\, dx\right)^{p/p^*} \le S_{n,p,\gamma} \int_{\mathbb{R}^n} \frac{|\nabla u|^p}{|x|^{\gamma p}}\, dx, \tag{1.7}$$

where $p^* = \frac{pN}{N-p}$.

It is quite natural to study the parabolic equations associated to inequality (1.3); namely, for the same values of p and γ we consider the problem

$$\text{(P)} \qquad \begin{cases} u_t - \operatorname{div}\left(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{\gamma p}}\right) = \lambda \dfrac{|u|^{p-2}u}{|x|^{(\gamma+1)p}}, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = \psi(x), & x \in \Omega, \end{cases}$$

where we assume that Ω is a bounded domain in $\mathbb{R}^n$ such that $0 \in \Omega$ and $\partial\Omega$ is a $\mathcal{C}^1$ submanifold.

It is clear that the constant (1.6) will play an essential role in what follows, since the behavior of the problem (P) will deeply depend on whether the parameter λ is smaller or greater than the value

$$\lambda_{n,p,\gamma} = \frac{1}{C_{n,p,\gamma}} = \left(\frac{N - p(\gamma+1)}{p}\right)^p. \tag{1.8}$$

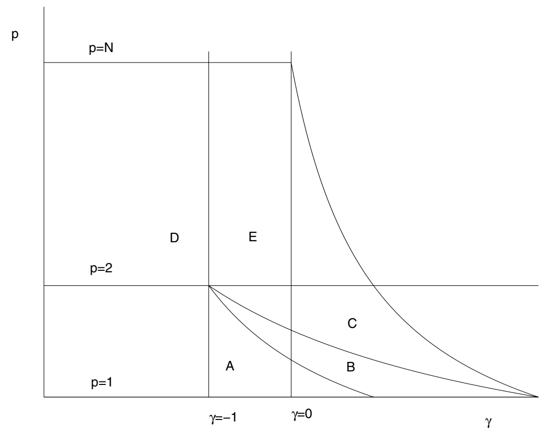

FIG. 1.1. *Summary of the existence and nonexistence results for* $\lambda > \lambda_{N,p,\gamma}$:
Region A: *Global existence of energy solutions.*
Region B: *Global existence of entropy solutions.*
Region C: *Global existence of very weak solutions.*
Region D: *Local existence of solutions.*
Region E: *Instantaneous complete blow-up.*

It could be expected that the behavior for problem (P) should be similar to the one obtained by Baras and Goldstein for (1.1). This conjecture is not completely true. Actually, there is another property which plays an important role in the spectral instantaneous and complete blow-up: a Harnack inequality for the homogeneous parabolic equation. This property is verified if $p \geq 2$ and $(1+\gamma) > 0$. The case $p = 2$ was proved by Chiarenza and Serapioni in [15], while the case $p > 2$ was proved by Abdellaoui and Peral in [1].

The main contribution of this paper is *to show that in the complementary range of the parameters* p *and* γ *we find solutions, even for large values of* λ. The case $p = 2$, $\gamma = 0$ has been studied in [7] and recently in [26]. The case $p \neq 2$, $\gamma = 0$ has been studied in [19] and [5].

The plan of this work is as follows. We begin with section 2, where some notation is provided and appropriate function spaces are defined. Section 3 is devoted to the existence results. In subsection 3.1 we obtain the existence of a global solution in the case $\lambda < \lambda_{N,p,\gamma}$ for all $1 < p < N$. This is the content of Theorem 3.1. In this case the solution belongs to the space $L^p(0, T; \mathcal{D}^{1,p}_{0,\gamma}(\Omega))$, which is naturally related to (P) (see section 2 for the definition). For this reason we will refer to this function u as an *energy solution*. In the proof of Theorem 3.1 we give the details of some convergence results that will be used thereafter. Subsection 3.2 deals with the case $\lambda > \lambda_{N,p,\gamma}$ and $1 < p \leq 2$. The existence of solutions according to the values of γ and p is investigated, and the main results are stated in Theorems 3.3, 3.6, and 3.8. Roughly speaking, as γ and p become larger, we find solutions which are less and less regular. More precisely, we show the following.

1. If $1 < p \leq 2$ and $\gamma + 1 < \frac{N(2-p)}{2p}$ (see region A in Figure 1.1), then we show the existence of energy solutions (see Theorem 3.3).

2. If $1 < p \leq 2$ and $\frac{N(2-p)}{2p} \leq \gamma + 1 < \frac{N(2-p)}{p}$ (see region B), we show the existence of a solution of (P) in the sense of distributions; however, this solution does not belong to the energy space (see Theorem 3.6). We will show that this is an *entropy*

solution in the sense introduced in [8], [22], and [23] for equations with L^1 data (see Definition 3.5 below).

3. If $1 < p \leq 2$ and $N(2-p)/(p) < \gamma + 1 < N/p$ (see region C), we show the existence of solutions of (P) in a very weak sense (see Theorem 3.8). We would like to point out that in this case we solve a problem where the right-hand side is not bounded in L^1.

Notice that, comparing the existence results with those contained in [3] for the case $p = 2$, we find that in the nonlinear case (i.e., $p \neq 2$) a very much different behavior of the solutions appears, depending on the parameters; namely, the behavior in cases 2 and 3 above is typical of the nonlinear setting and does not appear in the linear case.

In subsection 3.3, for completeness, we include the elementary local existence result for $p \geq 2$ and $\gamma \leq -1$ (see region D in Figure 1.1) in Theorem 3.10, which is also stated in [2].

In section 4 we study the blow-up when $p > 2$, $0 < 1 + \gamma < \frac{N}{p}$, and $\lambda > \lambda_{N,p,\gamma}$ (see region E in Figure 1.1), extending and improving the result of [19] for $\gamma = 0$. (See also [12].)

The case $p = 2$ is obtained in [3] by different kinds of techniques. The main result is Theorem 4.4 and its consequences. The results in Theorems 4.5 and 4.7 have also been stated in [2] and are included here for completeness. With regard to the proof of instantaneous blow-up that we give, it is interesting to point out that for $p > 2$ the blow-up is stronger than that obtained for $p = 2$. Indeed, even the solutions u_n of the problems with truncated potential, $W_n(x) = \min\{n, |x|^{-p(\gamma+1)}\}$, blow up in finite time, and the blow-up time tends to zero as $n \to \infty$.

Finally, in section 5 we study the extinction in finite time of the solution in the case $1 < p < 2$, according to the relation between λ and $\lambda_{N,p,\gamma}$. Roughly speaking, the role that $\lambda_{N,p,\gamma}$ plays in the case $p > 2$ for the blow-up is changed to be a threshold for the finite time extinction property in the case $1 < p < 2$.

2. Notation and function spaces. For $1 < p < \infty$ and $\gamma < \frac{N-p}{p}$, we define the weighted space

$$L^p_\gamma(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable, such that } \frac{u(x)}{|x|^\gamma} \in L^p(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{L^p_\gamma(\Omega)} = \left(\int_\Omega \frac{|u(x)|^p}{|x|^{\gamma p}}\, dx \right)^{1/p}.$$

It is easy to check that the dual space $(L^p_\gamma(\Omega))'$ of $L^p_\gamma(\Omega)$ is the space $L^{p'}_{-\gamma}(\Omega)$, where p' is defined by $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we define $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{\mathcal{D}^{1,p}_{0,\gamma}(\Omega)} = \|\nabla u\|_{L^p_\gamma(\Omega)} = \left(\int_\Omega \frac{|\nabla u(x)|^p}{|x|^{\gamma p}}\, dx \right)^{1/p}.$$

As $1 < p < \infty$, $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$ is reflexive, and we can define the dual space of $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$, which we will denote by $\mathcal{D}^{-1,p'}_{-\gamma}(\Omega)$, as

$$\mathcal{D}^{-1,p'}_{-\gamma}(\Omega) = \{G \in \mathcal{D}'(\Omega) : G = \operatorname{div} F,\ F \in L^{p'}_{-\gamma}(\Omega; \mathbb{R}^N)\}.$$

Let us point out that functions in $L^p_\gamma(\Omega)$ do not need to be distributions since they do not belong necessarily to $L^1(\Omega)$. If $\gamma+1\le -\frac{(p-1)N}{p}$, $\mathcal{D}^{1,p}_{0,\gamma}\not\subset L^1(\Omega)$. The meaning of the gradient in this case is understood as follows. If $u\in\mathcal{D}^{1,p}_{0,\gamma}$ and $\{\phi_n\}_{n\in\mathbb{N}}\subset\mathcal{C}^\infty_0(\Omega)$ is an approximating sequence, then we obtain

$$\nabla\phi_n\to\mathcal{V}\text{ in }L^p_\gamma(\Omega;R^N);$$

in fact, by density and duality we can justify the integration by parts, namely,

$$\int_\Omega\langle\mathcal{V},\psi\rangle dx=\lim_{n\to\infty}\int_\Omega\langle\nabla\phi_n,\psi\rangle dx=-\int_\Omega u\,\mathrm{div}(\psi)dx\quad\text{for all }\psi\in\mathcal{D}^{1,p'}_{0,-(\gamma+1)}.$$

As a consequence we define $\mathrm{grad}(u):=\mathcal{V}$. On the other hand, Theorem 1.18 in [20] shows that if $u\in\mathcal{D}^{1,p}_{0,\gamma}$, then the truncature $T_k(u)\in\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$, where $T_k(u)$ is defined by $T_k(u)=u$ if $|u|<k$ and $T_k(u)=k\frac{u}{|u|}$ if $|u|\ge k$. Since $T_k(u)\in L^\infty(\Omega)$, we can define $\nabla T_k(u)$ as a distribution and by Theorem 1.20 in [20] we have

$$\nabla T_k(u)=\mathrm{grad}(u)\chi_{\{|u|<k\}}. \tag{2.1}$$

Hereafter we will denote $\nabla u=\mathrm{grad}\,(u)$. Notice the relation of this concept of gradient with the one in Lemma 2.1 in [8].

