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VECTOR OPTIMIZATION PROBLEMS AND APPROXIMATED VECTOR OPTIMIZATION PROBLEMS

EUGENIA DUCA* and DOREL I. DUCA†

Abstract. In this paper, a so-called approximated vector optimization problem associated to a vector optimization problem is considered. The equivalence between the efficient solutions of the approximated vector optimization problem and efficient solutions of the original optimization problem is established.

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1. INTRODUCTION

We consider the vector optimization problem

$$(VOP)\qquad \begin{array}{ll} C-\min & f(x) \\ \text{s.t.} & x \in X \\ & g(x) \in -K, \end{array}$$

where X is a subset of $\mathbb{R}^n$, C is a convex cone in $\mathbb{R}^p$, K is a convex cone in $\mathbb{R}^m$, and $f : X \to \mathbb{R}^p$, $g : X \to \mathbb{R}^m$ are functions.

Let

$$\mathfrak{F}(VOP) := \{x \in X : \ g(x) \in -K\},$$

denote the set of all feasible solutions of Problem (VOP).

Let $L : X \times K^* \to \mathbb{R}^n$ be the **lagrangian** of Problem (VOP), i.e. the function defined by

$$L(x, v) := f(x) + \langle g(x), v\rangle e, \text{ for all } (x, v) \in X \times K^*,$$

where

$$K^* := \{u \in \mathbb{R}^m : \ \langle u, v\rangle \geqq 0, \text{ for all } v \in K\}$$

is the polar of the convex cone K, and $e = (1, ..., 1) \in \mathbb{R}^n$.

*Technical University, Department of Mathematics, Bariţiu Street, no. 25–28, 400027 Cluj-Napoca, Romania, e-mail: `jeniduca@yahoo.com, educa@math.utcluj.ro`.
† "Babeş-Bolyai" University, Faculty of Mathematics and Computer Science, 1 M. Kogălniceanu Street, 400084 Cluj-Napoca, Romania, e-mail: `dorelduca@yahoo.com`, `dduca@math.ubbcluj.ro`.

Definition 1.1. *Let x^0 be a point of* $\mathfrak{F}(VOP)$*. We say that x^0 is an* ***efficient solution*** *for Problem* (VOP) *if there exists no point $x \in \mathfrak{F}(VOP)$ such that*

$$f\left(x^0\right) - f(x) \in C \backslash \{0\}.$$

Remark 1.2. The point x^0 is an efficient solution for Problem (VOP) if and only if

$$f\left(x^0\right) - f(x) \notin C \backslash \{0\}, \text{ for all } x \in \mathfrak{F}(VOP). \quad \square$$

Definition 1.3. *Let x^0 be a point of* $\mathfrak{F}(VOP)$*. We say that x^0 is a* ***weak efficient solution*** *for Problem* (VOP) *if there exists no point $x \in \mathfrak{F}(VOP)$ such that*

$$f\left(x^0\right) - f(x) \in \operatorname{int} C.$$

Remark 1.4. The point x^0 is a weak efficient solution for Problem (VOP) if and only if

$$f\left(x^0\right) - f(x) \notin \operatorname{int} C, \text{ for all } x \in \mathfrak{F}(VOP). \quad \square$$

If C is the closed convex cone $\mathbb{R}^p_+$ and K is the closed convex cone $\mathbb{R}^m_+$, then problem (VOP), becomes the multicriteria optimization problem

$$\text{(MOP)} \qquad \begin{array}{ll} v\text{-}\min & f(x) \\ \text{s.t.} & x \in X \\ & g(x) \leqq 0. \end{array}$$

For solving vector optimization problem (VOP), there are various manners to approach. One of these manners is that for Problem (VOP) one attaches another optimization problem, problem whose solutions gives us the (information about) solutions of the initial problem (VOP).

