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NUMERICAL EXPERIMENTS WITH TWO SMOOTHING TYPE-ALGORITHMS FOR NONLINEAR COMPLEMENTARITY PROBLEMS

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Abstract. In this paper we deal with the numerical experiments of two smoothing descent-type algorithms for solving nonlinear complementarity problems (NCP). The first algorithm is due to Kanzow and the second one is due to Peng. These algorithms are both based on the reformulation of (NCP) as unconstrained minimization problems by using some smoothing merit functions including the so-called (NCP)-functions. Under suitable conditions they both showed that any stationary point of these problems are solutions of (NCP). For their numerical performances many strategies are used. Finally, these algorithms are applied to some problems of (NCP) found in the literature.

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1. INTRODUCTION

Consider the nonlinear complementarity problem (abbreviated NCP): find a vector $x \in \mathbb{R}^n$ such that:

$$x \geq 0,\ F(x) \geq 0, \text{ and } \quad x^{\mathrm{T}} F(x) = 0 \tag{1}$$

where F is a continuously differentiable vector valued mapping (function) from $\mathbb{R}^n$ into itself, the expression $x^{\mathrm{T}} F(x)$ denotes the usual inner product of the two vectors x and $F(x)$ in $\mathbb{R}^n$, and the inequalities are understood to be componentwise. The complementarity condition $x^{\mathrm{T}} F(x) = 0$ means that $x_i F_i(x) = 0$ for all $i = 1, 2, \ldots, n$.

This problem appears in many areas of application such as physics, economics, transport problems, game theory and mathematical programming. Also this problem can be seen as a special case of the well-known variational inequalities problem (VIP).

For solving the NCPs, there are different approaches in the literature notably the popular class of interior point methods and the class of smoothing

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and non-smoothing type methods or other class which is a hybrid between these two first classes. In this paper we are interested in the class of smoothing type methods and which is divided into two strategies. The first strategy is based on the reformulation of the (NCP) as an equivalent minimization problem with and without constrained, however the second one transforms the (NCP) into an equivalent nonlinear system of equations. Both reformulations are done by using some appropriate merit functions including the so-called (NCP)-functions. In general merit functions and (NCP)-functions are introduced and exploited by several authors (See Refs. [3, 7, 9, 10]). If these functions are differentiable everywhere then the corresponding methods are called smoothing methods, otherwise they called non-smoothing or generalized Jacobian's methods. Here, we reconsider two important smoothing descent-type algorithms to solve the (NCP). The first algorithm is due to Kanzow (Ref. [7]) and it is based on some smoothing (NCP)-functions for constructing equivalent unconstrained minimization problems and the second one is due to Peng (Ref. [10]) and it is constructed by the exploitation of some merit functions inspired from a general context for solving (VIP) problems. For their implementation and their numerical performances many strategies are used.

The rest of the paper is organized as follows. In the next section, we regroup in brief some definitions and basic results required in our study. In section 3, we give the connection of this problem with other mathematical problems followed by some comments on the used algorithms. In section 4, we give a short summary for illustrating the basic ideas behind the reformulation of (NCP) as unconstrained minimization problems. In section 5, the review of Kanzow's and Peng's algorithms are presented. In section 6, the implementation of these two algorithms and the numerical results on some problems of (NCP) are presented. In section 7, a conclusion and general comments are stated.

1.1. Notation. The notation used in this paper is: $\mathbb{R}^n$ denotes the space of real n-dimensional vectors and $\mathbb{R}^n_+$ is the orthant positive (cone) of $\mathbb{R}^n$. Given $x, y \in \mathbb{R}^n$, $x^{\mathrm{T}}y = \sum_{i=1}^{n} x_i y_i$ is their inner product, whereas $\|x\| = \sqrt{x^{\mathrm{T}}x}$ is the Euclidean norm of the vector x. Given a vector v in $\mathbb{R}^n$, $G = \mathrm{diag}(v)$ is the $n \times n$ diagonal matrix with $G_{ii} = v_i$ for all i. For a vector valued mapping F from $\mathbb{R}^n$ into itself, $F(x) = (F_1(x), F_2(x), \ldots, F_n(x))^{\mathrm{T}}$ and $F'(x)$ is the Jacobian matrix of F.

