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EXTENSION OF BOUNDED LINEAR FUNCTIONALS AND BEST APPROXIMATION IN SPACES WITH ASYMMETRIC NORM

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Abstract. The present paper is concerned with the characterization of the elements of best approximation in a subspace Y of a space with asymmetric norm, in terms of some linear functionals vanishing on Y. The approach is based on some extension results, proved in Section 3, for bounded linear functionals on such spaces. Also, the well known formula for the distance to a hyperplane in a normed space is extended to the nonsymmetric case.

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1. INTRODUCTION

Let X be a real vector space. An *asymmetric seminorm* on X is a positive sublinear functional $p : X \to [0, \infty)$, i.e. p satisfies the conditions:

(AN1) $p(x) \geq 0$,
(AN2) $p(tx) = tp(x), \ t \geq 0$,
(AN3) $p(x + y) \leq p(x) + p(y)$,

for all $x, y \in X$. The function $\bar{p} : X \to [0, \infty)$ defined by $\bar{p}(x) = p(-x), \ x \in X$, is another positive sublinear functional on X, called the *conjugate* of p, and

$$p^s(x) = \max\{p(x), p(-x)\}, \ x \in X,$$

is a seminorm on X. The inequalities

$$|p(x) - p(y)| \leq p^s(x - y) \quad \text{and} \quad |\bar{p}(x) - \bar{p}(y)| \leq p^s(x - y)$$

hold for all $x, y \in X$. If the seminorm p^s is a norm on X then we say that p is an *asymmetric norm* on X. This means that, beside (AN1)–(AN3), it satisfies also the condition

(AN4) $p(x) = 0$ and $p(-x) = 0$ imply $x = 0$.

The pair (X, p), where X is a linear space and p is an asymmetric seminorm on X is called a *space with asymmetric seminorm*, respectively a *space with asymmetric norm*, if p is an asymmetric norm.

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An asymmetric seminorm p generates a topology τ_p on X, having as a basis of neighborhoods of a point $x \in X$ the open p-balls

$$B'_p(x,r) = \{x' \in X : p(x'-x) < r\},\ r>0.$$

The family of closed p-balls

$$B_p(x,r) = \{x' \in X : p(x'-x) \le r\},\ r>0,$$

generates the same topology.

Denote by $B_p = B_p(0,1)$ the closed unit ball of (X,p) and by $B'_p = B'_p(0,1)$ its open unit ball.

The topology τ_p is translation invariant, i.e. the addition $+ : X \times X \to X$ is continuous, but the multiplication by scalars $\cdot : \mathbb{R} \times X \to X$ need not be continuous. For instance, in the space

$$C_0[0,1] = \Big\{x \in C[0,1] : \int_0^1 x(t)\mathrm{d}t = 0\Big\}$$

with the asymmetric seminorm $p(x) = \max x([0,1])$, the multiplication by scalars is not continuous at $t_0 = -1$ and $x_0 = 0$. Indeed, the ball $B_p(0,1)$ is a neighborhood of $0 = (-1)0$, but $-B(0,r) \nsubseteq B(0,1)$ for any $r>0$, because the functions x_n defined by

$$x_n(t) = \begin{cases} (n-1)(nt-1), & \text{for } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(t-\frac{1}{n}), & \text{for } \frac{1}{n} \le t \le 1, \end{cases}$$

is in $B_p(0,1)$ for all n, while $p(-x_n) = n-1 > r$ for large n (see [2]).

The topology τ_p could not be Hausdorff even if p is an asymmetric norm on X. Necessary and sufficient conditions in order that τ_p be Hausdorff were given in [8].

In this paper we shall study some best approximation problems in spaces with asymmetric seminorm. The significance of asymmetric norms for best approximation problems was first emphasized by Krein and Nudel′man (see [10, Ch. 9, § 5]). In the spaces $C(T)$ and L_r, $1 \le r < \infty$, one considers asymmetric norms defined through a pair $w = (w_+, w_-)$ of nonnegative upper semicontinuous functions, called weight functions, via the formula $\|f|_w = \max\{w_+(t)f_+(t) - w_-(t)f_-(t) : t \in T\}$, where f_+, f_- are the positive, respectively negative part of f. In the case of the spaces L_r the above formula is adapted to the corresponding integral norm. The approximation in such spaces is called sign-sensitive approximation and it is studied in a lot of papers, following the ideas from the symmetric case (see [1, 4, 5, 9, 18, 19, 20] and the references given in these papers). There are also papers concerning existence results, mainly generic, for best approximation in abstract spaces with asymmetric norms, see [3, 11, 12, 17].