If x^0 is a feasible solution for (MOP) and f is differentiable at x^0, C.R. Bector, S. Chandra and C. Singh [3], attached to Problem (MOP), the problem

$$\text{(LMOP)} \qquad \begin{array}{ll} v\text{-}\min & \left[\nabla f\left(x^0\right)\right](x) \\ \text{s.t.} & x \in X \\ & g(x) \leqq 0, \end{array}$$

and obtained connections of efficient solutions of the original problem (MOP) to the efficient solutions of the linearized multicriteria optimization problem (LMOP).

If $x^0 \in \mathfrak{F}(MOP)$ is an interior point of X, f is differentiable at x^0, Antczak [2], proposed the following approximated multicriteria optimization problem

$$(\eta MOP) \qquad \begin{array}{ll} v\text{-}\min & \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \\ \text{s.t.} & x \in X \\ & g(x) \leqq 0, \end{array}$$

where $\eta : X \times X \to \mathbb{R}^n$ is a function, and obtained results to connect (MOP) and (ηMOP).

In this paper, assuming that $x^0 \in \mathfrak{F}(VOP)$ is an interior point of X, and $\eta : X \times X \to \mathbb{R}^n$ is a function, we attach to Problem (VOP) the problems:

a) assuming that f is differentiable at x^0

$$\text{(FAVOP)} \qquad \begin{array}{ll} C\text{-}\min & f\left(x^0\right) + \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \\ \text{s.t.} & x \in X \\ & g(x) \in -K, \end{array}$$

and *b*) assuming that g is differentiable at x^0

$$\text{(CAVOP)} \qquad \begin{array}{ll} C-\min & f(x) \\ \text{s.t.} & x \in X \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \in -K. \end{array}$$

2. NOTIONS AND PRELIMINARY RESULTS

In the last few years, attempts have been made to weaken the convexity hypotheses and thus to explore the existence of optimality conditions applicability. Various classes of generalized convex functions have been suggested for the purpose of weakening the convexity limitation of the results. Among these, the concept of an invex function proposed by Hanson [11] has received more attention. The name of invex (invariant convex) function was given by Craven [5].

DEFINITION 2.1. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, C be a closed convex cone in $\mathbb{R}^p$, $f : X \to \mathbb{R}^p$ be a differentiable function at x^0, and $\eta : X \times X \to \mathbb{R}^n$ be a function.*

a) *We say that the function f is C-**invex** at x^0 with respect to (w.r.t.) η if*

$$f(x) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \in C, \text{ for all } x \in X. \tag{2.1}$$

b) *We say that the function f is C-**incave** at x^0 with respect to (w.r.t.) η if*

$$f(x) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \in -C, \text{ for all } x \in X.$$

REMARK 2.2. The function f is C-incave at x^0 w.r.t. η if anf only if the function f is $(-C)$-invex at x^0 w.r.t. η. □

EXAMPLE 2.3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ the function defined by

$$f(x) := \left(x_1^2 + \sin\tfrac{\pi x_2}{2}, x_2^2 + \sin\tfrac{\pi x_1}{3}\right), \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

a) The function f is $\mathbb{R}^2_+$-invex at $x^0 = (0,0)$ w.r.t. $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x, u) := \left(\tfrac{3}{\pi}\sin\tfrac{\pi x_1}{3}, \tfrac{2}{\pi}\sin\tfrac{\pi x_2}{2}\right),$$

for all $(x, u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$.

b) Also, the function f is $\mathbb{R}^2_+$-invex at $x^0 = (0,0)$ w.r.t. $\mu : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\mu(x, u) := \left(\tfrac{3}{\pi}\sin\tfrac{\pi x_1}{3} - 4, \tfrac{2}{\pi}\sin\tfrac{\pi x_2}{2} - 7\right),$$

for all $(x,u) = ((x_1,x_2),(u_1,u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Let's remark that $\mu(x,u) \neq (0,0)$, for all $(x,u) \in \mathbb{R}^2 \times \mathbb{R}^2$.

c) Also, the function f is $\mathbb{R}^2_+$-invex at $x^0 = (0,0)$ w.r.t. $\zeta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\zeta(x,u) := \left(\tfrac{3}{\pi}\sin\tfrac{\pi x_1}{3} - x_1^2, \tfrac{2}{\pi}\sin\tfrac{\pi x_2}{2} - x_2^2\right),$$

for all $(x,u) = ((x_1,x_2),(u_1,u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$. □

After the works of Hanson and Craven, other types of differentiable functions have appeared with the intent of generalizing invex function from different points of view.

