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ON CONVERGENCE OF CHLODOVSKY TYPE DURRMEYER POLYNOMIALS IN VARIATION SEMINORM

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Abstract. This paper deals with the variation detracting property and rate of approximation of the Chlodovsky type Durrmeyer polynomials in the space of functions of bounded variation with respect to the variation seminorm.

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1. INTRODUCTION

Let $X_{loc}[0,\infty)$ be the class of all complex-valued functions locally bounded on $[0,\infty)$. For $x \in X_{loc}[0,\infty)$, the Chlodovsky polynomials $C_n f$ are defined as:

$$(C_n f)(x) = \sum_{k=0}^{n} f\left(\tfrac{k}{n} b_n\right) \binom{n}{k} \left(\tfrac{x}{b_n}\right)^k \left(1 - \tfrac{x}{b_n}\right)^{n-k}, \ (0 \le x \le b_n) \tag{1}$$

where $n \in \mathbb{N}$ and (b_n) is an increasing sequence of positive numbers satisfying $\lim\limits_{n\to\infty} b_n = \infty$ and $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$.

These polynomials were introduced by I. Chlodovsky [1] in 1937 in generalization of the Bernstein polynomials, the case $b_n = 1$, $n \in \mathbb{N}$, which approximate the function f on the interval $[0,1]$. Some other generalizations of the Bernstein polynomials defined on unbounded sets can be found in [2], [3]. Works on Chlodovsky polynomials are fewer, since they are defined on an unbounded interval $[0,\infty)$.

This generalizes Chlodovsky polynomials by incorporating Durrmeyer operators [4], hence the name Chlodovsky-Durrmeyer operators

$$(D_n f)(x) = \tfrac{n+1}{b_n} \sum_{k=0}^{n} p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} f(t)\, p_{n,k}\left(\tfrac{t}{b_n}\right) dt, \quad 0 \le x \le b_n \tag{2}$$

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where (b_n) is a positive increasing sequence with the properties $\lim\limits_{n\to\infty} b_n = \infty$, $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$ and $p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis. We may also mention that some articles related to Chlodovsky-Durrmeyer operators and their different generalizations are given in [5]–[6].

The main motivation for this paper is to study the variation detracting property and rate of approximation of the Chlodovsky type Durrmeyer polynomials in the space of functions of bounded variation with respect to the variation seminorm. The first research devoted to the variation detracting property and the convergence in variation of a sequence of linear positive operators was due to Lorentz [7]. Later in [8], authors have introduced, developed in details and studied the deep interconnections between variation detracting property and the convergence in variation for Bernstein-type polynomials and singular convolution integrals. After this fundamental study, the convergence in variation seminorm has become a new research field in the theory of approximation. For further reading on different operators, we refer to readers to [9]–[15].

2. NOTATION AND AUXILIARY RESULTS

For the notation; let $I \subset \mathbb{R}$ be a bounded or unbounded interval. We denote by $V_{[I]}[f]$ the total Jordan variation of the function $f : I \to \mathbb{R}$. We deal with the class $BV(I)$ of all the functions of bounded variation on $I \subset \mathbb{R}$, endowed with norm $\|.\|_{BV(I)}$, where

$$\|f\|_{BV(I)} := V_{[I]}[f] + |f(a)|, \quad f \in BV(I),$$

a being any fixed point belonging to the interval I. If we remove the term $|f(a)|$, $V_{[I]}[f]$ turns into a seminorm, say $|.|_{BV(I)}$ on the same space. So we shall say, $TV(I)$ of all the functions of bounded variation on $I \subset \mathbb{R}$, endowed with seminorm

$$\|f\|_{BV(I)} := V_{[I]}[f].$$

Some interesting properties of the space $TV(I)$ are presented in [8].

In order to obtain a convergence result in the variation seminorm, it is necessary and important to state the variation detracting property. Let L be a linear operator acting on a given space S of real-valued functions defined on I such that $BV(I) \subset S$. The operator L possesses the variation detracting property if

$$V_{[I]}[Lf] \leq V_{[I]}[f] \ , f \in BV(I),$$

holds, *i.e.* positive linear operators from the space of functions of bounded variation into itself do not increase the total variation of functions.