2. DEFINITIONS AND BASIC RESULTS

Definition 1. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a multivalued vector mapping.*

- *F is monotone on $\mathbb{R}^n$ if for all $x, y \in \mathbb{R}^n$,*
$$(x - y)^{\mathrm{T}}(F(x) - F(y)) \geq 0.$$
- *F is strictly monotone if for all $x, y \in \mathbb{R}^n$, $x \neq y$,*
$$(x - y)^{\mathrm{T}}(F(x) - F(y)) > 0.$$

- *F is strongly monotone if there exists a positive constant $c > 0$ such that for all $x, y \in \mathbb{R}^n, x \neq y$,*

$$(x-y)^{\mathrm{T}}(F(x)-F(y)) > c\,\|x-y\|^2 .$$

REMARK 1. It is clear that: F strongly monotone $\Rightarrow$ F strictly monotone $\Rightarrow$ F monotone. □

DEFINITION 2. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be any function having the following property:*

$$\varphi(a,b) = 0 \Leftrightarrow a \geq 0,\ b \geq 0,\ ab = 0.$$

Then φ is called a complementarity function. (Abbreviated (NCP)*-function).*

EXAMPLE 1. The following functions are (NCP)-function.

1.
$$\varphi_1(a,b) = \tfrac{1}{2}\left\{\min(a,b)\right\}^2 .$$

2. Mangasarian and Solodov:

$$\varphi_2(a,b) = ab + \tfrac{1}{2\alpha}(\max\left(0, a-\alpha b\right))^2 - a^2 + (\max(0, b-\alpha a))^2 - b^2),\ \alpha > 1.$$

3. Fisher's function:
$$\varphi_3(a,b) = \sqrt{a^2+b^2} - (a+b).$$

4.
$$\varphi_4(a,b) = \tfrac{1}{2}(ab)^2 + (\min(0,a)^2 + (\min(0,b))^2.$$

5.
$$\varphi_5(a,b) = \tfrac{1}{2}\left(\sqrt{a^2+b^2} - a - b\right)^2 .$$

□

There exist other examples of (NCP)-function. The following few examples are cited in Refs. [7, 9] .

Next, we give few results concerning the existence and uniqueness of the solution of (NCP).

2.1. Existence and uniqueness of solutions of (NCP).

THEOREM 1. [8] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and strictly copositive function, i.e., there exists a function $S : \mathbb{R}_+ \to \mathbb{R}$ such that: $S(\lambda) \mapsto +\infty$ if $\lambda \mapsto +\infty$, and for all $\lambda \geq 1, x \geq 0$:*

$$(F(\lambda x) - F(0))^{\mathrm{T}} x \geq S(\lambda)(F(x) - F(0))^{\mathrm{T}} x.$$

Then the set of solutions of (1) *is not empty and compact.*

THEOREM 2. [8] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and strongly monotone function. Then* (1) *has a unique solution.*

THEOREM 3. [4] *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and monotone function. If the set of strictly feasible points*

$$X = \{x \in \mathbb{R}^n \ : \ x > 0, F(x) > 0\}$$

is not empty, then (1) *has a non empty compact solution set.*

3. RESOLUTION OF THE (NCP) AND SOME COMMENTS ON THE USED ALGORITHMS

The design of (iterative) algorithms for solving the problem (1), are systematically elaborated from attributed characterizations of this problem through its different reformulations. For that we can envisage a collective of algorithms, where each reformulation of (NCP) corresponds to a particular method. Here, we deal only with smoothing and non-smoothing algorithms. For interior point algorithms the reader consults other references concerning this subject.

Next, we give the connection of this problem with other mathematical problems followed by some comments on the used algorithms.

3.1. The linear complementarity problem (LCP). If the mapping is affine, i.e., $F(x) = Mx + q$ for some given $n \times n$ matrix M and $q \in \mathbb{R}^n$, then (1) is called the linear complementarity problem. The (LCP) is an important problem in mathematical programming which includes the linear and quadratic programming. We mention that there are other results of existence and uniqueness of the solution of (NCP), inspired from the theory of (LCP). (See Ref. [1]).

3.2. The (NCP) as a Variational inequalities problem (VIP). The variational inequalities problem (VIP) is defined as: Find $\bar{x} \in C$ such that

$$(x - \bar{x})^{\mathrm{T}} F(x) \geq 0, \text{ for all } x \in C,$$

where C is a non empty subset of $\mathbb{R}^n$ and F is a vector valued mapping from $\mathbb{R}^n$ into itself. If $C = \mathbb{R}^n_+$, then the (VIP) reduces to an (NCP) problem (See Refs. [2, 5]).

