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COMMON FIXED POINTS VERSUS INVARIANT APPROXIMATION FOR NONCOMMUTATIVE MAPPINGS

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Abstract. The aim of this paper is to obtain common fixed points as invariant approximation for noncommuting two pairs of mappings. As consequences, our works generalize the recent works of Nashine [9] by weakening commutativity hypothesis and by increasing the number of mappings involved.

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1. INTRODUCTION

Fixed-point theorems have been used at many places in approximation theory. One of them is existence of invariant approximants where it is applied. Number of results exists in the literature applying the fixed-point theorem to prove the existence of invariant approximation.

Meinardus [8] was the first who used fixed-point theorem to prove the existence of an invariant approximation. Later, Brosowski [1] generalized that result. Next, it was extended by Subrahmanyam [14]. Further, the linearity of mapping and convexity condition was dropped from the hypothesis of Brosowski [1] by Singh [11]. Subsequently Singh [12] underlying the necessity of nonexpansiveness of his own result [11].

Next, Hicks and Humpheries [3] demonstrated that if the mapping conditions satisfy only at the boundary of domain, then also the result given by Singh [11] remains true. Furthermore, Sahab et al. [10] extended the result of Hicks and Humpheries [3] and Singh [11] by considering two mappings. Interestingly, one linear and the other nonexpansive mappings. The result was further improved using weak and strong topology by Jungck and Sessa [5].

Recently, Nashine [9] obtained some existence results on common fixed points as invariant approximation for a class of contraction commutative three

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mappings in locally convex space. In this way, all the above mentioned results are summarized and extended by Nashine [9].

The purpose of this paper is to extend the result of Nashine [9] by employing compatible mappings in lieu of commuting mappings, and by using four mappings as opposed to three. For this purpose, we use the result of Jungck [5]. Finally, we derive some consequences from our main result.

2. PRELIMINARIES

Before we prove our main result, let us recall following definitions:

Definition 1. [7]. *In the sequel $(\mathcal{E},\tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in \mathcal{P}\}$ of seminorms defined on $\mathcal{E}$ is said to be an associated family of seminorms for τ if the family $\{\gamma\mathcal{U} : \gamma > 0\}$, where $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_{\alpha_i}$ and $\mathcal{U}_{\alpha_i} = \{x : p_{\alpha_i}(x) < 1\}$, forms a base of neighborhood of zero for τ. A family $\{p_\alpha : \alpha \in \mathcal{P}\}$ of seminorms defined on $\mathcal{E}$ is called an augmented associated family for τ if $\{p_\alpha : \alpha \in \mathcal{P}\}$ is an associated family with the property that the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in \mathcal{P}\}$ for any $\alpha, \beta \in \mathcal{P}$. The associated and augmented families of seminorms will be denoted by $\mathcal{A}(\tau)$ and $\mathcal{A}^*(\tau)$, respectively. It is well known that if given a locally convex space $(\mathcal{E},\tau)$, there always exists a family $\{p_\alpha : \alpha \in \mathcal{P}\}$ of seminorms defined of $\mathcal{E}$ such that $\{p_\alpha : \alpha \in \mathcal{P}\} = \mathcal{A}^*(\tau)$. A subset $\mathcal{M}$ of $\mathcal{E}$ is τ-bounded if and only if each p_α is bounded on $\mathcal{M}$.*

The following construction will be crucial. Suppose that $\mathcal{M}$ is a τ-bounded subset of $\mathcal{E}$. For this set $\mathcal{M}$, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in \mathcal{I}$ such that $\mathcal{M} \subset \lambda_\alpha \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \{x : p_\alpha(x) \leq 1\}$. Clearly, $\mathbf{B} = \bigcap_\alpha \lambda_\alpha \mathcal{U}_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains $\mathcal{M}$. The linear span $\mathcal{E}_{\mathbf{B}}$ of $\mathbf{B}$ in $\mathcal{E}$ is $\bigcup_{n=1}^{\infty} n\mathbf{B}$. The Minkowski functional of $\mathbf{B}$ is a norm $\|.\|_{\mathcal{B}}$ on $\mathcal{E}_{\mathbf{B}}$. Thus, $\mathcal{E}_{\mathbf{B}}$ is a normed space with $\mathbf{B}$ as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_{\mathbf{B}}$ for each $x \in \mathcal{E}_{\mathbf{B}}$.