Ben-Israel and Mond [4] defined the so-called pseudoinvex functions, generalizing pseudoconvex functions in the same way that invex functions generalize convex functions. Here we give the following notion of pseudoinvexity:

DEFINITION 2.4. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, K and L be two convex cones in $\mathbb{R}^p$, $f : X \to \mathbb{R}^p$ be a differentiable function at x^0, and $\eta : X \times X \to \mathbb{R}^n$ be a function. We say that f is (K,L)-**pseudoinvex** at x^0 with respect to (w.r.t.) η if, for each $x \in X \backslash \{x^0\}$ with the property that*

$$\left[\nabla f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right) \in K,$$

we have

$$f(x) - f\left(x^0\right) \in L.$$

REMARK 2.5. The notion of K-pseudoinvexity is equivalent with the notion of (K,K)-pseudoinvexity. □

DEFINITION 2.6. *Let X be a nonempty subset of $\mathbb{R}^n$, x^0 be an interior point of X, K and L be two convex cones in $\mathbb{R}^p$, $f : X \to \mathbb{R}$ be a differentiable function at x^0, and $\eta : X \times X \to \mathbb{R}^n$ be a function. We say that f is (K,L)-**quasiinvex** at x^0 with respect to (w.r.t.) η if, for each $x \in X \backslash \{x^0\}$ with the property that*

$$f\left(x^0\right) - f(x) \in K,$$

we have

$$\left[\nabla f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right) \in L.$$

REMARK 2.7. The notion of K-quasiinvexity is equivalent with the notion of $(K,-K)$-quasiinvexity. □

3. THE MODIFIED CRITERIA FUNCTION OF VECTOR OPTIMIZATION PROBLEMS

In this section, X is a subset of $\mathbb{R}^n$, x^0 is an interior point of X, $f : X \to \mathbb{R}^p$ is a differentiable function at x^0, C is a convex cone in $\mathbb{R}^p$, K is a convex cone in $\mathbb{R}^m$, and $g : X \to \mathbb{R}^m$ is a function.

For $\eta : X \times X \to \mathbb{R}^n$, we attach to Problem (VOP) the following optimization problem:

$$\text{(FAVOP)} \qquad \begin{array}{ll} C\text{-}\min & f\left(x^0\right) + \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) \\ \text{s.t.} & x \in X \\ & g\left(x\right) \in -K. \end{array}$$

Let

$$\mathfrak{F}\left(FAVOP\right) := \{x \in X : \ g\left(x\right) \in -K\},$$

denote the set of all feasible solutions of Problem (FAVOP).

Obviously

$$\mathfrak{F}\left(FAVOP\right) = \mathfrak{F}\left(VOP\right).$$

Let $F : X \to \mathbb{R}^p$ the function defined by

$$F\left(x\right) = f\left(x^0\right) + \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right), \text{ for all } x \in X,$$

i.e. the criteria function of Problem (FAVOP).

Theorem 3.1. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone in $\mathbb{R}^p$, $g : X \to \mathbb{R}^m$ be a function, $\eta : X \times X \to \mathbb{R}^n$ such that $\eta\left(x^0, x^0\right) = 0$ and $f : X \to \mathbb{R}^p$ be a differentiable function at x^0 and $(-C\backslash\{0\}, -C\backslash\{0\})$-pseudoinvex at x^0 w.r.t. η.*

If x^0 is an efficient solution for (VOP)*, then x^0 is an efficient solution for* (FAVOP) .