3.3. The (NCP) as a nonlinear system of equations. The general idea behind this reformulation is: given an (NCP)-function φ and setting $a = x_i$, and $b = F_i(x)$. Then we have:

$$\varphi(x_i, F_i(x)) = 0 \Leftrightarrow x_i \geq 0, \ F_i(x) \geq 0, \text{ and } \ x_i F_i(x) = 0.$$

Hence the problem (1) is equivalent to the following nonlinear system of equations:

$$\varphi(x_i, F_i(x)) = 0, \ i = 1, \ldots, n.$$

The above system can be written as a vectorial form:

$$\theta(x) = \begin{pmatrix} \varphi(x_1, F_1(x)) \\ \vdots \\ \varphi(x_n, F_n(x)) \end{pmatrix} = 0, \tag{2}$$

with $\theta : \mathbb{R}^n \to \mathbb{R}^n$ is a multivalued function and $x \in \mathbb{R}^n$. It is easy to see that the zeros of $\theta(x)$ are solutions of (1). Here 0 in (2) denotes the zero vector in $\mathbb{R}^n$.

3.4. Smoothing Newton's type algorithms for nonlinear system of equations. In the case when one of the (NCP)-functions φ is differentiable, then the corresponding nonlinear system (2) is also differentiable and we apply the classical principal of linearization (Newton's method) where we include a sequence of subproblems obtained by certain linearization of the problem (1). These algorithms have some nice theoretical properties such as the quadratic convergence (fast), but locally. However, in practice they have a great accuracy but with an expansive cost.

3.5. Generalized (or Non-smoothing) Newton's type algorithms. These methods are recent where the used (NCP)-function φ is not differentiable everywhere, hence a generalization of classical Newton's methods are applied, where they used the generalized Jacobian $\partial\theta(x)$ at points where φ is not differentiable. With these methods, we recuperate the global convergence which is superlinear and even quadratic. These methods do not have disadvantages cited for the first strategy, but it necessitates a supplementary effort for replacing the derivative. In addition, at each iteration, we need to solve subproblems which are linear complementarity problems and thus makes the task very hard.

3.6. The (NCP) as a constrained optimization problem. For the problem (1), we associate the following constrained minimization problem:

$$(\mathrm{OP})_C \quad m = \min_x \left[x^{\mathrm{T}} F(x)\right] \quad \text{subject to } \; x \geq 0, \; F(x) \geq 0.$$

Clearly, $\bar{x}$ is a solution of (1) if and only if

$$F(\bar{x}) = \min_x \left[x^{\mathrm{T}} F(x)\right] = m = 0.$$

The problem $(\mathrm{OP})_C$ is sometimes used in theory while in practice this problem does not often have the required properties for a convenient treatment, notably the convexity and differentiability. Thus other alternatives are proposed. Probably the simple way is to come down to an equivalent unconstrained minimization problem easy to apprehend in theory and in practice. This does not exclude the presence of some difficulties for any of alternatives, which gives to each approach its totally importance.

4. THE (NCP) AS AN UNCONSTRAINED OPTIMIZATION PROBLEM

In this section, we present a short summary about the main works that are developed in the literature in this context, in order to illustrate the image of the great and effected efforts given in this domain, with bringing out the principal difficulties under the form of some open questions that the researchers attempt to respond until to day. The work of Mangasarian and Solodov in (1993), [9], is one of the first work that have an encouraging respond, it consists a remarkable impulse for the later developments. The authors have used an implicit augmented Lagrangian $M(x, \alpha)$, which have important properties such as:

$$M(x, \alpha) \geq 0, \text{ for all } (x, \alpha) \in \mathbb{R}^n \times (1, +\infty)$$

and for $\alpha \in (1, +\infty)$,

$$M(x, \alpha) = 0 \Leftrightarrow x \text{ is a solution of (1).}$$

This nice property establishes a correspondence between the solution of (1), and the global minimum of the implicit augmented Lagrangian with out using any assumptions such that the monotonicity and differentiability of the function F. More precisely, we have the following result:

$$\bar{x} \text{ is a solution of (1)} \Leftrightarrow \bar{x} \text{ is a solution of } \min_x M(x, \alpha) : \text{s.t. } x \in \mathbb{R}^n .$$

In spite of the obtained preliminary results by the authors. These results are significant, but several judicious questions are arouse to respect to this approach and among them are [7]:

- Under what conditions, the local minimum of $M(x, \alpha)$ is global? Are the monotonicity and differentiability of F ?
- When a stationary point of $M(x, \alpha)$ is a solution of (1)?
- Under what conditions $M(x, \alpha)$ is convex or pseudoconvex in $\mathbb{R}^n$?

