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CHEBYSHEV-LIKE METHODS AND QUADRATIC EQUATIONS *

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Abstract. We show the classical Kantorovich technique to study the convergence of a new uniparametric family of third order iterative processes defined in Banach spaces. We obtain information about this family from the study of the same family in the real case. Besides we obtain, for a value of the parameter, a method which has order four when it is applied to quadratic equations.

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1. INTRODUCTION

Several scientific problems can be written in the form

$$F\left(x\right)=0. \tag{1}$$

In order to generalize as much as possible, let F be a nonlinear operator defined in an open convex domain Ω of a Banach space X with values in another Banach space Y. There are a lot of research works concerning the numerical solution of (1) by means of iterative processes, mainly by using Newton's method [8], [11].

But there are other iterative processes to solve (1). One of them is Chebyshev method [4], [5]:

$$x_{n+1}=x_n-\left[1+\tfrac{1}{2}L_F\left(x_n\right)\right]F'\left(x_n\right)^{-1}F\left(x_n\right), \tag{2}$$

where I is the identity operator on X and $L_F\left(x\right)$ is the linear operator on X given by

$$L_F\left(x\right)=F'\left(x\right)^{-1}F''\left(x\right)F'\left(x\right)^{-1}F\left(x\right),\quad x\in X,$$

assuming that $F'\left(x\right)^{-1}$ exists.

We notice that the only inverse operator we have to evaluate in each step of Chebyshev method is $F'\left(x_n\right)^{-1}$. This same inverse must be also calculate in each step of Newton's method. However, the expression of other third order iterative

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processes (Halley, super-Halley [7]) involves the computation of other different inverse operators. So, we can see Chebyshev method as the third order method with less computational cost.

In this paper we study a uniparametric family of third order iterations that only needs the computation of the inverse $F'(x_n)^{-1}$ in each step:

$$(3) \quad x_{n+1} = x_n - \left[1 + \tfrac{1}{2} L_F(x_n) + \alpha L_F(x_n)^2\right] F'(x_n)^{-1} F(x_n), \quad \alpha \in \left[0, \tfrac{1}{2}\right].$$

This family incudes the Chebyshev method as a particular case ($\alpha = 0$). We prove that the methods of (3) have got a faster convergence than Chebyshev method. Secondly, we find the value of the parameter α for which the convergence is four when the method is applied to quadratic equations. Finally, we compare the methods of the family (3), showing that for $\alpha = \frac{1}{2}$ we obtain the best method, from the standpoint of the speed of convergence and the error estimates.

The origin of the family (3) is a Gander's result [6] on third order iterative processes in the real case. In this work it is established that for a real function f of real variable, the iteration in the form:

$$x_{n+1} = x_n - H(L_f(x_n)) \tfrac{f(x_n)}{f'(x_n)},$$

where

$$L_f(x) = \tfrac{f(x) f''(x)}{f'(x)^2},$$

and H is a function satisfying $H(0) = 1$, $H'(0) = \frac{1}{2}$ and $|H''(0)| < +\infty$, are cubically convergent. In this paper we consider a function H with a finite Taylor's expansion and its generalization to Banach spaces. Of course, there are other functions H that give rise to third order methods. But, for instance, the functions that originate the Halley or the super-Halley methods, have got a non finite Taylor expansion. Hence the computational cost is higher.

2. A STUDY OF THE CONVERGENCE

To prove the convergence of the iterative processes given in (3), we assume that F satisfies the classical Kantorovich conditions [8]. These kind of conditions have been used by different authors in the study of third order iterative processes [1], [2], [4], [13]. So, throughout this paper we assume the following conditions:

(i) There exists $x_0 \in \Omega$ where $\Gamma_0 = F'(x_0)^{-1}$ is defined.