In [14, 16], there were studied the relations between the existence of best approximation and uniqueness of the extension of bounded linear functionals on spaces with asymmetric norm. In [13, 15] similar problems were considered within the framework of spaces of semi-Lipschitz functions on an asymmetric metric space (called quasi-metric space).

The present paper is concerned with the characterization of the elements of best approximation in a subspace Y of a space with asymmetric norm in terms of some linear functionals vanishing on Y. The approach is based on some extension results, proved in Section 3, for bounded linear functionals on such spaces. Also, the well known formula for the distance to a hyperplane in a normed space is extended to the nonsymmetric case. For the case of normed spaces see [21].

2. BOUNDED LINEAR MAPPINGS AND THE DUAL OF A SPACE WITH ASYMMETRIC SEMINORM

Let (X, p) and (Y, q) be spaces with asymmetric seminorms and $A : X \to Y$ a linear mapping. The mapping A is called *bounded* (or *semi-Lipschitz*) if there exists $L \geq 0$ such that

$$q(Ax) \leq Lp(x), \quad \text{for all} \ \ x \in X. \tag{2.1}$$

It was shown in [6] (see also [7]) that the boundedness of the linear mapping A is equivalent to its continuity with respect to the topologies τ_p and τ_q. Denoting by $L_b(X, Y)$ the set of all bounded linear mapping from (X, p) to (Y, q), it turns out that $L_b(X, Y)$ is not necessarily a linear space but rather a convex cone in the vector space $L_a(X, Y)$ of all linear mappings from X to Y, i.e.

$$\lambda \geq 0 \text{ and } A, B \in L_b(X, Y) \ \Rightarrow \ A + B \in L_b(X, Y) \text{ and } \lambda A \in L_b(X, Y).$$

For instance, in the space $X = C_0[0, 1]$ considered in the previous section, the linear functional $\varphi(x) = x(1),\ x \in C_0[0, 1],$ is bounded because $\varphi(x) \leq p(x),\ x \in X$, but the functional $-\varphi$ is not bounded. Taking $x_n(t) = 1 - nt^{n-1}$ we have $p(x_n) = 1$ for all n, but $-\varphi(x_n) = n - 1 \to \infty$ for $n \to \infty$ (see [2]).

As in the case of bounded linear mapping between normed linear spaces, one can define an asymmetric seminorm on $L_b(X, Y)$ by the formula

$$\|A| = \sup\{q(Ax) : x \in X,\ p(x) \leq 1\}. \tag{2.2}$$

It is not difficult to see that $\| \cdot |$ is an asymmetric seminorm on the cone $L_b(X, Y)$ which has properties similar to those of the usual norm:

Proposition 2.1. *Let* (X, p) *and* (Y, q) *be spaces with asymmetric seminorms and* $A \in L_b(X, Y)$. *Then*

1) $\forall x \in X \quad q(Ax) \leq \|A| \cdot p(x)$,
and $\|A|$ *is the smallest number* $L \geq 0$ *for which the inequality* (2.1) *holds.*
2) $\|A| = \sup\left\{\frac{q(Ax)}{p(x)} : x \in X,\ p(x) > 0\right\}$.

Proof. 1) If $p(x) = 0$ then, by the boundedness of A, $q(Ax) = 0 = \|A|p(x)$. If $p(x) > 0$ then $p((1/p(x))x) = 1$ and

$$q(A(\tfrac{1}{p(x)}x)) \le \|A| \iff q(Ax) \le \|A| \cdot p(x).$$

If $q(Ax) \le Lp(x)$, $\forall x \in X$, for some $L \ge 0$, then $q(Ax) \le L$ for all $x \in X$ with $p(x) \le 1$, implying $\|A| \le L$.