Therefore, inequality (1.4) implies the continuous imbedding

$$\mathcal{D}^{1,p}_{0,\gamma}(\Omega)\subset L^q_\delta(\Omega)\qquad\text{for }p,q,\gamma,\delta\text{ satisfying (1.5).} \tag{2.2}$$

This implies, by duality,

$$L^{q'}_{-\delta}(\Omega)\subset\mathcal{D}^{-1,p'}_{-\gamma}(\Omega)\qquad\text{for }p,q,\gamma,\delta\text{ satisfying (1.5).} \tag{2.3}$$

We now define the following "evolution" spaces which will be useful in what follows.

$$L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))=\{u(x,t):\Omega\times(0,T)\to\mathbb{R}\text{ measurable}:$$
$$u(\cdot,t)\in\mathcal{D}^{1,p}_{0,\gamma}(\Omega)\text{ for a.e. }t\in(0,T),\ \|u(\cdot,t)\|_{\mathcal{D}^{1,p}_{0,\gamma}(\Omega)}\in L^p(0,T)\},$$

endowed with the norm

$$\|u\|_{L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))}=\left(\int_0^T\|u(\cdot,t)\|^p_{\mathcal{D}^{1,p}_{0,\gamma}(\Omega)}dt\right)^{1/p}=\left(\iint_{Q_T}\frac{|\nabla u|^p}{|x|^{p\gamma}}\,dx\right)^{1/p}.$$

The dual space of $L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))$ is $L^{p'}(0,T;\mathcal{D}^{-1,p'}_{-\gamma}(\Omega))$. Let us point out that

$$\mathcal{D}^{1,p}_{0,\gamma}(\Omega)\subset L^q_\delta(\Omega)\quad\text{compactly}$$

for every p, q, γ, δ satisfying $\frac{1}{q}-\frac{\delta}{N}>\frac{1}{p}-\frac{\gamma+1}{N}$ with $\gamma\le\delta\le\gamma+1$ and $\delta q<N$, $\gamma p<N$.

Indeed, a sequence $\{u_n\}$ which is bounded in $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$ has a subsequence, again denoted by $\{u_n\}$, which converges almost everywhere in Ω to a function $u\in L^q_\delta(\Omega)$. Moreover, by Hölder's inequality and (1.7), for every measurable subset $E\subset\Omega$,

$$\int_E\frac{|u_n-u|^q}{|x|^{\delta q}}\,dx\le\left(\int_\Omega\frac{|u_n-u|^{p^*}}{|x|^{\gamma p^*}}\,dx\right)^{q/p^*}\left(\int_E\frac{1}{|x|^{(\delta-\gamma)\frac{qp^*}{p^*-q}}}\,dx\right)^{(p^*-q)/p^*}$$
$$\le c\left(\int_E\frac{1}{|x|^{(\delta-\gamma)\frac{qp^*}{p^*-q}}}\,dx\right)^{(p^*-q)/p^*}.$$

Since the function in the last integral is an L^1 function, we get the compactness result by Vitali's theorem.

It is easy to see that the operator defined by

$$-\Delta_{p,\gamma} u = -\text{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|x|^{p\gamma}}\right)$$

maps $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$ into its dual $\mathcal{D}^{-1,p'}_{-\gamma}(\Omega)$ and is hemicontinuous, coercive, and monotone. (See [21].)

In what follows, we will often use the following result, which is an easy application of Theorem 1.2 of [21] and the reference [24] for the continuity with respect to the time of the L^2-norm.

Proposition 2.1. *If $f \in L^{p'}(0,T;\mathcal{D}^{-1,p'}_{-\gamma}(\Omega))$, $\psi \in L^2(\Omega)$, then there exists a unique solution in the distribution sense, $u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap \mathcal{C}^0(0,T;L^2(\Omega))$, of the following problem:*

$$\begin{cases} u_t - \Delta_{p,\gamma} u = f & \text{in } \Omega \times (0,T), \\ u(x,t) = 0 & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = \psi(x) & \text{in } \Omega. \end{cases}$$

We have the following result about the boundedness of the solutions.

Lemma 2.2. *Let $u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap \mathcal{C}^0(0,T;L^2(\Omega))$ be a distributional solution of* (F) *(see section* 2*), with $\psi \in L^\infty(\Omega)$, and assume that there exist two constants q and β_0 such that*

$$q > \frac{N}{p}, \qquad \beta_0 < p\gamma, \qquad \operatorname*{ess\,sup}_t \int_\Omega |f(x,t)|^q |x|^{\beta_0 q}\,dx < +\infty. \tag{2.4}$$

Then $u \in L^\infty(Q_T)$.

The proof is a slight modification of the classical arguments and is omitted.

3. Existence results. We start with the simpler case $\lambda < \lambda_{N,p,\gamma}$, where $\lambda_{N,p,\gamma}$ is defined by (1.8).

3.1. The case $\boldsymbol{\lambda < \lambda_{N,p,\gamma}}$: Global existence. As usual we denote by $T_n(s)$ the truncation function, i.e., $T_n(s) = s$ if $|s| < n$, $T_n(s) = n \operatorname{sign} s$ if $|s| > n$. Let us observe that in this range for λ the operator $-\Delta_{p,\gamma} - \lambda\frac{|u|^{p-2}u}{|x|^{p(\gamma+1)}}$ is coercive in the space $\mathcal{D}^{1,p}_{0,\gamma}(\Omega)$. This essentially justifies the following.

Theorem 3.1. *If $1 < p < N$, $\gamma < \frac{N-p}{p}$, $\lambda < \lambda_{N,p,\gamma}$, $\psi(x) \in L^2(\Omega)$, there exists one distributional solution u for problem* (P). *Moreover, $u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap \mathcal{C}^0(0,T;L^2(\Omega))$.*

Proof. Define

$$w_n(x) = \begin{cases} |x|^{-p\gamma} & \text{if } \gamma \geq 0, \\ |x|^{-p\gamma} + \dfrac{1}{n} & \text{if } \gamma < 0, \end{cases}$$

$$f_n(x,u) = \begin{cases} \dfrac{T_n(|u|^{p-2}u)}{|x|^{p(\gamma+1)} + \frac{1}{n}} & \text{if } \gamma \geq 0, \\ \dfrac{T_n(|u|^{p-2}u)}{|x|^{p\gamma}(|x|^p + \frac{1}{n})} & \text{if } \gamma < 0. \end{cases}$$

Let us first consider the following approximate problems:

$$
(\mathrm{P}_n)\qquad \begin{cases} (u_n)_t - \operatorname{div}\,\big(w_n(x)|\nabla u_n|^{p-2}\nabla u_n\big) = \lambda f_n(x,u_n), & (x,t)\in\Omega\times(0,T),\\ u_n(x,t)=0, & (x,t)\in\partial\Omega\times(0,T),\\ u_n(x,0)=T_n(\psi(x)), & x\in\Omega. \end{cases}
$$

By Proposition 2.1 of section 2 and Schauder's fixed point theorem, it is quite easy to get existence of a solution $u_n \in W_0^{1,p}(\Omega)\cap L^\infty(Q_T)$. Let us multiply (P_n) by $u_n(x,t)$. Using inequality (1.4), one obtains

$$
\begin{aligned}
\iint_{Q_T} \frac{\partial u_n}{\partial t} u_n + \iint_{Q_T} w_n(x)|\nabla u_n|^p &\le \lambda \iint_{Q_T} f_n(x,u_n)u_n\\
&\le \lambda \iint_{Q_T} \frac{|u_n|^p}{|x|^{p(\gamma+1)}} \le \frac{\lambda}{\lambda_{N,p,\gamma}} \iint_{Q_T} \frac{|\nabla u_n|^p}{|x|^{p\gamma}},
\end{aligned}
$$

where the first integral is understood as a duality product. Since $\lambda<\lambda_{N,p,\gamma}$, we get the estimates

$$
\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le c_1, \tag{3.1}
$$

$$
\iint_{Q_T} \frac{|\nabla u_n|^p}{|x|^{p\gamma}}\,dx\,dt \le c_2, \tag{3.2}
$$

that is,

$$
\|u_n\|_{L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))} \le c_3. \tag{3.3}
$$

Therefore, there exist a function $u\in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ and a subsequence (still denoted by u_n) such that $u_n \rightharpoonup u$ weakly in $L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))$ and $*$-weakly in $L^\infty(0,T;L^2(\Omega))$.

Moreover, if B_ε is the sphere centered in the origin with radius ε, we also have

$$
\|u_n\|_{L^p(0,T;W^{1,p}(\Omega\setminus B_\varepsilon))} \le c_4(\varepsilon) \tag{3.4}
$$

for every $\varepsilon>0$. By (P_n) we also deduce

$$
\left\|\frac{\partial u_n}{\partial t}\right\|_{L^{p'}(0,T;W^{-1,p'}(\Omega\setminus B_\varepsilon))} \le c_5(\varepsilon). \tag{3.5}
$$

Using a compactness Aubin-type result (see, for instance, [24]), by (3.4) and (3.5) we can assume that $u_n\to u$ strongly in $L^p((\Omega_\varepsilon)\times(0,T))$ for every $\varepsilon>0$, and therefore, up to a subsequence,

$$
u_n \to u \quad \text{a.e. and in measure in } Q_T. \tag{3.6}
$$

Let us now prove that, for every $\varepsilon>0$, if we define

$$
Q_T^{(\varepsilon)} = (\Omega\setminus B_\varepsilon)\times(0,T),
$$

then

$$
\nabla u_n \to \nabla u \quad \text{in measure on } Q_T^{(\varepsilon)}. \tag{3.7}
$$

To do this, we follow a technique similar to the one introduced by Boccardo and Murat in [10]. Let us define, for $h > 0$, the set

$$H_h = H_{h,m,n} = \{(x,t) \in Q_T^{(\varepsilon)} : |\nabla u_n - \nabla u_m| > h\}.$$

We are going to prove that, for every $\delta > 0$, one has $\operatorname{meas} H_h < \delta$ for m and n large enough. Then, if we set, for positive A, k,

$$\begin{aligned}\Gamma(n,A) &= \{(x,t) \in Q_T^{(\varepsilon)} : |\nabla u_n| > A\},\\ \Lambda(k) &= \{(x,t) \in Q_T^{(\varepsilon)} : |u_n - u_m| > k\},\\ D(A,k,h) &= \{(x,t) \in Q_T^{(\varepsilon)} : |\nabla u_n - \nabla u_m| > h,\\ &\qquad |\nabla u_n| \le A,\ |\nabla u_m| \le A,\ |u_n - u_m| \le k\},\end{aligned}$$

then

$$H_h \subset \Gamma(n,A) \cup \Gamma(m,A) \cup \Lambda(k) \cup D(A,k,h).$$

For every $n \in \mathbb{N}$, $\operatorname{meas}\Gamma(n,A)$ is small for A large enough, uniformly in n, since $|\nabla u_n|^q$ is bounded in $L^1(Q_T)$ for every $q < Np/(N-\gamma p)$. Indeed