(ii) $\|\Gamma_0 (F''(x) - F''(y))\| \leq k \|x - y\|, \quad x, y \in \Omega, \ k \geq 0.$

(iii) $\|\Gamma_0 F(x_0)\| \leq a, \ \|\Gamma_0 F''(x_0)\| \leq b.$

(*iv*) The equation

$$p(t) \equiv \tfrac{k}{6}t^3 + \tfrac{b}{2}t^2 - t + a = 0 \tag{4}$$

has got a negative root and two positive roots r_1 and $r_2, (r_1 \le r_2)$ when $k > 0$. If $k = 0$, then (4) has got two positive roots r_1 and $r_2\,(r_1 \le r_2)$. Equivalently,

$$a \le \frac{b^2+4k-b\sqrt{b^2+2k}}{3k\left(b+\sqrt{b^2+2k}\right)}, \quad \text{if } k > 0,$$

or $ab \le \frac{1}{2}$ if $k = 0$.

First, we analyse the real sequence $\{t_n\}$ defined for

$$t_0 = 0, \quad t_{n+1} = G(t_n) = t_n - \left[1+\tfrac{1}{2}L_p(t_n)+\alpha L_p(t_n)^2\right]\frac{p(t_n)}{p'(t_n)}, \quad n \ge 0, \tag{5}$$

when $\alpha \in \left[0, \frac{1}{2}\right]$ and p is the polynomial (4).

The real sequences (5) allow us to establish the convergence of the sequences (3) defined in Banach spaces. So, under the hypothesis (**i**)–(**iv**), the sequences (5) and (3) are well defined and converge to r_1 and to x^*, a solution of (1), respectively. Moreover

$$\|x_{n+1} - x_n\| \le t_{n+1} - t_n, \quad \|x^* - x_n\| \le r_1 - t_n, \quad n \ge 0,$$

that is, $\{t_n\}$ majorizes $\{x_n\}$ (see [8], [12]).

LEMMA 2.1. *Let p be the polynomial defined in* (4). *Then if* $0 \le \alpha \le 0.5$, *the sequences* $\{t_n\}$ *given in* (5) *are well defined, are increasing and converge to* r_1.

Proof. It is easy to prove that $t_{n+1} \ge t_n$ because p is a decreasing convex polynomial in $[0, r_1]$.

On the other hand, let G be the function defined in (5). Then

$$G'(t) = L_p(t)^2\left[3\left(\tfrac{1}{2} - \alpha\right) + 5\alpha L_p(t) - \left(\tfrac{1}{2} + 2\alpha L_p(t)\right)L_{p'}(t)\right].$$

As $L_{p'}(t) \le 0$ for $t \in [0, r_1]$, we have $G'(t) \ge 0$, for $t \in [0, r_1]$, $0 \le \alpha \le \frac{1}{2}$. Thus $t_n \le r_1$. Consequently, $\{t_n\}$ converges to r_1. □

Notice that the sequences $\{t_n\}$ converge cubically to r_1 as a consequence of the Gander's result given in the introduction. Now, we study the sequences $\{x_n\}$, $\alpha \in \left[0, \frac{1}{2}\right]$, defined in Banach spaces.

LEMMA 2.2. *For* $\alpha \in \left[0, \frac{1}{2}\right]$ *and* $n \ge 0$ *we have:*

[I_n] *There exists* $\Gamma_n = F'(x_n)^{-1}$.

[II$_n$] $\|\Gamma_n F'(x_0)\| \le \frac{p'(t_0)}{p'(t_n)}$.
[III$_n$] $\|\Gamma_0 F''(x_n)\| \le -\frac{p''(t_n)}{p'(t_0)}$.
[IV$_n$] $\|\Gamma_0 F(x_n)\| \le -\frac{p(t_n)}{p'(t_0)}$.
[V$_n$] $\|x_{n+1} - x_n\| \le t_{n+1} - t_n$.

Proof. We use an inductive process. For $n = 0$, [I$_0$]–[IV$_0$] follow immediately from the hypothesis. To prove [V$_0$] we have

$$\|L_F(x_0)\| \le \left\|\Gamma_0 F''(x_0)\right\| \cdot \|\Gamma_0 F(x_0)\| \le \frac{p(t_0)p''(t_0)}{p'(t_0)^2} L_p(t_0),$$

and then,

$$\begin{aligned}\|x_1 - x_0\| &\le \left(1 + \tfrac{1}{2}\|L_F(x_0)\| + \alpha\|L_F(x_0)^2\|\right)\|\Gamma_0 F(x_0)\| \\ &\le -\left(1 + \tfrac{1}{2}L_p(t_0) + \alpha L_p(t_0)^2\right)\frac{p(t_0)}{p'(t_0)} = t_1 - t_0.\end{aligned}$$