2) Follows from the facts that $q(Ax) = 0$ if $p(x) = 0$ and

$$\tfrac{1}{p(x)}q(Ax) = q(A(\tfrac{1}{p(x)}))$$

if $p(x) > 0$. □

Bounded linear functionals on a space with asymmetric norm

As in the case of normed spaces, the cone of bounded linear functional on a space with asymmetric seminorm will play a key role in various problems concerning these spaces.

On the space $\mathbb{R}$ of real numbers, consider the asymmetric seminorm $u(\alpha) = \max\{\alpha, 0\}$ and denote by $\mathbb{R}_u$ the space $\mathbb{R}$ equipped with the topology τ_u generated by u. It is the topology generated by the intervals of the form $(-\infty, a)$, $a \in \mathbb{R}$. A neighborhood basis of a point $a \in \mathbb{R}_u$ is formed by the intervals $(-\infty, a+\epsilon)$, $\epsilon > 0$. The seminorm conjugate to u is $\bar{u}(\alpha) = u(-\alpha) = \max\{-\alpha, 0\}$, and $u^s(\alpha) = \max\{u(\alpha), u(-\alpha)\} = |\alpha|$. The continuity of a linear functional $\varphi : (X, p) \to (\mathbb{R}, u)$ with respect to the topologies τ_p and τ_u will be called (p, u)-*continuity.* It is easily seen that the (p, u)-continuity of a linear functional $\varphi : (X, \tau_p) \to (\mathbb{R}, u)$ is equivalent to its upper semi-continuity as a functional from (X, τ_p) to $(\mathbb{R}, |\,|)$. This is equivalent to the fact that for every $\alpha \in \mathbb{R}$ the set $\{x \in X : \varphi(x) \ge \alpha\}$ is closed in (X, τ_p) and has consequence the fact that, for every τ_p-compact subset Y of X, the functional φ is upper bounded on Y and there exists $y_0 \in Y$ such that $\varphi(y_0) = \sup \varphi(Y)$. Also, the linear functional φ is (p, u)-continuous if and only if it is p-bounded, i.e. there exists $L \ge 0$ such that

$$\forall x \in X \quad \varphi(x) \le Lp(x). \tag{2.3}$$

Denote by $X^\flat_p$ ($X^\flat$ when it is no danger of confusion) the cone of all bounded linear functionals on the space with asymmetric seminorm (X, p) and call it the *asymmetric dual* of (X, p). It follows that the functional

$$\|\varphi| = \|\varphi|_p = \sup\{\varphi(x) : x \in X,\ p(x) \le 1\}$$

is an asymmetric seminorm on $X^\flat$.

We shall need the following simple properties of this seminorm.

Proposition 2.2. *If φ is a bounded linear functional on a space with asymmetric seminorm (X, p), with $p \ne 0$, then:*

1) *$\|\varphi|$ is the smallest of the numbers $L \ge 0$ for which the inequality* (2.3) *holds;*

2) *We have:*

$$\begin{aligned}\|\varphi| &= \sup\{\varphi(x)/p(x) : x \in X,\ p(x) > 0\}\\ &= \sup\{\varphi(x) : x \in X,\ p(x) < 1\}\\ &= \sup\{\varphi(x) : x \in X,\ p(x) = 1\};\end{aligned}$$

3) *If* $\varphi \neq 0$ *then* $\|\varphi| > 0$.

Also, if $\varphi \neq 0$ *and* $\varphi(x_0) = \|\varphi|$ *for some* $x_0 \in B_p$, *then* $p(x_0) = 1$.

Proof. We shall prove the assertions 2) and 3), the first one being a particular case of the corresponding result for linear mappings.

Supposing $c := \sup\{\varphi(x) : p(x) < 1\} < \|\varphi|$, let $x_0 \in X,\ p(x_0) = 1$, be such that $c < \varphi(x_0) \leq \|\varphi|$. Then there is a number α, $0 < \alpha < 1$, such that $\varphi(\alpha x_0) = \alpha\varphi(x_0) > c$, in contradiction to the definition of c.