The works of Kanzow and Peng are motivated on some way by the question two. In fact, with adding other suitable supplementary conditions, they both succeed to repair some failure in Mangasarian's approach.

Next task, we review the Kanzow's and Peng's algorithms.

5. REVIEW OF KANZOW'S AND PENG'S ALGORITHMS

5.1. Kanzow's algorithm. In 1996, Kanzow gave an affirmative answer for the question 2, asked by Mangasarian. His approach aims to reformulate (1) as an equivalent unconstrained minimization problem by exploiting such (NCP)-functions, in the sense that both problems have the same solutions.

Let us consider again the nonlinear complementarity problem (1): Find $x \in \mathbb{R}^n$ such that

$$x \geq 0, \ F(x) \geq 0, \text{ and } x^{\mathrm{T}} F(x) = 0$$

and given an (NCP)-function φ. Let $F_i : \mathbb{R}^n \to \mathbb{R}$ be the i th component of F. Let us define the following functions: $\psi_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$\psi_i(x) = \varphi(x_i, F_i(x)),\ i = 1, \ldots, n, \tag{3}$$

and $\Psi : \mathbb{R}^n \to \mathbb{R}$ such that

$$\Psi(x) = \sum_{i=1}^{n} \psi_i(x). \tag{4}$$

The functions Ψ and $\psi_i,\ i = 1, \ldots, n$, depend on the selected (NCP) -function φ. Then, the associated minimization problem is defined as:

$$(\mathrm{OP})_K \quad \min_x \Psi(x) \text{ subject to } x \in \mathbb{R}^n.$$

Hence, solving (1) is equivalent to solve $(\mathrm{OP})_K$. This equivalence is justified by the following basic lemma.

Lemma 1. [7] *Assume that the problem* (1) *has at least one solution. Let* φ *be an (NCP)-function such that* $\varphi(a, b) \geq 0$, *for all* $(a, b)^{\mathrm{T}} \in \mathbb{R}^2$, *and let* $\psi_i,\ i = 1, \ldots, n$, *and* Ψ *be defined as in* (3) *and* (4), *respectively. Hence,* $\bar{x}$ *is a global minimum of* $\Psi \Leftrightarrow \bar{x}$ *is a solution of* (1), *with* $\Psi(\bar{x}) = 0$.

By this lemma, we can say that the problem of globalization is well-ordered, but there is an other question to ask: how to characterize the solution $\bar{x}$? For this purpose, Kanzow proves the following theorem.

Theorem 4. [7] *If* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is continuously differentiable such that the Jacobian* $F'(x)$ *is positive definite for all* $x \in \mathbb{R}^n$. *Assume that* (1) *has a solution. Let* φ *be a continuously differentiable (NCP)-function with:*

$$\varphi(a, b) \geq 0, \ \textit{ for all } (a, b) \in \mathbb{R}^2,$$

which satisfy:

- $\varphi(a, b) = 0 \Leftrightarrow \nabla\varphi(a, b) = 0$,
- $\frac{\partial\varphi}{\partial a}(a, b)\frac{\partial\varphi}{\partial b}(a, b) \geq 0$, *for all* $(a, b) \in \mathbb{R}^2$.

Then $\bar{x}$ *is a stationary point of* Ψ *if and only if* $\bar{x}$ *is a solution of* (1).

The (NCP)-functions: 2, 3, and 5 satisfied the hypothesis of the theorem 4, whereas 1 and 4 are not.

Next, the description of the algorithm is stated in the table below as follows.

THE ALGORITHM OF KANZOW

- **Step 0 (initialization):** let $\varepsilon > 0, \varphi : \mathbb{R}^2 \to \mathbb{R}, \psi_i : \mathbb{R}^n \to \mathbb{R}$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$, with $\sigma \in (0, 1)$, let $x^0 \in \mathbb{R}^n$ and $k := 0$;
- **Step 1 (stopping Criterion):** while $\Psi(x^k) > \varepsilon$ and $\left\|\nabla\Psi(x^k)\right\| > \varepsilon$ do
- **Step 2 (search direction):** compute a direction d_k from a linear system
  $d_k = -H_k^{-1}\nabla\Psi(x^k)$ where $H_k = \nabla^2\Psi(x^k)$ is the Hessian matrix of Ψ.