Definition 2. *Let $\mathcal{I}$ and $\mathcal{T}$ be selfmaps on $\mathcal{M}$. The map $\mathcal{T}$ is called*

(i) *$\mathcal{A}^*(\tau)$-nonexpansive if for all $x, y \in \mathcal{M}$*

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq p_\alpha(x - y),$$

for each $p_\alpha \in A^*(\tau)$.

(ii) *$A^*(\tau)$-$\mathcal{I}$-nonexpansive if for all $x, y \in \mathcal{M}$*

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq p_\alpha(\mathcal{I}x - \mathcal{I}y),$$

for each $p_\alpha \in \mathcal{A}^*(\tau)$.

For simplicity, we shall call $\mathcal{A}^*(\tau)$-nonexpansive ($\mathcal{A}^*(\tau)$-$\mathcal{I}$-nonexpansive) maps to be nonexpansive ($\mathcal{I}$-nonexpansive).

Following the concept of compatible due to Jungck [4], we have

DEFINITION 3. [4]. *A pair of self-mappings* $(\mathcal{T},\mathcal{I})$ *of a locally convex space* $(\mathcal{E},\tau)$ *is said to be compatible if* $p_\alpha(\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n) \to 0$, *whenever* $\{x_n\}$ *is a sequence in* $\mathcal{E}$ *such that* $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{E}$.

Every commuting pair of mappings is compatible but the converse is not true in general.

DEFINITION 4. [9]. *Let* $x_0 \in \mathcal{M}$. *We denote by* $\mathcal{P}_\mathcal{M}(x_0)$ *the set of best* $\mathcal{M}$*-approximant to* x_0, *i.e., if* $\mathcal{P}_\mathcal{M}(x_0) = \{y \in \mathcal{M} : p_\alpha(y - x_0) = d_{p_\alpha}(x_0, \mathcal{M})$ *for all* $p_\alpha \in \mathcal{A}^*(\tau)\}$, *where*

$$d_{p_\alpha}(x_0, \mathcal{M}) = \inf\{p_\alpha(x_0 - z) : z \in \mathcal{M}\}.$$

DEFINITION 5. [9]. *The map* $\mathcal{T} : \mathcal{M} \to \mathcal{E}$ *is said to be demiclosed at* 0 *if, for every net* $\{x_n\}$ *in* $\mathcal{M}$ *converging weakly to* x *and* $\{\mathcal{T}x_n\}$ *converging strongly to* 0, *we have* $\mathcal{T}x = 0$.

Throughout this paper $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$) denotes the set of fixed point of mapping $\mathcal{T}$ (resp. $\mathcal{I}$).

The following result of Jungck [5] is needed in the sequel:

THEOREM 6. [5]. *Let* $\mathcal{A}, \mathcal{B}, \mathcal{S}$ *and* $\mathcal{T}$ *be self mappings of a complete metric space* $(\mathcal{X}, d)$. *Suppose that* $\mathcal{S}$ *and* $\mathcal{T}$ *are continuous, the pairs* $(\mathcal{A}, \mathcal{S})$ *and* $(\mathcal{B}, \mathcal{T})$ *are compatible, and that* $\mathcal{A}(\mathcal{X}) \subset \mathcal{T}(\mathcal{X})$ *and* $\mathcal{B}(\mathcal{X}) \subset \mathcal{S}(\mathcal{X})$. *If there exists* $k \in (0, 1)$ *such that*

$$d(\mathcal{A}x,\mathcal{B}y) \le k \max\{d(\mathcal{A}x,\mathcal{S}x), d(\mathcal{B}y,\mathcal{T}y), d(\mathcal{S}x,\mathcal{T}y), \tfrac{1}{2}[d(\mathcal{A}x, \mathcal{T}y) + d(\mathcal{B}y, \mathcal{S}x)]\},$$

then there is a unique point z *in* $\mathcal{X}$ *such that* $z = \mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z$.