$AC(I)$ stands for the space of all absolutely continuous real-valued functions defined on I is a closed subspace of $TV(I)$ with respect to the convergence induced by the seminorm $\|f\|_{TV(I)}$. Moreover, if $\lim\limits_{n\to\infty} V_I[g_n - f] = 0$ for a sequence $(g_n)_{n\geq 1}$, $g_n \in AC(I)$, $n \in \mathbb{N}$, then also $f \in AC[0,1]$ and

$$V_I[g_n - f] = \int_I |g_n'(t) - f'(t)|\,dt = \|g_n' - f'\|,$$

where

$$\|f\| := \|f\|_{L_1(I)}.$$

So, convergence in variation of $(g_n)_{n\geq 1} \subset AC(I)$ to f, represents the convergence of the derivatives $(g_n')_{n\geq 1}$ to f' in the norm $L_1(I)$, the Banach space of all real-valued Lebesgue integrable functions defined on I.

Let us define the sum moments as in [11]:

$$T_{n,m}(x) = \sum_{k=0}^{n} [kb_n - nx]^m\, p_{n,k}\left(\tfrac{x}{b_n}\right) \tag{3}$$

where $m \in \mathbb{N}_0$ (the set of non-negative integers). Then there hold the following identities (see, e.g., [11])

$$T_{n,m}(x) = \begin{cases} 1 & m=0 \\ 0 & m=1 \\ nx(b_n - x) & m=2 \\ nx(b_n - x)(b_n - 2x) & m=3 \\ nx(b_n - x)\left(b_n^2 + 3(n-2)xb_n - 3(n-2)x^2\right) & m=4. \end{cases} \tag{4}$$

Let us define for the central moments of order $m \in \mathbb{N}_0$,

$$T_{n,m}^*(x) = \sum_{k=0}^{n} \left(\tfrac{kb_n}{n} - x\right)^m p_{n,k}\left(\tfrac{x}{b_n}\right)$$

and for any fixed $x \in [0,\infty)$

$$\left|T_{n,m}^*(x)\right| \leq A_m(x)\tfrac{x(b_n - x)}{b_n}\left(\tfrac{b_n}{n}\right)^{[(m+1)/2]} \quad (n \in \mathbb{N},\ n > b_n), \tag{5}$$

where $A_m(x)$ denotes a polynomial in x, of degree $[m/2]-1$, with non-negative coefficients independent of n, and $[a]$ denotes the integral part of a. For the proof see Butzer-Karsli [16].

Since

$$\tfrac{d}{dx}p_{n,k}\left(\tfrac{x}{b_n}\right) = \tfrac{(kb_n - nx)}{x(b_n - x)}p_{n,k}\left(\tfrac{x}{b_n}\right),$$

we can write the following representations for the first derivative of $(D_n f)(x)$;

$$(D_n f)'(x) = \tfrac{(n+1)}{b_n x(b_n - x)} \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} f(t)\, p_{n,k}\left(\tfrac{t}{b_n}\right) dt, \tag{6}$$

and

$$(D_n f)'(x) = \tfrac{n}{b_n} \sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x}{b_n}\right) \tfrac{n+1}{b_n} \int_0^{b_n} f(t)\left[p_{n,k+1}\left(\tfrac{t}{b_n}\right) - p_{n,k}\left(\tfrac{t}{b_n}\right)\right] dt. \tag{7}$$

3. VARIATION DETRACTING PROPERTY OF CHLODOVSKY-DURRMEYER OPERATORS

In this section, we prove the variation detracting properties of the Chlodovsky-Durrmeyer Operators.

THEOREM 1. *If* $f \in TV\left[0,b_n\right]$, *then*

$$V_{[0,b_n]}\left[D_n f\right] \leq V_{[0,b_n]}\left[f\right] \tag{8}$$

and

$$\left\|D_n f\right\|_{BV[0,b_n]} \leq \left\|f\right\|_{BV[0,b_n]} \tag{9}$$

hold true.

