REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 41 (2012) no. 2, pp. 149–156
ictp.acad.ro/jnaat

ABOUT BOUNDS FOR THE ELLIPTIC INTEGRAL OF THE FIRST KIND

PÁL A. KUPÁN* and RÓBERT SZÁSZ*

Abstract. We deduce an inequality using elementary methods which makes it possible to prove a conjecture regarding the upper bound of the elliptic integral of the first kind, furthermore we also improve the lower bound.

MSC 2000. 33C05
Keywords. Hypergeometric function; Elliptic integral; Inequality; Bounds.

1. INTRODUCTION

Legendre's complete elliptic integral of the first kind is defined for $r \in (0,1)$ by

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2 \sin t}} \mathrm{d}t.$$

This integral is a special case of Gauss's hypergeometric function

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad x \in (-1,1),$$

where $(a,n) = \prod_{k=0}^{n-1}(a+k)$. We have

$$\mathcal{K}(r) = \tfrac{\pi}{2}\, {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;r^2) = \tfrac{\pi}{2}\Big[1 + \sum_{n=1}^{\infty}\Big(\tfrac{1\cdot 3\cdot \ldots \cdot (2n-1)}{2\cdot 4 \cdot \ldots \cdot (2n)}\Big)^2 r^{2n}\Big]. \tag{1}$$

In [1] the authors posed the problem to determine the best values α^* and β^* such that

$$\tfrac{\pi}{2}\Big(\tfrac{\operatorname{arth}(r)}{r}\Big)^{3/4+\alpha^* r} < K(r) < \tfrac{\pi}{2}\Big(\tfrac{\operatorname{arth}(r)}{r}\Big)^{3/4+\beta^* r}, \quad r \in (0,1). \tag{2}$$

This problem is equivalent to the following: determine the best values α^* and β^* such that

$$\alpha^* < \Big[G(r) - \tfrac{3}{4}\Big]/r < \beta^*, \quad r \in (0,1), \quad \text{where} \quad G(r) = \tfrac{\log(2\mathcal{K}(r)/\pi)}{\log([\operatorname{arth}(r)]/r)}.$$

The first part of this problem had been solved by the authors in [1] showing that $\alpha^* = 0$. Concerning the second part they conjectured that the mapping $G : (0,1) \to \mathbb{R}$ is strictly increasing and convex. Since $\lim_{r \nearrow 1} G(r) = 1$,

*Department of Mathematics, Sapientia University, Sighişoarei st., no. 1C, Tg. Mureş, Romania, e-mail: `kupanp@ms.sapientia.ro, szasz_robert2001@yahoo.com`.

this conjecture would imply $\beta^* = 1/4$. The result $\beta^* = 1/4$ has been proved recently in [4]. The basic tool used in their proof is Theorem 1.25 from [2]. It seems very difficult to prove the conjecture regarding the monotonicity of G. In the following we will show that a different elementary approach leads to a result, which improves the upper bound conjectured in [1].
In order to prove our results, we need certain lemmas, which will be exposed in the next section.

2. PRELIMINARIES

LEMMA 1. *If* $a_n = \left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\right)^2$, $b_n = \frac{4}{\pi(4n+1)}$, $c_n = \frac{1}{\pi(n+\frac{4}{\pi}-1)}$, $x_n = \frac{a_n}{b_n}$, $n \in \mathbb{N}^*$, $y_n = \frac{a_n}{c_n}$, $n \in \mathbb{N}^*$, *then the sequence* $\left(x_n\right)_{n\in\mathbb{N}^*}$ *is strictly increasing, the sequence* $\left(y_n\right)_{n\in\mathbb{N}^*}$ *is strictly decreasing for* $n \geq 2$ *and* $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 1$.