Proof. Assume that x^0 is not an efficient solution for (FAVOP), then there exists a feasible solution $x^1 \in \mathfrak{F}\left(FAVOP\right)$ such that

$$F\left(x^0\right) - F\left(x^1\right) \in C\backslash\{0\}.$$

Since $\eta\left(x^0, x^0\right) = 0$, we have

$$F\left(x^0\right) - F\left(x^1\right) =$$

$$f\left(x^0\right) + \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^1, x^0\right)\right) =$$
$$= -\left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^1, x^0\right)\right),$$

hence

$$\left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^1, x^0\right)\right) \in -C\backslash\{0\}.$$

But f is $(-C\backslash\{0\}, -C\backslash\{0\})$-pseudoinvex at x^0 w.r.t η and then

$$f\left(x^1\right) - f\left(x^0\right) \in -C\backslash\{0\},$$

i.e. x^0 is not an efficient solution for (VOP). The theorem is proved. □

Remark 3.2. The hypothesis that f is $(-C\backslash\{0\}, -C\backslash\{0\})$-pseudoinvex at x^0 w.r.t. η is essential, as seen in the following example. □

EXAMPLE 3.3. Let's consider Problem (VOP) with $X := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $K := \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f(x) = \left(x_1^2 + \sin \tfrac{\pi x_2}{2}, x_2^2 + \sin \tfrac{\pi x_1}{3}\right),$$
$$g(x) = \left(x_1^2 - x_2, x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$\eta(x, u) = \left(\tfrac{3}{\pi} \sin \tfrac{\pi x_1}{3} - \tfrac{3}{\pi} x_2^2,\ \tfrac{2}{\pi} \sin \tfrac{\pi x_2}{2} - \tfrac{4}{\pi} x_1^2\right),$$

for all $(x, u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$.

The point $x^0 = (0, 0) \in \mathfrak{F}(VOP)$ is an efficient solution for Problem (VOP). On the other hand,

$$F(x) = f\left(x^0\right) + \left[\triangledown f(x^0)\right](\eta(x, x^0)) = \left(\sin \tfrac{\pi x_2}{2} - 2x_1^2, \sin \tfrac{\pi x_1}{3} - x_2^2\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, and for $x^1 = (1, 1) \in \mathfrak{F}(VOP) = \mathfrak{F}(FAVOP)$,

$$F\left(x^0\right) - F\left(x^1\right) = \left(1, \tfrac{1-\sqrt{3}}{2}\right) \in C\backslash\{0\} = \mathbb{R}^2_+\backslash\{0\}.$$

Consequently, x^0 is not an efficient solution for (FAVOP).

Let's remark that f is not $(-C\backslash\{0\}, -C\backslash\{0\})$-pseudoinvex at x^0 w.r.t. η, because

$$\left[\triangledown f(x^0)\right](\eta(x^1, x^0)) = \left(-1, \tfrac{-\sqrt{3}-1}{2}\right) \in -C\backslash\{0\} = -\mathbb{R}^2_+\backslash\{0\}$$

and

$$f\left(x^1\right) - f\left(x^0\right) = \left(2, 1 + \tfrac{\sqrt{3}}{2}\right) \notin -C\backslash\{0\} = -\mathbb{R}^2_+\backslash\{0\}.$$

$\square$

REMARK 3.4. The hypothesis that $\eta\left(x^0, x^0\right) = 0$ is essential, as seen in the following example. $\square$

EXAMPLE 3.5. Let's consider Problem (VOP) with $X := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $K := \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f(x) = (x_1, x_2), \quad g(x) = \left(x_1^2 - x_2, x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2_+$, and

$$\eta(x, u) = \left(x_1 + (x_1 - 1)^2, x_2 + (x_2 - 1)^2\right),$$

for all $(x, u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$. The point $x^0 = (0, 0)$ is an efficient solution for Problem (VOP). The function f is $(-C)$-invex at x^0 w.r.t. η, because, for all $(x_1, x_2) \in \mathbb{R}^2$, we have:

$$f(x) - f\left(x^0\right) - \left[\nabla f(x_0)\left(\eta\left(x, x^0\right)\right)\right] = -\left((1 - x_1)^2, (1 - x_2)^2\right) \in -C.$$

It follows that f is $(-C\backslash\{0\}, -C\backslash\{0\})$-pseudoinvex at x^0 w.r.t. η.