Proof. For convenience we write the Chlodovsky-Durrmeyer operators as:

$$\left(D_n f\right)(x) = \sum_{k=0}^{n} p_{n,k}\left(\tfrac{x}{b_n}\right) F_{k,n}$$

where

$$F_{k,n} := \tfrac{n+1}{b_n} \int_0^{b_n} f\left(t\right) p_{n,k}\left(\tfrac{t}{b_n}\right) dt.$$

As in (7), differentiating (2) and putting $\Delta F_{k,n} = F_{k+1,n} - F_{k,n}$

$$\begin{aligned}
\left(D_n f\right)'(x) &= \sum_{k=0}^{n} p'_{n,k}\left(\tfrac{x}{b_n}\right) F_{k,n} = \tfrac{1}{b_n}\sum_{k=1}^{n}\tbinom{n}{k}k\left(\tfrac{x}{b_n}\right)^{k-1}\left(1-\tfrac{x}{b_n}\right)^{n-k} F_{k,n} \\
&\quad -\tfrac{1}{b_n}\sum_{k=0}^{n-1}\tbinom{n}{k}\left(\tfrac{x}{b_n}\right)^{k}(n-k)\left(1-\tfrac{x}{b_n}\right)^{n-k-1} F_{k,n} \\
&= \tfrac{n}{b_n}\sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x}{b_n}\right) F_{k+1,n} - \tfrac{n}{b_n}\sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x}{b_n}\right) F_{k,n} \\
&= \tfrac{n}{b_n}\sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x}{b_n}\right)\left[F_{k+1,n} - F_{k,n}\right] \\
&= \tfrac{n}{b_n}\sum_{k=0}^{n-1} p_{n-1,k}\left(\tfrac{x}{b_n}\right)\Delta F_{k,n}. \qquad (10)
\end{aligned}$$

Considering the representation (10) of $\left(D_n f\right)'$, one has

$$\begin{aligned}
\left\|D_n f\right\|_{TV[0,b_n]} &= V_{[0,b_n]}\left[D_n f\right] = \int_0^{b_n}\left|\left(D_n f\right)'(x)\right| dx \\
&\leq \tfrac{n}{b_n}\sum_{k=0}^{n-1}\left|\Delta F_{k,n}\right| \int_0^{b_n} p_{n-1,k}\left(\tfrac{x}{b_n}\right) dx.
\end{aligned}$$

Since $\frac{n}{b_n}\int_0^{b_n} p_{n-1,k}\left(\frac{x}{b_n}\right)dx=1$, we get

$$\|D_n f\|_{TV[0,b_n]} \leq \sum_{k=0}^{n-1} |\Delta F_{k,n}|\,. \tag{11}$$

Now,

$$\begin{aligned}\Delta F_{k,n} &= \frac{n+1}{b_n}\int_0^{b_n} f(t)\left[p_{n,k+1}\left(\frac{t}{b_n}\right)-p_{n,k}\left(\frac{t}{b_n}\right)\right]dt\\ &= \frac{n+1}{b_n}\int_0^{b_n} f(t)\,\Delta p_{n,k}\left(\frac{t}{b_n}\right)dt.\end{aligned}$$

Since $\Delta p_{n,k}=-\frac{b_n}{n+1}p'_{n+1,k+1}$,

$$p'_{n,k}\left(\frac{t}{b_n}\right)=-\frac{n}{b_n}\Delta p_{n-1,k-1}\left(\frac{t}{b_n}\right),$$

and so we get

$$|\Delta F_{k,n}| = \left|\frac{n+1}{b_n}\int_0^{b_n} f(t)\left[-\frac{b_n}{n+1}p'_{n+1,k+1}\left(\frac{t}{b_n}\right)\right]dt\right|. \tag{12}$$

From (11) and (12), we obtain

$$\begin{aligned}V_{[0,1]}[D_n f] &= \sum_{k=0}^{n-1}|\Delta F_{k,n}| = \sum_{k=0}^{n-1}\left|-\int_0^{b_n} f(t)\,p'_{n+1,k+1}\left(\frac{t}{b_n}\right)dt\right|\\ &= \sum_{k=0}^{n-1}\left|\int_0^{b_n} f'(t)\,p_{n+1,k+1}\left(\frac{t}{b_n}\right)dt\right|\\ &\leq \sum_{k=0}^{n-1}\int_0^{b_n} p_{n+1,k+1}\left(\frac{t}{b_n}\right)|f'(t)|\,dt\\ &= \int_0^{b_n}\sum_{k=0}^{n-1}\binom{n+1}{k+1}\left(\frac{t}{b_n}\right)^{k+1}\left(1-\frac{t}{b_n}\right)^{n-k}|f'(t)|\,dt\\ &= \int_0^{b_n}\sum_{k=1}^{n}\binom{n+1}{k}\left(\frac{t}{b_n}\right)^{k}\left(1-\frac{t}{b_n}\right)^{n+1-k}|f'(t)|\,dt\\ &\leq \int_0^{b_n}\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{t}{b_n}\right)^{k}\left(1-\frac{t}{b_n}\right)^{n+1-k}|f'(t)|\,dt\end{aligned}$$