Proof. Since

$$\frac{x_{n+1}}{x_n} = \frac{(4n+5)(2n+1)^2}{(4n+1)(2n+2)^2} = \frac{16n^3+36n^2+24n+5}{16n^3+36n^2+24n+4} > 1$$

it follows that $x_{n+1} > x_n$, $n \in \mathbb{N}^*$. On the other hand, we have $a_n = \left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\right)^2 < \left(\frac{2\cdot 4\cdot 6\cdot\ldots\cdot(2n)}{3\cdot 5\cdot\ldots\cdot(2n+1)}\right)^2$. This implies $a_n < \frac{1}{2n+1}$ and finally we get $x_n < \frac{\pi(4n+1)}{4(2n+1)} < \frac{\pi}{2}$, $n \in \mathbb{N}^*$. Consequently, $(x_n)_{n\in\mathbb{N}^*}$ is convergent. Wallis product formula implies that $\lim_{n\to\infty} x_n = 1$. Thus we have

$$1 > x_{n+1} > x_n \geq x_1 = \frac{5\pi}{16} = 0.981\ldots.$$

An analogous calculation implies the assertion regarding $\left(y_n\right)_{n\in\mathbb{N}^*}$. □

LEMMA 2. *For all real numbers* $r \in (0,1)$, *we have*

$$\mathcal{K}(r) < \frac{\pi}{2}\left\{1 + \frac{1}{4}r^2 + \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{r^{2n}}{4n+1}\right\}. \tag{3}$$

Proof. We use the notations and the results of Lemma 1

$$\frac{2}{\pi}\mathcal{K}(r) = 1 + \sum_{n=1}^{\infty} a_n r^{2n} = 1 + \sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\right)^2 r^{2n},$$

and let

$$h(r) = 1 + a_1 r^2 + \sum_{n=2}^{\infty} b_n r^{2n} = 1 + \frac{1}{4}r^2 + \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{r^{2n}}{4n+1}.$$

We introduce the notations $\frac{2}{\pi}\mathcal{K}(r) = 1 + u(r)$ and $h(r) = 1 + v(r)$. Lemma 1 implies $a_n < b_n$, $n \in \mathbb{N}^*$, $n \geq 2$, and consequently $u(r) < v(r)$ for all $r \in (0,1)$. Thus, inequality (3) holds. □

Lemma 3. *For all real numbers* $r \in (0,1)$, *we have*

$$\frac{\pi}{2}\left\{1+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^{2n}}{n+\frac{4}{\pi}-1}\right\} < \mathcal{K}(r). \tag{4}$$

Proof. We use the notations and the results of Lemma1 in our proof again. We recall that

$$\frac{2}{\pi}\mathcal{K}(r) = 1+\sum_{n=1}^{\infty} a_n r^{2n} = 1+\sum_{n=1}^{\infty}\left(\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2\cdot 4\cdot\ldots\cdot(2n)}\right)^2 r^{2n},$$

and let

$$k(r) = 1+\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{r^{2n}}{n+\frac{4}{\pi}-1}.$$

Lemma 1 implies $c_n < a_n,\ n\in\mathbb{N}^*,\ n\geq 2$, and consequently $k(r) < \frac{2}{\pi}\mathcal{K}(r)$ for all $r\in(0,1)$. Thus, inequality (4) holds. □

Lemma 4. *(Bernoulli's inequality) If* $\alpha \geq 1$ *and* $a > -1$, *then*

$$(1+a)^{\alpha} \geq 1 + a\alpha. \tag{5}$$

If $b\in[0,1]$ *and* $\alpha\in(1,2)$, *then*

$$(1+b)^{\alpha} \geq 1+\alpha b+\frac{\alpha(\alpha-1)}{2}b^2+\frac{\alpha(\alpha-1)(\alpha-2)}{6}b^3. \tag{6}$$

Proof. We prove the second inequality. Let $g:[0,1]\to\mathbb{R}$ be the function defined by $g(b) = (1+b)^{\alpha}-1-\alpha b-\frac{\alpha(\alpha-1)}{2}b^2-\frac{\alpha(\alpha-1)(\alpha-2)}{6}b^3$. We have $g'(b) = \alpha(1+b)^{\alpha-1}-\alpha-\alpha(\alpha-1)b-\frac{\alpha(\alpha-1)(\alpha-2)}{2}b^2$, $g''(b) = \alpha(\alpha-1)[(1+b)^{\alpha-2}-1-(\alpha-2)b]$ and $g'''(b) = \alpha(\alpha-1)(\alpha-2)[(1+b)^{\alpha-3}-1]$. Thus $g''(0) = 0$ implies that $g''(b) > 0,\ b\in(0,1)$. An analogous argumentation shows that g' and g are strictly increasing on $(0,1)$ and so $g(0)=0$ implies inequality (6). □

Lemma 5. *Let* $w:(0,1)\to\mathbb{R}$ *be the function defined by* $\frac{\operatorname{arth}(r)}{r} = 1+\sum_{n=1}^{\infty}\frac{1}{2n+1}r^{2n} = 1+w(r)$. *If* $w(r)+w(r)v(r) = \sum_{n=1}^{\infty}\delta_n r^{2n}$, *then* $\delta_n\leq\frac{1}{3},\ n\in\mathbb{N},\ n\geq 1$.