Since, for each $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = f\left(x^0\right) + \left[\nabla f\left(x^0\right)\left(\eta\left(x, x^0\right)\right)\right] = \left(x_1 + (x_1 - 1)^2, x_2 + (x_2 - 1)^2\right),$$

we deduce that x^0 is not an efficient solution for Problem (FAVOP). Why? Because, for $x^1 = \left(\frac{1}{2}, \frac{1}{2}\right)$, we have

$$\begin{aligned} F\left(x^0\right) - F\left(x^1\right) &= f\left(x^0\right) + \left[\nabla f\left(x_0\right)\left(\eta\left(x^0, x^0\right)\right)\right] - f\left(x^0\right) - \\ &\quad - \left[\nabla f\left(x_0\right)\left(\eta\left(x^1, x^0\right)\right)\right] = \\ &= \left(\tfrac{3}{4}, \tfrac{3}{4}\right) \in C \backslash \{0\}. \end{aligned}$$

□

THEOREM 3.6. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone with nonempty interior in $\mathbb{R}^p$, $g : X \to \mathbb{R}^m$ be a function, $\eta : X \times X \to \mathbb{R}^n$ such that $\eta\left(x^0, x^0\right) = 0$ and $f : X \to \mathbb{R}^p$ be a differentiable function at x^0 and $(-\mathrm{int}C, -\mathrm{int}C)$-pseudoinvex at x^0 w.r.t. η.*

If x^0 is a weak efficient solution for (VOP)*, then x^0 is a weak efficient solution for* (FAVOP)*.*

Proof. The proof is similar to the proof of Theorem 3.1. □

THEOREM 3.7. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone in $\mathbb{R}^p$, $g : X \to \mathbb{R}^m$ be a function, $\eta : X \times X \to \mathbb{R}^n$ such that $\eta\left(x^0, x^0\right) = 0$ and $f : X \to \mathbb{R}^p$ be a differentiable function at x^0 and $(C\backslash\{0\}, C\backslash\{0\})$-quasiinvex at x^0 w.r.t. η.*

If x^0 is an efficient solution for (FAVOP)*, then x^0 is an efficient solution for* (VOP)*.*

Proof. Assume that x^0 is not an efficient solution for (VOP), then there exists a feasible solution $x^1 \in \mathfrak{F}(VOP)$ such that

$$f\left(x^0\right) - f\left(x^1\right) \in C\backslash\{0\}.$$

But f is $(C\backslash\{0\}, -C\backslash\{0\})$-quasiinvex at x^0 w.r.t η and hence

$$\left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^1, x^0\right)\right) \in -C\backslash\{0\}.$$

It follows that

$$\begin{aligned} &F\left(x^0\right) - F\left(x^1\right) = \\ &= f\left(x^0\right) + \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^0, x^0\right)\right) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x^1, x^0\right)\right) \in C\backslash\{0\}, \end{aligned}$$

because $\eta\left(x^0, x^0\right) = 0$. Consequently, x^0 is not an efficient solution for (FAVOP) which is a contradiction. The theorem is proved. □

REMARK 3.8. In Theorem 3.7, the hypothesis that f is C-invex at x^0 w.r.t. η is essential, as seen in the following example. □