- **Step 3 (line search):** compute $t_k = \arg\min_{t\geq 0} \Psi(x^k + td_k)$.
- **Step 4 (update)**: let $x^{k+1} = x^k + t_k d_k$, $k = k + 1$, and go back to **step 1**.

REMARK 2. [7] The algorithm of Kanzow converges quadratically, but locally. □

5.2. Peng's algorithm. In 1997, Peng also answered to the question 2, by considering (VIP) case. Here, Peng reformulates the (VIP) as an unconstrained differentiable minimization problem, by suggesting a class of merit functions $M(x, \alpha)$, where every stationary point of this problem is a solution of (VIP). The following merit function is suggested by Peng [10]:

$$M(x,\alpha) = \tfrac{1}{\alpha} f(x,\alpha) - \alpha f(x, \tfrac{1}{\alpha}), \alpha > 1 \tag{5}$$

with

$$f(x,\alpha) = -\langle F(x), H_\alpha(x) - x\rangle - \tfrac{1}{2}\,\|(H_\alpha(x) - x)\|_G^2$$

and

$$H_\alpha(x) = \underset{\mathbb{R}^n_+, G}{\Pr}(x - G^{-1}F(x)) = \max(0, x - G^{-1}F(x)),$$

where Pr is the projection of the vector $(x - G^{-1}F(x))$ over the cone $\mathbb{R}^n_+$ and G is a symmetric positive definite matrix with $\|y\|_G^2 = \langle y, Gy\rangle$.

Here, we adapt these results to (NCP) case. The following theorem shows that $M(x, \alpha)$ is a merit function for it.

THEOREM 5. *Let $M(x, \alpha)$ be defined as in* (5) *with $\alpha > 1$, hence $M(x, \alpha) \geq 0$ and $M(x, \alpha) = 0$, if and only if x is a solution of the* (1).

Now, we consider the unconstrained minimization problem associated with (1) and the merit function $M(x, \alpha)$:

$$(\mathrm{OP})_P : \ \min_x M(x,\alpha) \ \text{ subject to } x \in \mathbb{R}^n.$$

The following result establishes the equivalence between (1) and $(\mathrm{OP})_P$.

THEOREM 6. [10] *Let $F(x)$ be a differentiable mapping on $\mathbb{R}^n$ and x is a stationary point of $M(x, \alpha)$. If $\nabla F(x)$ is a positive definite matrix, then x is a solution of* (1).

Hence, solving (1) is equivalent to solve $(\mathrm{OP})_P$ and so the corresponding algorithm is as stated as follows.

THE ALGORITHM OF PENG

- **Step 0 (initialization):** Let $\varepsilon > 0$ and α ($\alpha > 2$) be a constant and G is a given $(n \times n)$ symmetric positive definite matrix. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}, \psi_i : \mathbb{R}^n \to \mathbb{R}$; and $\Psi : \mathbb{R}^n \to \mathbb{R}$, with $\sigma \in (0, 1)$. Let $x^0 \in \mathbb{R}^n$ and $k := 0$;

- **Step 1 (search direction):** compute the direction from: $d_k = H_\alpha(x^k) - H_{\frac{1}{\alpha}}(x^k)$, where

$$H_\alpha(x^k) = \underset{R^n_+,G}{\Pr}(x^k - \alpha G^{-1}F(x^k)) = \begin{cases} x^k - G^{-1}F(x^k) & \text{if } x^k - \alpha G^{-1}F(x^k) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

and

$$H_{\frac{1}{\alpha}}(x^k) = \underset{R^n_+,G}{\Pr}(x^k - \frac{1}{\alpha} G^{-1}F(x^k)) = \begin{cases} x^k - G^{-1}F(x^k) & \text{if } x^k - \frac{1}{\alpha} G^{-1}F(x^k) > 0; \\ 0 & \text{otherwise.} \end{cases}$$

- **Step 2 (stopping criterion):** while $|M(x,\alpha)| > \epsilon$ and $\|d_k\| > \epsilon$ do
- **Step 3 (line search):** compute $t_k = \arg\min_{t\in(0,\,1]} M(x^k + td_k)$.
- **Step 4 (update)**: let $x^{k+1} = x^k + t_k d_k$, $k = k+1$, and go back to **step 1** and **step 2**.