Proof. Indeed $\delta_1 = \frac{1}{3},\ \delta_2 = \frac{17}{60}$, and if $n\geq 3$, then

$$\begin{aligned}
\delta_n &= \frac{1}{2n+1}+\frac{1}{8n-4}+\frac{4}{\pi}\sum_{k=1}^{n-2}\frac{1}{(2k+1)(4(n-k)+1)}\\
&= \frac{1}{2n+1}+\frac{1}{8n-4}+\frac{4}{\pi(4n+3)}\sum_{k=1}^{n-2}\left(\frac{1}{2k+1}+\frac{2}{4(n-k)+1}\right)\\
&= \frac{1}{2n+1}+\frac{1}{8n-4}+\frac{4}{\pi(4n+3)}\left(\sum_{k=1}^{n-2}\frac{1}{2k+1}+\sum_{k=2}^{n-1}\frac{2}{4k+1}\right)
\end{aligned}$$

$$< \frac{1}{2n+1} + \frac{1}{8n-4} + \frac{4}{\pi(4n+3)} \frac{3(n-2)}{5}$$
$$\leq \frac{1}{5} + \frac{1}{20} + \frac{4}{25\pi} < \frac{1}{3}.$$

□

Lemma 6. [4] *Let* $(a_n)_{n\geq 1}$ *and* $(b_n)_{n\geq 1}$ *be two sequences of real numbers, and let the power series*

$$u(x) = \sum_{n=1}^{\infty} a_n x^n \quad \text{and} \quad v(x) = \sum_{n=1}^{\infty} b_n x^n$$

be convergent for $|x| < 1$. *If* $b_n > 0$, $n = 1, 2, 3, ...$, *and if the sequence* $\left(\frac{a_n}{b_n}\right)_{n\geq 1}$ *is strictly increasing (resp. decreasing), then the function* $\frac{u}{v} : (0, 1) \to \mathbb{R}$ *is strictly increasing (resp. decreasing).*

3. THE MAIN RESULT

Recall that

$$v(r) = \frac{1}{4}r^2 + \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{1}{4n+1} r^{2n},$$

and

$$w(r) = \frac{\operatorname{arth}(r)}{r} - 1 = \sum_{n=1}^{\infty} \frac{1}{2n+1} r^{2n}.$$

Theorem 7. *If* $r \in (0, 1)$, *then the following inequality holds:*

$$1 + v(r) < \big(1 + w(r)\big)^{\frac{3}{4}+\frac{1}{4}r^2}. \tag{7}$$

Proof. We begin with the remark that (7) is equivalent to

$$\left(1 + \frac{w(r)-v(r)}{1+v(r)}\right)^{\frac{4}{1-r^2}} > 1 + w(r), \quad r \in (0, 1). \tag{8}$$

The inequality (5) from Lemma 4 implies that

$$\left(1 + \frac{w(r)-v(r)}{1+v(r)}\right)^{\frac{4}{1-r^2}} > 1 + \frac{4}{1-r^2} \frac{w(r)-v(r)}{1+v(r)}, \quad r \in (0, 1).$$

Thus, in order to prove (8), we have to show that

$$\frac{4(w(r)-v(r))}{(1-r^2)(1+v(r))} > w(r), \ r \in (0, 1). \tag{9}$$

We have $w(r) - v(r) > \frac{1}{12}r^2$, $r \in (0, 1)$. Thus

$$\frac{r^2}{3(1-r^2)(1+v(r))} > w(r), \ r \in (0, 1)$$

implies (9). This inequality is equivalent to

$$\frac{r^2}{3(1-r^2)} > w(r) + w(r)v(r), \ r \in (0, 1).$$

According to Lemma 5, we have

$$\frac{r^2}{3(1-r^2)} = \sum_{n=1}^{\infty} \tfrac{1}{3} r^{2n} > \sum_{n=1}^{\infty} c_n r^{2n} = w(r) + w(r)v(r),$$

and the proof is completed. □

Theorem 1 and Lemma 2 imply the following result.