3. MAIN RESULT

LEMMA 7. *Let* $\mathcal{A}$ *and* $\mathcal{S}$ *be compatible self-maps of a* τ*-bounded subset* $\mathcal{M}$ *of a Hausdorff locally convex space* $(\mathcal{E}, \tau)$. *Then* $\mathcal{A}$ *and* $\mathcal{S}$ *be two compatible on* $\mathcal{M}$ *with respect to* $\|.\|_\mathbf{B}$.

Proof. By hypothesis for each $p_\alpha \in \mathcal{A}^*(\tau)$,

$$p_\alpha(\mathcal{A}\mathcal{S}x_n - \mathcal{S}\mathcal{A}x_n) \to 0, \tag{1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that $\mathcal{A}x_n, \mathcal{S}x_n \to t \in \mathcal{M}$.

Taking supremum on both sides, we get

$$\sup_\alpha p_\alpha\left(\tfrac{\mathcal{A}\mathcal{S}x_n - \mathcal{S}\mathcal{A}x_n}{\lambda_\alpha}\right) \to 0,$$

$$\|\mathcal{A}\mathcal{S}x_n - \mathcal{S}\mathcal{A}x_n\|_\mathbf{B} \to 0,$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that $\mathcal{A}x_n, \mathcal{S}x_n \to t \in \mathcal{M}$. □

We use a technique of Tarafdar [15] to obtain the following common fixed point theorem which generalize Theorem 6.

THEOREM 8. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete subset of a Hausdorff locally convex space $(\mathcal{E},\tau)$. Let $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$. If $(\mathcal{A},\mathcal{S})$ and $(\mathcal{B},\mathcal{T})$ are compatible pairs, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive and satisfying*

$$p_\alpha(\mathcal{A}x - \mathcal{B}y) \leq \mathbf{L}(x,y), \tag{2}$$

where

$$\mathbf{L}(x,y) = h\max\{p_\alpha(\mathcal{A}x - \mathcal{S}x), p_\alpha(\mathcal{B}y - \mathcal{T}y), p_\alpha(\mathcal{S}x - \mathcal{T}y),$$
$$\tfrac{1}{2}[p_\alpha(\mathcal{A}x - \mathcal{T}y) + p_\alpha(\mathcal{B}y - \mathcal{S}x)]\}$$

for all $x, y \in \mathcal{M}$, and $p_\alpha \in \mathcal{A}^*(\tau)$, where $h \in (0,1)$, then $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point in $\mathcal{M}$.

Proof. Since the norm topology on $\mathcal{E}_\mathbf{B}$ has a base of neighborhood of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially complete, therefore, $\mathcal{M}$ is a $\|.\|_\mathbf{B}$-sequentially complete subset of $(\mathcal{E}_\mathbf{B}, \|.\|_\mathbf{B})$ [15, Theorem 1.2]. By Lemma 7, $\mathcal{A}$ and $\mathcal{S}$ are $\|.\|_\mathbf{B}$-compatible maps of $\mathcal{M}$. Similarly, by the Lemma 7, $\mathcal{B}$ and $\mathcal{T}$ are $\|.\|_\mathbf{B}$-compatible maps of $\mathcal{M}$. From (2) we obtain for $x, y \in \mathcal{M}$,

$$\sup_\alpha p_\alpha(\tfrac{\mathcal{A}x-\mathcal{B}y}{\lambda_\alpha}) \leq h\max\{\sup_\alpha p_\alpha(\tfrac{\mathcal{A}x-\mathcal{S}x}{\lambda_\alpha}), \sup_\alpha p_\alpha(\tfrac{\mathcal{B}y-\mathcal{T}y}{\lambda_\alpha}), \sup_\alpha p_\alpha(\tfrac{\mathcal{S}x-\mathcal{T}y}{\lambda_\alpha}),$$
$$\tfrac{1}{2}[\sup_\alpha p_\alpha(\tfrac{\mathcal{A}x-\mathcal{T}y}{\lambda_\alpha}) + \sup_\alpha p_\alpha(\tfrac{\mathcal{B}y-\mathcal{S}x}{\lambda_\alpha})]\}.$$