$$\begin{aligned} &= \int_0^{b_n} \left(\tfrac{t}{b_n} + 1 - \tfrac{t}{b_n} \right)^{n+1} \left| f'(t) \right| dt = \int_0^{b_n} \left| f'(t) \right| dt \\ &\leq V_{[0,b_n]} [f] . \end{aligned}$$

The desired estimate (8) is now obvious.

Since

$$(D_n f)(0) = \tfrac{n+1}{b_n} \int_0^{b_n} \left(1 - \tfrac{t}{b_n} \right)^n f(t)\, dt$$

and

$$\|f\|_{BV[I]} := V_{[I]} [f] + |f(0)| ,$$

relation (9) is a result of (8). Indeed,

$$\begin{aligned} \|D_n f\|_{BV[0,b_n]} &= V_{[0,b_n]} [D_n f] + |(D_n f)(0)| \\ &\leq V_{[0,b_n]} [f] + \left| \tfrac{n+1}{b_n} \int_0^{b_n} \left(1 - \tfrac{t}{b_n} \right)^n f(t)\, dt \right| . \end{aligned}$$

Since $f \in TV[0, b_n]$ and

$$\left| \tfrac{n+1}{b_n} \int_0^{b_n} \left(1 - \tfrac{t}{b_n} \right)^n f(t)\, dt \right| = |f(0)| \leq |f(a)|$$

where a is any fixed point of $[0, b_n]$, we get

$$\|D_n f\|_{BV[0,b_n]} \leq \|f\|_{BV[0,b_n]} .$$

Thus, the proof of the theorem is complete. □

4. RATE OF APPROXIMATION IN TV-NORM

This section deals with the rates of approximation $D_n g$ to g in the variation seminorm.

In order to obtain a convergence result in variation seminorm, we assume that $\lim_{n\to\infty} b_n = \infty$ and $\lim_{n\to\infty} \frac{b_n^3}{n} = 0$

THEOREM 2. *Let $g'' \in AC[0, b_n]$, then*

$$V_{[0,b_n]} [D_n g - g] \leq \tfrac{B}{\delta^2} \tfrac{b_n^3}{n} \left\{ V_{[0,b_n]} [g] + V_{[0,b_n]} [g''] \right\}$$

holds true, where δ is sufficiently small positive real constant and a constant $B > 1$.

Proof. By Taylor's formula with integral remainder term, one has

$$g\left(\tfrac{k}{n}b_n\right) = g(x) + \left(\tfrac{k}{n}b_n - x\right)g'(x) + \left(\tfrac{k}{n}b_n - x\right)^2 \frac{g''(x)}{2} + \frac{1}{2}\int_x^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n - v\right)^2 g'''(v)\,dv. \tag{13}$$

From (13) we obtain

$$\begin{aligned}
&(D_n g)'(x) = \\
&= \tfrac{n+1}{b_n x(b_n-x)} \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} g\left(\tfrac{k}{n}b_n\right) p_{n,k}\left(\tfrac{t}{b_n}\right) dt \\
&= \tfrac{(n+1)}{b_n x(b_n-x)} g(x) \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} p_{n,k}\left(\tfrac{t}{b_n}\right) dt \\
&\quad + \tfrac{(n+1)}{b_n x(b_n-x)} g'(x) \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} \left(\tfrac{k}{n}b_n - x\right) p_{n,k}\left(\tfrac{t}{b_n}\right) dt \\
&\quad + \tfrac{(n+1)}{2b_n x(b_n-x)} g''(x) \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} \left(\tfrac{k}{n}b_n - x\right)^2 p_{n,k}\left(\tfrac{t}{b_n}\right) dt \\
&\quad + (R_n g)(x).
\end{aligned}$$

where

(14)

$$\begin{aligned}
&(R_n g)(x) = \\
&= \tfrac{n+1}{2b_n x(b_n-x)} \sum_{k=0}^{n} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int_0^{b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n - v\right)^2 g'''(v)\,dv\right] p_{n,k}\left(\tfrac{t}{b_n}\right) dt.
\end{aligned}$$

