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SPLINE APPROXIMATION FOR SYSTEM OF TWO THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS, II

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DESCRIPTION OF THE METHOD

Consider the system of nonlinear ordinary differential equations:

(1) $y''' = f_1(x,y,z),\ y(x_0) = y_0,\ y'(x_0) = y'_0,\ y''(x_0) = y''_0,$

(2) $z''' = f_2(x,y,z),\ z(x_0) = z_0,\ z'(x_0) = z'_0,\ z''(x_0) = z''_0$

where $f_1, f_2 \in C^r([0,1] \times \mathbf{R}^2)$.

Let Δ be the partition

$$\Delta : 0 = x_0 < x_1 < \ldots < x_k < x_{k+1} < \ldots < x_n = 1$$

where $x_{k+1} - x_k = h < 1$ and $k = 0(1)n-1$.

Let L_1 and L_2 be the Lipschitz constants satisfied by the functions $f_1^{(q)}$ and $f_2^{(q)}$ respectively, i.e.,

(3) $$\left|f_1^{(q)}(x,y_1,z_1) - f_1^{(q)}(x,y_2,z_2)\right| \le L_1\{|y_1 - y_2| + |z_1 - z_2|\}$$

and

(4) $$\left|f_2^{(q)}(x,y_1,z_1) - f_2^{(q)}(x,y_2,z_2)\right| \le L_2\{|y_1 - y_2| + |z_1 - z_2|\}$$

for all (x, y_1, z_1) and (x, y_2, z_2) in the domain of definition of f_1 and f_2 and all $q = 0(1)\,r$.

The functions $f_1^{(q)}$ and $f_2^{(q)}$, $q = 1(1)r$ are functions of x, y and z only and they are given from the following algorithm:
set

$$f_1^{(0)} = f_1(x,y,z),\ f_2^{(0)} = f_2(x,y,z)$$

and if $f_1^{(q-1)}$ and $f_2^{(q-1)}$ are defined, then

$$f_1^{(q)} = \frac{\partial f_1^{(q-1)}}{\partial x} + \frac{\partial f_1^{(q-1)}}{\partial y} y' + \frac{\partial f_1^{(q-1)}}{\partial z} z'$$

and

$$f_2^{(q)} = \frac{\partial f_2^{(q-1)}}{\partial x} + \frac{\partial f_2^{(q-1)}}{\partial y} y' + \frac{\partial f_2^{(q-1)}}{\partial z} z'.$$

Then, we define the spline functions approximating $y(x)$ and $z(x)$ by $S_\Delta(x)$ and $\overline{S_\Delta}(x)$ where

$$(5)\quad \begin{aligned} S_\Delta(x) \equiv S_k(x) &= S_{k-1}(x_k) + S'_{k-1}(x_k)(x-x_k) + S''_{k-1}(x_k)\frac{(x-x_k)^2}{2!} + \\ &+ \sum_{j=0}^{r} f_1^{(j)}[(x_k, s_{k-1}(x_k), \overline{S}_{k-1}(x_k)]\frac{(x-x_k)^{j+3}}{(j+3)!} \end{aligned}$$

and

$$(6)\quad \begin{aligned} \overline{S}_\Delta(x) \equiv \overline{S}_k(x) &= \overline{S}_{k-1}(x_k) + \overline{S}'_{k-1}(x_k)(x-x_k) + \overline{S}''_{k-1}(x_k)\frac{(x-x_k)^2}{2!} + \\ &+ \sum_{j=0}^{r} f_2^{(j)}[(x_k, s_{k-1}(x_k), \overline{S}_{k-1}(x_k)]\frac{(x-x_k)^{j+3}}{(j+3)!} \end{aligned}$$

where $x_{k-1} \le x \le x_k$, $k = 0(1)n-1$, $s_{-1}(x_0) = y_0$, $s'_{-1}(x_0) = y'_0$, $s''_{-1}(x_0) = y''_0$, $\overline{s}_{-1}(x_0) = z_0$, $\overline{s}'_{-1}(x_0) = z'_0$ and $\overline{s}''_{-1}(x_0) = z''_0$.

By construction it is clear that $s_\Delta(x), \overline{s}_\Delta(x) \in C^2[0,1]$.