Let's show now that $\|\varphi| = \sup\{\varphi(x) : p(x) = 1\}$. Suppose again that $\beta := \sup\{\varphi(x) : p(x) = 1\} < \|\varphi|$, and choose $x_0 \in X$ such that $p(x_0) < 1$ and $\varphi(x_0) > \beta$. Putting $x_1 = (1/p(x_0))x_0$, it follows $p(x_1) = 1$ and

$$\varphi(x_1) = \tfrac{1}{p(x_0)}\varphi(x_0) > \varphi(x_0) > \beta,$$

a contradiction.

3) Because $\varphi(x) \leq \|\varphi|p(x)$, the equality $\|\varphi| = 0$ implies $\varphi(x) \leq 0$ and $-\varphi(x) = \varphi(-x) \leq 0$, i.e. $\varphi(x) = 0$ for all $x \in X$.

Suppose now that that for $\varphi \neq 0$ there exists $x_0 \in X$, with $0 < p(x_0) < 1$, such that $\varphi(x_0) = \|\varphi|$. Then $\alpha := 1/p(x_0) > 1,\ x_1 = \alpha x_0 \in B_p$ and

$$\|\varphi| \geq \varphi(x_1) = \alpha\varphi(x_0) = \alpha\|\varphi|,$$

a contradiction, because $\|\varphi| > 0$. □

An immediate consequence of the preceding result is the following one. We agree to call a linear functional $(p,\bar{p})$-bounded if it is both p- and $\bar{p}$-bounded.

Proposition 2.3. *Let* $\varphi \neq 0$ *be a linear functional on a space with asymmetric seminorm* (X,p).

1) *If* φ *is* $(p,\bar{p})$*-bounded then*

$$\varphi(rB'_p) = (-r\|\varphi|_{\bar{p}}, r\|\varphi|_p) \quad \textit{and} \quad \varphi(rB'_{\bar{p}}) = (-r\|\varphi|_p, r\|\varphi|_{\bar{p}})$$

where $B'_p = \{x \in X : p(x) < 1\},\ B'_{\bar{p}} = \{x \in X : \bar{p}(x) < 1\}$ *and* $r > 0$.

2) *If* φ *is* p*-bounded but not* $\bar{p}$*-bounded then*

$$\varphi(rB'_p) = (-\infty, r\|\varphi|_p).$$

Proof. Obviously that it is sufficient to give the proof only for $r = 1$.

Suppose that φ is $(p,\bar{p})$-bounded. Then, by Proposition 2.2,

$$\sup\varphi(B'_p) = \|\varphi|_p$$

and

$$\inf\varphi(B'_p) = -\sup\{\varphi(-x) : p(x) < 1\} = -\sup\{\varphi(x') : p(-x') < 1\} = -\|\varphi|_{\bar{p}}.$$

Also, by the assertion 3) of Proposition 2.2, $\varphi(x) < \|\varphi|_p$ and $\varphi(x) > -\|\varphi|_{\bar{p}}$ for any $x \in B'_p$.

Because B'_p is convex, $\varphi(B'_p)$ will be a convex subset of $\mathbb{R}$, that is an interval, and the above considerations show that

$$\varphi(B'_p) = (\inf \varphi(B'_p), \sup \varphi(B'_p)) = (-\|\varphi|_{\bar{p}}, \|\varphi|_p).$$

If φ is p-bounded and

$$\sup\{\varphi(x); \bar{p}(x) < 1\} = \infty.$$

then

$$\inf\{\varphi(x') : p(x') < 1\} = -\sup\{\varphi(x) : p(-x) < 1\} = -\infty.$$

Reasoning like above, one obtains

$$\varphi(B'_p) = (-\infty, \|\varphi|_p). \qquad \square$$

3. EXTENSION RESULTS FOR BOUNDED LINEAR FUNCTIONALS

In this section we shall prove the analogs of some well known extension results for linear functional in normed spaces. The main tool is the Hahn-Banach extension theorem for linear functionals dominated by sublinear functionals.

Throughout this section (X, p) will be a space with asymmetric seminorm.