Let us assume now that [I$_n$]–[V$_n$] are true for a given n. Then

$$\begin{aligned}\left\|\Gamma_0 F'(x_{n+1}) - I - \Gamma_0 F''(x_0)(x_{n+1} - x_0)\right\| &= \left\|\int_{x_0}^{x_{n+1}} \Gamma_0\left[F''(x) - F''(x_0)\right] dx\right\| \\ &\le \tfrac{k}{2}\|x_{n+1} - x_0\|^2.\end{aligned}$$

Consequently,

$$\begin{aligned}\left\|I - \Gamma_0 F'(x_{n+1})\right\| &\le \tfrac{k}{2}\|x_{n+1} - x_0\|^2 + b\|x_{n+1} - x_0\| \\ &\le \tfrac{k}{2}t_{n+1}^2 + bt_{n+1} \\ &= 1 + p'(t_n + 1) < 1,\end{aligned}$$

and, by the Banach lemma on inversion of operators, there exists $[\Gamma_0 F'(x_{n+1})]^{-1}$ (and therefore $F'(x_{n+1})^{-1}$) and besides,

$$\left\|\Gamma_n F'(x_0)\right\| \le -\frac{1}{p'(t_n)} = \frac{p'(t_0)}{p'(t_n)}.$$

So we have shown [I$_{n+1}$] and [II$_{n+1}$].

To see [III$_{n+1}$], we notice that

$$\begin{aligned}\left\|\Gamma_0 F'' x_{n+1}\right\| &= \left\|\Gamma_0\left[F''(x_{n+1}) - F''(x_0)\right] + \Gamma_0 F''(x_0)\right\| \\ &\le k\|x_{n+1} - x_0\| + b \le p''(t_n + 1).\end{aligned}$$

To show $[\mathrm{IV}_{n+1}]$ we have, from (3) and the Taylor's formula,

$$\begin{aligned}
F(x_{n+1}) &= F(x_n) + F'(x_n)(x_{n+1}-x_n) + \tfrac{1}{2}F''(x_n)(x_{n+1}-x_n)^2 \\
&\quad + \int_{x_n}^{x_{n+1}} \left[F''(x) - F''(x_n)\right](x_{n+1}-x)\,dx \\
&= -\tfrac{1}{2}F''(x_n)(\Gamma_n F(x_n))^2 - \alpha F''(x_n)\Gamma_n F(x_n) L_F(x_n)\Gamma_n F(x_n) \\
&\quad + \tfrac{1}{2}F''(x_n)(x_{n+1}-x_n)^2 + \int_{x_n}^{x_{n+1}} \left[F''(x) - F''(x_n)\right](x_{n+1}-x)\,dx \\
&= \left(\tfrac{1}{2}-\alpha\right) F''(x_n)\Gamma_n F(x_n) L_F(x_n)\Gamma_n F(x_n) \\
&\quad + \alpha F''(x_n)\Gamma_n F(x_n) L_F(x_n)^2 \Gamma_n F(x_n) \\
&\quad + \tfrac{1}{8}F''(x_n) L_F(x_n)\Gamma_n F(x_n) L_F(x_n)\Gamma_n F(x_n) \\
&\quad + \tfrac{\alpha}{2}F''(x_n) L_F(x_n)\Gamma_n F(x_n) L_F(x_n)^2 \Gamma_n F(x_n) \\
&\quad + \tfrac{\alpha^2}{2}F''(x_n) L_F(x_n)^2\Gamma_n F(x_n) L_F(x_n)^2 \Gamma_n F(x_n) \\
&\quad + \int_{x_n}^{x_{n+1}} \left[F''(x) - F''(x_n)\right](x_{n+1}-x)\,dx.
\end{aligned}$$

The inductive procedure leads us to

$$\|L_F(x_n)\| \le L_p(t_n), \quad \|\Gamma_n F(x_n)\| \le -\tfrac{p(t_n)}{p'(t_n)}.$$