(3.8)
$$\iint_{Q_T} |\nabla u_n|^q = \iint_{Q_T} \frac{|\nabla u_n|^q}{|x|^{\gamma q}}|x|^{\gamma q} \le \left(\iint_{Q_T} \frac{|\nabla u_n|^p}{|x|^{\gamma p}}|x|^{\gamma q}\right)^{\frac{q}{p}} \left(\iint_{Q_T} |x|^{\frac{\gamma p q}{p-q}}\right)^{\frac{p-q}{p}},$$

and the last integral is finite. Moreover, by (3.6), for every fixed k, $\operatorname{meas}\Lambda(k)$ is small if n, m are large enough. We now consider the set $D(A,k,h)$. By multiplying by $\varphi(x)T_k(u_n - u_m)$ the equations satisfied by u_n and u_m, respectively, where $\varphi(x) \in C_0^\infty(\Omega)$, $\varphi(x) \equiv 0$ for $|x| \le \varepsilon/2$, and $\varphi(x) \equiv 1$ for $|x| \ge \varepsilon$, one obtains, since the integral involving the time-derivative is positive,

$$\begin{aligned}&\iint_{Q_T^{(\varepsilon/2)}} \frac{|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m}{|x|^{p\gamma}} \nabla T_k(u_n - u_m)\varphi(x)\\ &\quad\le \lambda k \iint_{Q_T^{(\varepsilon/2)}} \frac{|u_n|^{p-1} + |u_m|^{p-1}}{|x|^{p(\gamma+1)}} + k \iint_{Q_T^{(\varepsilon/2)}} \frac{|\nabla u_n|^{p-1} + |\nabla u_m|^{p-1}}{|x|^{p\gamma}} |\nabla\varphi|.\end{aligned} \tag{3.9}$$

Using Hölder's inequality, (1.4), and (3.3), one checks that the right-hand side of (3.9) is bounded by $c_6 k$, where c_6 is a constant which only depends on λ, ε, p, N. Since the left-hand side is greater than

$$\varepsilon^{-p\gamma} \iint_{Q_T^{(\varepsilon)} \cap \{|u_n - u_m| \le k\}} \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\right) \cdot \nabla(u_n - u_m),$$

we have proved that this last integral is small (uniformly in n and m) if k is sufficiently small. Observe now that by the monotonicity and continuity of $|\xi|^{p-2}\xi$, for every $h > 0$, there exists $\mu > 0$ such that

$$\begin{aligned}D(A,k,h) \subset G(A,k,\mu) = \{(x,t) \in Q_T^{(\varepsilon)} &: |\nabla u_n| \le A,\ |\nabla u_m| \le A,\ |u_n - u_m| \le k,\\ &(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m) \cdot \nabla(u_n - u_m) > \mu\}.\end{aligned}$$

It follows that

$$\operatorname{meas} D(A,k,h) \le \frac{1}{\mu} \iint_{Q_T^{(\varepsilon)} \cap \{|u_n - u_m| \le k\}} \big(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_m|^{p-2}\nabla u_m\big)\cdot\nabla(u_n - u_m),$$

so that meas $D(A,k,h)$ is small (uniformly in n and m) if k is sufficiently small. This proves (3.7). We can now pass to the limit in (P_n) in the sense of distributions. Indeed, if we multiply (P_n) by $\varphi(x,t) \in C_0^\infty(Q_T)$, we obtain

$$-\iint_{Q_T} u_n \frac{\partial \varphi}{\partial t} + \iint_{Q_T} \frac{|\nabla u_n|^{p-2}\nabla u_n}{|x|^{p\gamma}} \nabla\varphi = \lambda \iint_{Q_T} T_n\left(\frac{|u_n|^{p-2}u_n}{|x|^{p(\gamma+1)}}\right)\varphi. \tag{3.10}$$

One can easily pass to the limit in each term using the convergences (3.6) and (3.7), the estimates (3.1) and (3.3), the inequality (1.4), and Vitali's theorem. □

3.2. The case $\boldsymbol{\lambda > \lambda_{N,p,\gamma}}$, $\boldsymbol{p \le 2}$: Global existence. In this section we will suppose $\lambda > \lambda_{N,p,\gamma}$ and $p \le 2$. We will show the existence of solutions with different behaviors (see Theorems 3.3, 3.6, and 3.8 in subsections 3.2.1, 3.2.2, and 3.2.3 below), depending on the range for the parameters γ and p.

More precisely, we will find solutions which become weaker and weaker (from the point of view of regularity) as γ and p increase (see Figure 1.1).

First, let us prove the following lemma which will be useful in what follows. It gives the existence of self-similar solutions $S(x,t)$ of the equation in problem (P) for this range of the parameters.

LEMMA 3.2. *If $\lambda > \lambda_{N,p,\gamma}$ and $p < 2$, the function*

$$S(x,t) = A \cdot \left(\frac{t}{|x|^{p(\gamma+1)}}\right)^{\frac{1}{2-p}}, \tag{3.11}$$

where $A = A(\lambda,\gamma) > 0$, is such that

$$A^{p-2} = \frac{1}{(2-p)[(p-1)\delta^p - (N - p(\gamma+1))\delta^{p-1} + \lambda]} \quad \text{and} \quad \delta = \frac{p(\gamma+1)}{2-p} \tag{3.12}$$

satisfy the following:

1. *If $\gamma + 1 < \frac{N(2-p)}{2p}$, then $S(\cdot,t) \in \mathcal{D}^{1,p}_\gamma(\Omega)$ and verifies* (P) *in the sense of distributions.*
2. *If $\frac{N(2-p)}{2p} \le \gamma + 1 < \frac{N(2-p)}{p}$, then*
   (i) *$S(\cdot,t) \in L^q(\Omega)$ for every q such that $1 < q < \frac{N(2-p)}{p(\gamma+1)}$;*
   (ii) *$\nabla S(\cdot,t) \in L^{q_1}(\Omega)$ for every q such that $0 < q_1 < \frac{N(2-p)}{2+p\gamma}$;*
   (iii) *$\nabla S(\cdot,t) \in L^q_\gamma(\Omega)$ for every q such that $0 < q < \frac{N(2-p)}{2(\gamma+1)}$;*
   (iv) *$\frac{|\nabla S(\cdot,t)|^{p-1}}{|x|^{p\gamma}}$, $\frac{S(\cdot,t)^{p-1}}{|x|^{p(\gamma+1)}} \in L^1(\Omega)$;*
   (v) *S solves* (P) *in the sense of distributions.*
3. *If $N\frac{(2-p)}{p} \le (\gamma+1) < \frac{N}{p}$, then S solves* (P) *in $\mathcal{D}'(\mathbb{R}^N \setminus \{0\} \times (0,\infty))$ (and in some weighted Sobolev spaces that will be made precise later).*

Proof. We start by looking for solutions of (P) of the form

$$S(x,t) = t^\alpha f(r), \quad \text{with } r = |x|.$$

Choosing the exponent $\alpha = 1/(2-p)$, one can cancel the variable t from the equation, getting the following ordinary differential equation for $f(r)$:

(3.13)

$$\alpha f = (p-1)r^{-p\gamma}|f'|^{p-2}f'' + r^{-(p\gamma+1)}(N - (p\gamma+1))|f'|^{p-2}f' + \lambda r^{-p(\gamma+1)}|f|^{p-2}f.$$

Next we look for solutions $f(r)$ of the form

$$f(r) = Ar^{-\delta}, \quad A > 0.$$

It is easy to check that if we choose δ as in (3.12), we can cancel the terms involving powers of r in (3.13), getting solutions of the form (3.11), provided the constant A is defined as in (3.12) and is positive. This last assertion is true if

$$\lambda > \left(\frac{p(\gamma+1)}{2-p}\right)^p (s-1) = \mu_{p,\gamma},$$

where

$$s = \frac{N(2-p)}{p(\gamma+1)}.$$

Let us observe that the critical value $\lambda_{N,p,\gamma}$ can be rewritten as

$$\lambda_{N,p,\gamma} = \left(\frac{p-2+s}{2-p}(\gamma+1)\right)^p.$$

Moreover, if we regard the constants $\lambda_{N,p,\gamma}$ and $\mu_{N,p,\gamma}$ as functions of the variable s,

$$\lambda_{N,p,\gamma}(2) = \mu_{N,p,\gamma}(2), \qquad \lambda'_{N,p,\gamma}(2) = \mu'_{N,p,\gamma}(2), \qquad \lambda''_{N,p,\gamma}(s) > 0 \quad \text{for } s \geq 2-p,$$

which implies $\lambda_{N,p,\gamma} \geq \mu_{N,p,\gamma}$, since $s > 2-p$. Therefore, for $\lambda \geq \lambda_{N,p,\gamma}$ we have $A > 0$, and we obtain the existence of a positive solution $S(x,t)$. The regularity of S stated in the lemma is an easy calculation from the explicit expression of S. It is also easy to see that, if $\gamma+1 < N(2-p)/p$, then $S(x,t)$ is a solution of (P) in the sense of distributions. ☐

We can summarize the results about S for $1 < p < 2$ as follows.

(a) If $\gamma+1 < \frac{N(2-p)}{2p}$, $S(x,t)$ is an energy solution; i.e., $S(x,t) \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap \mathcal{C}^0(0,T;L^2(\Omega))$.

(b) If $\frac{N(2-p)}{2p} \leq \gamma+1 < \frac{N(2-p)}{p}$, $S(x,t)$ is an *entropy solution* (see Definition 3.5 in subsection 3.2.2).

(c) If $\frac{N(2-p)}{p} \leq \gamma+1 < \frac{N}{p}$, $S(x,t)$ is a *very weak solution* (see Theorem 3.8, below).

We will prove that the regularity of the self-similar solution gives the behavior of the solutions for the initial value problem in each interval of the parameters. Notice that behavior means that, if $1 < p < 2$, then, for all $\gamma \in (-\infty, \frac{N-p}{p})$, the *spectral instantaneous and complete blow-up* as in Baras–Goldstein does not occur. Namely, there exist solutions with different meanings for all λ.

Let us point out that, if $p = 2$, all the previous critical values collapse to $1+\gamma = 0$, and we will find that for $1+\gamma \leq 0$ there exist solutions in the energy sense. Note that in this case, by linearity, we obtain *global solutions*. Hence, also in this case, the *spectral instantaneous and complete blow-up* does not occur.

Moreover, if $p > 2$ and $1+\gamma \leq 0$, an argument of comparison allows us to conclude that there exists at least a local (in time) solution.

The remaining question about the behavior in the case $p \geq 2$, $\frac{N}{p} > 1+\gamma > 0$ will be discussed in section 4.

3.2.1. The case $\lambda > \lambda_{N,p,\gamma}$, $p \le 2$, $\gamma + 1 < N(2-p)/(2p)$: Global existence of solutions with finite energy.

