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GENERALIZED UNIMODAL MULTICRITERIA OPTIMIZATION

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Abstract. The aim of this paper is to characterize the sets of weakly-efficient solutions and efficient solutions for multicriteria optimization problem involving generalized unimodal objective functions. An implementable algorithm which completely determines these sets is given for the particular framework of discrete feasible domains.

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1. INTRODUCTION

Recall (see e.g. [1]) that a function $f : D \to \mathbb{R}$, defined on a nonempty subset D of $\mathbb{R}$, is called lower unimodal on a compact interval $[a, b] \subset D$ if there exists $u \in [a, b]$ such that f is decreasing on $[a, u]$ and increasing on $[u, b]$ (note that throughout this paper it will be convenient to denote $[\alpha, \beta] := \{x \in \mathbb{R} \mid \alpha \leq x \leq \beta\}$ for any $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$; in particular, $[\alpha, \alpha]$ means $\{\alpha\}$). An extension of the classical concept of unimodality was recently proposed in [3]. We present here a slightly modified version of it:

Definition 1. *Let $f : D \to \mathbb{R}$ be a function, defined on a nonempty set $D \subset \mathbb{R}$. We say that f is* lower unimodal *on $S \subset D$ if there exist $u, v \in S$ satisfying the following conditions:*

(LU1) $\quad f(u) = f(v)$;
(LU2) $\quad f(x) > f(y)$ *whenever* $x, y \in S$, $x < y \leq u$;
(LU3) $\quad f(x) < f(y)$ *whenever* $x, y \in S$, $v \leq x < y$;
(LU4) $\quad S \cap [u, v] = \{u, v\}$.

Remark 1. 1) By (LU1)–(LU2) it follows that $u \leq v$, so (LU4) makes sense. Indeed, supposing to the contrary that $v < u$, by letting $x := v$ and $y := u$ in (LU2), we derive $f(v) > f(u)$, which contradicts (LU1).

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2) As a direct consequence of (LU1)–(LU4) we can easily deduce that for any $x, y \in S$ the following implications hold:

$$x < y \leq v \implies f(x) \geq f(y);$$
$$u \leq x < y \implies f(x) \leq f(y).$$

3) If f is lower unimodal on S, then there exists a unique pair $(u, v) \in S \times S$ of numbers satisfying (LU1)–(LU4), implicitly defined by:

$$\operatorname*{Argmin}_{x \in S} f(x) = \{u, v\} \quad \text{and} \quad u \leq v.$$

4) When $S = D$ in Definition 1 is a compact interval, it follows by (LU4) that $u = v$ and we recover the classical notion of lower unimodality. In this case $\operatorname*{Argmin}_{x \in S} f(x) = \{u\}$.

5) If f is lower unimodal on $S = D \cap \mathbb{Z}$, then there exists an unique integer $u \in D$ such that either $\operatorname*{Argmin}_{x \in S} f(x) = \{u\}$ or $\operatorname*{Argmin}_{x \in S} f(x) = \{u, u + 1\}$.

6) It is easily seen that if f is lower unimodal (in the classical sense) on a compact interval D then f is lower unimodal on every nonempty finite subset S of D. $\square$

Example 1. Consider the function $f : \mathbb{R} \to \mathbb{R}$, defined for all $x \in \mathbb{R}$ by $f(x) = |x^2 - x|$. It is easily seen that f is lower unimodal on $\mathbb{Z}$ and $\operatorname*{Argmin}_{x \in \mathbb{Z}} f(x) = \{0, 1\}$. $\square$

2. UNIMODAL MULTICRITERIA OPTIMIZATION

Let $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ ($m \in \mathbb{N}$, $m \geq 2$) be a vector-valued function defined on a nonempty set $D \subset \mathbb{R}$, such that the scalar components $f_1, \ldots, f_m$ are lower unimodal on a nonempty subset S of D. Consider the multicriteria optimization problem:

$$\text{(1)} \qquad \begin{cases} \text{Minimize} & f(x) \\ \text{subject to} & x \in S, \end{cases}$$

where the partial ordering in the image space of the objective function is understood to be induced by the standard ordering cone $\mathbb{R}^m_+$. More precisely, denoting $I := \{1, \ldots, m\}$, we have for any $a = (a_1, \ldots, a_m)$, $b = (b_1, \ldots, b_m) \in \mathbb{R}^n$:

$$a \leq b \;:\Longleftrightarrow\; a_i \leq b_i, \quad \text{for all} \quad i \in I,$$
$$a < b \;:\Longleftrightarrow\; a_i < b_i, \quad \text{for all} \quad i \in I.$$