Thus

$$\|\mathcal{A}x - \mathcal{B}y\|_\mathbf{B} \leq h\max\{\|\mathcal{A}x - \mathcal{S}x\|_\mathbf{B}, \|\mathcal{B}y - \mathcal{T}y\|_\mathbf{B}, \|\mathcal{S}x - \mathcal{T}y\|_\mathbf{B},$$
$$\tfrac{1}{2}[\ \|\mathcal{A}x - \mathcal{T}\mathcal{S}y\|_\mathbf{B} + \|\mathcal{B}y - \mathcal{S}x\|_\mathbf{B}]\}. \tag{3}$$

Note that, if $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive on a τ-bounded, τ-sequentially compact subset $\mathcal{M}$ of $\mathcal{E}$, then $\mathcal{S}$ and $\mathcal{T}$ are also nonexpansive with respect to $\|.\|_\mathbf{B}$ and hence $\|.\|_\mathbf{B}$-continuous ([7]). A comparison of our hypothesis with that of Theorem 6 tells that we can apply Theorem 6 to $\mathcal{M}$ as a subset of $(\mathcal{E}_\mathbf{B}, \|.\|_\mathbf{B})$ to conclude that there exists a unique $z \in \mathcal{M}$ such that $z = \mathcal{A}z = \mathcal{B}z = \mathcal{S}z = \mathcal{T}z$. □

THEOREM 9. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and q-starshaped subset of a Hausdorff locally convex space $(\mathcal{E},\tau)$. Let $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M}) \subset \mathcal{S}(\mathcal{M})$. Suppose $(\mathcal{A},\mathcal{S})$ and $(\mathcal{B},\mathcal{T})$ are compatible pairs, $\mathcal{A}$ and $\mathcal{B}$ are continuous, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive and affine, $\mathcal{S}(\mathcal{M}) = \mathcal{M} = \mathcal{T}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. If $\mathcal{A},\mathcal{B},\mathcal{S}$ and $\mathcal{T}$ satisfy the following:*

$$p_\alpha(\mathcal{A}x - \mathcal{B}y) \leq \mathbf{L}(x,y), \tag{4}$$

where

$$\mathbf{L}(x,y) = h\max\{p_\alpha(\mathcal{A}x - \mathcal{S}x), p_\alpha(\mathcal{B}y - \mathcal{T}y), p_\alpha(\mathcal{S}x - \mathcal{T}y),$$
$$\tfrac{1}{2}[p_\alpha(\mathcal{A}x - \mathcal{T}y) + p_\alpha(\mathcal{B}y - \mathcal{S}x)]\}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, where $h \in (0,1)$, then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{M}$ provided one of the following conditions hold:

(i) *$\mathcal{M}$ is τ-sequentially compact;*
(ii) *$\mathcal{A}, \mathcal{B}$ are compact maps;*
(iii) *$\mathcal{M}$ is weakly compact in $(\mathcal{E}, \tau)$, $\mathcal{S}$ and $\mathcal{T}$ are weakly continuous and $\mathcal{S} - \mathcal{A}$ and $\mathcal{T} - \mathcal{B}$ are demiclosed at 0.*

Proof. Choose a monotonically nondecreasing sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. For each $n \in \mathbb{N}$, define $\mathcal{A}_n, \mathcal{B}_n : \mathcal{M} \to \mathcal{M}$ as follows:

$$\mathcal{A}_n x = k_n \mathcal{A} x + (1 - k_n) p, \quad \mathcal{B}_n x = k_n \mathcal{B} x + (1 - k_n) p. \tag{5}$$