EXAMPLE 3.9. Let's consider Problem (VOP) with $X := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $K := \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f(x) = (x_1, x_2),$$
$$g(x) = \left(x_1^2 - x_2, x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$\eta(x,u) = \left(x_1 + (x_1 - 1)^2,\ x_2 + (x_2 - 1)^2\right),$$

for all $(x,u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$. For $x^0 = (0,0) \in \mathfrak{F}(VOP)$, we have

$$f(x) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x, x^0\right)\right) =$$
$$= -\left((x_1 - 1)^2, (x_2 - 1)^2\right) \in -C = -\mathbb{R}^2_+,$$

for all $x \in \mathbb{R}^2$. Consequently, the function f is not C-invex at x^0 w.r.t. η.

Since, for each $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = f\left(x^0\right) + \left[\triangledown f(x^0)\right](\eta(x, x^0)) = \left(x_1 + (x_1 - 1)^2, x_2 + (x_2 - 1)^2\right) =$$
$$= \left(\left(x_1 - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}, \left(x_2 - \tfrac{1}{2}\right)^2 + \tfrac{3}{4}\right)$$

it follows that $x^1 = \left(\frac{1}{2}, \frac{1}{2}\right) \in \mathfrak{F}(VOP) = \mathfrak{F}(FAVOP)$ is an efficient solution for Problem (FAVOP).

On the other hand,

$$f\left(x^1\right) - f\left(x^0\right) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \in C\backslash\{0\}.$$

Consequently, x^1, which is an efficient solution for Problem (FAVOP), is not an efficient solution for problem (VOP); the function f is not C-invex at x^0 w.r.t. η. □

REMARK 3.10. In Theorem 3.7, the hypothesis that $\eta\left(x^0, x^0\right) = 0$ is essential, as seen in the following example. □

EXAMPLE 3.11. Let's consider Problem (VOP) with $X := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $K := \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f(x) = \left(x_1^2 + x_1, x_2^2 + x_2\right),$$
$$g(x) = \left(x_1^2 - x_2, x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$\eta(x,u) = \left(x_1 - \left(x_1^2 + 1\right)^2,\ x_2 - \left(x_2^2 + 1\right)^2\right),$$

for all $(x,u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$.

For $x^0 = (0,0) \in \mathfrak{F}(VOP)$, we have

$$f(x) - f\left(x^0\right) - \left[\nabla f\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right) = \\ = \left(x_1^2 + \left(x_1^2+1\right), x_2^2 + \left(x_2^2+1\right)\right) \in C,$$

for all $x \in \mathbb{R}^2$. Consequently, the function f is C-invex at x^0 w.r.t. η.

Since, for each $x = (x_1, x_2) \in \mathbb{R}^2$,

$$F(x) = f\left(x^0\right) + \left[\bigtriangledown f(x^0)\right](\eta(x,x^0)) = \left(x_1 - \left(x_1^2+1\right)^2, x_2 - \left(x_2^2+1\right)^2\right) = \\ = \left(-x_1^4 - 4x_1^3 - 6x_1^2 - 3x_1 - 1, -x_2^4 - 4x_2^3 - 6x_2^2 - 3x_2 - 1\right),$$

it follows that $x^1 = (1,1) \in \mathfrak{F}(VOP) = \mathfrak{F}(FAVOP)$ is an efficient solution for Problem (FAVOP).

On the other hand,

$$f\left(x^0\right) - f\left(x^1\right) = (2,2) \in C\backslash\{0\}.$$

Consequently, x^1, which is an efficient solution for Problem (FAVOP), is not an efficient solution for problem (VOP).