REMARK 3. [10] The convergence of the algorithm is globally linear. □

REMARK 4. There are other functions of merit suggested by Peng's. (See [10]). □

6. IMPLEMENTATION AND NUMERICAL RESULTS

In this section, we present some numerical results applied to some problems of (NCP). Our algorithms were programmed in Pascal 7, on a Pentium III where the tolerance in Kanzow's algorithm is $\varepsilon = 10^{-8}$ with $\sigma = 10^{-10}$, and for Peng's algorithm the tolerance is $\varepsilon = 10^{-6}$.

6.1. Computation of the direction and the steplength in Kanzow's algorithm. The descent direction d_k is computed by solving the following linear system:

$$H_k d_k = -g_k, \tag{6}$$

where $H_k = \nabla^2\Psi(x^k)$ and $g_k = \nabla\Psi(x^k)$. Since H_k is symmetric positive definite matrix, a Cholesky factorization is used to solve (6). The steplength t_k is computed thanks to the Armijo's rule:

$$\Psi(x^k + t_k d_k) \leq \Psi(x^k) + t_k \sigma d_k.$$

There exist other rules that we can use to compute the steplength such as Goldstein, constant step, etc.

The first and the second derivatives of the function Ψ are computed thanks to the finite difference procedure, since the direct calculus of this last is very

difficult and with an expensive cost. For that, we have evaluated the first order and the second derivatives, respectively, by the following formulas:

$$\frac{\partial\Psi}{\partial x_i}(x^{(0)}) = \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}+h_i,x_{i+1}^{(0)},\ldots,x_n^{(0)})}{2h_i} - \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}-h_i,x_{i+1}^{(0)},\ldots,x_n^{(0)})}{2h_i},$$

$$\begin{aligned}\frac{\partial^2\Psi}{\partial x_i^2}(x^{(0)}) = {} & \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}+h_i,x_{i+1}^{(0)},\ldots,x_{j-1}^{(0)},x_j^{(0)}+h_j,x_{j+1}^{(0)},\ldots,x_n^{(0)})}{4h_ih_j} \\ & - \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}-h_i,x_{i+1}^{(0)},\ldots,x_{j-1}^{(0)},x_j^{(0)}-h_j,x_{j+1}^{(0)},\ldots,x_n^{(0)})}{4h_ih_j} \\ & + \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}+h_i,x_{i+1}^{(0)},\ldots,x_{j-1}^{(0)},x_j^{(0)}-h_j,x_{j+1}^{(0)},\ldots,x_n^{(0)})}{4h_ih_j} \\ & + \frac{\Psi(x_1^{(0)},\ldots,x_{i-1}^{(0)},x_i^{(0)}-h_i,x_{i+1}^{(0)},\ldots,x_{j-1}^{(0)},x_j^{(0)}-h_j,x_{j+1}^{(0)},\ldots,x_n^{(0)})}{4h_ih_j}\end{aligned}$$

with $H = (h_i = \frac{1}{100}),\ i = 1,\ldots,n$. With these formulas, we obtain good results.

6.2. Computation of the direction and the steplength in Peng's algorithm. In this algorithm, the computation of the projection the direction is linked with the computation of H_α which it is easy to handled. The choice of the matrix G and the steplength have a great influence on its performance. Here G is chosen as a diagonal positive definite matrix. For the steplength, we have adapted two strategies namely the constant and the linearsearch procedure steps where we have taken the better of them.

6.3. Numerical results. The numerical results of two problems with these algorithms is stated in the arrays below where we display the following quantities: x^0 is the initial starting point, x^*_{app} is the approximate optimal solution, t is the steplength, **iter** meant the iterations number and **cpu** is the time produced by the machine. Our testing problems are inspired in some of them from nonlinear programming problems. The (NCP)-functions used for the implementation in Kanzow's algorithm is:

$$\varphi_2(a,b) = ab + \tfrac{1}{2\alpha}\left[\max\left(0,a-\alpha b)\right)^2 - a^2 + \left(\max(0,b-\alpha a)\right)^2 - b^2\right],$$

with different suitable choices of the parameter $\alpha\,(\alpha > 1)$. For Peng's algorithm, we have used the following merit function:

$$M(x,\alpha,\beta) = \frac{1}{\alpha}f(x,\alpha) - \beta f(x,\frac{1}{\beta}),\ (\beta > \alpha > 1),$$

with also different suitable choices of the parameters α and β.

