REVUE D'ANALYSE NUMÉRIQUE ET DE THÉORIE DE L'APPROXIMATION
Rev. Anal. Numér. Théor. Approx., vol. 32 (2003) no. 2, pp. 193–202
ictp.acad.ro/jnaat

ON THE CONVERGENCE ORDER OF SOME AITKEN–STEFFENSEN TYPE METHODS*

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Abstract. In this note we make a comparative study of the convergence orders for the Steffensen, Aitken and Aitken–Steffensen methods. We provide some conditions ensuring their local convergence. We study the case when the auxiliary operators used have convergence orders $r_1, r_2 \in \mathbb{N}$ respectively. We show that the Steffensen, Aitken and Aitken–Steffensen methods have the convergence orders $r_1 + 1$, $r_1 + r_2$ and $r_1 r_2 + r_1$ respectively.

MSC 2000. 47J25.
Keywords. Steffensen, Aitken and Aitken–Steffensen iterations.

1. INTRODUCTION

It is well known that the Aitken–Steffensen type methods are meant to accelerate the convergence of some sequences converging to the solutions of operational equations [1], [2], [6]–[9], [12], [13] and [16].

Let X be a Banach space and $F : D \subseteq X \to X$ a nonlinear mapping. Consider the equation

$$F(x) = \theta, \tag{1.1}$$

where θ is the null element of X.

Additionally, consider the equations

$$x = \varphi_1(x), \tag{1.2}$$

$$x = \varphi_2(x), \tag{1.3}$$

which are assumed to be equivalent to (1.1), i.e., they have the same solutions.

As usually, $\mathcal{L}(X)$ stands for the set of linear operators from X into itself. For $x, y, z \in X$ denote by $[x, y; F] \in \mathcal{L}(X)$ the first order divided difference of F at the nodes x and y and by $[x, y, z; F]$ the second order divided difference of F at x, y, z ([7]–[9]).

For solving (1.1) we consider the sequences $(x_n)_{n\geq 0}$ generated by the following methods:

1. The Steffensen method:

$$x_{n+1} = x_n - [x_n, \varphi_1(x_n); F]^{-1} F(x_n), \tag{1.4}$$

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$n = 0, 1, \ldots, \ x_0 \in D$;

2. The Aitken method:

$$x_{n+1} = \varphi_1(x_n) - [\varphi_1(x_n), \varphi_2(x_n); F]^{-1} F(\varphi_1(x_n)), \tag{1.5}$$

$n = 0, 1, \ldots, \ x_0 \in D$;

3. The Aitken–Steffensen method

$$x_{n+1} = \varphi_1(x_n) - [\varphi_1(x_n), \varphi_2(\varphi_1(x_n)); F]^{-1} F(\varphi_1(x_n)), \tag{1.6}$$

$n = 0, 1, \ldots, \ x_0 \in D$;

Assume there exists $x^* \in D$, the common solution for (1.1), (1.2) and (1.3), and that the mappings φ_1, φ_2 are Fréchet differentiable at x^* up to the orders r_1 and r_2 respectively, with

$$\varphi_1^{(i)}(x^*) = \theta_i, \quad i = 1, \ldots, r_1 - 1, \ \varphi_1^{(r_1)}(x^*) \neq \theta_{r_1}, \tag{1.7}$$

where θ_i, $i = 1, \ldots, r_1$ are the null i-linear mapping. Analogously, assume

$$\varphi_2^{(i)}(x^*) = \theta_i, \quad i = 1, \ldots, r_2 - 1, \ \varphi_2^{(r_2)}(x^*) \neq \theta_{r_2}. \tag{1.8}$$

It is well known that relations (1.7) and (1.8) ensure that the iteration processes of the form

$$x_{n+1} = \varphi_1(x_n), \quad n = 0, 1, \ldots, \quad x_0 \in X, \tag{1.9}$$

and

$$x_{n+1} = \varphi_2(x_n), \quad n = 0, 1, \ldots, \ x_0 \in X, \tag{1.10}$$

have the convergence orders r_1, resp. r_2.

In this note we show that the Steffensen method has the convergence order $r_1 + 1$, the Aitken method $r_1 + r_2$, while the Aitken–Steffensen method $r_1(r_2 + 1)$. We also provide conditions ensuring the local convergence of these sequences.