As $0 \le \alpha \le \frac{1}{2}$, we deduce

$$\begin{aligned}
\|\Gamma_0 F(x_{n+1})\| &\le \tfrac{k}{6}(t_{n+1}-t_n)^3 - \\
&- \tfrac{p(t_n)}{p'(t_0)}\left[\left(\tfrac{1}{2}-\alpha\right)L_p(t_n)^2 + \alpha L_p(t_n)^3 + \tfrac{1}{8}L_p(t_n)^3 + \tfrac{\alpha}{2}L_p(t_n)^4 + \tfrac{\alpha^2}{2}L_p(t_n)^5\right].
\end{aligned}$$

Repeating the same process with the polynomial p, we obtain

$$\begin{aligned}
p(t_{n+1}) &= \tfrac{k}{6}(t_{n+1}-t_n)^3 + \\
&+ p(t_n)\left[\left(\tfrac{1}{2}-\alpha\right)L_p(t_n)^2 + \alpha L_p(t_n)^3 + \tfrac{1}{8}L_p(t_n)^3 + \tfrac{\alpha}{2}L_p(t_n)^4 + \tfrac{\alpha^2}{2}L_p(t_n)^5\right],
\end{aligned}$$

and consequently,

$$\|\Gamma_0 F(x_{n+1})\| \le -\tfrac{p(t_{n+1})}{p'(t_0)}. \tag{6}$$

Finally, to see $[\mathrm{V}_{n+1}]$, we proceed as in the case $n = 0$. □

The following result gives us conditions on the convergence of the methods or the family (3).

THEOREM 2.3. *Assume that conditions* (i)–(iv) *hold and*

$$\bar{B} = \overline{B(x_0, r_1)} = \{x \in X; \|x - x_0\| \leq r_1\} \subseteq \Omega.$$

Then:

(a) *The sequences* (3) *are well-defined for* $\alpha \in \left[0, \frac{1}{2}\right]$, *lie in* B *(interior of* $\bar{B}$*) and converge to a solution* x^* *of the equation* (1).
(b) x^* *is the only solution of* (1) *in* $B(x_0, r_2) \cap \Omega$.
(c) *We have the following error bounds:*

$$\|x^* - x_n\| \leq r_1 - t_n.$$

Proof. The convergence follows immediately from Lemmas 2.1 and 2.2. Besides, x^* is the solution of (1) as a consequence of (6). To show the uniqueness, we assume that y^* is another solution of (1) in $\overline{B(x_0, r)}$ for some $r > 0$. Then, following Argyros and Chen [4], we deduce

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\, dt\, (y^* - x^*).$$

We have to see that $\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ has got an inverse. So, we take into account that

$$\begin{aligned} I - \Gamma_0 &\int_0^1 F'(x^* + t(y^* - x^*))\, dt = \\ &= -\Gamma_0 \int_0^1 \int_{x_0}^{x^*+t(y^*-x^*)} F''(z)\, dz dt \\ &= -\Gamma_0 \int_0^1 \int_{x_0}^{x^*+t(y^*-x^*)} \left[F''(x_0) + \left(F''(z) - F''(x_0)\right)\right] dz dt, \end{aligned}$$

and if $\gamma(t) = \|x^* - x_0 + t(y^* - x^*)\|$, then

$$\begin{aligned} &\left\| I - \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))\, dt \right\| \leq \\ &\leq \int_0^1 \left(b + \tfrac{k}{2}\gamma(t)\right) \gamma(t)\, dt \\ &< \int_0^1 \left(b + \tfrac{k}{2}\left[tr + (1-t) r_1\right]\right) \left[tr - (1-t) r_1\right] dt \\ &= \tfrac{k}{6} r_2^2 + \left(\tfrac{k}{6} r_1 + \tfrac{b}{2}\right)(r_2 + r_1). \end{aligned}$$

Now, we define the polynomial

$$q(r) = \tfrac{k}{6} r^2 + \left(\tfrac{k}{6} r_1 + \tfrac{b}{2}\right) r + \left(\tfrac{k}{6} r_1^2 + \tfrac{k}{2} r_1 - 1\right).$$

If $q(r) \le 0$, then

$$\left\| t - \Gamma_0 \int_0^1 F'(x^* + t(y^* - x^*))\, dt \right\| < 1$$

and the inverse $\int_0^1 F'(x^* + t(y^* - x^*))\, dt$ exists. Consequently, we deduce the uniqueness of the solution x^* in $B(x_0, r) \cap \Omega$.