Problem 1. This problem arises as (**KKT**) optimality conditions for the following nonlinear programming problem:

$$\min_x \left[g(x) = x_1^2 + x_2^3 + x_3^3 + x_4^2 - 2x_1 - 3x_4\right]$$

subject to:

$$\begin{aligned} g_1(x) &= 2x_1 + x_2^2 + x_3^2 + 4x_4 - 8 \leq 0, \\ g_2(x) &= x_1 + x_2^2 + 2x_3^2 + x_4 - 7 \leq 0, \\ g_3(x) &= 3x_1 + 4x_2^2 + 2x_3^2 + x_4 - 10 \leq 0, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{aligned}$$

Hence the **K.K.T**-reformulation gives the following (NCP):

$$\begin{aligned} F : \mathbb{R}^7 \to \mathbb{R}^7 & \\ F_1(x) &= 2x_1 + 2x_2 + x_6 + 3x_7 - 2, \\ F_2(x) &= 3x_2^2 + 2x_1x_5 + 2x_2x_6 + 8x_2x_7, \\ F_3(x) &= 3x_2^3 + 2x_3x_5 + 4x_3x_6 + 4x_3x_7, \\ F_4(x) &= 2x_4 + 4x_5 + x_6 + x_7 - 3, \\ F_5(x) &= -2x_1 - x_2^2 - x_3^2 - 4x_4 + 6, \\ F_6(x) &= -x_1 - x_2^2 - 2x_3^2 - x_4 + 7, \\ F_7(x) &= -3x_1 - 4x_2^2 - 2x_3^2 - x_4 + 10, \\ x_i &\geq 0, i = 1, \ldots, 7. \end{aligned}$$

The exact solution is:

$$(1, 0, 0, \tfrac{3}{2}, 0, 0, 0)^{\mathrm{T}}.$$

Table 1. Kanzow's numerical results with $\alpha = 1.1$ and for different starting points

x^0	x^*_{app}	t	**Iter**	**cpu**
$(0, 1, 1, 0, 1, 1, 1)$	$(0.999997, 0.00832,$ $0.000832, 1.499996,$ $0.000001, 0.000001,$ $0.000001)$.	1	1	0.16
$(0, 0.25, 0.25, 0,$ $0.25, 0.25, 0.25)$	$(1, 0, 0, 1.499999,$ $0.000001, 0, 0)$	1	1	0.16
$(0.5, 0.5, 0.5, 0.5,$ $0.5, 0.5, 0.5)$	$(0.999996, 0.00833,$ $0.000834, 1.499981,$ $0.000007, 0.000002,$ $0.000004)$.	1	2	0.22
$(0.75, 0.75, 0.75, 0.75,$ $0.75, 0.75, 0.75)$	$(0.999994, 0.00834,$ $0.000833, 1.499993,$ $0.000003, 0.000001,$ $0.000002)$.	1	2	0.22
$(0.25, 0.25, 0.25, 0.25,$ $0.25, 0.25, 0.25)$	$(1.000002, 0, 0,$ $1.500004, 0.000002,$ $0.000001, 0)$.	1	1	0.16

Table 2. Peng's algorithm ($\alpha = 3, \beta = 10^5, \varepsilon = 10^{-6}$).

G= diag(v)	t	x^0	x^*_{app}	**iter**	**cpu**
$(2,1,2,1,2,1,2)$	0.3	$(2,2,2,$ $2,2,2,2)$	$(0.999936, 0.000352,$ $0.000980, 1.499965,$ $0, 0.000045, 0.000028)$	62	0.05
$(2,2,1,3,3,3,3)$	0.3	$(2,2,2,$ $2,2,2,2)$	$(0.999961, 0.000904,$ $0.001762, 1.499980,$ $0, 0.00023, 0.000018)$	49	0.05
$(2,2,1,3,3,3,3)$	0.4	$(4,3,4,$ $3,2,2,1)$	$(0.999958, 0.000015,$ $0.000016, 1.499979,$ $0, 0.000016, 0.000020)$	42	0.00
$(2,2,1,3,3,3,3)$	0.5	$(4,3,4,$ $3,2,2,1)$	$(0.999961, 0.000006,$ $0.000007, 1.499981,$ $0, 0.000015, 0.000020)$	33	0.00
$(2,2,1,3,3,3,3)$	0.6	$(3,6,2,$ $7,4,1,1)$	$(0.999962, 0,$ $0, 1.499983, 0,$ $0.000015, 0.000018)$	64	0.05