Let's remark that $\eta\left(x^0,x^0\right) = (-1,-1) \neq (0,0)$. □

THEOREM 3.12. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone in $\mathbb{R}^p$, $g: X \to \mathbb{R}^m$ be a function, $\eta: X \times X \to \mathbb{R}^n$ such that $\eta\left(x^0,x^0\right) = 0$ and $f: X \to \mathbb{R}^p$ be a differentiable function at x^0 and* $(\mathrm{int}C, \mathrm{int}C)$*-quasiinvex at x^0 w.r.t. η.*

If x^0 is a weak efficient solution for (FAVOP)*, then x^0 is a weak efficient solution for* (VOP)*.*

Proof. The proof is similar to the proof of Theorem 3.7. □

4. THE MODIFIED CONSTRAINT FUNCTION OF VECTOR OPTIMIZATION PROBLEMS

In this section, X is a subset of $\mathbb{R}^n$, x^0 is an interior point of X, $f: X \to \mathbb{R}^p$ is a function, C is a convex cone in $\mathbb{R}^p$, K is a convex cone in $\mathbb{R}^m$, and $g: X \to \mathbb{R}^m$ is a differentiable function at x^0.

For $\eta: X \times X \to \mathbb{R}^n$, we attach to Problem (VOP) the following vector optimization problem:

$$\text{(CAVOP)} \qquad \begin{array}{ll} C\text{-}\min & f(x) \\ \text{s.t.} & x \in X \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right) \in -K \end{array}$$

Let

$$\mathfrak{F}(CAVOP) := \{x \in X : \ g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x,x^0\right)\right) \in -K\},$$

denote the set of all feasible solutions of Problem (CAVOP).

THEOREM 4.1. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone in $\mathbb{R}^p$, $\eta : X \times X \to \mathbb{R}^n$ and $f : X \to \mathbb{R}$ be two functions and $g : X \to \mathbb{R}^m$ be a differentiable function at x^0.*

If the function g is K-incave at x^0 w.r.t. η, then every feasible solution for Problem (CAVOP) *is a feasible solution for Problem* (VOP)*, i. e.*

$$\mathcal{F}(CAVOP) \subseteq \mathcal{F}(VOP).$$

Proof. Let $x^1 \in \mathcal{F}(CAVOP)$, i.e.

$$g(x^0) + \left[\triangledown g(x^0)\right](\eta(x^1, x^0)) \in -K.$$

Since g is K-incave at x^0 w.r.t. η, we have

$$g(x^1) - g(x^0) - \left[\triangledown g(x^0)\right](\eta(x^1, x^0)) \in -K$$

From this, it follows

$$g(x^1) \in -K + \{g\left(x^0\right) + \left[\triangledown g(x^0)\right](\eta(x^1, x^0))\} \subseteq -K + (-K) = -K,$$

hence $x \in \mathcal{F}(VOP)$ □

EXAMPLE 4.2. Let's consider Problem (VOP) with $X = \mathbb{R}^2$, $C = K = \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f\left(x\right) = \left(\sin \tfrac{(x_1+x_2)\pi}{4},\ x_1^2\left(x_2-7\right)^2\right),$$

$$g\left(x\right) = \left(x_1^2 - x_2,\ x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$ and

$$\eta\left(x, u\right) = \left(x_1 - u_1,\ x_2 - u_2\right)$$

for all $(x, u) = \left(\left(x_1, x_2\right), \left(u_1, u_2\right)\right) \in \mathbb{R}^2 \times \mathbb{R}^2$.

The function g is not $\mathbb{R}^2_+$-incave at $x^0 = (0, 0)$ w.r.t. η, because

$$g\left(1,1\right) - g\left(x^0\right) - \left[\nabla g\left(x^0\right)\right]\left(\eta\left(\left(1,1\right), x^0\right)\right) = (1,1) \notin -\mathbb{R}^2_+$$

Since

$$g\left(x^0\right) + \left[\triangledown g(x^0)\right](\eta(x, x^0)) = \left(-x_2, -x_1\right),\ \text{for all } x = (x_1, x_2) \in \mathbb{R}^2,$$

the set of feasible solutions for Problem (CAVOP) is $\mathfrak{F}\left(CAVOP\right) = \mathbb{R}^2_+$.