2. LOCAL CONVERGENCE

Let $S = \{x \in X : \|x - x^*\| \leq r\}$ be the ball with center at x^* and with radius r, and suppose $S \subseteq D$.

The mappings φ_1, φ_2 are assumed to obey the following hypotheses:

i) the mapping φ_1 admits Fréchet derivatives up to the order $r_1 \geq 1$ on S, and

$$\sup_{x \in S} \|\varphi_1^{(r_1)}(x)\| = L_1 < +\infty,$$

ii) the mapping φ_2 admits Fréchet derivatives up to the order $r_2 \geq 1$ on S, and

$$\sup_{x \in S} \|\varphi_2^{(r_2)}(x)\| = L_2 < +\infty.$$

The mapping F is assumed to obey

i$_1$) the linear mapping $[x,y;F]$ is invertible for all $x,y\in S$, and

$$\sup_{x,y\in S}\|\,[x,y;F]^{-1}\,\|=m<+\infty;$$

ii$_1$) the bilinear mapping $[x,y,z;F]$ is bounded for all $x,y,z\in S$:

$$\sup_{x,y,z\in S}\|\,[x,y,z;F]\,\|=M<+\infty.$$

Regarding the convergence of the Steffensen method, we obtain the following result.

THEOREM 2.1. *Assume that φ_1 obeys* i) *and* (1.7), *F obeys* i$_1$) *and* ii$_1$), *and the initial approximation $x_0\in S$ is chosen such that*

a) $mM\,|x^*-x_0|<1$;
b) $\frac{L_1}{r_1!}\,\|x^*-x_0\|^{r_1-1}\le 1$.

Then the sequence $(x_n)_{n\ge1}$ generated by (1.4) *remains in S and converges to x^*, satisfying*

j$_1$) $\|x^*-x_n\|\le\big(\frac{MmL_1}{r_1!}\big)^{-\frac{1}{r_1}}\rho_0^{p^n},\quad n=0,1,\ldots,$ *where*

$$\rho_0=\big(\tfrac{MmL_1}{r_1!}\big)^{\frac{1}{r_1}}\,\|x^*-x_0\|<1$$

and $p=r_1+1$;

jj$_1$) $\lim_{n\to\infty}x_n=x^*$.

Proof. Since $x_0\in S$, then by the Taylor formula and by (1.7) and i) we get

$$\|\varphi_1(x_0)-x^*\|=\|\varphi_1(x_0)-\varphi(x^*)\|\le\tfrac{L_1}{r_1!}\,\|x_0-x^*\|^{r_1}\,,\tag{2.1}$$

whence, by b) it follows that

$$\|\varphi_1(x_0)-x^*\|\le\|x^*-x_0\|\le r,$$

i.e., $\varphi_1(x_0)\in S$. By the Newton identity,

$$\begin{aligned}\theta=F(x^*)=&F(x_0)+[x_0,\varphi(x_0);F](x^*-x_0)\\&+[x^*,x_0,\varphi(x_0);F](x^*-\varphi(x_0))(x^*-x_0)\,,\end{aligned}$$

and taking into account i$_1$), ii$_1$) and (1.4) for $n=0$, we obtain

$$x^*-x_1=-[x_0,\varphi(x_0);F]^{-1}[x^*,x_0,\varphi(x_0);F](x^*-\varphi(x_0))(x^*-x_0)$$

whence, using (2.1), a) and b), it follows that

$$\|x^*-x_1\|\le\tfrac{MmL_1}{r_1!}\,\|x^*-x_0\|^{r_1+1}\le\|x^*-x_0\|\le r,\tag{2.2}$$

i.e., $x_1\in S$.

Assume now that $x_1,\ldots,x_n\in S$ and denote

$$\rho_i=\big(\tfrac{MmL_1}{r_1!}\big)^{\frac{1}{r_1}}\,\|x^*-x_i\|\,,\quad i=0,1,\ldots,n.\tag{2.3}$$

Relations a) and b) imply $\rho_0<1$, while (2.2) attracts

$$\rho_1\le\rho_0^p.\tag{2.4}$$

Assume now that

$$\rho_{i+1} \le \rho_i^p, \quad i = 0, \dots, n-1. \tag{2.5}$$

From (2.4) and (2.5) we get

$$\rho_i \le \rho_0^{p^i}, \quad i = 1, \dots, n.$$

We prove now that $x_{n+1} \in S$ and $\rho_{n+1} \le \rho_n^p$. Applying the Taylor formula and taking into account the hypotheses,

$$\|\varphi_1(x_n) - \varphi_1(x^*)\| \le \tfrac{L_1}{r_1!} \|x_n - x^*\|^{r_1}. \tag{2.6}$$