Notice that $q(0) < 0$ and $q(r_1) = p'(r_1) < 0$ and then, the uniqueness holds in $B(x_0, r) \cap \Omega$ with $r > r_1$. Moreover, by using Cardano's formulas, for $k > 0$ we have $r_1 + r_2 = r_0 - \frac{3b}{k}$ and $r_1 r_2 = \frac{6a}{k r_0}$, where $-r_0, r_1$ and r_2 are the roots of the equation (4). So $q(r_2) = -\frac{2a + b r_0^2}{r_0} < 0$. If $k = 0$, then $q(r_2) = 0$. In both cases, the uniqueness in $B(x_0, r_2) \cap \Omega$ holds (if $k > 0$ the uniqueness holds in $B(x_0, r_3) \cap \Omega$ where r_3 is the positive root of $q(r) = 0$).

Finally, by Lemma 2.2 and for $m \ge 0$, we have

$$\| x_{n+m} - x_n \| \le t_{n+m} - t_n.$$

By letting $m \to \infty$, we conclude the result. □

3. ERROR ESTIMATES

This section is devoted to the study of the error estimates for the real sequences (5). As a result we obtain error bounds for the sequences defined in Banach spaces.

We distinguish two situations:

- If the polynomial p given in (4) is quadratic, we follow the technique developed by Ostrowski [9], to obtain error bounds.
- If p is a cubic polynomial, we use the technique introduced in [7]. This provides error estimates instead of error bounds.

Let p be the polynomial given in (4) with $k = 0$:

$$p(t) = \tfrac{b}{2} t^2 + a.$$

Assume that p has two positive roots r_1 and $r_2\,(r_1 \le r_2)$. Let $\{t_n\}$ be the sequence defined in (5) and $a_n = r_1 - t_n, b_n = r_2 - t_n$, $n \ge 0$. Then

$$p(t_n) = \tfrac{b}{2} a_n b_n, \quad p'(t_n) = -\tfrac{b}{2}(a_n + b_n).$$

By (5), we have

$$a_{n+1} = r_1 - t_{n+1} = a_n^3 \frac{a_n^3 + 4a_n^2 b_n + 5a_n b_n^2 + 2(1-2\alpha) b_n^3}{(a_n+b_n)^5}$$

and

$$b_{n+1} = r_2 - t_{n+1} = b_n^3 \frac{b_n^3 + 4b_n^2 a_n + 5b_n a_n^2 + 2(1-2\alpha) a_n^3}{(a_n+b_n)^5}.$$

If $r_1 < r_2$, let us write $\theta = \frac{r_1}{r_2} < 1$ and $\mu_n = \frac{a_n}{b_n}$. Then $\mu_{n+1} = \mu_n^3 h(\mu_n)$ where

$$h(x) = \frac{x^3 + 4x^2 + 5x + 2(1-2\alpha)}{2(1-2\alpha)x^3 + 5x^2 + 4x + 1}, \quad 0 \le x \le 1.$$

If $\alpha = \frac{1}{2}$, notice that

$$\mu_n^4 \le \mu_{n+1} = \mu_n^4 \frac{\mu_n^2 + 4\mu_n + 5}{5\mu_n^2 + 4\mu_n + 1} \le 5\mu_n^4.$$

So $\mu_n \le 5\mu_{n-1}^4 \le \ldots \le 5^{-\frac{1}{3}} \left[5^{\frac{1}{3}}\theta\right]^{4^n}$ and $\mu_n \ge \mu_{n-1}^4 \ge \ldots \ge \theta^{4^n}$. Hence

$$(r_2 - t_n)\, \theta^{4^n} \le r_1 - t_n \le (r_2 - t_n)\, 5^{-\frac{1}{3}} [5^{\frac{1}{3}}\theta]^{4^n}.$$

Consequently,

$$(r_2 - r_1) \frac{\theta^{4^n}}{1-\theta^{4^n}} r_1 - t_n \le 5^{-\frac{1}{3}} \frac{(r_2 - r_1)[5^{\frac{1}{3}}\theta]^{4^n}}{1 - 5^{-\frac{1}{3}} [5^{\frac{1}{3}}\theta]^{4^n}}.$$

The second inequality holds if $\sqrt[3]{5}\theta < 1$.