Recall (see e.g. [2]) that the sets of *efficient solutions* and *weakly-efficient solutions* of problem (1) are given, respectively, by:

$$\begin{aligned} \operatorname{Eff}(S; f) &:= \{x \in S \mid (f(x) - \mathbb{R}^m_+) \cap f(S) = \{f(x)\}\} \\ &= \{x \in S \mid \nexists\, y \in S \text{ such that } f(y) \leq f(x) \neq f(y)\}, \end{aligned}$$

$$\begin{aligned}\mathrm{WEff}(S;f) &:= \{x\in S \mid (f(x)-\operatorname{int}\mathbb{R}^m_+)\cap f(S)=\emptyset\}\\ &= \{x\in S\mid \nexists\, y\in S \text{ such that } f(y)<f(x)\}\,.\end{aligned}$$

The principal aim of this paper is to show that both the sets $\mathrm{Eff}(S;f)$ and $\mathrm{WEff}(S;f)$ can be completely determined by only using the numbers $u_1, v_1, \ldots, u_m, v_m$, implicitly defined (in view of Remark 1.3) by

$$\operatorname*{Argmin}_{x\in S} f_i(x)=\{u_i,v_i\} \quad\text{and}\quad u_i\le v_i, \quad \text{for all} \quad i\in I.$$

To this end, we firstly introduce the following notations:

$$\underline{u}:=\min_{i\in I}u_i,\quad \underline{v}:=\min_{i\in I}v_i,\quad \overline{u}:=\max_{i\in I}u_i \quad\text{and}\quad \overline{v}:=\max_{i\in I}v_i.$$

Remark 2. 1) Recalling that $u_i\le v_i$ for all $i\in I$, it is easily seen that $\underline{u}\le\min\{\underline{v},\overline{u}\}\le\max\{\underline{v},\overline{u}\}\le\overline{v}$.

2) If $\overline{u}<\underline{v}$ then for each $i\in I$ we have $u_i\le\overline{u}<\underline{v}\le v_i$, which in view of (LU4) implies that $u_i=\overline{u}$ and $\underline{v}=v_i$. In this case we have $\underline{u}=\overline{u}=u_i$ and $\underline{v}=\overline{v}=v_i$, for all $i\in I$. □

Theorem 2. *The set of weakly efficient solutions of problem* (1) *admits the following representation:*

$$\mathrm{WEff}(S;f) \;=\; [\underline{u},\overline{v}]\cap S. \tag{2}$$

Proof. Suppose to the contrary that there exists $x\in\mathrm{WEff}(S;f)$ such that $x\notin[\underline{u},\overline{v}]\cap S$. Since $x\in S$, it follows that either $x<\underline{u}$ or $x>\overline{v}$. In the first case we have $x<\underline{u}\le u_i$, hence $f_i(x)>f_i(\underline{u})$ for each $i\in I$, which yields $f(\underline{u})<f(x)$, contradicting the weak efficiency of x. In the second case we should have $v_i\le\overline{v}<x$, hence $f_i(\overline{v})<f_i(x)$ for each $i\in I$, which yields $f(\overline{v})<f(x)$, contradicting again the weak efficiency of x. Thus the inclusion $\mathrm{WEff}(S;f)\subset[\underline{u},\overline{v}]\cap S$ holds.

In order to prove the converse inclusion, let us suppose to the contrary that there exists $x\in[\underline{u},\overline{v}]\cap S\setminus\mathrm{WEff}(S;f)$. Then we can find some $y\in S$ such that

$$f_i(y)<f_i(x), \quad \text{for all} \quad i\in I. \tag{3}$$

Obviously $x\ne y$. If $x<y$ then (in view of the second implication in Remark 1.2) it follows by (3) that $x<u_i$ for every $i\in I$, which yields $x<\underline{u}$, a contradiction. Similarly, if $y<x$ then (in view of the first implication in Remark 1.2) we can deduce by (3) that $x>v_i$ for every $i\in I$, which yields $x>\overline{v}$, contradicting again the choice of x. Thus the inclusion $[\underline{u},\overline{v}]\cap S\subset\mathrm{WEff}(S;f)$ holds and relation (2) is proven. □

