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A COMMENT ON EWALD QUAK'S "ABOUT B-SPLINES"

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Abstract. The early contributions to B-spline theory by Tiberiu Popoviciu and by Liubomir Chakalov are recalled.

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The publication of a paper on the many aspects of B-splines in this journal suggests adding a comment on the contributions to B-splines made by this journal's founder, Tiberiu Popoviciu.

Briefly, in his paper [P34b] on n-convex functions (a terminology he introduced), Povoviciu proves the recurrence relation for B-splines (see [Q, (42)]), Marsden's identity (see [Q, p.46]), and mentions without proof that the B-splines form a generating set for what he calls **elementary functions of degree n with m vertices** by which he means smooth functions whose $(n-1)$st derivative is a continuous broken line with $m-2$ interior breaks. In addition, you can find in [P34a] Boehm's formula for knot insertion (if you know what to look for).

To be a bit more explicit (as laid out in [BP03]), Popoviciu introduces (see [P34b, p.89]), for a given strictly increasing sequence $x_1 < \cdots < x_m$, certain piecewise polynomial functions Ψ_i with support in the interval [1] $(x_i \,..\, x_{i+n+1})$, $i = 1, \ldots, m-n-1$, of degree $\leq n$, and proves (see [P34b, p.93]) the relation

$$(x_{n+i+1} - x_i)\Psi_i(x) = (x - x_i)\Psi_i'(x) + (x_{n+i+1} - x)\Psi_{i+1}'(x), \tag{1}$$

with, as he writes, the Ψ_i' defined just like the Ψ_i except that n is replaced by $n-1$. Compare (1) with the well-known recurrence relation (see [Q, (42)])

$$B_{i,n+1} = \omega_{i,n+1}B_{in} + (1 - \omega_{i+1,n+1})B_{i+1,n}, \qquad \omega_{i,n+1}(x) := \tfrac{x - x_i}{x_{i+n} - x_i}$$

in which B_{jk} is the B-spline with knots $x_j, \ldots, x_{j+k}$ (normalized to be part of a partition of unity), and add to that the fact that Popoviciu's formula for

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[1] I was pleased to see already in Popoviciu's article this handy use of .. in the description of an interval.

Ψ_i readily reduces to $\Psi_i = \chi_{(x_i..x_{i+1})}$ for $n = 0$, to conclude that, for given n, $(x_{i+n+1} - x_i)\Psi_i = B_{i,n+1}$. In fact, since Popoviciu's formula for Ψ_i involves ratios of Vandermonde determinants, it is not that hard to derive directly that

$$\Psi_i(x) = \Delta(x_i, \ldots, x_{i+n+1})(\cdot - x)_+^n \tag{2}$$

with $\Delta(x_i, \ldots, x_{i+n+1}) = [x_i, \ldots, x_{i+n+1}]$ the divided difference[2] functional at the nodes $x_i, \ldots, x_{i+n+1}$ (provided one knows the handy notation $(\cdot - a)_+^n$ for the nth truncated power). The simple formula (2) could have been of help in simplifying some of Popoviciu's arguments concerning n-convexity. It could also have readily supplied the fact (not proved in the paper) that the Ψ_i have all derivatives of order $< n$ continuous.

Popoviciu uses the recurrence relation (1) to prove the positivity of the Ψ_i on their support, as well as the following formula (see [P34b, p.93])

$$\sum_{i=1}^{n+1}(x_{i+n+1} - x_i)\Psi_i(\xi)(x - x_{i+1}) \cdots (x - x_{i+n}) = (x - \xi)^n \tag{3}$$

which we now call Marsden's identity because of [Ma70].

While the sequence $x_1, \ldots, x_m$ starts out strictly increasing, in the last section of [P34b] all is specialized to the case

$$x_1 = \cdots = x_{n+1} = a < b = x_{n+2} = \cdots = x_{2n+2}$$

which we now associate with the names Bernstein and Bézier.

Finally, on page 7 of [P34a], Popoviciu uses induction to derive from the formula

$$(t_n - t_0)\Delta(t_0, \ldots, t_n) = (t_n - \xi)\Delta(\xi, t_1, \ldots, t_n) + (\xi - t_0)\Delta(t_0, \ldots, t_{n-1}, \xi) \tag{4}$$

the fact that, for an increasing refinement σ of the increasing sequence τ,

$$\Delta(\tau_0, \ldots, \tau_n) = \sum_j \alpha_j(\tau, \sigma)\Delta(\sigma_j, \ldots, \sigma_{j+n}) \tag{5}$$

with the coefficients $\alpha_j(\tau, \sigma)$ nonnegative and summing to 1. Given formula (2), you will recognize in (4) Boehm's now standard formula for knot insertion, and in (5) a formula for expressing the B-spline with knots $\tau_0, \ldots, \tau_n$ in terms of B-splines of the same order on a finer knot sequence σ.

Another early contributor to B-splines is Liubomir Chakalov who studied (see [C38] or its discussion in [BP03]) B-splines in the sense that he focused on the Peano kernel u in the integral representation

$$\Delta(x_1, \ldots, x_{n+1})f = \int u(s)D^n f(s)\,\mathrm{d}s$$

of the divided difference. Necessarily,

$$u(x) = \Delta(x_1, \ldots, x_{n+1})(\cdot - x)_+^{n-1}/(n-1)!$$

[2] I was pleased to see this literal notation for divided differences already in [S64].

hence, by (2) (see also [Q, p.22]), u is a B-spline,

$$u = B_{1n}/((x_{n+1} - x_1)(n-1)!).$$

Chakalov provides the B-spline recurrence relation in a form that involves a derivative and, more importantly, provides the representation

$$u(x) = \frac{1}{2\pi\mathrm{i}} \int_C \frac{(z-x)^{n-1}\,\mathrm{d}z}{(n-1)!\prod_{j=1}^{n+1}(z-x_j)}$$

of this B-spline in terms of a contour integral (with the contour C dependent on x and the x_i) which was rediscovered many years later by Meinardus and put to good use in [Me74].

[Q] mentions Sommerfeld, and could have mentioned Favard, and even Laplace, as people who, like Popoviciu and Chakalov, came across B-splines in their work well before (Curry and) Schoenberg, although Popoviciu and Chakalov are the only ones I am aware of who developed the B-spline recurrence relations. It seems that it is not enough to have a good idea or insight. One needs, like Schoenberg, the appreciation and courage to develop the idea systematically, make its objects mathematically presentable by giving them names, and give them much exposure in many papers.

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