Consequently

$$\mathfrak{F}\left(CAVOP\right) = \mathbb{R}^2_+ \supseteq \mathfrak{F}\left(VOP\right) = \{(x_1, x_2) :\ x_1^2 - x_2 \leqq 0,\ x_2^2 - x_1 \leqq 0\}.$$

Obviously, the point

$$x^1 = (0, 7) \in \mathfrak{F}\left(CAVOP\right)\ \setminus \mathfrak{F}\left(VOP\right).$$

The point $x^0 = (0, 0)$ is an efficient solution for Problem (VOP) and x^0 is not an efficient solution for Problem (CAVOP) because

$$f\left(x^0\right) - f\left(0, 7\right) = (1, 0) \in C \backslash \{0\} = \mathbb{R}^2_+ \backslash \{0\}.$$

□

Theorem 4.3. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, K be a closed convex cone in $\mathbb{R}^m$, C be a closed convex cone in $\mathbb{R}^p$, $\eta : X \times X \to \mathbb{R}^n$ and $f : X \to \mathbb{R}$ be two functions and $g : X \to \mathbb{R}^m$ be a differentiable function at x^0.*

If the function g is K-invex at x^0 w.r.t. η, then every feasible solution for Problem (VOP)*, is a feasible solution for Problem* (CAVOP)*, i. e.*

$$\mathcal{F}(VOP) \subseteq \mathcal{F}(CAVOP).$$

Proof. Let $x^1 \in \mathcal{F}(VOP)$, i.e. $g(x^1) \in -K$. Since g is K-invex at x^0 w.r.t. η we have

$$g(x^1) - g(x^0) - \left[\triangledown g(x^0)\right](\eta(x^1, x^0)) \in K$$

From this, it follows

$$g(x^0) + \left[\triangledown g(x^0)\right](\eta(x^1, x^0)) \in -K + \{g\left(x^1\right)\} \subseteq -K + (-K) = -K,$$

hence $x^1 \in \mathcal{F}(CAVOP)$. □

Example 4.4. Let's consider Problem (VOP) with $X := \mathbb{R}^2$, $C := \mathbb{R}^2_+$, $K := \mathbb{R}^2_+$, and $f : \mathbb{R}^2 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and $\eta : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ the functions defined by

$$f(x) = \left(\sin \tfrac{(x_1+x_2)\pi}{4},\ x_1^2 (x_2 - 7)^2\right),$$

$$g(x) = \left(x_1^2 - x_2,\ x_2^2 - x_1\right),$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, and

$$\eta(x, u) = (x_1 - u_1,\ x_2 - u_2)$$

for all $(x, u) = ((x_1, x_2), (u_1, u_2)) \in \mathbb{R}^2 \times \mathbb{R}^2$.

We have

$$\mathfrak{F}(VOP) = \{(x_1, x_2) :\ x_1^2 - x_2 \leqq 0,\ x_2^2 - x_1 \leqq 0\} \subseteq [0, 1] \times [0, 1];$$

the point $x^0 = (0, 0)$ is an efficient solution for (VOP) and the function g is $\mathbb{R}^2_+$-invex at x^0 w.r.t. η.

Since

$$g\left(x^0\right) + \left[\triangledown g(x^0)\right](\eta(x, x^0)) = (-x_2,\ -x_1),\ \text{for all } x = (x_1, x_2) \in \mathbb{R}^2,$$

the set of feasible solutions for Problem (CAVOP) is

$$\mathfrak{F}(CAVOP) = \mathbb{R}^2_+ \supseteq \mathfrak{F}(VOP).$$

Easy to remark that x^0 is not an efficient solution for Problem (CAVOP) because

$$f\left(x^0\right) - f(0, 7) = (1, 0) \in C\backslash\{0\} = \mathbb{R}^2_+\backslash\{0\}.$$

□

5. CONCLUSIONS

In this paper one shows how, under some hypotheses, in order to obtain a solution for a vector optimization problem it is sufficient to solve another vector optimization problem.

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