Theorem 3. *The set of efficient solutions of problem* (1) *is given by the following representation:*

$$\mathrm{Eff}(S;f) \;=\; [\min\{\underline{v},\overline{u}\},\max\{\underline{v},\overline{u}\}]\cap S. \tag{4}$$

Proof. We will distinguish two cases:

Case 1: $\overline{u} < \underline{v}$. In this case, according to Remark 2.2, we have $\underline{u} = \overline{u} = u_i$ and $\underline{v} = \overline{v} = v_i$ for all $i \in I$. In view of Remark 1.3 and (LU4) we infer that

$$\bigcap_{i\in I} \operatorname*{Argmin}_{x\in S} f_i(x) \quad = \quad [\overline{u}, \underline{v}] \cap S,$$

which actually means (4), since in this case the efficient solutions become ideal efficient (cf. Proposition 2.2.2 in [2]).

Case 2: $\underline{v} \leqslant \overline{u}$. In this case relation (4) can be rewritten as:

$$\operatorname{Eff}(S; f) \quad = \quad [\underline{v}, \overline{u}] \cap S. \tag{5}$$

Suppose to the contrary that there exists $x \in \operatorname{Eff}(S; f)$ such that $x \notin [\underline{v}, \overline{u}] \cap S$. Since $x \in S$, it follows that either $x < \underline{v}$ or $x > \overline{u}$. If $x < \underline{v}$, then for each $i \in I$ we have $x < \underline{v} \leq v_i$, which (in view of Remark 1.2) implies that $f_i(x) \geq f_i(\underline{v})$. Hence $f(x) \geq f(\underline{v})$. Since $x \in \operatorname{Eff}(S; f)$ it follows that $f(x) = f(\underline{v})$. Taking into account that $x < \underline{v} \leq v_i$ and $f_i(\underline{v}) = f_i(x)$ for each $i \in I$, we can deduce (in view of (LU1)–(LU4)) that $u_i = x < \underline{v} = v_i$ for all $i \in I$. In particular, it follows that $\overline{u} < \underline{v}$, a contradiction. Similarly, if $x > \overline{u}$, then for each $i \in I$ we have $u_i \leq \overline{u} < x$, which (in view of Remark 1.2) implies that $f_i(\overline{u}) \leq f_i(x)$. Hence $f(\overline{u}) \leq f(x)$. Since $x \in \operatorname{Eff}(S; f)$ it follows that $f(\overline{u}) = f(x)$. Taking into account that $u_i \leq \overline{u} < x$ and $f_i(\overline{u}) = f_i(x)$ for each $i \in I$, we can deduce that $u_i = \overline{u} < x = v_i$ for all $i \in I$. In particular, it follows that $\overline{u} < \underline{v}$, which gives again a contradiction. Thus inclusion $\operatorname{Eff}(S; f) \subset [\underline{v}, \overline{u}] \cap S$ holds.

In order to prove the converse inclusion, let us suppose to the contrary that there exists $x \in [\underline{v}, \overline{u}] \cap S \setminus \operatorname{Eff}(S; f)$. Then we can find some $y \in S$ such that $f(y) \leq f(x)$ and $f(y) \neq f(x)$. It follows that

$$f_i(y) \leq f_i(x), \quad \text{for all} \quad i \in I \tag{6}$$

and $y \neq x$. If $y < x$ then we have $y < x \leq \overline{u}$. In particular, by choosing $h \in I$ such that $u_h = \overline{u}$, we obtain $y < x \leq u_h$, which in view of (LU2) yields $f_h(y) > f_h(x)$, contradicting (6). Similarly, if $x < y$ then we have $\underline{v} \leq x < y$. In particular, by choosing $k \in I$ such that $v_k = \underline{v}$, we obtain $v_k \leq x < y$, which in view of (LU3) yields $f_k(x) < f_k(y)$, contradicting again (6). Thus the inclusion $[\underline{v}, \overline{u}] \cap S \subset \operatorname{Eff}(S; f)$ also holds.

Consequently, relation (5) is proven.