The induction hypotheses also imply

$$\|x_n - x^*\| = \frac{\rho_n}{\left(\frac{MmL_1}{r_1!}\right)^{\frac{1}{r_1}}} \le \frac{\rho_0^{p^n}}{\left(\frac{MmL_1}{(r_1!)}\right)^{\frac{1}{r_1}}} = \|x^* - x_0\|. \tag{2.7}$$

Replacing (2.7) in (2.6) and taking into account b) leads to

$$\|\varphi_1(x_n) - x^*\| \le \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \le r,$$

which shows that $\varphi_1(x_n) \in S$.

The fact that x_n, $\varphi_1(x_n)$, $x^* \in S$ implies

$$\begin{aligned}\theta = F(x^*) = &F(x_n) + [x_n, \varphi_1(x_n); F](x^* - x_n) \\ &+ [x^*, x_n, \varphi_1(x_n); F](x^* - \varphi_1(x_n))(x^* - x_n)\end{aligned}$$

whence

$$\|x^* - x_{n+1}\| \le \tfrac{MmL_1}{r_1!} \|x^* - x_n\|^p. \tag{2.8}$$

Denoting $\rho_{n+1} = \left(\frac{MmL_1}{r_1!}\right)^{\frac{1}{r_1}} \|x^* - x_{n+1}\|$, then by (2.8) we get

$$\rho_{n+1} \le \rho_n^p \le \rho_0^{p^{n+1}}. \tag{2.9}$$

It remains to show that $x_{n+1} \in S$, which follows by (2.9) and (2.8):

$$\|x^* - x_{n+1}\| \le \frac{\rho_0^{p^{n+1}}}{\left(\frac{MmL_1}{r_1!}\right)^{\frac{1}{r_1}}} < \frac{\rho_0}{\left(\frac{MmL_1}{r_1}\right)^{\frac{1}{r_1}}} \le \|x^* - x_0\| \le r.$$

Relation (2.9) implies conclusion (jj$_1$). □

We obtain the following result regarding the Aitken method.

THEOREM 2.2. *Assume that the mappings* φ_1, φ_2 *obey* i) *and* (1.7), *resp.* ii) *and* (1.8), *the mapping* F *obeys* i$_1$), i$_2$) *and the initial approximation* $x_0 \in S$ *is chosen such that*

a′) $\frac{L_1}{r_1!} \|x^* - x_0\|^{r_1 - 1} \le 1$;
b′) $\frac{L_2}{r_2!} \|x^* - x_0\|^{r_2 - 1} \le 1$;
c′) $Mm \|x^* - x_0\| < 1$.

Then the sequence generated by (1.5) *remains in* S, *converges to* x^* *and, moreover, the following relations are true:*

j$_2$) $\|x_n - x^*\| \leq \left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{1-q}} \rho_0^{q^n}$, $n = 1, 2, \ldots,$ *where*

$$\rho_0 = \left(\tfrac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}} \|x^* - x_0\| < 1$$

and $q = r_1 + r_2$;

jj$_2$) $\lim_{n\to\infty} x_n = x^*$.

Proof. From i), ii), (1.7) and (1.8) it results

$$\|x^* - \varphi_1(x_0)\| \leq \tfrac{L_1}{r_2!} \|x^* - x_0\|^{r_1}, \tag{2.10}$$

whence, by a$'$) it follows that $\varphi_1(x_0) \in S$. Analogously, for φ_2 we get

$$\|x^* - \varphi_2(x_0)\| \leq \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_2}, \tag{2.11}$$

i.e., $\varphi_2(x_0) \in S$.