Obviously, for each n, $\mathcal{A}_n$ and $\mathcal{B}_n$ map $\mathcal{M}$ into itself since $\mathcal{M}$ is q-starshaped. As $\mathcal{S}$ is affine, $(\mathcal{A}, \mathcal{S})$ are compatible and $p \in \mathcal{F}(\mathcal{S})$, so

$$\mathcal{A}_n \mathcal{S} x = k_n \mathcal{A} \mathcal{S} x + (1 - k_n) p,$$

$$\mathcal{S} \mathcal{A}_n x_n = \mathcal{S}(k_n \mathcal{A} x + (1 - k_n) p) = k_n \mathcal{S} \mathcal{A} x + (1 - k_n) \mathcal{S} p.$$

Since $(\mathcal{A}, \mathcal{S})$ are compatible, we have

$$\begin{aligned} 0 &\le \lim_n p_\alpha(\mathcal{A}_m \mathcal{S} x_n - \mathcal{S} \mathcal{A}_m x_n) \\ &\le \lim_n p_\alpha(\mathcal{A} \mathcal{S} x_n - \mathcal{S} \mathcal{A} x_n) + \lim_n (1 - k_m) p_\alpha(p - \mathcal{S} p) \\ &= 0, \end{aligned}$$

whenever $\lim_n \mathcal{S} x_n = \lim_n \mathcal{A} x_n = t \in \mathcal{M}$ for all n and for each $x \in \mathcal{M}$. Hence $\{\mathcal{A}_n\}$ and $\mathcal{S}$ are compatible on $\mathcal{M}$ for each n and $\mathcal{A}_n(\mathcal{M}) \subseteq \mathcal{M} = \mathcal{T}(\mathcal{M})$. Similarly, we can prove $\mathcal{B}_n$ and $\mathcal{T}$ are compatible for each n and $\mathcal{B}_n(\mathcal{M}) \subseteq \mathcal{M} = \mathcal{S}(\mathcal{M})$.

For all $x, y \in \mathcal{M}, p_\alpha \in \mathcal{A}^*(\tau)$ and for all $j \ge n$ (n fixed), we obtain from (4) and (5) that

$$\begin{aligned} p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y) &= k_n \;\; p_\alpha(\mathcal{A} x - \mathcal{S} y) \le k_j \;\; p_\alpha(\mathcal{A} x - \mathcal{B} y) \\ &\le p_\alpha(\mathcal{A} x - \mathcal{B} y) \\ &\le h \max\{p_\alpha(\mathcal{A} x - \mathcal{S} x), p_\alpha(\mathcal{B} y - \mathcal{T} y), p_\alpha(\mathcal{S} x - \mathcal{T} y), \\ &\qquad \tfrac{1}{2}[p_\alpha(\mathcal{A} x - \mathcal{T} y) + p_\alpha(\mathcal{B} y - \mathcal{S} x)]\} \\ &\le h \max\{p_\alpha(\mathcal{A} x - \mathcal{A}_n x) + p_\alpha(\mathcal{A}_n x - \mathcal{S} x), \\ &\qquad p_\alpha(\mathcal{B} y - \mathcal{B}_n y) + p_\alpha(\mathcal{B}_n y - \mathcal{T} y), p_\alpha(\mathcal{S} x - \mathcal{T} y), \\ &\qquad \tfrac{1}{2}[p_\alpha(\mathcal{A} x - \mathcal{A}_n x) + p_\alpha(\mathcal{A}_n x - \mathcal{T} y) + \\ &\qquad p_\alpha(\mathcal{B} y - \mathcal{B}_n y) + p_\alpha(\mathcal{B}_n y - \mathcal{S} x)]\}, \end{aligned}$$

$$\begin{aligned} p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y) \le h \max\{&(1 - k_n) p_\alpha(\mathcal{A} x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{S} x), \\ &(1 - k_n) p_\alpha(\mathcal{B} y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{T} y), p_\alpha(\mathcal{S} x - \mathcal{T} y), \\ &\tfrac{1}{2}[(1 - k_n) p_\alpha(\mathcal{A} x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{T} y) + \\ &(1 - k_n) p_\alpha(\mathcal{B} y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{S} x)]\}. \end{aligned}$$