COROLLARY 8. *If* $r \in (0,1)$, *then*

$$\mathcal{K}(r) < \tfrac{\pi}{2} \left(\frac{\operatorname{arth}(r)}{r} \right)^{\frac{3}{4} + \frac{1}{4} r^2}.$$

THEOREM 9. *If* $r \in (0,1)$, *then*

$$1 + \tfrac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n + \frac{4}{\pi} - 1} > \left(\frac{\operatorname{arth}(r)}{r} \right)^{\frac{3}{4} + \frac{r^4}{200}}. \tag{10}$$

Proof. We introduce the notations $\mu_1 = \frac{4}{\pi} - 1$ and $z(r) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n + \frac{4}{\pi} - 1} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n + \mu_1}$. Using this notation, (10) will be equivalent to

$$\big(1 + z(r)\big)^{\frac{200}{r^4 + 150}} > 1 + w(r). \tag{11}$$

We shall prove this inequality in three steps. First assume that $r \in [0, \frac{1}{5}]$. In this case we use the second inequality of Lemma 4 putting $\alpha = \frac{200}{r^4 + 150}$ and $b = z(r)$, and we obtain

$$(1 + z(r))^{\alpha} \geq 1 + \alpha z(r) + \tfrac{\alpha(\alpha-1)}{2} (z(r))^2 + \tfrac{\alpha(\alpha-1)(\alpha-2)}{6} (z(r))^3. \tag{12}$$

On the other hand we have $\frac{\alpha(\alpha-1)(2-\alpha)}{6} < \frac{1}{20}$, $\frac{\alpha(\alpha-1)}{2} > 0.22$, $(z(r))^3 < \frac{r^6}{50}$, $r \in (0, \frac{1}{5})$. Thus, inequality (12) implies

$$(1 + z(r))^{\alpha} \geq 1 + \alpha z(r) + 0.22 (z(r))^2 - \tfrac{r^6}{1000}, \quad r \in [0, \tfrac{1}{5}],$$

and consequently, in order to prove (11) we have to show that

$$1 + \alpha z(r) + 0.22 (z(r))^2 - \tfrac{r^6}{1000} \geq 1 + w(r), \quad r \in [0, \tfrac{1}{5}].$$

This inequality is equivalent to

$$0.22 \Big(\tfrac{1}{\pi} \sum_{n=1}^{\infty} \tfrac{r^{2n}}{n + \mu_1} \Big)^2 > \tfrac{r^6}{1000} + \sum_{n=1}^{\infty} \Big(\tfrac{1}{2n+1} - \tfrac{\alpha}{\pi(n + \mu_1)} \Big) r^{2n}, \quad r \in [0, \tfrac{1}{5}]. \tag{13}$$

Let us denote the coefficient of r^{2n} in $0.22 \Big(\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n + \mu_1} \Big)^2$ by d_n, $n \geq 2$. In order to prove inequality (13), we will show that

$$d_2 r^4 \geq \tfrac{r^6}{1000} + \Big(\tfrac{1}{3} - \tfrac{\alpha}{\pi(1 + \mu_1)} \Big) r^2 + \Big(\tfrac{1}{5} - \tfrac{\alpha}{\pi(2 + \mu_1)} \Big) r^4, \quad r \in [0, \tfrac{1}{5}], \tag{14}$$

and

$$d_n \geq \tfrac{1}{2n+1} - \tfrac{\alpha}{\pi(n+\mu_1)}, \quad n \geq 3. \tag{15}$$

The inequality (14) holds, because

$$\begin{aligned} d_2 r^4 = 0.22\tfrac{1}{16}r^4 &\geq \left[\tfrac{1}{25000} + \tfrac{1}{450\cdot 25} + \left(\tfrac{1}{5} - \tfrac{200}{(4+\pi)(\frac{1}{625}+150)}\right)\right] r^4 \\ &\geq \tfrac{r^6}{1000} + \tfrac{1}{450+3r^4}r^6 + \left(\tfrac{1}{5} - \tfrac{200}{(4+\pi)(r^4+150)}\right) r^4 \\ &= \tfrac{r^6}{1000} + \left(\tfrac{1}{3} - \tfrac{\alpha}{\pi(1+\mu_1)}\right) r^2 + \left(\tfrac{1}{5} - \tfrac{\alpha}{\pi(2+\mu_1)}\right) r^4, \quad r \in [0, \tfrac{1}{5}]. \end{aligned}$$