Proposition 3.1. *Let Y be a subspace of X and $\varphi_0 : Y \to \mathbb{R}$ a bounded linear functional. Then there exists a bounded linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\varphi|_Y = \varphi_0 \quad \textit{and} \quad \|\varphi| = \|\varphi_0|.$$

Proof. The functional $q(x) = \|\varphi_0|p(x),\ x \in X,$ is sublinear and $\varphi_0(y) \leq q(y),\ y \in Y$. By the Hahn-Banach extension theorem there exists a linear functional $\varphi : X \to \mathbb{R}$ such that

$$\varphi|_Y = \varphi_0 \quad \text{and} \quad \forall x \in X\ \varphi(x) \leq \|\varphi_0|p(x).$$

The second of the above relations implies that φ is bounded and $\|\varphi| \leq \|\varphi_0|$. Since

$$\|\varphi| = \sup\{\varphi(x) : x \in X,\ p(x) \leq 1\} \geq \sup\{\varphi(y) : y \in Y,\ p(y) \leq 1\} = \|\varphi_0|,$$

it follows $\|\varphi| = \|\varphi_0|$. $\square$

We agree to call a functional φ satisfying the conclusions of Proposition 3.1 a *norm preserving extension* of φ_0.

Proposition 3.2. *If x_0 is a point in X such that $p(x_0) > 0$ then there exists a bounded linear functional $\varphi : X \to \mathbb{R}$ such that*

$$\|\varphi| = 1 \quad \textit{and} \quad \varphi(x_0) = p(x_0).$$

Proof. Let $Y := \mathbb{R}x_0$ and let $\varphi_0 : Y \to \mathbb{R}$ be defined by $\varphi_0(tx_0) = tp(x_0)$, $t \in \mathbb{R}$. It follows that φ_0 is linear and

$$\varphi_0(tx_0) = tp(x_0) = p(tx_0)$$

for $t > 0$ and

$$\varphi_0(tx_0) = tp(x_0) \le 0 \le p(tx_0)$$

for $t \le 0$. Again, the Hahn-Banach extension theorem yields a linear functional $\varphi : X \to \mathbb{R}$, such that

$$\varphi|_Y = \varphi_0 \quad \text{and} \quad \forall x \in X \ \varphi(x) \le p(x).$$

It follows $\|\varphi| \le 1,\ \varphi(x_0) = p(x_0)$, and, since $p((1/p(x_0))x_0) = 1$,

$$\|\varphi| \ge \varphi(\tfrac{1}{p(x_0)}x_0) = 1,$$

i.e. $\|\varphi| = 1$. □

This last proposition has as consequence the following useful result.

Corollary 3.3. *If $p(x_0) > 0$, then*

$$p(x_0) = \sup\{\varphi(x_0) : \varphi \in X^\flat,\ \|\varphi| \le 1\}.$$

Proof. Denote by s the supremum in the right hand side of the above formula. Since $\varphi(x_0) \le \|\varphi|\, p(x_0) \le p(x_0)$ for every $\varphi \in X^\flat$, $\|\varphi| \le 1$, it follows $s \le p(x_0)$. Choosing $\varphi \in X_p^\flat$ as in Proposition 3.2, it follows $p(x_0) = \varphi(x_0) \le s$. □

The next extension result involves the distance from a point to a set in an asymmetric seminormed space. Let Y be a nonempty subset of an asymmetric seminormed space (X, p). Due to the asymmetry of the seminorm p we have to consider two *distances* from a point $x \in X$ to Y, namely

$$d_p(x, Y) = \inf\{p(y - x) : y \in Y\} \tag{3.1}$$

and

$$d_p(Y, x) = \inf\{p(x - y) : y \in Y\}. \tag{3.2}$$

Observe that $d_p(Y, x) = d_{\bar{p}}(x, Y)$, where $\bar{p}$ is the seminorm conjugate to p.

