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NUMERICAL EVALUATION OF CAUCHY PRINCIPAL VALUE INTEGRALS BY MEANS OF NODAL SPLINE APPROXIMATION

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1. INTRODUCTION

In this paper we investigate the convergence properties of some quadrature rules for evaluating Cauchy principal value (CPV) integrals

$$J(kf;\lambda)=\int_{-1}^{1} k(x)\frac{f(x)}{x-\lambda}\,dx,\quad -1<\lambda<1. \tag{1}$$

The quadrature rules considered here are based on optimal nodal interpolatory splines (o.n.s), studied by De Villiers and Rohwer [5–8].

More recently, Rabinowitz [13] has investigated convergence properties of product integration rules based on o.n.s. These splines have many of the desirable properties of quasi-interpolatory splines studied in [9] and used for constructing integration rules in [1], [2] and [14]. However, o.n.s have the advantage of being interpolatory, but they present a certain complexity in their definition.

After the necessary definitions and properties of o.n.s have been given, we consider the following approach for approximating (1) by quadrature rules.

By subtracting the singularity from (1), and assuming that $J(k;\lambda)$ exists for all $\lambda\in(-1,1)$, we can write the CPV integral in the form

$$J(kf;\lambda)=\int_{-1}^{1} k(x)\,g_\lambda(x)\,dx+f(\lambda)\,J(k;\lambda)=I(kg_\lambda)+f(\lambda)\,J(k;\lambda), \tag{2}$$

where

$$g_\lambda(x)=g(x;\lambda)=\begin{cases}\dfrac{f(x)-f(\lambda)}{x-\lambda} & x\neq\lambda\\ f'(\lambda) & x=\lambda \text{ and } f'(\lambda) \text{ exists}\\ 0 & \text{otherwise,}\end{cases} \tag{3}$$

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and k is an arbitrary weight function subject to certain conditions ensuring that (1) exists for some classes of functions f, for all $\lambda \in (-1,1)$.

If we approximate $I(kg_\lambda)$ in (2) by the rules

$$I(kg_\lambda) \cong I(kW_n g_\lambda) = \sum_{i=0}^{n} v_{in}(k)\, g_\lambda(\xi_{in}) \tag{4}$$

defined in [13], we can write

$$J(kf;\lambda) = J_n(kf;\lambda) + E_n(kf;\lambda), \tag{5}$$

where

$$J_n(kf;\lambda) = \sum_{i=0}^{n} v_{in}(k)\, g_\lambda(\xi_{in}) + f(\lambda)\, J(k;\lambda). \tag{6}$$

We observe, from (2) and (5), the quadrature error $E_n(kf;\lambda)$ is the truncation error of the rules (4) are applied, i.e.,

$$E_n(kf;\lambda) = I(kg_\lambda) - I(kW_n g_\lambda).$$

Therefore, the rules (6) are convergent if, and only if the corresponding rules (4) converge. In Section 2, we shall introduce the o.n.s. using the notation of [13] and report on some convergence results for product integration of piecewise-continuous and unbounded integrand functions.

Making use of these results, in Section 3, we shall investigate the convergence of rules (6) whenever $f \in H_\mu(\Im), \Im \equiv [-1,1], 0<\mu\le 1$[1] and their uniform convergence if $f \in C^1(\Im)$.

2. OPTIMAL NODAL INTERPOLATING SPLINES

We shall now give the necessary definitions and properties of nodal splines which appear in [13].

Let m be an integers ≥ 3, the order of the spline, and let $\prod_n$, $n = m, m+1, \ldots$, be a sequence of partitions in $\Im$, where

$$\prod_n ; \xi_{0n} = -1 < \xi_{1n} < \ldots < \xi_{nn} = 1. \tag{7}$$

[1] $H_\mu(S) = \{g \in C(S) : |g(x_1) - g(x_2)| \le M |x_1 - x_2|^\mu,\ \forall x_1, x_2 \in S\}$, where M is a positive constant, and $0 < \mu \le 1$.