It is sufficient to prove (15) for $r = \frac{1}{5}$. We have

$$d_n = \tfrac{0.22}{\pi^2} \sum_{k=1}^{n-1} \tfrac{1}{(n-k+\mu_1)(k+\mu_1)} = \tfrac{0.44}{\pi^2} \tfrac{1}{n+2\mu_1} \sum_{k=1}^{n-1} \tfrac{1}{k+\mu_1}.$$

If $r = \frac{1}{5}$, inequality (15) is equivalent to

$$t_n = \tfrac{2\cdot 62500(n+\frac{1}{2})}{46876\pi(n+\mu_1)} + \tfrac{0.88}{\pi^2} \tfrac{n+\frac{1}{2}}{n+2\mu_1} \sum_{k=1}^{n-1} \tfrac{1}{k+\mu_1} > 1, \ n \in \mathbb{N}^*, \ n \geq 3. \tag{16}$$

We prove now that the sequence $(t_n)_{n\geq 3}$ is strictly increasing.

$$\begin{aligned} &t_{n+1} - t_n \\ &> \tfrac{2\cdot 62500}{46876\pi} \left(\tfrac{n+\frac{3}{2}}{n+\frac{4}{\pi}} - \tfrac{n+\frac{1}{2}}{n+\frac{4}{\pi}-1} \right) + \tfrac{0.88}{\pi^2} \tfrac{n+\frac{3}{2}}{n+\frac{8}{\pi}-1} \left(\sum_{k=1}^{n} \tfrac{1}{k+\mu_1} - \sum_{k=1}^{n-1} \tfrac{1}{k+\mu_1} \right) \\ &= \tfrac{0.88}{\pi^2} \tfrac{n+\frac{3}{2}}{(n+\frac{8}{\pi}-1)(n+\frac{4}{\pi}-1)} - \tfrac{2\cdot 62500}{46876} \tfrac{\frac{3}{2\pi}-\frac{4}{\pi^2}}{(n+\frac{4}{\pi})(n+\frac{4}{\pi}-1)} \\ &= \tfrac{1}{n+\frac{4}{\pi}-1} \left(\tfrac{0.88}{\pi^2} \tfrac{n+\frac{3}{2}}{n+\frac{8}{\pi}-1} - \tfrac{2\cdot 62500}{46876} \tfrac{\frac{3}{2\pi}-\frac{4}{\pi^2}}{n+\frac{4}{\pi}} \right) > 0, \ n \geq 3. \end{aligned}$$

Consequently, inequality (16) holds, and the proof of inequality (10) is done for $r \in [0, \frac{1}{5}]$.

In the second step we will prove that inequality (10) holds if $r \in [\frac{1}{5}, \frac{97}{100}]$. Let $r_k = \frac{1}{5} + \frac{k}{200000}$, $k = \overline{0, 154000}$. The functions $1+z(r)$ and $\big(1+w(r)\big)^{\frac{3}{4}+\frac{r^4}{200}}$ are strictly increasing on $[\frac{1}{5}, \frac{97}{100}]$. Thus, if the inequalities

$$1 + z(r_{k-1}) \geq \big(1 + w(r_k)\big)^{\frac{3}{4}+\frac{r_k^4}{200}}, \quad k = \overline{1, 154000} \tag{17}$$

hold, then the inequality-chains

$$1+z(r) \geq 1+z(r_{k-1}) \geq \big(1+w(r_k)\big)^{\frac{3}{4}+\frac{r_k^4}{200}} \geq \big(1+w(r)\big)^{\frac{3}{4}+\frac{r^4}{200}}, \quad r \in [r_{k-1}, r_k],$$

imply (10) for $r \in [\frac{1}{5}, \frac{97}{100}]$. The inequalities (17) can be verified easily using a computer program.