Problem 2. $F : \mathbb{R}^9 \to \mathbb{R}^9$

$$
\begin{aligned}
F_1(x) &= x_2(x_1 + 1),\\
F_2(x) &= x_3(\tfrac{1}{2}x_2 - 1),\\
F_3(x) &= x_3^2 - x_5,\\
F_4(x) &= x_4 + x_7^2 + 2x_8 - 1,\\
F_5(x) &= x_5 - 1,\\
F_6(x) &= x_5x_6 + x_7 - 1,\\
F_7(x) &= x_3(x_2 - x_7) + x_1x_7,\\
F_8(x) &= x_6 - x_7 + 3x_8 + 1,\\
F_9(x) &= -3x_1 + x_2 + 3x_3 - 2x_4 - 2x_5 + 3x_6 - 2x_7 + 3x_8 + 2x_9,\\
& x_i \geq 0, i = 1, \ldots, 9.
\end{aligned}
$$

The exact solution is:

$$x^* = (0, 2, 1, 1, 1, 1, 0, 0, 0)^{\mathrm{T}}.$$

7. CONCLUSION AND GENERAL COMMENTS

In this paper, we have dealt with a computational experience of two smoothing-type algorithms for solving (NCP)-problems. Our preliminary numerical results are encouraging and show with consulting the arrays above and with doing a small comparison between them that the number of iterations given by Kanzow's algorithm is better than one given by Peng's algorithm. However,

Table 3. The numerical results of Kanzow's algorithm for different starting points x^0

x^0	x^*_{app}	t	**iter**	**cpu**
$(0,1,1,0,1,1,0,1,0)$	$(0, 1.999999, 1.000005,$ $0.999980, 1.000005,$ $0.999993, 0, 0.000006, 0, 000000)$	0.7	6	1.32
$(6,5,4,3,2,1,2,2,2)$	$(0.000003, 1.999999, 1.000001,$ $1.000001, 1.000002, 1.000003,$ $0.000005, 0.000001, 0.000009)$	0.7	8	1.75
$(7,6,5,4,3,1,1,1,1)$	$(0.000003, 1.999998, 1.000007,$ $1.000001, 1.000003, 1.000008,$ $0.999981, 0.000005, 0.000001)$	0.49	12	2.58
$(10,7,6,5,4,2,2,2,2)$	$(0.000002, 2.000001, 1.000010,$ $1.999999, 1.0000013, 1.999974,$ $0.000003, 0, 0.000008)$	0.7	8	2.08

Table 4. For Peng's algorithm with $\alpha = 1.1, \beta = 10^5,$ and $\varepsilon = 10^{-6}$

$G =$diag(v)	t	x^0	x^*_{app}	**iter**	**cpu**
$(2,1,3,6,$ $5,4,3,3,3)$	1	$(4,4,3,3,2$ $2,1,1,1)$	$(0.00004, 2, 1.000002,$ $1.000032, 0.999984,$ $0.999966, 0.000001,$ $0.000006, 0.000005)$	51	0.05
$(4,2,2,5,$ $4,3,4,4,4)$	1	$(4,4,3,3,2$ $2,1,1,1)$	$(0.00005, 2.000001,$ $1.000001,$ $1.000036, 1.000036,$ $0.999990, 0.000010,$ $0.000005, 0.000003)$	41	0.05
$(6,5,4,3,$ $2,1,2,2,2)$	0.7	$(4,4,4,4,4$ $4,4,4,4)$	$(0.000003, 2.000088, 1,$ $0.999980, 1,$ $0.999990, 0.000010,$ $0.000010, 0.000020)$	118	0.11
$(6,5,4,3,$ $2,1,2,2,2)$	0.5	$(4,4,4,4,4$ $4,4,4,4)$	$(0.000003, 2.000089,$ $1, 0.999980,$ $1, 0.999990, 0.000010,$ $0.000010, 0.000020)$	167	0.11
$(6,5,4,3$ $2,1,2,2,2$	0.3	$(4,4,4,4,4$ $4,4,4,4)$	$(0.000003, 2.000091,$ $1, 0.999980,$ $1, 0.999990, 0.000010,$ $0.000010, 0.000020)$	281	0.17

for the time produced by the Peng's algorithm is better than in Kanzow's. The convergence of Peng's algorithm is guaranteed only with high values of

β and with limited maximal value of α. A good choice of merit functions in both algorithms has a great influence on their numerical performances, also the starting point stays a crucial task for their initialization and it needs an adequate treatment.

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