Calculating (3) and (4), we obtain

$$(D_n g)'(x) = g'(x) + \tfrac{b_n-2x}{2n} g''(x) + (R_n g)(x). \tag{15}$$

As in the proof of [15], we choose δ a sufficiently small positive real number, let's divide $(R_n g)(x)$ in to two parts as follows;

$$(R_n g)(x) = (R_{n,1} g)(x) + (R_{n,2} g)(x), \tag{16}$$

where

(17)

$$(R_{n,1}g)(x) = = \frac{n+1}{2b_n x(b_n-x)} \cdot \sum_{\left|\frac{k}{n}-x\right|\le\delta} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_0^{b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\,dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

and

(18)

$$(R_{n,2}g)(x) = = \frac{n+1}{2b_n x(b_n-x)} \cdot \sum_{\left|\frac{k}{n}-x\right|>\delta} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_0^{b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\,dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt.$$

In order to estimate the integration domain of the double integral in the remainder term (14), we divide the summation into different sums as following;

$(R_{n,1}g)(x) = A_{2,n}g + A_{5,n}g$ and $(R_{n,2}g)(x) = A_{1,n}g + A_{3,n}g + A_{4,n}g + A_{6,n}g$.

Here $A_{i,n}g$ for $i = 1, \ldots, 6$,

$$A_{1,n}g = = \frac{n+1}{2b_n x(b_n-x)} \cdot \sum_{\delta < x - \frac{k}{n}b_n \le x} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_0^{\frac{k}{n}b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\,dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt,$$

$$A_{2,n}g = = \frac{n+1}{2b_n x(b_n-x)} \cdot \sum_{0 \le x - \frac{k}{n}b_n \le \delta} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_{\frac{k}{n}b_n}^{x} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\,dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt,$$

$$A_{3,n}g = \\ = \frac{n+1}{2b_n x(b_n-x)} \cdot \\ \cdot \sum_{\delta < x - \frac{k}{n}b_n \le x} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_x^{b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\, dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt,$$

$$A_{4,n}g = \\ = \frac{n+1}{2b_n x(b_n-x)} \cdot \\ \cdot \sum_{\delta < \frac{k}{n}b_n - x \le 1-x} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_0^{x} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\, dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt,$$

$$A_{5,n}g = \\ = \frac{n+1}{2b_n x(b_n-x)} \cdot \\ \cdot \sum_{0 < \frac{k}{n}b_n - x \le \delta} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_x^{\frac{k}{n}b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\, dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt,$$

and

$$A_{6,n}g = \\ = \frac{n+1}{2b_n x(b_n-x)} \cdot \\ \cdot \sum_{\delta < \frac{k}{n}b_n - x \le 1-x} (kb_n - nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int_{\frac{k}{n}b_n}^{b_n} \left[\int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\, dv \right] p_{n,k}\left(\frac{t}{b_n}\right) dt.$$

It is easy to see that, $A_{1,n}g + A_{2,n}g = -A_{4,n}g$ and $A_{5,n}g + A_{6,n}g = -A_{3,n}g$. So one has

$$|A_{1,n}g + A_{2,n}g| = |-A_{4,n}g| \le |A_{1,n}g| + |A_{2,n}g|$$

and

$$|A_{5,n}g + A_{6,n}g| = |-A_{3,n}g| \le |A_{5,n}g| + |A_{6,n}g|.$$

So, we get

$$|(R_n g)(x)| \le 2\left(|A_{1,n}g| + |A_{2,n}g| + |A_{5,n}g| + |A_{6,n}g|\right)$$

or

$$|(R_n g)(x)| \le 2\left(|-A_{3,n}g| + |-A_{4,n}g|\right).$$

Now we only estimate $A_{i,n}g$ for $i=1,2,5,$ and 6 respectively. Firstly, let us estimate $A_{1,n}g$ as follows;