Assuming that the sequence is locally uniform (l.u), i.e.,

$$\frac{\xi_{i+1,n} - \xi_{in}}{\xi_{j+1,n} - \xi_{jn}} \le A \quad \text{for all } i,\ n \text{ and all } j = \pm 1, \tag{8}$$

where A is some constant ≥ 1, we suppose that the norm

$$\Delta_n = \max_{1\le i\le n} (\xi_{i+1,n} - \xi_{in}) \to 0 \quad \text{as} \quad n \to \infty. \tag{9}$$

We introduce now two integers $\rho = \left[\frac{m+1}{2}\right]$ and $\mu = (m+1) - \rho$, and two integer functions

$$p_j = \begin{cases} 0 & j = 0,\ldots,\mu-2 \\ j-\mu+1 & j = \mu-1,\ldots,n-\rho \\ n-m+1 & j = n-\rho+1,\ldots,n-1 \end{cases}, \quad q_j = \begin{cases} m-1 & j = 0,\ldots,\mu-2 \\ j+\rho & j = \mu-1,\ldots,n-\rho \\ n & j = n-\rho+1,\ldots,n-1. \end{cases}$$

Then, for any real-valued function f on $\Im$, $(f \in B(\Im))$, we can define the approximating function

$$W_n f(x) = \sum_{p_j}^{q_j} f(\xi_{in})\, w_{in}(x) \quad x \in [\xi_{jn}, \xi_{j+1,n}],\ j = 0,1,\ldots,n-1, \tag{10}$$

where the functions w_{in} are given by

$$w_{in}(x) = \begin{cases} \prod\limits_{\substack{k=0 \\ k\ne i}}^{m-1} \dfrac{x - \xi_{kn}}{\xi_{in} - \xi_{kn}}, & x \in [-1, \xi_{\mu-1,n}], \quad 0 \le i < m \\ S_{in}(x), & x \in [\xi_{\mu-1,n}, \xi_{n-\rho+1,n}] \\ \prod\limits_{\substack{k=0 \\ k\ne n-i}}^{m-1} \dfrac{x - \xi_{n-k,n}}{\xi_{in} - \xi_{n-k,n}}, & x \in [\xi_{n-\rho+1,n}, 1] \quad n-m < i \le n. \end{cases} \tag{11}$$

The functions s_{in} belong to the set of nodal splines studied in [5]. Each s_{in} has the compact support $[\xi_{i-p,n}, \xi_{i+\mu,n}]$ and is nodal with respect to $\prod_n$, i.e., $s_{in}(\xi_{jn}) = \delta_{ij}$ so that $W_n f$ interpolates to f at the points ξ_{in}. The functions s_{in} are linear combinations of B-splines of order m, and each B-spline has its $m+1$ knots chosen from $m+1$ consecutive points from the set of the points consisting of the ξ_{in} plus $m-2$ distinct points arbitrarily placed in each of the open intervals $(\xi_{jn}, \xi_{j+1,n})$ [5–7].

Therefore, the nodal spline approximation $W_n : B(\Im) \to S_{\Pi_n}$ has the following properties:

i. W_n is local in the sense that, for every $f \in B(\Im)$, for a fixed $x \in \Im$ and j such that $x \in [\xi_{jn}, \xi_{j+1,n}]$, the value of $W_n f(x)$ depends only on the values of f at most $m+1$ neighbouring points of x;

ii. $W_n f(\xi_{in}) = f(\xi_{in})$;

iii. $W_n g = g$ whenever $g \in P_m$, where P_m is the class of polynomials of order m (degree $m-1$).

We remark that, in order to obtain the above properties, we give the defining formula for $W_n f(x)$ in a slightly different way from that given in [6].

If (8) holds, then [7]

(12) $$\|s_{in}\|_\infty < B_1, \quad \forall i, n$$

and, using (8), we have that for all i, k by (11), it follows that

(13) $$\|w_{in}\|_\infty < B \quad \forall i, n.$$

De Villiers [7] proved the following

THEOREM 1. *Let $g \in C(\Im)$; we define $r_n^0 = g - W_n g$. Then*

(14) $$\|r_n^0\|_\infty \le C\omega(g; m\Delta_n),$$

where C is a constant independent of n and ω is the usual modulus of continuity.