Hence for all $j \geq n$, we have

$$
\begin{aligned}
(6)\qquad p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y) \leq h \max\{&(1-k_j)p_\alpha(\mathcal{A}x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{S}x),\\
&(1-k_j)p_\alpha(\mathcal{B}y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{T}y), p_\alpha(\mathcal{S}x - \mathcal{T}y),\\
&\tfrac{1}{2}[(1-k_j)p_\alpha(\mathcal{A}x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{T}y)+\\
&(1-k_j)p_\alpha(\mathcal{B}y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{S}x)]\}.
\end{aligned}
$$

As $\lim k_j = 1$, from (6), for every $n \in \mathbb{N}$, we have

$$
\begin{aligned}
(7)\qquad p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y) &= \lim\nolimits_j p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y)\\
&\leq h \lim\nolimits_j\{\max\{(1-k_j)p_\alpha(\mathcal{A}x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{S}x),\\
&\quad (1-k_j)p_\alpha(\mathcal{B}y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{T}y), p_\alpha(\mathcal{S}x - \mathcal{T}y),\\
&\quad \tfrac{1}{2}[(1-k_j)p_\alpha(\mathcal{A}x - p) + p_\alpha(\mathcal{A}_n x - \mathcal{T}y)+\\
&\quad (1-k_j)p_\alpha(\mathcal{B}y - p) + p_\alpha(\mathcal{B}_n y - \mathcal{S}x)]\}.
\end{aligned}
$$

This implies that, for every $n \in \mathbb{N}$,

$$
\begin{aligned}
(8)\qquad p_\alpha(\mathcal{A}_n x - \mathcal{B}_n y) \leq h \max\{&p_\alpha(\mathcal{A}_n x - \mathcal{S}y), p_\alpha(\mathcal{B}_n x - \mathcal{T}x), p_\alpha(\mathcal{S}x - \mathcal{T}y),\\
&\tfrac{1}{2}[p_\alpha(\mathcal{A}_n x - \mathcal{T}y) + p_\alpha(\mathcal{B}_n y - \mathcal{S}x)]\},
\end{aligned}
$$

for all $x, y \in \mathcal{M}$ and for all $p_\alpha \in \mathcal{A}^*(\tau)$.

Moreover, $\mathcal{S}$ and $\mathcal{T}$ being nonexpansive on $\mathcal{M}$, implies that $\mathcal{S}$ and $\mathcal{T}$ are $\|.\|_{\mathbf{B}}$-nonexpansive and, hence, $\|.\|_{\mathbf{B}}$-continuous. Since the norm topology on $\mathcal{E}_{\mathbf{B}}$ has a base of neighborhood of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially complete, then $\mathcal{M}$ is a $\|.\|_{\mathbf{B}}$-sequentially complete subset of $(\mathcal{E}_{\mathbf{B}}, \|.\|_{\mathbf{B}})$ (see proof of Theorem 1.2 in [15]). Thus, from Theorem 8, for every $n \in \mathbb{N}$, $\mathcal{A}_n, \mathcal{B}_n$, $\mathcal{S}$ and $\mathcal{T}$ have unique common fixed point x_n in $\mathcal{M}$, i.e.,

$$
(9)\qquad x_n = \mathcal{A}_n x_n = \mathcal{B}_n x_n = \mathcal{S}x_n = \mathcal{T}x_n,
$$

for each $n \in \mathbb{N}$.

(i) As $\mathcal{M}$ is τ-sequentially compact and $\{x_n\}$ is a sequence in $\mathcal{M}$, then $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in \mathcal{M}$. As $\mathcal{A}, \mathcal{B}$ and $\mathcal{S}, \mathcal{T}$ are continuous and

$$
x_m = \mathcal{S}x_m = \mathcal{A}_m x_m = k_m \mathcal{A}x_m + (1-k_m)p,
$$

$$
x_m = \mathcal{T}x_n = \mathcal{B}_m x_m = k_m \mathcal{B}x_m + (1-k_m)p,
$$

then it follows that $y = \mathcal{T}y = \mathcal{S}y = \mathcal{A}y = \mathcal{B}y$.