$$|A_{1,n}g| \leq$$

$$\leq \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n\leq x} |kb_n-nx|\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int\limits_0^{\frac{k}{n}b_n} \left| \int\limits_x^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n-v\right)^2 g'''(v)\,dv \right| p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$\leq \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n\leq x} |kb_n-nx|\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int\limits_0^{\frac{k}{n}b_n} \left| \int\limits_x^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n-v\right)^2 |g'''(v)|\,dv \right| p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$= \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int\limits_0^{\frac{k}{n}b_n} \left[\int\limits_x^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n-v\right)^2 |g'''(v)|\,dv \right] p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$\leq \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int\limits_0^{\frac{k}{n}b_n} \left(\tfrac{k}{n}b_n-x\right)^2 \left[\int\limits_{\frac{k}{n}b_n}^{x} |g'''(v)|\,dv \right] p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$\leq \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right) x^2 \int\limits_0^{\frac{k}{n}b_n} \left[\int\limits_{\frac{k}{n}b_n}^{x} |g'''(v)|\,dv \right] p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$\leq \tfrac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right) x^2 \int\limits_0^{\frac{k}{n}b_n} \left[\int\limits_0^{b_n} |g'''(v)|\,dv \right] p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$\leq \tfrac{(n+1)x^2}{2b_n x(b_n-x)} \left\|g'''\right\| \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \int\limits_0^{b_n} p_{n,k}\left(\tfrac{t}{b_n}\right) dt$$

$$= \tfrac{x^2}{2x(b_n-x)} \left\|g'''\right\| \sum_{\delta<x-\frac{k}{n}b_n} (nx-kb_n)\, p_{n,k}\left(\tfrac{x}{b_n}\right).$$

Since

$$\sum_{\delta<x-\frac{k}{n}b_n}(nx-kb_n)\,p_{n,k}\left(\frac{x}{b_n}\right)=\sum_{\frac{k}{n}b_n-x<-\delta}(kb_n-nx)\,p_{n,k}\left(\frac{x}{b_n}\right),$$

we get

$$|A_{1,n}g|\leq\frac{x^2}{2x(b_n-x)}\left\|g'''\right\|\sum_{\frac{k}{n}b_n-x<-\delta}(kb_n-nx)\,p_{n,k}\left(\frac{x}{b_n}\right). \tag{19}$$

Analogously, $A_{2,n}g$ can be estimated by

$$\begin{aligned}
&|A_{2,n}g|\leq\\
&\leq\frac{n+1}{2b_nx(b_n-x)}\\
&\quad\cdot\sum_{0\leq x-\frac{k}{n}b_n\leq\delta}|kb_n-nx|\,p_{n,k}\left(\frac{x}{b_n}\right)\int\limits_{\frac{k}{n}b_n}^{x}\left|\int\limits_{x}^{\frac{k}{n}b_n}\left(\frac{k}{n}b_n-v\right)^2g'''(v)\,dv\right|p_{n,k}\left(\frac{t}{b_n}\right)dt\\
&\leq\frac{n+1}{2b_nx(b_n-x)}\cdot\\
&\quad\cdot\sum_{0\leq x-\frac{k}{n}b_n\leq\delta}|kb_n-nx|\,p_{n,k}\left(\frac{x}{b_n}\right)\int\limits_{\frac{k}{n}b_n}^{x}\left|\int\limits_{x}^{\frac{k}{n}b_n}\left(\frac{k}{n}b_n-v\right)^2|g'''(v)|\,dv\right|p_{n,k}\left(\frac{t}{b_n}\right)dt\\
&\leq\frac{n+1}{2b_nx(b_n-x)}\cdot\\
&\quad\cdot\sum_{x-\frac{k}{n}b_n\leq\delta}(nx-kb_n)\,p_{n,k}\left(\frac{x}{b_n}\right)\int\limits_{\frac{k}{n}b_n}^{x}\left[\int\limits_{\frac{k}{n}b_n}^{x}\left(\frac{k}{n}b_n-v\right)^2|g'''(v)|\,dv\right]p_{n,k}\left(\frac{t}{b_n}\right)dt\\
&\leq\frac{n+1}{2b_nx(b_n-x)}\cdot\\
&\quad\cdot\sum_{x-\frac{k}{n}b_n\leq\delta}(nx-kb_n)\,p_{n,k}\left(\frac{x}{b_n}\right)\int\limits_{\frac{k}{n}b_n}^{x}\left(\frac{k}{n}b_n-x\right)^2\left[\int\limits_{\frac{k}{n}b_n}^{x}|g'''(v)|\,dv\right]p_{n,k}\left(\frac{t}{b_n}\right)dt\\
&\leq\frac{n+1}{2b_nx(b_n-x)}\cdot\\
&\quad\cdot\sum_{x-\frac{k}{n}b_n\leq\delta}(nx-kb_n)\,p_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{k}{n}b_n-x\right)^2\int\limits_{\frac{k}{n}b_n}^{x}\left[\int\limits_{\frac{k}{n}b_n}^{x}|g'''(v)|\,dv\right]p_{n,k}\left(\frac{t}{b_n}\right)dt
\end{aligned}$$