THEOREM 3.3. *If $\lambda > \lambda_{N,p,\gamma}$, $1 < p \le 2$, $\gamma + 1 < \frac{N(2-p)}{2p}$, $\psi(x) \in L^2(\Omega)$, then there exists a distributional solution u of problem* (P) *such that*

$$u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)).$$

Proof. Let us consider the approximate problems (P_n) defined in the proof of Theorem 3.1. Using $u_n(x,t)$ as test function in (P_n), we get

$$\frac{1}{2}\int_\Omega u_n^2(x,\tau)\,dx + \iint_{Q_\tau} \frac{|\nabla u_n|^p}{|x|^{p\gamma}} \le \lambda \iint_{Q_\tau} \frac{|u_n|^p}{|x|^{p(\gamma+1)}} - \frac{1}{2}\int_\Omega \psi^2(x)\,dx.$$

If $p < 2$, one has

$$\iint_{Q_\tau} \frac{|u_n|^p}{|x|^{p(\gamma+1)}} \le \iint_{Q_\tau} u_n^2 + c_1 T \int_\Omega \frac{dx}{|x|^{2p(\gamma+1)/(2-p)}},$$

where $c_1 = c_1(p)$. The last integral is finite by the hypotheses on γ. If $p = 2$, then necessarily $\gamma + 1 < 0$, and therefore

$$\iint_{Q_\tau} \frac{|u_n|^p}{|x|^{p(\gamma+1)}} \le c_2 \iint_{Q_\tau} u_n^2,$$

with $c_2 = c_2(\Omega,\gamma)$. In both cases, by Gronwall's lemma, we obtain the estimates (3.1)–(3.3), and we can conclude the proof exactly as for Theorem 3.1. ∎

Remark 3.4. Note that actually, in the proof of this theorem, λ can be any real number, since the principal part of the operator is never used to obtain estimates.

3.2.2. The case $\lambda > \lambda_{N,p,\gamma}$, $p \le 2$, $N(2-p)/(2p) < \gamma+1 < N(2-p)/p$: Global existence of entropy solutions. We will specify the sense in which we consider solutions in this case.

DEFINITION 3.5. *Assume that $\psi \in L^1(\Omega)$. We say that $u \in C([0,T];L^1(\Omega))$ is an entropy solution to problem* (P) *if $\frac{|u|^{(p-1)}}{|x|^{p(\gamma+1)}} \in L^1(Q_T)$, $T_k(u) \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))$ for all $k > 0$, and*

$$\int_\Omega \Theta_k(u(T) - v(T))\,dx + \int_0^T \langle v_t, T_k(u-v)\rangle\,dt + \iint_{Q_T} \frac{|\nabla u|^{p-2}}{|x|^{p\gamma}} \nabla u \cdot \nabla(T_k(u-v))$$
$$\le \int_\Omega \Theta_k(\psi - v(0))\,dx + \lambda \iint_{Q_T} \frac{|u|^{p-2}u}{|x|^{p(\gamma+1)}} T_k(u-v) \tag{3.14}$$

for all $k > 0$ and $v \in L^p((0,T),\mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap L^\infty(Q_T) \cap C([0,T];L^1(\Omega))$ such that $v_t \in L^{p'}((0,T);\mathcal{D}^{-1,p'}_{-\gamma}(\Omega))$, where Θ_k is given by

$$\Theta_k(s) = \int_0^s T_k(t)dt. \tag{3.15}$$

For a general definition and basic properties of entropy solutions, see, for instance, the references [9], [23], and [22].

THEOREM 3.6. *If* $\lambda \geq \lambda_{N,p,\gamma}$, $1 < p < 2$, $\frac{N(2-p)}{2p} \leq \gamma + 1 < \frac{N(2-p)}{p}$, *while the initial datum* $\psi(x)$ *satisfies*

$$\psi \in L^q(\Omega) \quad \text{for every } q \text{ such that } 1 < q < \frac{N(2-p)}{p(\gamma+1)},$$

then there exists a distributional solution u *of problem* (P) *such that*

$$u \in L^\infty(0,T;L^q(\Omega)) \quad \text{for every } q \text{ such that } 1 < q < \frac{N(2-p)}{p(\gamma+1)}, \tag{3.16}$$

$$\frac{|\nabla u|^{q_1}}{|x|^{\gamma q_1}} \in L^1(Q_T) \quad \text{for every } q_1 \text{ such that } 0 < q_1 < \frac{N(2-p)}{2(\gamma+1)}, \tag{3.17}$$

$$\frac{|\nabla u|^{p-1}}{|x|^{p\gamma}}, \ \frac{u^{p-1}}{|x|^{p(\gamma+1)}} \in L^1(Q_T). \tag{3.18}$$

Moreover, u *is an entropy solution to problem* (P).

Proof. Once again, we consider the approximate problems (P_n), and we multiply them by the test function $\Phi(u_n) = [(1+|u_n|)^{1-\mu} - 1] \operatorname{sign} u_n$, with $\mu \in (0,1)$ to be chosen hereafter. If we define

$$\Psi(s) = \int_0^s \Phi(\sigma)\, d\sigma = \frac{(1+|s|)^{2-\mu} - 1}{2-\mu} - |s|,$$

we have

$$\Psi(s) \geq c_1(\mu)|s|^{2-\mu} - c_2(\mu). \tag{3.19}$$

Therefore,

$$\begin{aligned} &\int_\Omega \Psi(u(x,\tau))\, dx + (1-\mu) \iint_{Q_\tau} \frac{|\nabla u_n|^p}{|x|^{\gamma p}} \frac{1}{(1+|u_n|)^\mu} \\ &\quad \leq \int_\Omega \Psi(\psi(x))\, dx + \lambda \iint_{Q_\tau} \frac{|u_n|^{p-1}}{|x|^{p(\gamma+1)}} (1+|u_n|)^{1-\mu} \\ &\quad \leq \int_\Omega \Psi(\psi(x))\, dx + c_3 \iint_{Q_\tau} \frac{|u_n|^{p-\mu}+1}{|x|^{p(\gamma+1)}}, \end{aligned} \tag{3.20}$$

where c_3 depends on λ, μ, p. Note that $\Psi(\psi)$ is integrable by the hypothesis on the initial datum. Since $p < 2$, we can estimate the last integral as

$$\iint_{Q_\tau} \frac{|u_n|^{p-\mu}+1}{|x|^{p(\gamma+1)}} \leq c_4 \iint_{Q_\tau} |u_n|^{2-\mu} + c_5 \iint_{Q_\tau} \frac{1}{|x|^{p(\gamma+1)(2-\mu)/(2-p)}}, \tag{3.21}$$

where c_4 and c_5 depend on μ and p. Now we choose μ in such a way that

$$2 - \frac{(2-p)N}{p(\gamma+1)} < \mu < 1. \tag{3.22}$$

This implies that the last integral in (3.21) converges. Using (3.19)–(3.22) and Gronwall's lemma, we obtain the following estimates:

$$\|u_n\|_{L^\infty(0,T;L^q(\Omega))} \le c_6 \quad \text{for every } q \text{ such that } 1 < q < \frac{(2-p)N}{p(\gamma+1)}, \tag{3.23}$$

$$\iint_{Q_T} \frac{|\nabla u_n|^p}{|x|^{\gamma p}} \frac{1}{(1+|u_n|)^\mu} \le c_7 \quad \text{for every } \mu \text{ such that (3.22) holds}, \tag{3.24}$$

$$\iint_{Q_T} \frac{|\nabla u_n|^{q_1}}{|x|^{\gamma q_1}} \le c_8 \quad \text{for every } q_1 \text{ such that } 0 < q_1 < \frac{(2-p)N}{2(\gamma+1)}, \tag{3.25}$$

$$\iint_{Q_T} \frac{|u_n|^{p-\mu}}{|x|^{p(\gamma+1)}} \le c_9 \quad \text{for every } \mu \text{ such that (3.22) holds.} \tag{3.26}$$

Indeed,

$$\iint_{Q_T} \frac{|\nabla u_n|^{q_1}}{|x|^{\gamma q_1}} \le \left(\iint_{Q_T} \frac{|\nabla u_n|^p}{|x|^{\gamma p}} \frac{1}{(1+|u_n|)^\mu}\right)^{q_1/p} \left(\iint_{Q_T} (1+|u_n|)^{\mu q_1/(p-q_1)}\right)^{(p-q_1)/p}.$$

The estimate (3.25) follows from (3.24) and (3.22).

We now show that the sequence $\{u_n\}$ satisfies

$$\iint_{Q_T} \frac{|u_n|^{(p-1)r}}{|x|^{p(\gamma+1)}} \le c_{10} \quad \text{for all } r \text{ such that } 1 \le r < \frac{2-p}{p-1}\left[\frac{N}{p(\gamma+1)} - 1\right], \tag{3.27}$$

$$\iint_{Q_T} \frac{|\nabla u_n|^{(p-1)s}}{|x|^{p\gamma}} \le c_{11} \quad \text{for all } s \text{ such that } 1 \le s < \frac{(N-p\gamma)(2-p)}{(p-1)(2+p\gamma)}. \tag{3.28}$$

Inequality (3.27) follows from (3.26) and (1.4), while (3.28) follows easily from (3.25). We can now pass to the limit in the distributional formulation, as we have done in the proof of Theorem 3.1, using the estimate in $L^{q_1}(0,T;W^{1,q_1}(\Omega \setminus B_\varepsilon))$, which follows from (3.17), for every $\varepsilon > 0$.

The function u is an entropy solution. Indeed, it is easy to prove (taking $T_k(u_n)$ as test function in (P_n)) that $T_k(u_n)$ is bounded in $L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma}(\Omega))$ and (using Vitali's theorem and (3.28)) that $f_n(x,u_n)$ converges to $\frac{u^{p-1}}{|x|^{p(\gamma+1)}}$ strongly in $L^1(Q_T)$.

Then, if we take $T_k(u_n - v)$ as test function in (P_n), with v as in Definition 3.5, we can easily pass to the limit and get the result with the same techniques as in [22]. ∎

Remark 3.7. As far as the sharpness of the regularity of the solutions found in Theorem 3.6, let us observe that any function of the form $S_{t_0}(x,t) = S(x,t+t_0)$, where S is defined by (3.11), is a solution in the distribution sense of problem (P), with initial data $\psi(x) = S(x,t_0)$, and its regularity is exactly the one we quoted in Theorem 3.6.

3.2.3. The case $\lambda > \lambda_{N,p,\gamma}$, $p \le 2$, $N(2-p)/(p) < \gamma+1 < N/p$: Global existence of very weak solutions. We point out that for every $t > 0$ the singular solution $S(x,t)$ is continuous with respect to t with values in $L^2_{-\alpha p/2}(\Omega)$ for every α such that

$$\alpha > \frac{2(\gamma+1)}{2-p} - \frac{N}{p}. \tag{3.29}$$

This, together with the previous estimates on S, suggests the definition of the following space:

$$\mathcal{Y}_\alpha = \{u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma-\alpha}(\Omega)) \cap C^0([0,T];L^2_{-\frac{\alpha p}{2}}(\Omega)) : u' \in L^{p'}(0,T;\mathcal{D}^{-1,p'}_{-\beta}(\Omega))\}, \tag{3.30}$$

where

$$\beta = \gamma + \alpha(p-1). \tag{3.31}$$

The following theorem specifies the meaning of a very weak solution.