On the other hand, as k is decreasing in α, the best error bound is attained for $\alpha = \frac{1}{2}$ and the worse for $\alpha = 0$ (Chebyshev method).

Then, the error bounds obtained for the method of the family (5) $\left(0 < \alpha < \frac{1}{2}\right)$ are better than the bonds given by Chebyshev method and worse than the ones given by the method (5) with $\alpha = \frac{1}{2}$.

The definition of R-order of convergence given by Potra [10] allows us to establish that the methods of the family (5) and hence the methods of (3), have at least R-order three for $0 \le \alpha \le \frac{1}{2}$ and four for $\alpha = \frac{1}{2}$.

Finally, we analyze the case $r_1 = r_2$. Then $a_n = b_n$ and

$$a_{n+1} = a_n \frac{3-\alpha}{8},$$
$$r_1 - t_n = r_1 \left[\frac{3-\alpha}{8}\right]^n.$$

We see that the convergence of the methods of (5) is linear, but it is faster for increasing values of the parameter α.

Let us now consider the polynomial p, defined in (4), with $k > 0$. Assume that p has two positive roots r_1 and r_2 $(r_1 \leq r_2)$ and a negative root, $-r_0$, that is

$$p(t) = \tfrac{k}{6}(r_1 - t)(r_2 - t)(r_0 + t).$$

First we study the case $r_1 < r_2$. We denote again $a_n = r_1 - t_n, b_n = r_2 - t_n$ and define

$$Q(t_n) = \tfrac{b_n^3 a_{n+1}}{a_n^3 b_{n+1}} = \tfrac{(r_1 - G(t_n))(r_2 - t_n)^3}{(r_2 - G(t_n))(r_1 - t_n)^3},$$

with G defined in (5).

As $G(r_1) = r_1$, $G'(r_1) = G''(r_1) = 0$, we have for t close enough to r_1

$$\begin{aligned} Q(t) \sim (r_2 - r_1)^2 \lim_{t \to r_1} \tfrac{r_1 - G(t)}{(r_1 - t)^3} &= \tfrac{G'''(r_1)}{6}(r_2 - r_1)^2 \\ &= \tfrac{3(1-2\alpha)p''(r_1)^2 + p'''(r_1)p'(r_1)}{6p'(r_1)^2}(r_2 - r_1)^2 \\ &= \tfrac{(r_2 - r_1)}{(r_0 + r_1)} + 2(1 - 2\alpha)\tfrac{(r_0 - 2r_1 - r_2)^2}{(r_0 + r_1)^2} = \lambda. \end{aligned}$$

By letting $n \to \infty$, we have $t_n \to r_1$ and it follows

$$\tfrac{a_n}{b_n} \sim \left(\tfrac{a_{n-1}}{b_{n-1}}\right)^3 \lambda \sim \ldots \sim \left(\sqrt{\lambda}\tfrac{r_1}{r_2}\right)^{3^n} \tfrac{1}{\sqrt{\lambda}},$$

and if $\sqrt{\lambda}\theta < 1$, then

$$r_1 - t_n \sim \frac{(r_2 - r_1)\left(\sqrt{\lambda}\theta\right)^{3^n}}{\sqrt{\lambda} - \left(\sqrt{\lambda}\theta\right)^{3^n}}, \qquad n \geq 0.$$

When $r_1 = r_2$, we have

$$\widetilde{Q}(t_n) = \tfrac{a_{n+1}}{a_n} = \tfrac{r_1 - G(t_n)}{r_1 - t_n}.$$

Then, when t approaches r_1,

$$\widetilde{Q}(t) \sim \widetilde{Q}(r_1) = \tfrac{3-\alpha}{8},$$

and

$$r_1 - t_n \sim r_1\left(\tfrac{3-\alpha}{8}\right)^n.$$

If p is a cubic polynomial, we obtain better error estimates for increasing values of $\alpha \in \left[0, \frac{1}{2}\right]$, because λ is decreasing as a function of α. We obtain again the best error estimates for the method defined in (3) with $\alpha = \frac{1}{2}$.