$$\leq \frac{n+1}{2b_n x(b_n-x)}\cdot$$

$$\cdot \sum_{x-\frac{k}{n}b_n\leq\delta} (nx-kb_n)\, p_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{k}{n}b_n-x\right)^2 \int\limits_{\frac{k}{n}b_n}^{x} \left[\int\limits_0^{b_n} |g'''(v)|\, dv\right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \left\|g'''\right\| \cdot$$

$$\cdot \sum_{x-\frac{k}{n}b_n\leq\delta} (nx-kb_n)\, p_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{k}{n}b_n-x\right)^2 \int\limits_{\frac{k}{n}b_n}^{x} p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{\|g'''\|}{2n^2 x(b_n-x)}\cdot$$

$$\cdot \sum_{x-\frac{k}{n}b_n\leq\delta} |kb_n-nx|\, p_{n,k}^{1/2}\left(\frac{x}{b_n}\right)(kb_n-nx)^2\, p_{n,k}^{1/2}\left(\frac{x}{b_n}\right).$$

In view of Hölder inequality, (4) and (5), we get

$$|A_{2,n}g| \leq$$

$$\leq \frac{\|g'''\|}{2n^2 x(b_n-x)} \left(\sum_{k=0}^{n} (kb_n-nx)^2\, p_{n,k}\left(\frac{x}{b_n}\right)\right)^{\frac{1}{2}} \left(\sum_{k=0}^{n} (kb_n-nx)^4\, p_{n,k}\left(\frac{x}{b_n}\right)\right)^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{b_n}}{n}\left\|g'''\right\|$$

which implies that

$$\|A_{2,n}g\| \leq \frac{\sqrt{b_n}}{n}\left\|g'''\right\|. \tag{20}$$

As to the term $A_{5,n}g$, noting (4) and (5), we have

$$|A_{5,n}g| \leq$$

$$\leq \frac{n+1}{2b_n x(b_n-x)}\cdot$$

$$\cdot \sum_{0<\frac{k}{n}b_n-x\leq\delta} |kb_n-nx|\, p_{n,k}\left(\frac{x}{b_n}\right) \int\limits_x^{\frac{k}{n}b_n} \left|\int\limits_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n-v\right)^2 g'''(v)\, dv\right| p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)}\cdot$$

$$\cdot \sum_{\frac{k}{n}b_n-x\leq\delta} (kb_n-nx)\, p_{n,k}\left(\frac{x}{b_n}\right) \int\limits_x^{\frac{k}{n}b_n} \left[\int\limits_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n-v\right)^2 |g'''(v)|\, dv\right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\frac{k}{n}b_n - x \leq \delta} (kb_n - nx) p_{n,k}\left(\frac{x}{b_n}\right)\left(\frac{k}{n}b_n - x\right)^2 \int_x^{\frac{k}{n}b_n} \left[\int_x^{\frac{k}{n}b_n} |g'''(v)|\, dv\right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{\|g'''\|}{2n^2 x(b_n-x)} \sum_{k=0}^{n} |kb_n - nx|^3 p_{n,k}\left(\frac{x}{b_n}\right)$$