The Newton identity for F,

$$\begin{aligned}\theta = F(x^*) =& F(\varphi_1(x_0)) + [\varphi_1(x_0), \varphi_2(x_0); F](x^* - \varphi_1(x_0)) \\ &+ [x^*, \varphi_1(x_0), \varphi_2(x_0); F](x^* - \varphi_2(x_0))(x^* - \varphi_1(x_0))\end{aligned}$$

by (1.5) for $n = 0$, (2.10), (2.11), implies

$$\begin{aligned}\|x_1 - x^*\| \leq& Mm \|x^* - \varphi_2(x_0)\| \|x^* - \varphi_1(x_0)\| \\ \leq& \tfrac{MmL_1L_2}{r_1!r_2!} \|x^* - x_0\|^{r_1+r_2}.\end{aligned} \tag{2.12}$$

Denoting $r_1 + r_2 = q$ and

$$\rho_k = \left(\tfrac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}} \|x^* - x_k\|, \quad k = 0, 1, \tag{2.13}$$

then, by (2.12) we get

$$\rho_1 \leq \rho_0^q. \tag{2.14}$$

From a$'$), b$'$) and c$'$) we obtain that $x_1 \in S$ and

$$\rho_0 < 1. \tag{2.15}$$

The fact that $x_1 \in S$ is implied by relations (2.14) and (2.15):

$$\|x_1 - x^*\| \leq \frac{\rho_0^q}{\left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}}} < \frac{\rho_0}{\left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}}} = \|x_0 - x^*\| \leq r. \tag{2.16}$$

It remains to show that $\varphi_1(x_1)$, $\varphi_2(x_1) \in S$. Using a$'$) and (2.16), we have

$$\begin{aligned}\|x^* - \varphi_1(x_1)\| \leq& \tfrac{L_1}{r_1!} \|x^* - x_1\|^{r_1} \\ \leq& \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \\ \leq& \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1-1} \|x^* - x_0\| \\ \leq& r\end{aligned}$$

and, analogously, (2.16) and b$'$) imply

$$\|x^* - \varphi_2(x_1)\| \leq \tfrac{L_2}{r_2!} \|x^* - x_1\|^{r_2} \leq \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_2} \leq r.$$

Assume $x_0, x_1, \ldots, x_k \in S$ and $\varphi_i(x_s) \in S,\ i = 1, 2,\ s = 0, \ldots, k$.

Denote

$$\rho_i = \left(\tfrac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}} \|x^* - x_i\|, \quad i = 1, \ldots, k \tag{2.17}$$

and assume that

$$\rho_{i+1} \leq \rho_i^q, \quad i = 1, \ldots, k-1. \tag{2.18}$$

Relations (2.18), together with (2.14) show that the following inequalities are verified:

$$\rho_i \leq \rho_0^{q^i}, \quad i = 1, \ldots, k. \tag{2.19}$$

We show now that $x_{k+1} \in S$. Similarly to (2.12), one immediately deduces that

$$\|x_{k+1} - x^*\| \leq \tfrac{MmL_1L_2}{r_1!r_2!} \|x^* - x_k\|^q. \tag{2.20}$$

Denoting

$$\rho_{k+1} = \left(\tfrac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}} \|x^* - x_k\|, \tag{2.21}$$

then by (2.20) and (2.19) we deduce

$$\rho_{k+1} \leq \rho_0^{q^{k+1}}, \tag{2.22}$$

i.e., (2.19) holds also for $i = k+1$.

From (2.20), (2.21) and (2.22) we get

$$\|x_{k+1} - x^*\| \leq \frac{\rho_0^{q^{k+1}}}{\left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}}} \leq \frac{\rho_0}{\left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}}} \leq \|x^* - x_0\| \leq r, \tag{2.23}$$

i.e., $x_{k+1} \in S$. It remains to prove that $\varphi_1(x_{k+1})$, $\varphi_2(x_{k+1}) \in S$.

The Taylor formula leads to the relations

$$\|x^* - \varphi_1(x_{k+1})\| \leq \tfrac{L_1}{r_1!} \|x^* - x_{k+1}\|^{r_1}$$

and

$$\|x^* - \varphi_2(x_{k+1})\| \leq \tfrac{L_2}{r_2!} \|x^* - x_{k+1}\|^{r_2}.$$

By (2.23), we get $\|x^* - x_{k+1}\| \leq \|x^* - x_0\|$ and so, from a′) and b′),

$$\|x^* - \varphi_1(x_{k+1})\| \leq \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \leq r$$

and, analogously

$$\|x^* - \varphi_2(x_{k+1})\| \leq \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_1} \leq r$$

i.e., $\varphi_1(x_{k+1}), \varphi_2(x_{k+1}) \in S$.