THEOREM 3.8. *Assume that* $\lambda > \lambda_{N,p,\gamma}$, $1 < p < 2$, $\frac{N(2-p)}{p} \le \gamma + 1 < \frac{N}{p}$, *and that the initial data* $\psi(x)$ *belongs to* $L^2_{-\frac{\alpha p}{2}}(\Omega)$ *for some* α *satisfying* (3.29). *Then there exists a function* $u \in \mathcal{Y}_\alpha$ *which is a distributional solution of* (P) *away from the origin* (*that is, in* $\mathcal{D}'((\Omega \setminus \{0\}) \times (0,T))$). *Moreover,* u *is a solution of* (P) *in the following sense:*

$$\begin{aligned} &-\int_0^\tau \langle v', |x|^{\alpha p} u\rangle\,dt + \int_\Omega u(\tau)v(\tau)|x|^{\alpha p}\,dx - \int_\Omega \psi v(0)|x|^{\alpha p}\,dx \\ &+ \iint_{Q_\tau} \frac{|\nabla u|^{p-2}\nabla u\cdot\nabla(v|x|^{\alpha p})}{|x|^{\gamma p}}\,dx\,dt = \iint_{Q_\tau} \frac{|u|^{p-2}uv|x|^{\alpha p}}{|x|^{(\gamma+1)p}}\,dx\,dt \end{aligned} \tag{3.32}$$

for every $\tau \in [0,T]$ *and for every* $v \in \mathcal{Y}_\alpha$.

Proof. Step 1: *A priori estimate.* Let u_n be a solution of problem (P_n). We use $|x|^{\alpha p}u_n(x,t)$ as test function in (P_n). Then, by Young's inequality,

$$\begin{aligned} &\int_\Omega u_n^2(x,T)|x|^{\alpha p}\,dx + \iint_{Q_T} |\nabla u_n|^p |x|^{(\alpha-\gamma)p} \\ &\quad\le c_1 \iint_{Q_T} |\nabla u_n|^{p-1}|x|^{(\alpha-\gamma)p-1} + \lambda \iint_{Q_T} \frac{|u_n|^p}{|x|^{p(\gamma+1-\alpha)}} + \frac{1}{2}\int_\Omega \psi^2(x)|x|^{\alpha p}\,dx \\ &\quad\le \frac{1}{2}\iint_{Q_T} |\nabla u_n|^p|x|^{(\alpha-\gamma)p} + c_3 \iint_{Q_T} |u_n|^2|x|^{\alpha p} + c_3\int_\Omega |x|^{p\left(\alpha - \frac{2(\gamma+1)}{2-p}\right)} \\ &\qquad + \frac{1}{2}\int_\Omega \psi^2(x)|x|^{\alpha p}\,dx. \end{aligned}$$

Under the hypotheses on α and on the initial datum, the last two integrals are finite. Therefore, we can use Gronwall's lemma to conclude that

$$u_n \text{ is bounded in } L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma-\alpha}(\Omega)) \cap C^0([0,T];L^2_{-\frac{\alpha p}{2}}(\Omega)).$$

By (P_n), one can easily check that

$$u_n' \text{ is bounded in } L^{p'}(0,T;\mathcal{D}^{-1,p'}_{-\beta}(\Omega)).$$

Step 2: *Passage to the limit.* By weak convergence, and following the same argument as in the proof of Theorem 3.1 for the pointwise convergence of the gradients, we obtain a function $u \in L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma-\alpha}(\Omega)) \cap L^\infty(0,T;L^2_{-\frac{\alpha p}{2}}(\Omega))$, with

$u' \in L^{p'}(0,T;\mathcal{D}^{-1,p'}_{-\beta}(\Omega))$, such that

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;\mathcal{D}^{1,p}_{0,\gamma-\alpha}(\Omega)),$$

$$u_n \rightharpoonup u \quad *\text{-weakly in } L^\infty(0,T;L^2_{-\frac{\alpha p}{2}}(\Omega)),$$

$$\nabla u_n \to \nabla u \quad \text{almost everywhere in } Q_T,$$

$$u_n(\cdot,\tau) \to u(\cdot,\tau) \quad \text{a.e. in } \Omega \text{ and weakly in } L^2_{-\frac{\alpha p}{2}}(\Omega) \text{ for every } \tau \in [0,T].$$

Using these convergences, one can take $|x|^{\alpha p} v$ as test function in (P_n) and pass to the limit as $n \to \infty$, obtaining the weak formulation (3.32). Since the functions of the form $|x|^{\alpha p} v$ include smooth test functions in $\mathcal{D}(Q_T)$ which are zero in a neighborhood of the origin, we have also proved that u is a solution in the distributional sense far from the origin.

We now prove that $u \in C^0([0,T];L^2_{-\frac{\alpha p}{2}}(\Omega))$. According to the uniform estimates for the approximate solutions, we find that $u_n(\cdot,t)$ is an equicontinuous sequence in $L^2_{-\frac{\alpha p}{2}}(\Omega))$. By the Ascoli–Arzelà lemma, we conclude. □

Remark 3.9.

(i) The previous result, in the case where $\gamma = 0$, improves the result contained in [19] and specifies the meaning of the solution given in that paper; more precisely, it gives us that the solution is in $L^p(0,T;\mathcal{D}^{1,p}_{-\alpha}(\Omega))$ for some $\alpha > 2/(2-p) - N/p$.

(ii) If we define the operator $\Gamma v = |x|^{\alpha p} v$, then Γ is an isomorphism from $\mathcal{D}^{1,p}_{0,\gamma-\alpha}(\Omega)$ to $\mathcal{D}^{1,p}_{0,\beta}(\Omega)$, where $\beta = (p-1)\alpha + \gamma$. Therefore, the weak formulation (3.32) could be rewritten as

$$-\int_0^\tau \langle w', u\rangle\, dt + \int_\Omega u(\tau)w(\tau)\, dx - \int_\Omega \psi w(0)\, dx + \iint_{Q_\tau} \frac{|\nabla u|^{p-2}\nabla u \cdot \nabla w}{|x|^{\gamma p}}\, dx\, dt$$
$$= \iint_{Q_\tau} \frac{|u|^{p-2} u w}{|x|^{(\gamma+1)p}}\, dx\, dt$$

for every $\tau \in [0,T]$ and for every $w \in L^p(0,T;\mathcal{D}^{1,p}_{0,\beta}(\Omega)) \cap C^0([0,T];L^2_{\frac{\alpha p}{2}}(\Omega))$ such that $w' \in L^{p'}(0,T;\mathcal{D}^{-1,p'}_{\alpha-\gamma}(\Omega))$.

(iii) In the case where the initial data $\psi(x)$ is nonnegative and satisfies

$$\psi(x) \le S(x, t+t_0) \quad \text{for some positive } t_0,$$

it is possible to obtain an alternative (constructive) proof by a monotone iteration argument, using $S(x,t+t_0)$ as a supersolution and solving, by induction, the sequence of problems

$$(\tilde{\mathrm{P}}_n) \qquad \begin{cases} \dfrac{\partial u_n}{\partial t} - \Delta_{p,\gamma} u_n = \lambda T_n\left(\dfrac{1}{|x|^{p(\gamma+1)}}\right) u_{n-1}^{p-1}, & (x,t) \in \Omega \times (0,T),\\ u_n(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T),\\ u_n(x,0) = \psi(x), & x \in \Omega, \end{cases}$$

with $u_0 \equiv 0$.

(iv) The solution found in Theorem 3.10 satisfies the equation in a very weak sense because the right-hand side of the equation does not even belong to L^1.

3.3. The case $\lambda > \lambda_{N,p,\gamma}$, $p \geq 2$, $\gamma \leq -1$: Existence for small times. This subsection deals with existence for small values of t in the case $\lambda > \lambda_{N,p,\gamma}$, $p > 2$, $\gamma \leq -1$. The result of this subsection can be compared with the ones of section 4: an instantaneous blow up will occur for the solutions of the approximate problems for the same values of λ and p when $\gamma > -1$.

THEOREM 3.10. *If $\lambda > \lambda_{N,p,\gamma}$, $p \geq 2$, $\gamma \leq -1$, while the initial data $\psi(x)$ satisfies $\psi(x) \in L^\infty(Q_T)$ and $\psi(x) \geq 0$, then there exist $T^* = T^*(N, p, \gamma, \lambda, \|\psi\|_{L^\infty(\Omega)}) > 0$ and a distributional solution u in Q_{T^*} of our problem with $u \in L^p(0, T; \mathcal{D}^{1,p}_{0,\gamma}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ for every $T < T^*$. Moreover, if $p = 2$, T^* is any positive value.*

Proof. Let us define the problems $(\tilde{\mathrm{P}}_n)$ as in the previous subsection and let $y(t)$ be the solution of the ordinary differential equation

$$\begin{cases} y'(t) = d\, y^{p-1}, \\ y(0) = \|\psi\|_{L^\infty(\Omega)}, \end{cases}$$

where

$$d \geq \lambda \sup_{x \in \Omega} |x|^{-p(\gamma+1)}. \tag{3.33}$$

An immediate calculation shows the following.

(α) If $p > 2$, the solution is

$$y(t) = \frac{\|\psi\|_{L^\infty(\Omega)}}{(1 - (p-2)d\|\psi\|^{p-2}_{L^\infty(\Omega)} t)^1/(p-2)},$$

which blows up in $t = T^* = \frac{1}{(p-2)d\|\psi\|^{p-2}_{L^\infty(\Omega)}}$.