(ii) As $\mathcal{A}$ is compact and $\{x_n\}$ is bounded, then $\{\mathcal{A}x_n\}$ has a subsequence $\{\mathcal{A}x_m\}$ such that $\{\mathcal{A}x_m\} \to z \in \mathcal{M}$. Now we have

$$
x_m = \mathcal{A}_m x_m = k_m \mathcal{A}x_m + (1-k_m)p
$$

Proceeding to the limit as $m \to \infty$ and using the continuity of $\mathcal{S}$ and $\mathcal{A}$, we have $\mathcal{S}z = z = \mathcal{A}z$. Similarly, we can show $\mathcal{B}z = z = \mathcal{T}z$.

(iii) The sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converges to $u \in \mathcal{M}$. Since $\mathcal{S}$ is weakly continuous and so as in (i), we have $\mathcal{S}u = u$. Now,

$$
x_m = \mathcal{S}x_m = \mathcal{A}_m x_m = k_m \mathcal{A}x_m + (1-k_m)p
$$

implies that

$$\mathcal{S}x_m - \mathcal{A}x_m = (1 - k_m)[p - \mathcal{A}x_m] \to 0$$

as $m \to \infty$. The demiclosedness of $\mathcal{S}$–$\mathcal{A}$ at 0 implies that $(\mathcal{S} - \mathcal{A})u = 0$. Hence $\mathcal{S}u = u = \mathcal{A}u$. Similarly, we can show $\mathcal{T}u = u = \mathcal{B}u$, when $\mathcal{T} - \mathcal{B}$ is demiclosed at 0. This completes the proof.

□

An immediate consequence of the Theorem 9 is as follows:

Corollary 10. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and q-starshaped subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be self mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M}) \subset \mathcal{S}(\mathcal{M})$. Suppose $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible pairs, $\mathcal{A}$ and $\mathcal{B}$ are continuous, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive, and affine, $\mathcal{S}(\mathcal{M}) = \mathcal{M} = \mathcal{T}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. If $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ satisfy the following:*

$$p_\alpha(\mathcal{A}x - \mathcal{B}y) \le \mathbf{L}(x, y), \tag{10}$$

where

$$\mathbf{L}(x, y) = h \max\{p_\alpha(\mathcal{A}x - \mathcal{S}x), p_\alpha(\mathcal{B}y - \mathcal{T}y), p_\alpha(\mathcal{S}x - \mathcal{T}y),$$
$$\tfrac{1}{2}p_\alpha(\mathcal{A}x - \mathcal{T}y), \tfrac{1}{2}p_\alpha(\mathcal{B}y - \mathcal{S}x)\}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, where $h \in (0, 1)$, then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{M}$ under each of the conditions (i)–(iii) of Theorem 9.

Next, in the Theorem 9 and Corollary 10, if $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ are commuting mappings, then we get the following result:

Corollary 11. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and q-starshaped subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be self commuting mappings of $\mathcal{M}$ with $\mathcal{A}(\mathcal{M}) \subset \mathcal{T}(\mathcal{M})$ and $\mathcal{B}(\mathcal{M}) \subset \mathcal{S}(\mathcal{M})$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are continuous, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive, and affine, $\mathcal{S}(\mathcal{M}) = \mathcal{M} = \mathcal{T}(\mathcal{M})$, $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. If $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ satisfy* (4) *or* (10) *for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, where $h \in (0, 1)$, then $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{M}$ under each of the conditions* (i)–(iii) of Theorem 9.

As application of Theorem 9, we prove the following more general result in invariant approximation theory:

Theorem 12. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be self-mappings of a Hausdorff locally convex space $(\mathcal{E}, \tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A}, \mathcal{B}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are continuous, $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible pairs, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A}, \mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ satisfy* (4) *for each $x, y \in \mathcal{D}, p_\alpha \in A^*(\tau)$ where $h \in (0, 1)$. If $\mathcal{D}$ is nonempty q-starshaped with $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$ and $\mathcal{S}(\mathcal{D}) = \mathcal{D} = \mathcal{T}(\mathcal{D})$, then*

$\mathcal{A}, \mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{D}$ provided one of the following conditions hold:

(i) *$\mathcal{D}$ is τ-sequentially compact;*
(ii) *$\mathcal{A}, \mathcal{B}$ are compact mappings;*
(iii) *$\mathcal{D}$ is weakly compact in $(\mathcal{E}, \tau)$, $\mathcal{S}$ and $\mathcal{T}$ are weakly continuous and $\mathcal{S} - \mathcal{A}$ and $\mathcal{T} - \mathcal{B}$ are demiclosed at* 0.