ERROR ESTIMATIONS AND CONVERGENCE

For all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, the exact solutions of (1) and (2) can be written, by Taylor's expansion, in the following forms:

$$(7)\quad y(x) = \sum_{j=0}^{r+2} \frac{y_k^{(j)}}{j!}(x-x_k)^j + \frac{y^{(r+3)}(\xi_k)}{(r+3)!}(x-x_k)^{r+3}$$

and

$$(8)\quad z(x) = \sum_{j=0}^{r+2} \frac{z_k^{(j)}}{j!}(x-x_k)^j + \frac{z^{(r+3)}(\eta_k)}{(r+3)!}(x-x_k)^{r+3}$$

where $\xi_k, \eta_k \in (x_k, x_{k+1})$, $k = 0(1)n-1$.

The following natation will be used along the discussion of the convergence of those spine approximants:

$$(9)\quad \begin{aligned} &e(x) = |y(x) - S_\Delta(x)|, \\ &e_k = |y_k - S_\Delta(x_k)|, \\ &\overline{e}(x) = |z(x) - \overline{S}_\Delta(x)|, \\ &\overline{e}_k = |z_k - \overline{S}_\Delta(x_k)|, \\ &f_{1,k}^{(j)} = f_1^{(j)}[x_k, s_{k-1}(x_k), \overline{s}_{k-1}(x_k)] \end{aligned}$$

and

$$f_{2,k}^{(j)} = f_2^{(j)}[x_k, s_{k-1}(x_k), \overline{S}_{k-1}(x_k)]$$

where $j = 0(1)r$ and $k = 0(1)n-1$.

Along this work, we will deal with the general subinterval

$$I_k = [x_k, x_{k-1}],\ k = 0(1)n-1.$$

Now, we are going to estimate $|y_k - S_k(x)|$

Using (5), (7) the Lipschitz condition (3) and the notation (9), we get:

$$(10)\quad \begin{aligned} |y(x) - S_k(x)| &\le |y_k - S_{k-1}(x_k)| + |y'_k - S'_{k-1}(x_k)||x-x_k| + \\ &+ |y''_k - S''_{k-1}(x_k)|\frac{|x-x_k|^2}{2!} + \sum_{j=0}^{r-1}\left|y_k^{(j+3)} - f_{1,k}^{(j)}\right|\frac{|x-x_k|^{j+3}}{(j+3)!} + \\ &+ \left|y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}\right|\frac{|x-x_k|^{r+3}}{(r+3)!} \le \\ &\le e_k + he'_k + \frac{h^2}{2!}e''_k + \sum_{j=0}^{r-1}\left|y_k^{(j+3)} - f_{1,k}^{(j)}\right|\frac{h^{j+3}}{(j+3)!} + \\ &+ \left|y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}\right|\frac{h^{r+3}}{(r+3)!} \end{aligned}$$

Now, let

$$U = \left|y_k^{(j+3)} - f_{1,k}^{(j)}\right|$$

then using the Lipschitz condition (3), we get:

(11) $U \le L_1(e_k + \bar{e}_k)$

Also let

$$v = \left|y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}\right|$$

then, using (3), we get:

$$v \le \left|y^{(r+3)}(\xi_k) - y_k^{(r+3)}\right| + \left|f_1^{(r)}(x_k, y_k, z_k) - f_{1,k}^{(r)}\right| \le$$

(12) $$\le \omega(y^{(r+3)}, h) + L_1(e_k + \bar{e}_k)$$

where $\omega(y^{(r+3)}, h)$ is the modulus of continuity of the function $y^{(r+3)}$.

From (10-12) and noting that

(13) $$\sum_{j=0}^{r-1} \frac{h^{j+3}}{(j+3)!} < h^2(e-1) < h^2 e$$

we can see that:

(14) $$e(x) \le (1 + c_0 h^2) e_k + c_0 h^2 \bar{e}_k + h e_k' + \frac{h^2}{2!} e_k'' + \frac{h^{r+3}}{(r+3)!} \omega(y^{(r+3)}, h)$$

where $C_0 = L_1\left(e + \frac{1}{(r+3)!}\right)$ is a constant independent of h.

Similarly, using (6), (8), the Lipschitz condition (4) and the notation (9), we can see that:

(15) $$\bar{e}(x) \le C_1 h^2 e_k + (1 + C_1 h^2)\bar{e}_k + h\bar{e}_k' + \frac{h^2}{2!}\bar{e}_k'' + \frac{h^{r+3}}{(r+3)!}\omega(z^{(r+3)}, h)$$

Where $C_1 = L_2\left(e + \frac{1}{(r+3)!}\right)$, is a constant independent of h and $\omega(z^{(r+3)}, h)$ is the modulus of continuity of the function $z^{(r+3)}$.