(β) If $p = 2$, then the global solution is

$$y(t) = \|\psi\|_{L^\infty(\Omega)} e^{dt}.$$

Since $y(t)$ is a supersolution of (P), by the comparison principle we have

$$u_1 \leq u_2 \leq \cdots \leq u_n \leq \cdots \leq y.$$

If we multiply problem $(\tilde{\mathrm{P}}_n)$ by $u_n \chi_{(0,\tau)}$, we obtain

$$\frac{1}{2} \int_\Omega u_n^2(x,\tau)\, dx + \iint_{Q_T} \frac{|\nabla u|^p}{|x|^{p\gamma}} \leq \lambda \iint_{Q_T} |y|^{p-1} |x|^{-p(\gamma+1)} + \frac{1}{2} \int_\Omega \psi^2(x)\, dx.$$

By condition (3.33),

$$\lambda \iint_{Q_T} |y|^{p-1} |x|^{-p(\gamma+1)} \leq \operatorname{meas} \Omega (y(\tau) - \|\psi\|_{L^\infty(\Omega)}).$$

Therefore, we get the estimates

$$\|u_n\|_{L^\infty(0,\tau; L^2(\Omega))} \leq c_1, \quad \|u_n\|_{L^p(0,\tau; \mathcal{D}^{1,p}_{0,\gamma}(\Omega))} \leq c_2 \quad \text{for every } \tau < T^*.$$

In the case $p = 2$, we can fix any $T^* > 0$ to get the same estimates. Now the conclusion follows exactly as in the proof of Theorem 3.1. ∎

4. Blow-up: $p > 2$, $N/p > (1+\gamma) > 0$, and $\lambda > \lambda_{n,p,\gamma}$. We consider in this section the *spectral, instantaneous, and complete blow-up* in the case $p > 2$ and $(1+\gamma) > 0$. The case $p = 2$ has been obtained in [3] and requires a different method. We would like to point out that in the case $p > 2$ a stronger result than in the linear case is obtained. This behavior is given because even the problem with the truncated potential blows up in finite time. We will assume that the initial data verifies that $\psi \in L^2(\Omega)$ and there exists $\delta > 0$ such that $\psi > 0$ in $B_\delta(0)$. Notice that for the equation

$$u_t - \Delta_{p,\gamma} u = 0 \tag{4.1}$$

and by direct calculations we can find Barenblatt-type solutions; precisely,

$$\mathcal{B}(x,t) = t^{-N\beta(N,p,\gamma)} \left[M - \frac{(p-2)\beta(N,p,\gamma)^{\frac{1}{p-1}}}{p(\gamma+1)} \xi^{\frac{p(\gamma+1)}{p-1}} \right]_+^{\frac{(p-2)}{(p-1)}}, \tag{4.2}$$

where M is a positive arbitrary constant,

$$\beta(N,p,\gamma) = \frac{1}{N(p-2)+p(\gamma+1)}, \quad \text{and} \quad \xi = \frac{|x|}{t^{\beta(N,p,\gamma)}}.$$

This property could be understood as some kind of *finite speed of propagation* for the equation with zero right-hand side. It is necessary to point out that if $\gamma \neq 0$, the equation is not invariant by translation, and then the corresponding translated Barenblatt functions are not solutions to the equation.

The lack of homogeneity in (4.1) provides the following weak Harnack inequality.

LEMMA 4.1. *Let u be a nonnegative weak solution to* (4.1)*, and assume that $u(x_0, t_0) > 0$ for some $(x_0, t_0) \in \Omega_T$; then there exists $B(N, p, \gamma) > 1$ such that, for all $\theta, \rho > 0$ satisfying $B_{4\rho}(x_0) \times (t_0 - 4\theta, t_0 + 4\theta) \subset \Omega_T$, we have*

$$\begin{aligned} &\frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} u(x,t_0)dx \\ &\quad \le B \left[\left(\frac{\rho^{p(\gamma+1)}}{\theta} \right)^{\frac{1}{p-2}} + \left(\frac{\theta}{\rho^{p(\gamma+1)}} \right)^{\frac{N}{p(\gamma+1)}} \left(\inf_{B_\rho(x_0)} u(.,t_0+\theta) \right)^{\frac{\lambda_\gamma}{p(\gamma+1)}} \right], \end{aligned} \tag{4.3}$$

where $\lambda_\gamma = N(p-2) + p(\gamma+1) = \frac{1}{\beta(N,p,\gamma)}$.

The proof is similar to the one by DiBenedetto in [17] for the case $\gamma = 0$. The details can be found in [1] in the case $(1+\gamma) > 0$, where some counterexamples to the Harnack inequality if $(1+\gamma) \le 0$ are shown.

We consider problem (P), and we make the following assumptions:

(H1) $p > 2$, $0 < 1+\gamma < N/p$, and $\lambda > \lambda_{n,p,\gamma}$.

(H2) $\psi \in L^\infty(\Omega)$, $\psi(x) \ge 0$, and moreover, there exists $\rho, \delta > 0$ such that $\psi(x) > \delta$ for every $x \in B_\rho(0)$.

We will prove that problem (P) has no solution. We start by studying, for $n \in \mathbb{N}$, the approximate problems

$$\begin{cases} (u_n)_t - \Delta_{p,\gamma} u_n = \lambda W_n(x) |u_n|^{p-2} u_n & \text{in } Q_T, \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = \psi(x) & \text{in } \Omega, \end{cases} \tag{4.4}$$

where $W_n(x) = T_n(\frac{1}{|x|^{p(\gamma+1)}})$. Note that for every fixed n, problem (4.4) has a solution at least for small times (depending on n and λ), as one can easily see using a convenient supersolution independent of x.

By separation of variables we look for solutions of (4.4) of the form $\Phi(x,t) = \Theta(t)X(x)$, to use as a subsolution. The equation becomes

$$\Theta' X - \Theta^{p-1}\Delta_{p,\gamma} X = \lambda W_n(x)\Theta^{p-1}X^{p-1}.$$

We take the $\Theta(t)$ solution of

$$\begin{cases} \Theta'(t) = \mu\Theta^{p-1}(t), \\ \Theta(0) = A, \end{cases} \tag{4.5}$$

that is,

$$\Theta(t) = \frac{A}{[1-(p-2)\mu A^{p-2}t]^{1/(p-2)}}$$

with $\mu, A > 0$ to be chosen. Note that $\lim_{t\to\tau}\Theta(t) = \infty$ for $\tau = \frac{1}{\mu(p-2)A^{p-2}}$.

On the other hand, $X(x)$ must solve the elliptic problem

$$\begin{cases} -\Delta_{p,\gamma} X = \lambda W_n(x) X^{p-1} - \mu X & \text{in } \Omega, \\ \quad X(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.6}$$

Defining $\alpha X = Y$ with $\mu\alpha^{p-2} = \lambda$ the problem above becomes

$$\begin{cases} -\Delta_{p,\gamma} Y = \lambda(W_n(x) Y^{p-1} - Y) & \text{in } \Omega, \\ \quad Y(x) = 0 & \text{in } \partial\Omega. \end{cases} \tag{4.7}$$

Problem (4.7) fails in the hypotheses for bifurcation from infinity as in [6]; see [16] for details.

Let $\lambda_1(n)$ be the first eigenvalue for the problem

$$\begin{cases} -\Delta_{p,\gamma}\varphi = \lambda W_n(x)|\varphi|^{p-2}\varphi & \text{in } \Omega, \\ \quad \varphi(x) = 0 & \text{in } \partial\Omega. \end{cases}$$

Then (i) $\lambda_1(n) > 0$; (ii) $\lambda_1(n)$ is isolated and simple; (iii) the first eigenfunction does not change sign; (iv) $\lambda_1(n)$ is decreasing in n, and $\lambda_1(n) \searrow \lambda_{N,p,\gamma}$. The properties (i), (ii), and (iii) are similar to the p-*laplacian* case and are detailed in [16]; (iv) is easily checked following the proof for the p-laplacian in [19].

THEOREM 4.2. *If $\lambda > \lambda_{N,p,\gamma}$, then there exists n_0 such that, for every $n > n_0$, there exists a bounded positive solution $Y(x)$ to* (4.7).

Proof. As $\lambda > \lambda_{N,p,\gamma}$ there exists n_0 such that, for $n > n_0$, $\lambda > \lambda_1(n)$. Now $\lambda_1(n)$ is the unique bifurcation point of positive solutions from infinity for problem (4.7). Moreover, as $(1+\gamma) > 0$, the solutions in the branch are bounded; see [16] and [6]. Moreover, if $Y > 0$ is a solution to (4.7), then $\|Y\|_\infty \geq R_n > 0$ for some constant R_n, because if a positive solution Y is such that $\|Y\|_\infty < \varepsilon$, then we have $-\Delta_{p,\gamma} Y \leq \lambda Y(n\varepsilon^{p-2} - 1) < 0$, and for ε small we reach a contradiction with the maximum principle. ◻

As a consequence we can find a subsolution to problem (4.4) that shows the finite time blow-up. Precisely, we have the following result.

LEMMA 4.3. *Let u be a solution to problem* (4.4)*, where $\lambda > \lambda_1(n)$ and $\psi(x) > 0$ in every $x \in \Omega$. Then there exists $T > 0$ depending on the data and there exists a subsolution Φ such that $u(x,t) \geq \Phi(x,t)$ and $\lim_{t\to T} \Phi(x,t) = \infty$ for every $x \in \Omega$.*

Proof. The solution u is positive and, by regularity (see [1]), is bounded for small times. Therefore, we fix a small time $\tau > 0$, and we look for a subsolution of the form $\Phi(x,t) = X(x)\Theta(t)$, with $X(x)$ the solution of (4.6), and

$$\Theta(t) = \epsilon(1 - (p-2)\varepsilon^{p-2}(t-\tau))^{-1/(p-2)},$$

with $\varepsilon > 0$ such that $\varepsilon X(x) \leq u(x,\tau)$. By the weak comparison principle we conclude. ☐

In order to show the instantaneous complete blow-up, we need to rescale the problem, using the following property. Define

$$Z_n(x) = \left(\frac{n_0}{n}\right)^{\frac{1}{p-2}} X\left(\left(\frac{n}{n_0}\right)^{\frac{1}{p(\gamma+1)}} x\right). \tag{4.8}$$

Then Z_n solves

$$\begin{cases} -\Delta_{p,\gamma} Z_n = \lambda W_n(x) Z_n^{p-1} - \mu Z_n & \text{if } |x| < \left(\frac{n_0}{n}\right)^{\frac{1}{p(\gamma+1)}}, \\ \quad Z_n(x) = 0 & \text{if } |x| = \left(\frac{n_0}{n}\right)^{\frac{1}{p(\gamma+1)}} \end{cases}$$

since $(\frac{n}{n_0}) W_{n_0}((\frac{n}{n_0})^{\frac{1}{p(\gamma+1)}} x) = W_n(x)$. Moreover, the radius of the ball goes to zero and $\|Z_n\|_\infty \to 0$ as $n \to \infty$. Therefore, for prescribed $R, \eta > 0$ we can choose n such that

$$\left(\frac{n_0}{n}\right)^{\frac{1}{p(\gamma+1)}} < R, \qquad Z_n(x) \leq \eta \quad \text{on } B_R. \tag{4.9}$$

THEOREM 4.4. *Assume that* (H1), (H2) *hold. Then for every $\varepsilon > 0$ there exist $r(\varepsilon) > 0$ and n_ε such that if u_n is the minimal solution to* (4.4) *$\forall n > n_\varepsilon$*

$$u_n(x,t) \equiv +\infty \quad \textit{for} \quad t > \varepsilon \textit{ and } |x| < r(\varepsilon).$$

Proof. Take n_0 such that $\lambda > \lambda_1(n_0)$. We prescribe the blow-up time $T = \varepsilon$ and choose $\mu = [(p-2)\varepsilon]^{-1}$. For such μ and $n > n_0$, the scaled solution (4.8) to (4.4), X_n, satisfies (4.9) with $R = \rho$ and $\eta = \delta$. Consider $\Theta(t)$ solution to (4.5) with μ as above and A $= 1$. Then $\phi_n(x,t) = \Theta(t) X_n(x)$ blows up in $T = \varepsilon$. By weak comparison in the ball $|x| < (\frac{n_0}{n})^{\frac{1}{p(\gamma+1)}}$, the minimal solution to (4.4) blows up in $T_0 < \varepsilon$. ☐

We point out that in order to obtain blow-up in a prescribed small time we have to take the index n large enough. We will use the concept of entropy solution introduced in Definition 3.5 and a straightforward modification of the comparison arguments for entropy solutions (see [23]).