From this theorem, Rabinowitz immediately obtained [13] the following

THEOREM 2. *Let $g \in C(\Im)$, and $k \in L_1(\Im)$. If $\{\Pi_n\}$ is a sequence of partitions satisfying* (8) *and* (9), *then*

(15) $$I(kW_n g) \to I(kg) \quad as \quad n \to \infty,$$

where

(16) $$I(kg) = \int_{-1}^{1} k(x)\, g(x)\, dx$$

and

(17) $$I(kW_n g) = \sum_{i=0}^{n} v_{in}(k)\, g(\xi_{in}).$$

The following theorem is a generalization of the convergence result (15) for functions $g \in PC(\Im)$, the set of piecewise continuous functions on $\Im$ [13].

THEOREM 3. *Let $g \in PC(\Im)$, and $k \in L_1(\Im)$. If $\{\Pi_n\}$ is a sequence of partitions satisfying* (8) *and* (9), *then* (15) *holds.*

We prove now that (15) holds for all $g \in \Re(\Im)$, the set of Riemann-integrable functions on $\Im$. For this purpose we need the following definition given in [4] and the lemma proved in [11].

Let D designate the union of a finite number of intervals (disjoint or not) located in $\Im$ and let $l(D)$ be the sum of the lengths of the individual intervals of D.

The notation $\sum_D |v_{in}|$ will designate the sum taken over those v_{in} for which $\xi_{in} \in D$.

We define the set function $\Delta(D)$ as

(18) $$\Delta(D) = \lim_{n\to\infty} \sup \sum_D |v_{in}|.$$

The set function $\Delta(D)$ is called *semicontinuous* if, for any sequence $D_1 \supset D_2 \supset \dots$ with $l(D) \to 0$, for $n \to \infty$,

$$\lim_{n\to\infty} \Delta(D_n) = 0, \quad \text{where} \quad \Delta(D_n) = \lim_{n\to\infty} \sup \sum_{D_n} |v_{in}(k)|.$$

LEMMA 1. *If $\lim_{n\to\infty} I(kW_n g) = I(kg)$ for all $g \in C(\Im)$, then the quadrature rules converge to the integral, for all $g \in \Re(\Im)$ if and only if $\Delta(D)$ is semicontinuous.*

THEOREM 4. *Let $g \in \Re(\Im)$ and $k \in L_1(\Im)$. If $\{\Pi_n\}$ is a sequence of partitions satisfying* (8) *and* (9), *then* (15) *holds.*

Proof. We can write

(19) $$I(kW_n g) = \sum_{i=0}^{n} v_{in}(k)\, g(\xi_{in}),$$

where

(20) $$v_{in}(k) = \int_{-1}^{1} k(x)\, w_{in}(x)\, dx.$$

We first consider the terms $v_{in}(k)$ in (19) for $m \le i \le n-m-1$. Then [7]

(21) $$|v_{in}(k)| \le \int_{-1}^{1} |k(x)||s_{in}(x)|\, dx \le$$

$$\le \left[\sum_{h=1}^{\rho} A^h\right]^{m-1} \int_{-1}^{1} |k(x)| \sum_{j=-(m-1)\rho}^{(m-1)\mu-m} B_{(m-1)i+j}(x)\, dx \le$$

$$\le \left[\sum_{h=1}^{\rho} A^h\right]^{m-1} (m-1)(\mu+\rho-1)|\omega_i|,$$

where $\omega_i = \int_{-1}^{1} |k(x)| B_i(x)\,dx$ are the weights of quadrature rules based on approximating splines considered by Rabinowitz [11]. Since these rules converge for Riemann-integrable functions, it follows that the set function

$$\Delta_1(D) = \lim_{n\to\infty} \sum_D |\omega_i|,$$

is semicontinuous. Hence, the set function

$$\Delta(D) = \lim_{n\to\infty} \sup \sum_{\substack{D \\ m\le i\le n-m-1}} |v_{in}(k)| \le \left[\sum_{h=1}^{\rho} A^h\right]^{m-1} (m-1)(\mu+\rho-1)\,\Delta_1(D)$$

is semicontinuous.