As in the proof of $A_{2,n}g$, one has

$$|A_{5,n}g| \leq \frac{\sqrt{b_n}}{n} \|g'''\|,$$

which yields

$$\|A_{5,n}g\| \leq \frac{\sqrt{b_n}}{n} \|g'''\|. \tag{21}$$

Finally to the next term $A_{6,n}g$, we have

$$|A_{6,n}g| \leq$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta < \frac{k}{n}b_n - x \leq 1-x} |kb_n - nx| p_{n,k}\left(\frac{x}{b_n}\right) \int_{\frac{k}{n}b_n}^{b_n} \left| \int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 g'''(v)\, dv \right| p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta < \frac{k}{n}b_n - x \leq 1-x} |kb_n - nx| p_{n,k}\left(\frac{x}{b_n}\right) \int_{\frac{k}{n}b_n}^{b_n} \left| \int_x^{\frac{k}{n}b_n} \left(\frac{k}{n}b_n - v\right)^2 |g'''(v)|\, dv \right| p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta < \frac{k}{n}b_n - x} (kb_n - nx) p_{n,k}\left(\frac{x}{b_n}\right) \int_{\frac{k}{n}b_n}^{b_n} \left(\frac{k}{n}b_n - x\right)^2 \left[\int_x^{\frac{k}{n}b_n} |g'''(v)|\, dv\right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

$$\leq \frac{n+1}{2b_n x(b_n-x)} \cdot$$

$$\cdot \sum_{\delta < \frac{k}{n}b_n - x} (kb_n - nx) p_{n,k}\left(\frac{x}{b_n}\right) (b_n - x)^2 \int_{\frac{k}{n}b_n}^{b_n} \left[\int_0^{b_n} |g'''(v)|\, dv\right] p_{n,k}\left(\frac{t}{b_n}\right) dt$$

(22)

$$\leq \frac{(b_n-x)^2}{2x(b_n-x)} \|g'''\| \cdot \sum_{\delta < \frac{k}{n}b_n - x} (kb_n - nx) p_{n,k}\left(\frac{x}{b_n}\right).$$

Collecting (19) and (22), we obtain

$$\begin{aligned}
|A_{1,n}g| + |A_{6,n}g| &\le \tfrac{x^2+(b_n-x)^2}{2x(b_n-x)} \|g'''\| \sum_{\left|\frac{k}{n}b_n - x\right| > \delta} (kb_n - nx)\, p_{n,k}\left(\tfrac{x}{b_n}\right) \\
&\le \tfrac{x^2+(b_n-x)^2}{2x(b_n-x)} \|g'''\|\, n \sum_{\left|\frac{k}{n}b_n - x\right| > \delta} \left(\tfrac{k}{n}b_n - x\right) \tfrac{\left(\frac{k}{n}b_n - x\right)^2}{\delta^2} p_{n,k}\left(\tfrac{x}{b_n}\right) \\
&= \tfrac{x^2+(b_n-x)^2}{2x(b_n-x)} \|g'''\| \frac{n}{\delta^2} \sum_{\left|\frac{k}{n}b_n - x\right| > \delta} \left(\tfrac{k}{n}b_n - x\right)^3 p_{n,k}\left(\tfrac{x}{b_n}\right) \qquad (23) \\
&\le \tfrac{b_n^3}{n\delta^2} \|g'''\| . \qquad (24)
\end{aligned}$$

In view of (24), one obtains

$$\|A_{1,n}g\| + \|A_{6,n}g\| \le \tfrac{b_n^3}{n\delta^2} \|g'''\| . \qquad (25)$$

Thus, altogether with the results in (20), (21) and (25), we have for (16)

$$\|R_n g\| \le 2\left(\tfrac{\sqrt{b_n}}{n} + \tfrac{\sqrt{b_n}}{n} + \tfrac{b_n^3}{n\delta^2}\right) \|g'''\| \le \tfrac{6b_n^3}{\delta^2 n} \|g'''\| .$$

Finally we obtain by using (15)

$$\left\|(D_n g)' - g'\right\| \le \tfrac{b_n}{2n} \|g''\| + \tfrac{6b_n^3}{\delta^2 n} \|g'''\| .$$

According to Stein's inequality (see, e.g., [17, Th. A10.1]) one has

$$\begin{aligned}
\|g''(x)\|_{L_1(0,1)} &\le C\sqrt{\|g'(x)\|_{L_1(0,1)} \|g'''\|_{L_1(0,1)}} \\
&\le C\left(\|g'\|_{L_1(0,1)} + \|g'''\|_{L_1(0,1)}\right),
\end{aligned}$$

where $C > 1$ is a constant. So, we have

$$\left\|(D_n g)' - g'\right\| \le \tfrac{B}{\delta^2} \tfrac{b_n^3}{n} \left(\|g'\| + \|g'''\|\right) \qquad (26)$$

where $B > 1$ is a constant. This finally establishes the theorem. □

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