THEOREM 4.5. *Assume that* (H1), (H2) *hold. Then problem* (P) *has no entropy solution, even for small times, and moreover, if $u_n(x,t)$ is the minimal solution to* (4.4)*, we have that $\lim_{n\to\infty} u_n(x,t) = +\infty$ for all $(x,t) \in \Omega \times (0,\infty)$.*

Proof. By contradiction, assume that there exists an entropy solution $u(x,t) > 0$ of problem (P). Then u is a supersolution to problem (4.4) for all n. As a consequence the minimal solution to (4.4) satisfies $u_n(x,t) \leq u(x,t)$; hence $u(x,t)$ blows up at least in the time in which u_n blows up, so we conclude.

By using Theorem 4.4 we obtain a region E_∞ such that

$$E_\infty \supset \{|x| < r(t)\} \times (0, \infty),$$

such that

$$\lim_{n\to\infty} u_n(x,t) = +\infty \quad \text{for all } (x,t) \in E_\infty.$$

Next we use the Harnack inequality (4.3), assume that there exists a point $(x_0, t_0) \in \Omega \times (0,\infty)$ such that $0 \le u_n(x_0,t_0) \le M < \infty$, and call

$$\rho(x_0,t_0) = \operatorname{dist}\{x_0, \partial\Omega\} > 0.$$

Then, if $B_r(x_0) \times \{t = t_1\} \cap E_\infty$ has N-dimensional positive measure for some $r < \rho(x_0,t_0)$ and $t_1 < t_0$, we consider the problem

$$\begin{cases} (v_n)_t - \Delta_{p,\gamma} v_n = 0 & \text{in } B_r(x_0) \times (t_1, t_0), \\ v_n(x,t) = 0 & \text{on } \partial B_r(x_0) \times (t_1,t_0), \\ v_n(x,t_1) = u_n(x,t_1) & \text{in } B_r(x_0); \end{cases} \tag{4.10}$$

then $v_n(x,t) \le u_n(x,t)$, and this is a contradiction to the Harnack inequality (4.3). If for all $r < \rho(x_0)$ and all $t_1 < t_0$, $|B_r(x_0) \times \{t = t_1\} \cap E_\infty| = 0$, then for all $\delta > 0$ we can find in a finite number of steps a point $(x_1, t_0 - \delta) \in \Omega \times (0, t_0)$ such that

$$|B_r(x_0) \times \{t = t_1\} \cap E_\infty| > 0,$$

and then we reach a contradiction as above. ◻

Remark 4.6. Notice that this result is stronger, in some sense, than the result by Baras and Goldstein (see [7]) for the heat equation; if $p > 2$, even the solution to the equation with truncated potential blows up in finite time.

Next we will prove that even if we truncate the whole nonlinearity, we find spectral instantaneous complete blow-up. More precisely, we have the following result.

THEOREM 4.7. *Consider the truncated problem*

$$\begin{cases} (v_n)_t - \Delta_{p,\gamma} v_n = \lambda\, W_n(x)\, T_n(v_n^{p-1}) & \text{in } \Omega \times \mathbb{R}^+, \\ v(x,t) = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ v(x,0) = \psi(x) & \text{in } \Omega, \end{cases} \tag{4.11}$$

where (H1) *and* (H2) *hold. Then*

$$\lim_{n\to\infty} v_n(x,t) = +\infty \quad \text{for every } (x,t) \in \Omega \times \mathbb{R}^+.$$

Proof. Using the same argument as in [2], we find that if $B_{4r}(0) \subset \Omega$, then

$$\lim_{n\to\infty} \int_{B_r(0)} v_n(x,t)\, dx = +\infty \quad \text{for every } t > 0.$$

Then by the Harnack inequality and a strategy which is similar to the one in Theorem 4.5, we obtain the complete blow-up. ◻

Remark 4.8.

(i) An alternative method to the one described above can be seen in [1]. The separation of variables gives a more transparent view of the behavior but uses in a strong way the presence of exactly two homogeneities. In the linear case (see [3]), or if the second member is not eigenvalues-like (see [2]), different arguments are needed.

(ii) If instantaneous and complete blow-up happens without hypothesis (H2), this seems to be an open problem. If $\gamma = 0$, we can take as a subsolution a convenient scaled and translated Barenblatt function that allows us to conclude that there exists a $T^* > 0$ such that for $t > T^*$ the same result as in Theorem 4.7 holds.

5. Behavior of solutions in the case $1 < p < 2$ and $\lambda < \lambda_{N,p,\gamma}$. In this section we will try to explain how the optimal constant in the Hardy inequality becomes the threshold for extinction in finite time of the solution.

5.1. Finite time extinction.

THEOREM 5.1. *Assume that*

$$\max\left\{\frac{2N}{N+2}, \frac{2N}{N+2(\gamma+1)}\right\} < p < 2,$$

$\lambda < \lambda_{N,p,\gamma}$, *and* $\psi \in L^2(\Omega)$. *Then there exists a constant*

$$T^* = T^*(N,p,\gamma,\lambda,\Omega) \le c_1(N,p,\gamma,\lambda,\Omega)\|\psi\|_{L^2(\Omega)}^{2-p}$$

such that any solution of problem (P) *satisfies*

$$u(\cdot,t) \equiv 0 \quad \textit{for } t \ge T^*. \tag{5.1}$$

Proof. Taking u as a test function in (P), and using inequalities (1.3) and (1.7), we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2(t)\,dx + \frac{1}{S_{N,p,\gamma}}\left(1 - \frac{\lambda}{\lambda_{N,p,\gamma}}\right)\left[\int_\Omega \frac{|u(t)|^{p^*}}{|x|^{\gamma p^*}}\,dx\right]^{\frac{p}{p^*}} \le 0.$$

Using the assumptions on p and γ, by Hölder's inequality we obtain

$$\int_\Omega u^2(t)\,dx \le \left[\int_\Omega \frac{|u(t)|^{p^*}}{|x|^{\gamma p^*}}\,dx\right]^{\frac{2}{p^*}}\left[\int_\Omega |x|^{\frac{2\gamma p^*}{p^*-2}}\,dx\right]^{\frac{p^*-2}{\gamma p^*}} \le c_1\left[\int_\Omega \frac{|u(t)|^{p^*}}{|x|^{\gamma p^*}}\,dx\right]^{\frac{2}{p^*}},$$

where $c_1 = c_1(N,p,\gamma,\Omega)$ is a positive constant. Therefore, setting

$$\phi(t) = \int_\Omega u^2(t)\,dx,$$

one has

$$\phi'(t) + c_2[\phi(t)]^{\frac{p}{2}} \le 0,$$

with $c_2 > 0$. Since $p < 2$, this implies

$$\phi(t) \le \left([\phi(0)]^{\frac{2-p}{2}} - c_3 t\right)_+^{\frac{2}{2-p}},$$

from which the statement follows. ◻

THEOREM 5.2. *Assume that*

$$\gamma \ge 0, \qquad 1 < p < \frac{2N}{N+2},$$
$$\lambda < \eta_{N,p,\gamma} = \left(\frac{N(2-p)}{p} - 1\right)\left(\frac{[N-p(\gamma+1)]p}{(2-p)(N-p)}\right)^p,$$

and

$$\int_\Omega |\psi|^{\frac{N(2-p)}{p}}\, dx < \infty.$$

Then there exists a constant

$$T^* = T^*(N,p,\gamma,\lambda,\Omega) \le c_1(N,p,\gamma,\lambda,\Omega)\|\psi\|^{2-p}_{L^{\frac{N}{p}(2-p)}(\Omega)}$$

such that any solution of problem (P) *found by approximation as in Theorem* 3.1 *satisfies*

$$u(\cdot,t) \equiv 0 \quad \text{for } t \ge T^*. \tag{5.2}$$

Proof. We take $v_n = |u_n|^{\alpha-2}u_n$ as test function in (P_n), with $\alpha \ge 2$ to be chosen hereafter. We obtain

$$\frac{1}{\alpha}\frac{d}{dt}\int_\Omega u_n^\alpha(t)\, dx + (\alpha-1)\int_\Omega \frac{|\nabla u_n(t)|^p |u_n(t)|^{\alpha-2}}{|x|^{\gamma p}}\, dx = \lambda \int_\Omega \frac{|u_n(t)|^{\alpha-(2-p)}}{|x|^{(\gamma+1)p}}\, dx.$$

Since

$$\int_\Omega \frac{|\nabla u_n(t)|^p |u_n(t)|^{\alpha-2}}{|x|^{\gamma p}}\, dx = \left(\frac{p}{\alpha-(2-p)}\right)^p \int_\Omega \frac{\left|\nabla\left(|u_n(t)|^{\frac{\alpha-(2-p)}{p}}\right)\right|^p}{|x|^{\gamma p}}\, dx$$

and, by Hardy's inequality,

$$\int_\Omega \frac{|u_n(t)|^{\alpha-(2-p)}}{|x|^{(\gamma+1)p}}\, dx \le \lambda_{N,p,\gamma}^{-1}\int_\Omega \frac{\left|\nabla\left(|u_n(t)|^{\frac{\alpha-(2-p)}{p}}\right)\right|^p}{|x|^{\gamma p}}\, dx,$$

we obtain

$$\frac{1}{\alpha}\frac{d}{dt}\int_\Omega u_n^\alpha(t)\, dx + c_1\int_\Omega \frac{\left|\nabla\left(|u_n(t)|^{\frac{\alpha-(2-p)}{p}}\right)\right|^p}{|x|^{\gamma p}}\, dx \le 0,$$

where

$$c_1 = (\alpha-1)\left(\frac{p}{\alpha-(2-p)}\right)^p - \lambda\left(\frac{p}{N-p(\gamma+1)}\right)^p > 0.$$

Therefore, by (1.7),

$$\frac{1}{\alpha}\frac{d}{dt}\int_\Omega u_n^\alpha(t)\, dx + c_1 S_{N,p,\gamma}\left[\int_\Omega \frac{|u_n(t)|^{\frac{[\alpha-(2-p)]p^*}{p}}}{|x|^{\gamma p^*}}\, dx\right]^{\frac{p}{p^*}} \le 0. \tag{5.3}$$

Choosing

$$\alpha = \frac{N(2-p)}{p},$$

the two powers of u_n become equal. Since $\gamma \geq 0$, if we define

$$\phi(t) = \int_\Omega u_n^\alpha(t)\,dx,$$

we obtain

$$\phi'(t) + c_2[\phi(t)]^{\frac{p}{p^*}} \leq 0,$$

where $c_2 = c_2(N, p, \gamma, \Omega) > 0$, and we obtain the result for the approximate solutions u_n as in the previous theorem. The result on u follows by taking the limit on n. ∎

Remark 5.3. Note that $\eta_{N,p,\gamma} = \lambda_{N,p,\gamma}$ for $p = \frac{2N}{N+2}$.