Consider now the sums

(22) $$S_1 = \sum_{i=0}^{m-1} v_{in}(k)\, g(\xi_{in}) \quad \text{and} \quad S_2 = \sum_{i=n-m}^{n} v_{in}(k)\, g(\xi_{in}).$$

For $1 \le i \le m-1$, from (8), (11), (13) and by the finite support of s_{in}, we have

(23) $$|v_{in}(k)| < C \int_{-1}^{\xi_{in}} |k(x)|\,dx.$$

By (9) and the hypothesis on $k(x), |v_{in}(k)| \to 0$ as $n\to\infty$, for $1 \le i \le m-1$. It follows that $S_1 \to 0$ as $n\to\infty$. Similarly, $S_2 \to 0$ as $n\to\infty$.

Therefore, the sequence of rules

(24) $$\mathring{I}(kW_n g) = I(kW_n g) - S_1 - S_2$$

converges to $I(kg)$ for all continuous functions since the sequence $I(kW_n g)$ does, and $S_1, S_2 \to 0$ as $n\to\infty$. By Lemma 1 $\{\mathring{I}(kW_n g)\}$ and, consequently, $\{I(kW_n g)\}$ converge to $I(kg)$ for all $g \in \Re(\Im)$ [11].

We now discuss the convergence of $I(kW_n g)$ to $I(kg)$ when g is unbounded in $\Im$ but $kg \in L_1(\Im)$. For this purpose, we start by defining, for $-1<\zeta<1$, the family of functions

(25) $$M_d(\zeta,k) = \left\{\begin{matrix} g \in C(\Im\setminus\zeta),\ \exists G{:}\, G \ \textit{is continuous nondecreasing in}\, [-1,\zeta), \\ \textit{continuous nonincreasing in}\, (\zeta,1];\ kG \in L_1(\Im), |g| \le G \ \textit{in}\ \Im \end{matrix}\right\}.$$

For such functions we state the following lemma [10]:

LEMMA 2. *Let* $-1<\zeta<1$, $f \in M_d(\zeta;k)$ *and assume that*

(26) $$\lim_{n\to\infty} V_n g = I(kg)$$

for all $g \in PC(\Im)$, *where* $V_n g$ *is any numerical integration rule of the form*

(27) $$V_n g = \sum_{i=0}^{n} u_{in}\, g(\xi_{in}) \quad \xi_{in} \in \Im.$$

Then a necessary and sufficient condition for having $\lim_{n\to\infty} V_n g = I(kg)$ *is that, given* $\varepsilon>0$, *there exist* $n_0 = n_0(\varepsilon), \beta_1 \in (-1,\zeta), \beta_2 \in (\zeta,1)$ *such that*

(28) $$|V_n(\beta_1,\beta_2;f)| < \varepsilon \quad \forall n > n_0,$$

where

(29) $$V_n(\beta_1,\beta_2;g) = \sum_{\substack{\beta_1 \le \xi_{in} < \zeta \\ \zeta < \xi_{in} \le \beta_2}} u_{in}\, g(\xi_{in}).$$

We shall apply this lemma for the case

(28) $$V_n g = I(kW_n g) - v_{hn}(k)\, g(\xi_{hn}) - v_{pn}(k)\, g(\xi_{pn}) = \mathring{I}(kW_n g),$$

where h is the greatest integer such that $\xi_{hn} < \zeta$ and p is the smallest integer such that $\xi_{pn} > \zeta$ so that in $\mathring{I}(kW_n g)$ we avoid the singularity.