We are going to estimate $|y'(x) - S_\Delta'(x)|$. For this purpose we use equations (5), (7), the Lipschitz condition (3), the notation (9) and the inequalities (11), (12) and (13) and we get:

(16) $$e'(x) \le C_2 h e_k + C_2 h \bar{e}_x + e'_k + h e_k'' + \frac{h^{r+2}}{(r+2)!}\omega(y^{(r+3)}, h)$$

where $C_2 = L_1\left(e + \frac{1}{(r+2)!}\right)$, is a constant independent of h.

Similarly, we estimate $|z'(x) - \bar{s}_\Delta'(x)|$. Thus using (6), (8), the Lipschitz condition (4), the notation (9), it can be easily shown that:

(17) $$\bar{e}'(x) \le C_3 h e_k + C_3 h \bar{e}_k + \bar{e}_k' + h\bar{e}_k'' + \frac{h^{r+2}}{(r+2)!}\omega(z^{(r+3)}, h)$$

where $C_3 = L_2\left(e + \frac{1}{(r+2)!}\right)$, is a constant independent of h.

We now estimate $|y''(x) - S_k''(x)|$ and $|z''(x) - \bar{s}_k''(x)|$. Thus, using equations (5-8), the Lipschitz conditions (3-4) and the notation (9), we get:

(18) $$e''(x) \le C_4 h e_k + C_4 h \bar{e}_k + e_k'' + \frac{h^{r+1}}{(r+1)!}\omega(y^{(r+3)}, h)$$

and

(19) $$\bar{e}''(x) \le C_5 h e_k + C_5 h \bar{e}_k + \bar{e}_k'' + \frac{h^{r+1}}{(r+1)!}\omega(z^{(r+3)}, h)$$

where $C_4 = L_1\left(e + \frac{1}{(r+1)!}\right)$ and $C_5 = L_2\left(e + \frac{1}{(r+1)!}\right)$ are constants independent of h.

To complete the convergence proof, we use the matrix inequality which is given in the following definition:

DEFINITION 1. *Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two matrices of the same order, then we say that $A \le B$ iff*

(i) *both a_{ij} and b_{ij} are nonnegative,*

(ii) *$a_{ij} \le b_{ij}$ for all i, j.*

According to this definition, and if we use the matrix notation:

$$E(x) = [e(x) \quad \bar{e}(x) \quad e'(x) \quad \bar{e}'(x) \quad e''(x) \quad \bar{e}''(x)]^T$$

and

$$E_k = (e_k \quad \bar{e}_k \quad e_k' \quad \bar{e}_k' \quad e_k'' \quad \bar{e}_k'')^T$$

then, we can write the estimations (14-19) in the following form:

(20) $$E(x) \le (I + hA)E_k + h^{r+1}\omega(h)B$$

$$\text{where}\quad A = \begin{bmatrix} C_0 & C_0 & 1 & 0 & \frac{1}{2!} & 0 \\ C_1 & C_1 & 0 & 1 & 0 & \frac{1}{2!} \\ C_2 & C_2 & 0 & 0 & 1 & 0 \\ C_3 & C_3 & 0 & 0 & 0 & 1 \\ C_4 & C_4 & 0 & 0 & 0 & 0 \\ C_5 & C_5 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{(r+3)!} \\ \frac{1}{(r+3)!} \\ \frac{1}{(r+2)!} \\ \frac{1}{(r+2)!} \\ \frac{1}{(r+1)!} \\ \frac{1}{(r+1)!} \end{bmatrix},$$

I is the identity matrix of order 6 and

$$\omega(h) = \max\{\omega(y^{(r+3)}, h),\ \omega(z^{(r+3)}, h)\}.$$

Then, we give the following definition of the matrix norm.