The third case is $r \in [\frac{97}{100}, 1)$. In this case we will prove the following inequality, which is stronger than (10):

$$1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n+\frac{4}{\pi}-1} > \left(\frac{arth(r)}{r} \right)^{\frac{151}{200}}, \quad r \in [\tfrac{97}{100}, 1).$$

We define the function $m : [\frac{97}{100}, 1) \to \mathbb{R}$ by $m(r) = 1 + z(r) - \big(1 + w(r)\big)^{\frac{151}{200}}$. We have

$$m'(r) = w'(r) \Big(\frac{z'(r)}{w'(r)} - \frac{151}{200} \frac{1}{(1+w(r))^{\frac{49}{200}}} \Big).$$

According to Lemma 6 the function $\frac{z'}{w'} : (0,1) \to \mathbb{R}$ is strictly decreasing, and $\lim_{r \nearrow 1} \frac{z'(r)}{w'(r)} = \frac{2}{\pi}$. Thus

$$\frac{z'(r)}{w'(r)} > \frac{2}{\pi} > \frac{151}{200} \frac{1}{(1+w(\frac{97}{100}))^{\frac{49}{200}}} \geq \frac{151}{200} \frac{1}{(1+w(r))^{\frac{49}{200}}}, \ r \in [\tfrac{97}{100}, 1),$$

and it follows that the mapping m is strictly increasing. Consequently, the inequality $m(\frac{97}{100}) > 0$ implies $m(r) > 0, \ r \in [\frac{97}{100}, 1)$ and the proof is complete. □

Remark 10. In order to prove the inequalities (17) we used the estimations

$$0 < z(r) - \frac{1}{\pi} \sum_{n=1}^{p} \frac{r^{2n}}{n+\mu_1} < \frac{r^{2p+2}}{\pi(p+\mu_1+1)(1-r^2)},$$

$$0 < w(r) - \sum_{n=1}^{p} \frac{1}{2n+1} r^{2n} < \frac{r^{2p+2}}{(2p+3)(1-r^2)},$$

and applied numerical methods using the Matlab program.

4. FINAL COMMENTS

Theorem 2 and Corollary 1 imply the inequalities

$$\frac{\pi}{2} \left(\frac{\text{arth}(r)}{r} \right)^{\frac{3}{4}+\frac{r^4}{200}} < \frac{\pi}{2} \Big(1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n+\frac{4}{\pi}-1} \Big) < \mathcal{K}(r) < \frac{\pi}{2} \left(\frac{\text{arth}(r)}{r} \right)^{\frac{3}{4}+\frac{1}{4}r^2},$$

for $r \in (0,1)$. Since

$$\frac{\pi}{2} \Big(1 + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{r^{2n}}{n+\frac{4}{\pi}-1} \Big) = \frac{\pi}{2} + \frac{1}{2\mu_1} [{}_2F_1(1, \mu_1, \mu_1+1, r^2) - 1]$$

it follows that the first inequality implies

$$\frac{\pi}{2} \left(\frac{\text{arth}(r)}{r} \right)^{3/4} < \frac{\pi}{2} + \frac{1}{2\mu_1} [{}_2F_1(1, \mu_1, \mu_1+1, r^2) - 1], \ r \in (0,1),$$

which was conjectured in [3]. The second inequality has been established for the first time in [3]. The third inequality implies a conjecture from [1]. The authors of [4] proved that the following inequalities hold

$$\frac{\pi}{2}\left(\frac{\operatorname{arth}(r)}{r}\right)^{\frac{3}{4}+\alpha^* r} < \mathcal{K}(r) < \frac{\pi}{2}\left(\frac{\operatorname{arth}(r)}{r}\right)^{\frac{3}{4}+\beta^* r}, \ r \in (0,1),$$

with the best possible constants $\alpha^* = 0$ and $\beta^* = 1/4$. Our results are improvements of these inequalities.

REFERENCES

[1] H. Alzer and S.-L. Qiu, *Monotonicity theorems and inequalities for the complete elliptic integrals*, Journal of Comp. Appl. Math., **172** (2004), pp. 289–312.

[2] G.D. Anderson, M.K. Vamanamurthy and M. Vourinen, *Inequalities for quasiconformal mappings in spaces*, Pacific J. Math., **160** (1993), pp. 1–18.

[3] Sz. András and Á. Baricz, *Bounds for complete elliptic integrals of the first kind*, Expo. Math., **28** (2010) no. 4, pp. 357–364.

[4] Y.-M. Chu, M.-K. Wang and Y.-F. Qiu, *On Alzer and Qiu's conjecture for complete elliptic integral and inverse hyperbolic tangent function*, Abstract and Applied Analysis, article ID. 697547, 2011, pp. 1–7.

[5] S. Ponnusamy and M. Vuorinen, *Asymptotic expansions and inequalities for hypergeometric functions*, Mathematika, **44** (1997) no. 2, pp. 278–301.

Received by the editors: February 6, 2012.