We have the following convergence result:

THEOREM 5. *Let* $-1<\zeta<1$ *and* $g \in M_d(\zeta;k)$. *Suppose that* $k \in L_1(\Im) \cap C(N_\delta(\zeta))$, *where* $N_\delta(\zeta)$ *is the neighbourhood of the point* ζ *thus defined*

(31) $$N_\delta(\zeta) = \{x \mid \zeta-\delta \le x \le \zeta+\delta,\ \delta>0\},$$

and δ *is such that* $N_\delta(\zeta) \subset \Im$. *Then, if* $\{\Pi_n\}$ *is a sequence of partitions satisfying* (8) *and* (9),

(32) $$\mathring{I}(kW_n g) \to I(kg) \quad \textit{as} \quad n\to\infty.$$

Let ξ^* *be the node closest to* ζ *defined by*

(33) $$\xi^* = \begin{cases} \xi_{hn} & \textit{if}\ \zeta-\xi_{hn} \le \xi_{pn}-\zeta \\ \xi_{pn} & \textit{if}\ \zeta-\xi_{hn} > \xi_{pn}-\zeta, \end{cases}$$

where $\xi_{hn}(\xi_{pn})$ *is the node closest to* ζ *from the left (right), and suppose that*

(34) $$|\xi^*-\zeta| > C\max\{(\xi_{hn}-\xi_{h-1,n}),\ (\xi_{p+1,n}-\xi_{pn})\}$$

for some positive constant C; then

(35) $$I(kW_n g) \to I(kg) \quad as \quad n \to \infty.$$

Proof. Since $\mathring{I}(kW_n g)$ results from $I(kW_n g)$ by dropping a finite number of terms, the convergence behaviour of both rules is the same for $g \in \Re(\Im)$. Hence, in order to prove (32), we need only to show that (28) holds when V_n is $\mathring{I}(kW_n g)$.

For this purpose, it is sufficient to prove that

(36) $$\left| \sum_{\zeta < \xi_{pn} < \xi_{in} \le \beta_2} v_{in}(k)\, g(\xi_{in}) \right| < \varepsilon, \quad \beta_2 \in (\zeta, \zeta+\delta],$$

since we can prove, in a similar way, that for $\beta_1 \in [\zeta-\delta, \zeta)$, the sum over the ξ_{in}, such that $\beta_1 \le \xi_{in} < \xi_{hn} < \zeta$, is less than ε.

We have

(37) $$\left| \sum_{\zeta < \xi_{pn} < \xi_{in} \le \beta_2} v_{in}(k)\, g(\xi_{in}) \right| \le \left| \sum_{\zeta < \xi_{pn} \le \xi_{in} \le \xi_{p+\rho-1,n}} v_{in}(k)\, g(\xi_{in}) \right| +$$

$$\left| \sum_{\xi_{p+\rho n} < \xi_{in} \le \beta_2} v_{in}(k)\, g(\xi_{in}) \right| = \sum\nolimits_1 + \sum\nolimits_2 .$$

Suppose that n is such that $p \ge m$, and ξ_m is the greatest node $\le \beta_2$. Then

(38) $$\sum\nolimits_2 \le \sum_{i=p+\rho} \left| g(\xi_{in}) \int_{\xi_{i-\rho n}}^{\xi_{i+\rho n}} k(x)\, s_{in}(x)\, dx \right|.$$

Since $k \in C(N_\delta(\zeta))$, $|k(x)| \le L$ in $N_\delta(\zeta)$, so that $|k(x)\, g(x)| \le LG$ in $[\zeta-\delta, \zeta) \cup (\zeta, \zeta+\delta]$ with G defined in (25). Since G is nonincreasing in $(\zeta, \zeta+\delta]$,

(39) $$\sum\nolimits_2 \le \mathring{C} \sum_{i=p+\rho}^{r} \int_{\xi_{i-\rho n}}^{\xi_{i-\rho+1,n}} k(x)\, G(x)\, dx < C^* \int_{\zeta}^{\beta_2} LG(x)\, dx < \varepsilon$$

by choosing β_2 sufficiently close to ζ.

If $\xi_{in} \in (\xi_{pn}, \xi_{p+\rho-1,n}]$, then

$$|v_{in}(k)\, g(\xi_{in})| \le C_1 \int_{\xi_{pn}}^{\xi_{p+1,n}} |k(x)|\, G(x)\, dx.$$

Since there are at most ρ values, we have that:

(40) $$\sum\nolimits_1 \le \rho C_1 \int_{\zeta}^{\beta_2} LG(x)\, dx < \varepsilon$$

and the thesis (32) is verified.