According to the induction principle, relations (2.19) are true for all $i \in \mathbb{N}$, so

$$\|x_n - x^*\| \leq \frac{1}{\left(\frac{MmL_1L_2}{r_1!r_2!}\right)^{\frac{1}{q-1}}} \rho_0^{q^n}$$

whence, since $\rho_0 < 1$, $\lim_{n\to\infty} \|x_n - x^*\| = 0$, i.e., $\lim_{n\to\infty} x_n = x^*$. $\square$

Finally, the following result holds for the Aitken–Steffensen method.

THEOREM 2.3. *Under the hypotheses of Theorem* 2.2, *the sequence* $(x_n)_{n\geq 0}$, *generated by* (1.6) *remains in the ball* S, *and converge to* x^* *such that:*

j$_3$) $\|x^* - x_n\| \leq L^{\frac{1}{1-s}} \rho_0^{s^n}$,
where $\rho_0 = L^{\frac{1}{s-1}} \|x^* - x_0\|$, $L = mM\left(\frac{L_1}{r_1!}\right)^{r_2+1} \frac{L_2}{r_2!}$, $n = 0, 1, \ldots,$ *and* $s = r_1 r_2 + r_1$;

jj$_3$) $\lim_{n\to\infty} x_n = x^*$.

Proof. Assume that $x_0 \in S$ verifies a$'$)–c$'$). We show first that $\varphi_1(x_0)$, $\varphi_2(\varphi_1(x_0)) \in S$. For φ_1 we have

$$\|x^* - \varphi_1(x_0)\| \leq \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \leq \|x^* - x_0\| \leq r \tag{2.24}$$

i.e., $\varphi_1(x_0) \in S$. Next, taking into account (2.24), we get

$$\begin{aligned}
\|x^* - \varphi_2(\varphi_1(x_0))\| \leq& \tfrac{L_2}{r_2!} \|x^* - \varphi_1(x_0)\| \\
\leq& \tfrac{L_2}{r_2!} \tfrac{L_1^{r_2}}{(r_1!)^{r_2}} \|x^* - x_0\|^{r_1 r_2} \\
=& \tfrac{L_2}{r_2!} \left[\tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1-1}\right]^{r_2} \|x^* - x_0\|^{r_2} \\
\leq& \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_2} \\
\leq& \|x^* - x_0\| \\
\leq& r.
\end{aligned}$$

We prove now that $x_1 \in S$. Analogously to (2.11), we get

$$\begin{aligned}
\|x_1 - x^*\| \leq& Mm \|x^* - \varphi_1(x_0)\| \|x^* - \varphi_2(\varphi_1(x_0))\| \\
\leq& Mm\left(\tfrac{L_1}{r_1!}\right)^{r_2+1} \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_1 r_2 + r_1}.
\end{aligned} \tag{2.25}$$

As it can be easily noticed, the previous relation may also be written as

$$\begin{aligned}
\|x_1 - x^*\| \leq& Mm \|x^* - x_0\| \left[\tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1-1}\right]^{r_2} \\
&\cdot \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1-1} \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_2-1} \|x^* - x_0\|,
\end{aligned} \tag{2.26}$$

whence, by a$'$)–c$'$), it follows that $\|x_1 - x^*\| \leq r$, i.e., $x_1 \in S$. Further, denote $s = r_1 r_2 + r_1$, $L = Mm\left(\frac{L_1}{r_1!}\right)^{r_2+1} \frac{L_2}{r_2!}$, and so inequality (2.25) becomes

$$\|x_1 - x^*\| \leq L \|x^* - x_0\|^s,$$

and, for $L^{\frac{1}{s-1}} \|x^* - x_0\| = \rho_0$, it reads as

$$\rho_1 \leq \rho_0^s, \tag{2.27}$$

with $\rho_1 = L^{\frac{1}{s-1}} \|x^* - x_1\|$.