Now we are ready to give the following result.

LEMMA 3.1. *With the previous notations, let $\theta = \frac{r_1}{r_2}$. Then for the methods of the family* (5) *applied to the polynomial* (4) *with $k = 0$, we have the following error bounds:*

- *When $r_1 < r_2$ and $\alpha = \frac{1}{2}$.*

$$(r_2 - r_1)\,\frac{\theta^{4^n}}{1-\theta^{4^n}} \le r_1 - t_n \le \frac{r_2-r_1}{1-5^{-\frac{1}{3}}\left[5^{\frac{1}{3}}\theta\right]^{4^n}}\,5^{-\frac{1}{3}}\left[5^{\frac{1}{3}}\theta\right]^{4^n}, \qquad \textit{if } \sqrt[3]{5}\theta < 1.$$

- *When $r_1 < r_2$ and $0 \le \alpha < \frac{1}{2}$:*

$$(r_2 - r_1)\,\frac{\theta^{3^n}}{1-\theta^{3^n}} \le r_1 - t_n \le (r_2 - r_1)\,\frac{\left[\sqrt{2}\theta\right]^{3^n}}{\sqrt{2}-\left[\sqrt{2}\theta\right]^{3^n}}, \qquad \textit{if } \sqrt{2}\theta < 1.$$

- *When $r_1 = r_2$:*

$$r_1 - t_n = r_1\left[\frac{3-\alpha}{8}\right]^n.$$

If we apply the methods of (5) *to a polynomial* (4) *with $k > 0$, the error estimates are:*

- *When $r_1 < r_2$:*

$$r_1 - t_n \sim \frac{(r_2-r_1)\left(\sqrt{\lambda}\theta\right)^{3^n}}{\sqrt{\lambda}-\left(\sqrt{\lambda}\theta\right)^{3^n}}, \qquad \textit{if } \sqrt{\lambda}\theta < 1,$$

with $\lambda = \frac{(r_2-r_1)}{(r_0+r_1)} + 2\,(1-2\alpha)\,\frac{(r_0+2r_1-r_2)^2}{(r_0+r_1)^2}$.

- *When $r_1 = r_2$:*

$$r_1 - t_n \sim r_1\left(\frac{3-\alpha}{8}\right)^n.$$

4. EXAMPLES

We give two examples to illustrate the previous results.

EXAMPLE 1. Let us consider the space $X = C\,[(0,1)]$ of all continuous functions defined in the interval $[0,1]$ with the norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|,$$

and the equation $f(x) = 0$, where

$$F(x)(s) = x(s) - s + \tfrac{1}{2}\int_0^1 s\cos(x(t))\,dt, \qquad x \in C([0,1]),\ s \in [0,1].$$

With the previous notations and for $x_0 = x_0(s) = s$, we calculate the first and second Fréchet derivatives of F to obtain

$$a = b = \tfrac{\sin 1}{2-\sin 1+\cos 1},\ k = \tfrac{1}{2-\sin 1+\cos 1}.$$

So, in this case, the polynomial (4) is

$$p(t) = \tfrac{1}{6(2-\sin 1+\cos 1)}\left[t^3 + 3(\sin 1)t^2 - 6(2-\sin 1+\cos 1)t + 6\sin 1\right],$$

which has two positive roots

$$r_1 = 0.6095694860276291,\ r_2 = 1.70990829134757.$$

Then, by Theorem 2.3, we know that $F(x) = 0$ has a solution in $\overline{B(x_0, r_1)}$ and this solution is unique in $B(x_0, r_2)$.

Moreover, we have the following error estimates for $10^{11}\|x^* - x_2\|$ when $\alpha = 0$ and $\alpha = \frac{1}{2}$:

If $\alpha = 0$, $10^{11}\|x^* - x_2\| \leq 10^{11}(r_1 - t_2) \sim 48601696.08024329$.

If $\alpha = \frac{1}{2}$, $10^{11}\|x^* - x_2\| \leq 10^{11}(r_1 - t_2) \sim 16958.95309621804$. □

Observe that the previous results do agree quite well with our previous analysis which showed that better error estimates are obtained for the method (3) with $\alpha = \frac{1}{2}$.