In order to demonstrate (35), it remains to prove that $v_{pn}(k)\, g(\xi_{pn}) \to 0$ as $n \to \infty$. However, this follows from the fact that, by (34), the above term is bounded by $C_2 \int_{\zeta}^{\xi_{pn}} LG(x)\, dx$ for some positive constant C_2. Since this quantity converges to 0 as $n \to \infty$, the theorem is therefore completely proved. □

Remark. Theorem 5 holds with a weaker hypothesis on k, but we have specified that k must be at least continuous in a neighbourhood of ζ in order that CPV integrals exist.

3. ON THE CONVERGENCE OF RULES (6) BASED ON QUADRATURE (4)

In this section we investigate the convergence of sequences of rules (6) based on quadrature (4). We know that rules (6) converge to $J(kf;\lambda)$ or diverge as rules (4) do, when they are applied to the integral

(41) $$I(kg_\lambda) = \int_{-1}^{1} k(x)\, g_\lambda(x)\, dx,$$

where the function g_λ has been defined in (3).

We can state the following theorems.

THEOREM 6. *For any* $\lambda \in (-1,1)$, *let* $f \in H_1(N_\delta(\lambda)) \cap \Re(\Im)$ *and* $k \in L_1(\Im)$. *If* $\{\Pi_n\}$ *is a sequence of partitions satisfying* (8) *and* (9), *then* $E_n(kf;\lambda) \to 0$ *as* $n \to \infty$.

Proof. If $f \in H_1(N_\delta(\lambda)) \cap \Re(\Im)$, then $g_\lambda \in PC(\Im)$. Therefore we can apply Theorem 3 to prove the thesis. □

THEOREM 7. *Let* $f \in H_\mu(\Im)$, $0<\mu<1$ *and suppose that* $k \in L_1(\Im) \cap C(N_\delta(\lambda))$ *and* $\{\Pi_n\}$ *is a sequence of partitions satisfying* (8), (9) *and* (34) *with* $\zeta = \lambda$. *Then*

(42) $$E_n(kf;\lambda) \to 0 \quad as \quad n \to \infty.$$

Proof. If $f \in H_\mu(\Im)$, $0<\mu<1$, the function g_λ in (41) is not greater than $M|x-\lambda|^{\mu-1}$ for some constant M independent of n. Therefore, by Theorem 4, where we consider $\zeta = \lambda$, it follows that $I(kW_n g_\lambda) \to I(kg_\lambda)$ as $n \to \infty$. □

THEOREM 8. *Let $k \in L_1(\Im)$ and $f \in C^1(\Im)$, then $E_n(kf;\lambda) \to 0$ uniformly in λ as $n \to \infty$. Hence, if $f \in C^1(\Im)$ and $k \in L_1(\Im) \cap DT(-1,1)$,*[2] *then rules* (6) *converge uniformly to the* CPV *integral $J(kf;\lambda)$.*

Proof. If $f \in C^1(\Im)$, then g_λ is uniformly continuous for all pairs $(x,\lambda) \in \Im \times \Im$. For any $\lambda \in (-1,1)$, by Theorem 1, we have

$$|g_\lambda(x) - W_n g_\lambda(x)| \le C\omega(g_\lambda; m\Delta_n). \tag{43}$$

By the uniform continuity of g_λ in $\lambda, \omega(g_\lambda; m\Delta_n)$ is independent of λ. Hence

$$|E_n(kf;\lambda)| \le C\omega(g_\lambda; m\Delta_n)\int_{-1}^{1} |k(x)|\,\mathrm{d}x = o(1) \tag{44}$$

uniformly in λ. If $k \in DT(-1,1)$, then $I(kf;\lambda)$ exists for all $\lambda \in (-1,1)$, which yields the uniform convergence of $J_n(kf;\lambda)$. □

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