Proposition 3.4. *Let Y be a subspace of a space with asymmetric seminorm (X, p) and $x_0 \in X$. Denote by $\bar{d}$ the distance $d_{\bar{p}}(x_0, Y)$ and suppose $\bar{d} > 0$.*

Then there exists a p-bounded linear functional $\varphi : X \to \mathbb{R}$ such that

$$(i)\ \varphi|_Y = 0, \quad (ii)\ \|\varphi| = 1, \quad \textit{and} \quad (iii)\ \varphi(x_0) = \bar{d}.$$

If $d = d_p(x_0, Y) > 0$ then there exists $\psi \in X_p^\flat$ such that

$$(j)\ \psi|_Y = 0, \quad (jj)\ \|\psi| = 1, \quad (jjj)\ \psi(-x_0) = d.$$

Proof. Suppose that $\bar{d} = d_{\bar{p}}(x_0, Y) > 0$, so that $x_0 \notin Y$. Let $Z := Y \dot{+} \mathbb{R}x_0$ ($\dot{+}$ stands for the direct sum) and let $\psi_0 : Z \to \mathbb{R}$ be defined by

$$\psi_0(y + tx_0) = t,\ y \in Y,\ t \in \mathbb{R}.$$

Then ψ_0 is linear, $\psi_0(y) = 0,\ \forall y \in Y,$ and $\psi_0(x_0) = 1$. For $t > 0$ we have

$$p(y + tx_0) = tp(x_0 + t^{-1}y) \geq t\bar{d} = \bar{d} \cdot \psi_0(y + tx_0),$$

so that

$$\psi_0(y + tx_0) = t \leq \tfrac{1}{\bar{d}} p(y + tx_0).$$

Since this inequality obviously holds for $t \leq 0$, it follows $\|\psi_0| \leq 1/\bar{d}$. Let (y_n) be a sequence in Y such that $p(x_0 - y_n) \to \bar{d}$ for $n \to \infty$ and $p(x_0 - y_n) > 0$ for all $n \in \mathbb{N}$. Then

$$\|\psi_0| \geq \psi_0\big(\tfrac{x_0 - y_n}{p(x_0 - y_n)}\big) = \tfrac{1}{p(x_0 - y_n)} \to \tfrac{1}{\bar{d}},$$

implying $\|\psi_0| \geq 1/\bar{d}$. Therefore $\|\psi_0| = 1/\bar{d}$.

If $\bar{\psi} : X \to \mathbb{R}$ is a linear functional such that

$$\bar{\psi}|_Z = \psi_0 \quad \text{and} \quad \|\bar{\psi}| = \|\psi_0|$$

then the linear functional $\varphi = \bar{d} \cdot \bar{\psi}$ fulfills all the requirements of the proposition.

Suppose now $d = d_p(x_0, Y) > 0$, and let $Z := Y \dot{+} \mathbb{R}x_0$. Define $\psi_0 : Z \to \mathbb{R}$ by

$$\psi_0(y + tx_0) = -t \iff \psi_0(y - tx_0) = t \quad \text{for } y \in Y \text{ and } t \in \mathbb{R}.$$

Then ψ_0 is linear and, for $t > 0$, we have

$$p(y - tx_0) = tp(\tfrac{1}{t}y - x_0) \geq td = d \cdot \psi_0(y - tx_0),$$

so that

$$\psi_0(y - tx_0) \leq \tfrac{1}{d} p(y - tx_0),$$

for $t > 0$. Since this inequality is obviously true if $\psi_0(y - tx_0) = t \leq 0$, it follows that ψ_0 is bounded and $\|\psi_0| \leq 1/d$. Choosing a sequence (y'_n) in Y such that $p(y'_n - x_0) \to d$ and $p(y'_n - x_0) > 0$ for all n, and reasoning like above one obtains the inequality $\|\psi_0| \geq 1/d$, so that $\|\psi_0| = 1/d$. Extending ψ_0 to a functional $\psi_1 \in X_p^\flat$ of the same norm, and letting $\psi = d \cdot \psi_1$, one obtains the wanted functional ψ. □

4. APPLICATIONS TO BEST APPROXIMATION

Let (X, p) be a space with asymmetric seminorm and Y a nonempty subset of X. By the asymmetry of the seminorm p we have to distinct two "distances" from a point $x \in X$ to the subset Y, as given by (3.1) and (3.2).

Since $d_p(Y, x) = d_{\bar{p}}(x, Y)$, we shall use the notation $d_{\bar{p}}(x, Y)$ for the distance (3.2).