Example 2. Let us consider again the space $X = C[(0,1)]$, where now we define the equation $F(x) = 0$, with

$$F(x)(s) = 1 - x(s) + \tfrac{x(s)}{4}\int_0^1 \tfrac{s}{s+t}x(t)\,dt, \qquad x \in C([0,1]),\ s \in [0,1].$$

This is a quadratic equation that is known as Chandrasekhar's equation [3].

For $x_0 = x_0(s) = 1$, we have $a = 0.2652$, $b = 0.5303$ and $k = 0$. In that case the polynomial (4) has two roots:

$$r_1 = 0.287042, \quad r_2 = 3.48512.$$

So Chandrasekhar's equation has a solution that lies in $\overline{B(x_0, r_1)}$ and is unique in $B(x_0, r_2)$.

In Tables 1 and 2 we show some error expressions.

$$\|x^* - x_n\| \le r_1 - t_n$$

for two methods of the family (3) and other well known third order iterative processes, like the Halley method ($\{u_n\}$) or the super-Halley method ($\{v_n\}$).

n	$r_1 - t_n\,(\alpha = 0)$	$r_1 - t_n\,\left(\alpha = \frac{1}{2}\right)$
0	0.2870424876072915	0.2870424876072915
1	0.0035775864318520	0.0007359920056491
2	$8.92413 \cdot 10^{-9}$	$4.48122 \cdot 10^{-14}$
3	$1.38979 \cdot 10^{-25}$	$6.16436 \cdot 10^{-55}$

Table 1. Methods of (3) for $\alpha = 0$ and $\alpha = \frac{1}{2}$

n	$r_1 - u_n$	$r_1 - v_n$
0	0.2870424876072915	0.2870424876072915
1	0.0017877932422075	0.0001471713056166
2	$5.577583 \cdot 10^{-10}$	$1.433989 \cdot 10^{-17}$
3	$1.696526 \cdot 10^{-29}$	$1.292759 \cdot 10^{-69}$

Table 2. Halley and super-Halley methods

As we can see the method corresponding to $\alpha = \frac{1}{2}$ provides better error estimates than the Chebyshev and the Halley methods, but worse than the super-Halley method. The cause is that the super-Halley method also has convergence of order four when it is applied to quadratic equations. □

REFERENCES

[1] G. Alefeld, On the convergence of Halley's method, *Amer. Math. Monthly*, **88** (1981), 530–536.

[2] M. Altman, Concerning the method of tangent hyperbolas for operator equations, *Bull. Acad. Pol. Sci., Serie Sci. Math., Ast. et Phys.*, **9**(1961), 633–637.

[3] I.K. Argyros, Quadratic equations and aplications to Chandrasekhar's and related equations, *Bull. Austral. Math. Soc.*, **38** (1988), 275–292.

[4] I.K. Argyros and D. Chen, Results on the Chebyshev method in Banach spaces, *Proyecciones*, **12**, 2 (1993), 119–128.

[5] V. Candela and A. Marquina, Recurrence relations for rational cubic methods II: the Chebyshev method, *Computing*, **45** (1990), 355–367.

[6] W. Gander, On Halley's iteration method, *Amer. Math. Monthly,* **92** (1985), 131–134.

[7] J.M. Gutiérrez and M.A. Hernández, A family of Chebyshev-Halley type methods in Banach spaces, *Bull. Austral. Math. Soc.,* **55** (1997), 113–130.

[8] L.V. Kantorovich and G.P. Akilov, Functional Analysis, Pergamon Press, 1982.

[9] A.M. Ostrowski, *Solution of Equations in Euclidean and Banach Spaces,* Academic Press, 1963.

[10] F.A. Potra and V. Ptak, *Nondiscrete Induction and Iterative Processes*, Pitman Pub., 1983.

[11] L.B. Rall, *Computational Solution of Nonlinear Operator Equations*, Robert E. Krieger Publishing Company, Inc., 1979.

[12] W.C. Rheinboldt. A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.*, **5** (1968), 42–63.

[13] T. Yamamoto, On the method of tangent hyperbolas in Banach spaces, *J. Comput. Appl. Math.* **21** (1988), 75–86.

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