Proof. First, we show that $\mathcal{A}$ and $\mathcal{B}$ are self map on $\mathcal{D}$, i.e., $\mathcal{A}, \mathcal{B} : \mathcal{D} \mapsto \mathcal{D}$. Let $y \in \mathcal{D}$, then $\mathcal{S}y$ and $\mathcal{T}y \in \mathcal{D}$, since $\mathcal{S}(\mathcal{D}) = \mathcal{D} = \mathcal{T}(\mathcal{D})$. Also, if $y \in \partial\mathcal{M}$, then $\mathcal{A}y \in \mathcal{M}$, since $\mathcal{A}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{A}x_0 = \mathcal{B}x_0 = x_0 = \mathcal{S}x_0 = \mathcal{T}x_0$, so for each $p_\alpha \in \mathcal{A}^*(\tau)$, we have from (4)

$$p_\alpha(\mathcal{A}y - x_0) = p_\alpha(\mathcal{A}y - \mathcal{B}x_0) \leq \mathbf{L}(y, x_0).$$

This imply that $\mathcal{A}y$ is also closest to x_0, so $\mathcal{A}y \in \mathcal{D}$. Similarly $\mathcal{B}y \in \mathcal{D}$. Consequently $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ are selfmaps on $\mathcal{D}$. The conditions of Theorem 9 (i)–(iii) are satisfied and, hence, there exists a $w \in \mathcal{D}$ such that $\mathcal{A}w = \mathcal{B}w = \mathcal{S}w = w = \mathcal{T}w$. This completes the proof. □

An immediate consequence of the Theorem 12 is as follows:

Corollary 13. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be self-mappings of a Hausdorff locally convex space $(\mathcal{E}, \tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A}, \mathcal{B}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are continuous, $(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are compatible pairs, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A}, \mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ satisfy* (4) for each $x, y \in \mathcal{D}, p_\alpha \in A^*(\tau)$ where $h \in (0, 1)$. If $\mathcal{D}$ is nonempty q-starshaped with $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$ and $\mathcal{S}(\mathcal{D}) = \mathcal{D} = \mathcal{T}(\mathcal{D})$, then $\mathcal{A}, \mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{D}$ under each of the conditions (i)–(iii) of Theorem 12.

Next, in the Theorem 12 and Corollary 13, if $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ are commuting mappings, then we get the following result:

Corollary 14. *Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$ be self commuting mappings of a Hausdorff locally convex space $(\mathcal{E}, \tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{A}, \mathcal{B}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{A}) \cap \mathcal{F}(\mathcal{B}) \cap \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are continuous, $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive and affine on $\mathcal{D} = \mathcal{P}_{\mathcal{M}}(x_0)$. Further, suppose $\mathcal{A}, \mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ satisfy* (4) *or* (10) *for each $x, y \in \mathcal{D}, p_\alpha \in \mathcal{A}^*(\tau)$ where $h \in (0, 1)$. If $\mathcal{D}$ is nonempty q-starshaped with $p \in \mathcal{F}(\mathcal{S}) \cap \mathcal{F}(\mathcal{T})$ and $\mathcal{S}(\mathcal{D}) = \mathcal{D} = \mathcal{T}(\mathcal{D})$, then $\mathcal{A}$, $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{T}$ have a common fixed point in $\mathcal{D}$ under each of the conditions* (i)–(iii) of Theorem 12.

Remark 15. In the light of the comment given by Jungck [4], that every commuting pair of mappings is compatible but the converse is not true in general, and by using four mappings as opposed to three, our results generalize the results of Nashine [9] by weakening commutativity hypothesis and by increasing the number of mappings involved and consequently other related results, also [1, 2, 3, 6, 8, 10, 11, 12, 13, 14]. □

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