An element $y_0 \in Y$ such that $p(x - y_0) = \bar{p}(y_0 - x) = d_{\bar{p}}(x, Y)$ is called a *$\bar{p}$-nearest point* to x in Y, and an element $y_1 \in Y$ such that $p(y_1 - x) = d_p(x, Y)$ will be called a *p-nearest point* to x in Y.

By Proposition 3.4, we obtain the following characterization of $\bar{p}$-nearest points.

Proposition 4.1. *Let (X,p) be a space with asymmetric seminorm, Y a subspace of X and x_0 a point in X such that $\bar{d} = d_{\bar{p}}(x_0, Y) > 0$.*

An element $y_0 \in Y$ is a $\bar{p}$-nearest point to x_0 in Y if and only if there exists a bounded linear functional $\varphi : X \to \mathbb{R}$ such that

$$\text{(i) } \varphi|_Y = 0, \quad \text{(ii) } \|\varphi| = 1, \quad \text{(iii) } \varphi(x_0) = p(x_0 - y_0).$$

Proof. Suppose that $y_0 \in Y$ is such that $p(x_0 - y_0) = d = d_{\bar{p}}(x_0, Y) > 0$. By Proposition 3.4, there exists $\varphi \in X_p^\flat$, $\|\varphi| = 1$, such that $\varphi|_Y = 0$ and $\varphi(x_0) = d = p(x_0 - y_0)$.

Conversely, if for $y_0 \in Y$ there exists $\varphi \in X_p^\flat$ satisfying the conditions (i)–(iii), then for every $y \in Y$,

$$p(x_0 - y) \geq \varphi(x_0 - y) = \varphi(x_0 - y_0) = p(x_0 - y_0),$$

implying $p(x_0 - y_0) = d_{\bar{p}}(x_0, Y)$. □

Another consequence of Proposition 3.4 is the following duality formula for best approximation:

Proposition 4.2. *Let Y be a subspace of a space with asymmetric seminorm (X,p). If $d_p(Y, x_0) > 0$ then the following duality formula holds:*

$$d_p(Y, x_0) = \sup\{\psi(x_0) : \psi \in Y^\perp,\ \|\psi| \leq 1\},$$

where $Y^\perp = \{\varphi \in X_p^\flat : \varphi|_Y = 0\}$.

Proof. For any $\psi \in Y^\perp$, $\|\psi| \leq 1$ and any $y \in Y$, we have:

$$\psi(x_0) = \psi(x_0 - y) \leq p(x_0 - y),$$

implying $\sup\{\psi(x_0) : \psi \in Y^\perp, \|\psi| \leq 1\} \leq d_p(Y, x_0)$. If we choose ψ to be the functional φ given by Proposition 3.4, then we obtain the reverse inequality: $d_p(Y, x_0) = \varphi(x_0) \leq \sup\{\psi(x_0) : \psi \in Y^\perp, \|\psi| \leq 1\}$. □

The distance to a hyperplane

The well known formula for the distance to a closed hyperplane in a normed space has an analog in spaces with asymmetric seminorm. Remark that in this case we have to work with both of the distances d_p and $d_{\bar{p}}$ given by (3.1) and (3.2).

Proposition 4.3. *Let (X,p) be a space with asymmetric seminorm, $\varphi \in X_p^\flat$, $\varphi \neq 0$, $c \in \mathbb{R}$,*

$$H = \{x \in X : \varphi(x) = c\}$$

the hyperplane corresponding to φ and c, and

$$H^< = \{x \in X : \varphi(x) < c\} \quad \textit{and} \quad H^> = \{x \in X : \varphi(x) > c\},$$

the open half-spaces determined by H.

1) *We have*

$$d_{\bar{p}}(x_0, H) = \tfrac{\varphi(x_0)-c}{\|\varphi|} \tag{4.1}$$

for every $x_0 \in H^{>}$, *and*

$$d_p(x_0, H) = \tfrac{c-\varphi(x_0)}{\|\varphi|} \tag{4.2}$$

for every $x_0 \in H^{<}$.

2) *If there exists an element* $z_0 \in X$ *with* $p(z_0) = 1$ *such that* $\varphi(z_0) = \|\varphi|$, *then every element in* $H^{>}$ *has a* $\bar{p}$*-nearest point in* H *and every element in* $H^{<}$ *has a* p*-nearest point in* H.