□

Remark 3. 1) By Theorem 2 and Theorem 3 it follows that in the particular case when $\overline{u} < \underline{v}$ we have

$$\operatorname{Eff}(S; f) = \operatorname{WEff}(S; f) = \{u_i, v_i\}, \quad \text{for all} \quad i \in I.$$

2) As a direct consequence of Theorem 2 and Theorem 3, we can also deduce that in the particular case when S is an interval (i.e. the functions $f_1, \dots, f_m$ are lower unimodal in the classical sense) we have

$$\mathrm{Eff}(S;f) = \mathrm{WEff}(S;f) = [\underline{u}, \overline{v}]\,.$$

Note that this representation was already obtained in [5], by using a quite different approach, based on some results from [4]. □

3. NUMERICAL APPROACH

In what follows we consider the particular case where the feasible set S of problem (1) is discrete. More precisely, we assume that all objective functions $f_1, \dots, f_m$ are lower unimodal on a finite set $S = \{x_1, \dots, x_n\} \subset D$, where $n \in \mathbb{N}$, $n \geq 2$, and $x_1 < \cdots < x_n$. In this case, Theorem 2 and Theorem 3 can be used to design a numerical method for solving problem (1). Throughout the algorithm below, the elements $\underline{u}$, $\overline{u}$, $\underline{v}$ and $\overline{v}$, needed for generating the sets $\mathrm{WEff}(S;f)$ and $\mathrm{Eff}(S;f)$, will be $x_{j_{\underline{u}}}$, $x_{j_{\overline{u}}}$, $x_{j_{\underline{v}}}$ and $x_{j_{\overline{v}}}$, respectively.

Algorithm

1° Set $j := 1$.
2° Set $i := 1$.
3° If $f_i(x_j) \leq f_i(x_{j+1})$ then go to step 4°, else set $i := i+1$ and go to step 7°.
4° If $f_i(x_j) = f_i(x_{j+1})$ then go to step 5°, else set $j_{\underline{v}} := j$ and go to step 9°.
5° Set $i := i+1$.
6° If $i \leq m$ then go to step 4°, else set $j_{\underline{v}} := j+1$ and go to step 9°.
7° If $i \leq m$ then go to step 3°, else set $j := j+1$.
8° If $j < n$ then go to step 2°, else set $j_{\underline{v}} := j$.
9° Set $j_{\underline{u}} := j$.
10° Set $j := n$.
11° Set $i := 1$.
12° If $f_i(x_{j-1}) \geq f_i(x_j)$ then go to step 13°, else set $i := i+1$ and go to step 16°.
13° If $f_i(x_{j-1}) = f_i(x_j)$ then go to step 14°, else set $j_{\overline{u}} := j$ and go to step 18°.
14° Set $i := i+1$.
15° If $i \leq m$ then go to step 13°, else set $j_{\overline{u}} := j-1$ and go to step 18°.
16° If $i \leq m$ then go to step 12°, else set $j := j-1$.
17° If $j > 1$ then go to step 11°, else set $j_{\overline{u}} := j$.
18° Set $j_{\overline{v}} := j$.
19° Generate $\mathrm{WEff}(S;f) := \{x_j \mid j_{\underline{u}} \leq j \leq j_{\overline{v}}\}$.

20° If $j_{\underline{v}} \leq j_{\overline{u}}$ then generate $\mathrm{Eff}(S;f) := \{x_j \mid j_{\underline{v}} \leq j \leq j_{\overline{u}}\}$, else generate $\mathrm{Eff}(S;f) := \{x_j \mid j_{\overline{u}} \leq j \leq j_{\underline{v}}\}$.
21° Stop.

REMARK 4. 1) The above algorithm may be used for solving integer multicriteria optimization problems of type (1), where the objective functions $f_1, \ldots, f_m$ are lower unimodal on a finite feasible domain $S = D \cap \mathbb{Z} = \{x_1, \ldots, x_n\}$ with cardinality $n \geq 2$.

2) If the objective functions $f_1, \ldots, f_m$ are lower unimodal (in the classical sense) on a compact interval D then, by choosing S as being a division of D with small enough norm, the above algorithm can serve for approximating the sets $\mathrm{Eff}(D;f)$ and $\mathrm{WEff}(D;f)$ by $\mathrm{Eff}(S;f)$ and $\mathrm{WEff}(S;f)$, respectively. □

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