Next, we show that $\varphi_1(x_1)$, $\varphi_2(\varphi_1(x_1)) \in S$. From (2.26) we have that

$$\|x^* - x_1\| \leq \|x^* - x_0\|. \tag{2.28}$$

For $\varphi_1(x_1)$ we obtain

$$\|x^* - \varphi_1(x_1)\| \le \tfrac{L_1}{r_1!} \|x^* - x_1\|^{r_1} \le \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \le r,$$

while for $\varphi_2(\varphi_1(x_1))$ we deduce

$$\begin{aligned}\|x^* - \varphi_2(\varphi_1(x_1))\| \le & \tfrac{L_2}{r_2!} \big(\tfrac{L_1}{r_1!}\big)^{r_2} \|x^* - x_1\|^{r_1 r_2} \\ = & \tfrac{L_2}{r_2!} \|x^* - x_1\|^{r_2 - 1} \big[\tfrac{L_1}{r_1!} \|x^* - x_1\|^{r_1 - 1}\big]^{r_2} \|x^* - x_1\|\end{aligned}$$

whence, taking into account (2.28) and a′), b′), it follows

$$\|x^* - \varphi_2(\varphi_1(x_1))\| \le r$$

i.e., $\varphi_2(\varphi_1(x_1)) \in S$.

Assume now that $x_0, \ldots, x_k \in S$, $\varphi_1(x_0), \ldots, \varphi_1(x_k) \in S$, $\varphi_2(\varphi_1(x_0)), \ldots,$ $\varphi_2(\varphi_1(x_k)) \in S$. Denote $\rho_i = L^{\frac{1}{s-1}} \|x^* - x_i\|$, $i = 0, \ldots, k$ and also suppose that

$$\rho_{i+1} \le \rho_i^s, \quad i = 0, \ldots, k-1 \tag{2.29}$$

which, by (2.27), becomes

$$\rho_i \le \rho_0^{s^i}, \quad i = 1, \ldots, k. \tag{2.30}$$

We show now that $x_{k+1} \in S$. Using the hypotheses of the theorem and the Newton identity, it can be easily shown that

$$\begin{aligned}\|x^* - x_{k+1}\| \le & Mm \|x^* - \varphi_1(x_k)\| \|x^* - \varphi_2(\varphi_1(x_k))\| \\ \le & Mm \big(\tfrac{L_1}{r_1!}\big)^{r_2+1} \tfrac{L_2}{r_2!} \|x^* - x_0\|^{r_1 r_2 + r_1}\end{aligned}$$

i.e., by denoting $\rho_{k+1} L^{\frac{1}{s-1}} \|x^* - x_{k+1}\|$,

$$\rho_{k+1} \le \rho_k^s.$$

By (2.30), this leads to

$$\rho_{k+1} \le \rho_0^{s^{k+1}},$$

and since $\rho_0 < 1$, we get

$$\rho_{k+1} \le \rho_0.$$

Further,

$$\|x_{k+1} - x^*\| \le \|x_0 - x^*\| \le r,$$

which shows that $x_{k+1} \in S$. Now, we show that $\varphi_1(x_{k+1}), \varphi_2(\varphi_1(x_{k+1})) \in S$. First, taking into account a′),

$$\|x^* - \varphi_1(x_{k+1})\| \le \tfrac{L_1}{r_1!} \|x^* - x_k\|^{r_1} \le \tfrac{L_1}{r_1!} \|x^* - x_0\|^{r_1} \le r,$$

and, finally,

$$\begin{aligned}\|x^* - \varphi_2(\varphi_1(x_{k+1}))\| \le & \tfrac{L_2}{r_2!} \big(\tfrac{L_1}{r_1!}\big)^{r_2} \|x^* - x_{k+1}\|^{r_1 r_2} \\ \le & \tfrac{L_2}{r_2!} \big(\tfrac{L_1}{r_1!}\big)^{r_2} \|x^* - x_0\|^{r_1 r_2} \\ \le & r.\end{aligned}$$

$\square$

Inequalities j_1), j_2), j_3) from this conclusions of Theorems 2.1, 2.2, resp. 2.3 characterize the convergence orders of the methods (1.4), (1.5), resp. (1.6).

The numbers r_1, resp. r_2 represent as we have already specified, the convergence orders of the iteration methods given by (1.9), resp. (1.10).

We also notice the following facts.

Remark 2.1. If $r_1 = r_2 = 1$, then the three studied methods have the same convergence order: $p = q = s = 2$.

Remark 2.2. Regarding method (1.6), we may also consider the following iterations instead:

$$x_{n+1} = \varphi_2(x_n) - [\varphi_2(x_n), \varphi_1(\varphi_2(x_n)); F]^{-1} F(\varphi_2(x_n)) \tag{2.31}$$

having the same convergence order $r_1 r_2 + r_2$. Consequently, if $r_2 > r_1$, then (2.31) is preferred instead of (1.6).

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Received by the editors: April 13, 2003.