If there is an element $x_0 \in H^{>}$ *having a* $\bar{p}$*-nearest point in* H, *or there is an element* $x_0' \in H^{<}$ *having a* p*-nearest point in* H, *then there exists an element* $z_0 \in X,\ p(z_0) = 1,$ *such that* $\varphi(z_0) = \|\varphi|$. *It follows that, in this case, every element in* $H^{>}$ *has a* $\bar{p}$*-nearest point in* H, *and every element in* $H^{<}$ *has a* p*-nearest point in* H.

Proof. Let $x_0 \in H^{>}$. Then, for every $h \in H,\ \varphi(h) = c$, so that

$$\varphi(x_0) - c = \varphi(x_0 - h) \le \|\varphi|p(x_0 - h),$$

implying

$$d_{\bar{p}}(x_0, H) \ge \tfrac{\varphi(x_0)-c}{\|\varphi|}.$$

By the assertion 2) of Proposition 2.2, there exists a sequence (z_n) in X with $p(z_n) = 1$, such that $\varphi(z_n) \to \|\varphi|$ and $\varphi(z_n) > 0$ for all $n \in \mathbb{N}$. Then

$$h_n := x_0 - \tfrac{\varphi(x_0)-c}{\varphi(z_n)} z_n$$

belongs to H and

$$d_{\bar{p}}(x_0, H) \le p(x_0 - h_n) = \tfrac{\varphi(x_0)-c}{\varphi(z_n)} \to \tfrac{\varphi(x_0)-c}{\|\varphi|}.$$

It follows $d_{\bar{p}}(x_0, H) \ge (\varphi(x_0) - c)/\|\varphi|$, so that formula (4.1) holds.

To prove (4.2), observe that for $h \in H$,

$$c - \varphi(x_0') = \varphi(h - x_0') \le \|\varphi|p(h - x_0'),$$

implying

$$d_p(x_0', H) \ge \tfrac{c-\varphi(x_0')}{\|\varphi|}.$$

If the sequence (z_n) is as above then

$$h_n' := \tfrac{c-\varphi(x_0')}{\varphi(z_n)} z_n + x_0'$$

belongs to H and

$$d_p(x_0', H) \le p(h_n' - x_0') = \tfrac{c-\varphi(x_0')}{\varphi(z_n)} \to \tfrac{c-\varphi(x_0')}{\|\varphi|},$$

so that $d_p(x_0', H) \ge (c - \varphi(x_0'))/\|\varphi|$, and formula (4.2) holds too.

2) Let $z_0 \in X$ be such that $p(z_0) = 1$ and $\varphi(z_0) = \|\varphi|$. Then, for $x_0 \in H^>$ and $x'_0 \in H^<$, the elements

$$h_0 := x_0 - \tfrac{\varphi(x_0)-c}{\varphi(z_0)} z_0 \quad \text{and} \quad h'_0 := \tfrac{c-\varphi(x'_0)}{\varphi(z_0)} z_0 + x'_0$$

belong to H,

$$p(x_0 - h_0) = \tfrac{\varphi(x_0)-c}{\|\varphi|} = d_{\bar{p}}(x_0, H) \quad \text{and} \quad p(h'_n - x'_0) = \tfrac{c-\varphi(x'_0)}{\|\varphi|} = d_p(x'_0, H).$$

If an element $x_0 \in H^>$ has a $\bar{p}$-nearest point $h_0 \in H$, then

$$p(x_0 - h_0) = d_{\bar{p}}(x_0, H) = \tfrac{\varphi(x_0)-c}{\|\varphi|} = \tfrac{\varphi(x_0-h_0)}{\|\varphi|}.$$

It follows that $z_0 = (x_0 - h_0)/p(x_0 - h_0)$ satisfies the conditions $p(z_0) = 1$ and $\varphi(z_0) = \|\varphi|$.

If an element $x'_0 \in H^<$ has a p-nearest point h'_0 in H, then $z'_0 = (h'_0 - x'_0)/p(h'_0 - x'_0)$ satisfies $p(z'_0) = 1$ and $\varphi(z'_0) = \|\varphi|$. $\square$

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