THEOREM 5.4. *Assume that*

$$(5.4)\qquad \begin{gathered} 0 < \gamma + 1 < \frac{N(2-p)}{2p}, \\ \lambda < \mu_{N,p,\gamma} = \left(\frac{N(2-p)}{p(\gamma+1)} - 1\right)\left(\frac{p(\gamma+1)}{2-p}\right)^p, \end{gathered}$$

and that there exists

$$\bar\alpha > \frac{(2-p)N}{p(\gamma+1)}$$

such that $\psi \in L^{\bar\alpha}(\Omega)$. *Then there exists a constant*

$$T^* = T^*(N, p, \gamma, \lambda, \Omega, \bar\alpha, \psi) \leq c_1(N, p, \gamma, \lambda, \Omega, \bar\alpha)\|\psi\|^{2-p}_{L^\alpha(\Omega)}$$

such that any solution of problem (P) *found by approximation as in Theorem* 3.1 *satisfies*

$$(5.5)\qquad u(\cdot, t) \equiv 0 \quad \textit{for } t \geq T^*.$$

Proof. We use $|u_n|^{\alpha-2}u_n$ as a test function in (P_n), where α is such that

$$(5.6)\qquad \frac{(2-p)N}{p(\gamma+1)} < \alpha \leq \bar\alpha$$

and

$$(5.7)\qquad \lambda < (\alpha - 1)\left(\frac{p(\gamma+1)}{\alpha-(2-p)}\right)^p.$$

Note that this is always possible, since assumption (5.4) implies that (5.7) is true for $\alpha = \frac{(2-p)N}{p(\gamma+1)}$. As in the previous proof, we obtain inequality (5.3), where the constant c_1 is positive by (5.7). Now observe that condition (5.6) implies

$$\alpha > \frac{N(2-p)}{p}$$

and

$$\frac{\gamma\alpha pp^*}{p^*[\alpha-(2-p)]-\alpha p} > -N;$$

therefore, by Hölder's inequality,

$$\begin{aligned}\int_\Omega u_n^\alpha(t)\,dx &\le \left[\int_\Omega |x|^{\frac{\gamma\alpha pp^*}{p^*[\alpha-(2-p)]-\alpha p}}\,dx\right]^{\frac{p^*[\alpha-(2-p)]-\alpha p}{p^*[\alpha-(2-p)]}} \left[\int_\Omega \frac{|u_n(t)|^{\frac{[\alpha-(2-p)]p^*}{p}}}{|x|^{\gamma p^*}}\,dx\right]^{\frac{\alpha p}{p^*[\alpha-(2-p)]}}\\ &\le c_2(N,p,\gamma,\alpha,\Omega)\left[\int_\Omega \frac{|u_n(t)|^{\frac{[\alpha-(2-p)]p^*}{p}}}{|x|^{\gamma p^*}}\,dx\right]^{\frac{\alpha p}{p^*[\alpha-(2-p)]}}.\end{aligned}$$

Hence one has

$$\frac{d}{dt}\int_\Omega u_n^\alpha(t)\,dx + c_3\left[\int_\Omega u_n^\alpha(t)\,dx\right]^{\frac{\alpha-(2-p)}{\alpha}} \le 0,$$

with $c_3 > 0$. Since $\frac{\alpha-(2-p)}{\alpha} < 1$, we conclude as before. ◻

Remark 5.5. Note that condition $0 < \gamma + 1 < \frac{N(2-p)}{2p}$ in Theorem 5.4 means that $1 < p < \frac{2N}{N+2(\gamma+1)}$, which implies, for $\gamma \ge 0$, that p also satisfies $1 < p < \frac{2N}{N+2}$. Therefore, we can compare the results of Theorems 5.2 and 5.4 in the region where $1 < p < \frac{2N}{N+2}$ and $\gamma \ge 0$. An easy calculation shows that in that region we have $\eta_{N,p,\gamma} < \mu_{N,p,\gamma}$, where $\eta_{N,p,\gamma}$ and $\mu_{N,p,\gamma}$ are given in the statements of Theorems 5.2 and 5.4, respectively. Since $\frac{N(2-p)}{p} > \frac{N(2-p)}{p(\gamma+1)}$, Theorem 5.4 gives a better result than Theorem 5.2 in the above region. Let us also point out that the value $\mu_{N,p,\gamma}$ is the same value we find in Lemma 3.2, which gives the esistence of self-similar solutions of the equation in problem (P).

5.2. Nonextinction results. If $p > 2$ and ψ verifies the hypothesis (H2), by using the Barenblatt-type functions one can easily prove that there is no extinction in finite time. Indeed, for any fixed time $T > 0$, consider the function $B(x, t+1)$, where B is the function defined in (4.2). One can easily check that, if the constant M in (4.2) is sufficiently small, then this function is a subsolution of problem (P). Since T is arbitrary, the result follows.

In this section we will prove that solutions to problem (P) with $1 < p < 2$, $\gamma + 1 \ge 0$, and $\lambda > \lambda_{n,p,\gamma}$ are nonzero for all time. The key of the proof is the construction of a nonnegative subsolution to the problem

$$(5.8)\qquad \begin{cases} u_t - \Delta_{p,\gamma}(u) = \lambda\dfrac{|u|^{p-2}u}{|x|^{(\gamma+1)p}}, & (x,t)\in\Omega\times(0,T),\\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = 0, & x\in\Omega,\end{cases}$$

following the ideas in [18] (see also [19]). Consider the eigenvalue problem

$$(5.9)\qquad \begin{cases} -\Delta_{p,\gamma}(\phi_1) = \mu_1(n)W_n(x)\phi_1^{p-1}, & x\in\Omega,\\ u(x) = 0, & x\in\partial\Omega,\end{cases}$$

where $W_n(x) = \min\{n, |x|^{-(\gamma+1)p}\}$. The principal eigenvalue is isolated and simple. Moreover, it is easy to check that the sequence of principal eigenvalues, $\{\mu_1(n)\}$, is decreasing, that $\lim_{n\to\infty} \mu_1(n) = \lambda_{n,p,\gamma}$, and that the corresponding eigenfunction ϕ_1 has constant sign (see, for instance, [16]). In this way, if $\lambda > \lambda_{n,p,\gamma}$, there exists n_0 such that for $n > n_0$, one has $\lambda > \mu_1(n)$. Hence, for $n > n_0$, let $\Theta(t)$ be the positive solution to the problem $\Theta'(t) = \Theta^{p-1}(t)$, $\Theta(0) = 0$.

Define

$$v(x,t) = \Theta(\varepsilon t)\phi_1(x),$$

where $\varepsilon > 0$ will be chosen later, and ϕ_1 is a positive eigenfunction of (5.9) such that $\|\phi_1\|_\infty = 1$. We have that

$$\frac{v_t - \Delta_{p,\gamma}(v)}{\lambda v(x,t)^{p-1}} < \frac{\varepsilon\phi_1^{2-p}}{\lambda} + \frac{\mu_1(n)}{\lambda} W_n(x);$$

hence, as $2 - p > 0$, $\gamma + 1 \geq 0$, and $\frac{\mu_1(n)}{\lambda} < 1$, for a suitable $\varepsilon > 0$ we obtain that

$$\frac{v_t - \Delta_{p,\gamma}(v)}{\lambda v(x,t)^{p-1}} < W_n(x).$$

Then $v(x,t)$ is a subsolution to the truncated problem obtained from (5.8) and therefore to problem (5.8) with $1 < p < 2$, $\psi(x) \geq 0$, $(1+\gamma) > 0$, and $\lambda > \lambda_{n,p,\gamma}$. For the truncated equation we obtain a flat supersolution by solving the ordinary differential equation $y'(t) = n\lambda[y(t)]^{p-1}$, $1 < p < 2$, with data $y(0) = a$, whose solution is $y(t) = [a^{2-p} + n\lambda(2-p)t]^{1/(2-p)}$. Given a $T > 0$ we find a value of a for which $v(x,t) < y(t)$ in $\Omega \times (0,T)$ and $y(0) \geq \psi(x)$. Iterating from v, we obtain as a conclusion that in these hypotheses the minimal solution to the truncated equation of (5.8) has no finite time extinction. And therefore the same result holds for (5.8).

Remark 5.6. If $1 + \gamma < 0$, the weights are flat at the origin. If we use the eigenvalue analysis as in [16], i.e., for $\beta_n = (1+\gamma) - \frac{1}{n}$, then we define, for instance,

$$\alpha_n(x) = \begin{cases} |x|^{-p\beta_n} & \text{if } x \in \Omega \cap B_1(0), \\ |x|^{-p(\gamma+1)} & \text{if } x \in \Omega \setminus B_1(0). \end{cases}$$

In this way $\alpha_n(x) \leq |x|^{-p(\gamma+1)}$ for all $x \in \Omega$, and moreover, the eigenvalue problems

$$\begin{cases} -\operatorname{div}\left(\dfrac{|\nabla\psi_1|^{p-2}\nabla\psi_1}{|x|^{\gamma p}}\right) = \nu_1(n)\alpha_n(x)\psi_1^{p-1}, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{5.10}$$

verify the following:

1. The principal eigenvalue is isolated and simple.
2. We can choose the corresponding eigenfunction ψ_1 positive.
3. The sequence of principal eigenvalues satisfies $\nu_1(n) \searrow \lambda_{N,p,\gamma}$ as $n \to \infty$.

However, the final construction does not work.

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