DEFINITION 2. *Let $T = [t_{ij}]$ be an $m \times n$ matrix, then we define*

$$\| T \| = \max_i \sum_{j=1}^{n} |t_{ij}|.$$

According to this definition, we get:

$$\|E(x)\| = \max\{e(x), \bar{e}(x), e'(x), \bar{e}'(x), e''(x), \bar{e}''(x)\}. \tag{21}$$

Since (20) is valid for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, then the following inequalities hold true:

$$\|E(x)\| \le (1 + h\|A\|)\|E_k\| + h^{r+1}\,\omega(h)\|B\|$$

$$(1 + h\|A\|)\|E_k\| \le (1 + h\|A\|)^2\|E_{k-1}\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)$$

$$(1 + h\|A\|)^2\|E_{k-1}\| \le (1 + h\|A\|)^3\|E_{k-2}\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)^2$$

$$\dots \qquad \dots \qquad \dots$$

$$(1 + h\|A\|)^k\|E_1\| \le (1 + h\|A\|)^{k+1}\|E_0\| + h^{r+1}\omega(h)\|B\|(1 + h\|A\|)^k$$

Adding L.H.S. and R.H.S. of these inequalities and noting that $\|E_0\| = 0$, we get:

$$\|E(x)\| < C_6 h^r \omega(h)$$

Where $C_6 = \frac{\|B\|}{\|A\|}\left(e^{\|A\|} - 1\right)$, is a constant independent of h.

Then applying (21), we get:

$$e(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right),$$

$$\bar{e}(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right),$$

$$e'(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right), \tag{22}$$

$$\bar{e}'(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right),$$

$$e''(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right)$$

and

$$\bar{e}''(x) \le C_6 h^r \omega(h) = O\left(h^{r+\alpha}\right)$$

Now, we estimate $\left|y^{(q)}(x) - s_k^{(q)}(x)\right|$ where $q = 3(1)r + 2$.

Using (3), (5), (7), (9), (11), (12), and (22), we get:

$$\left|y^{(q)}(x) - s_k^{(q)}(x)\right| = \sum_{j=q-3}^{r-1} \left|y_k^{(j+3)} - f_{1,k}^{(j)}\right| \frac{|x - x_k|^{j+3-q}}{(j+3-q)!} +$$

$$+\left|y^{(r+3)}(\xi_k) - f_{1,k}^{(r)}\right| \frac{|x - x_k|^{r+3-q}}{(r+3-q)!} \le \tag{23}$$

$$\le C_7 h^{r+3-q}\omega(h) = O\left(h^{\alpha+r+3-q}\right)$$

where C_7 is a constant independent of h.

For the case $q = r + 3$, we use (5), (7), (12) and (22) we get:

$$\left|y^{(r+3)}(x) - s_k^{(r+3)}(x)\right| = \left|y^{(r+3)}(\xi) - f_{1,k}^{(r)}\right| \le$$

$$\le C_8\omega(h) = O\left(h^{\alpha}\right)$$

where $C_8 = 1 + 2\, L_1 C_6$, is a constant independent of h.

In a similar manner, using (4), (6), (8), (9) and (22), it can be shown that:

$$\left| z^{(q)}(x) - \bar{s}_k^{(q)}(x) \right| \le C_9 h^{r+3-q} \omega(h) = O\left(h^{\alpha+r+3-q}\right)$$

and

$$\left| z^{(r+3)}(x) - \bar{s}_k^{(r+3)}(x) \right| \le C_{10} \omega(h) = O(h^{\alpha})$$

where $q = 3(1)r+2$ and C_9, C_{10} are constants independent of h.

Thus, we have proved the following theorem:

THEOREM. *Let $S_\Delta(x)$ and $\bar{S}_\Delta(x)$ be the approximate solutions to problem (1)–(2) given by equations (5–6), and let $f_1, f_2 \in C^r([0,1] \times \mathbf{R}^2)$. Then for all $x \in [x_k, x_{k+1}]$, $k = 0(1)n-1$, we have:*

$$\left| y^{(i)}(x) - s_k^{(i)}(x) \right| \le C h^r \omega(h) \quad , i = 0(1)2,$$

$$\left| z^{(i)}(x) - \bar{s}_k^{(i)}(x) \right| \le C h^r \omega(h) \quad , i = 0(1)2,$$

$$\left| y^{(i)}(x) - s_k^{(i)}(x) \right| \le k h^{r+3-j} \omega(h)$$

and

$$\left| z^{(i)}(x) - \bar{s}_k^{(i)}(x) \right| \le k^* h^{r+3-j} \omega(h)$$

where $j = 3(1)r+3$, c, k and k^* are constants independent of h.

NUMERICAL EXAMPLE

Consider the following system of differential equations:

$$y''' = y - z + 2x + e^{-x}, \; y(0) = 1, \; y'(0) = 0, \; y''(0) = 1,$$

$$z''' = y - z + 2x + e^{x}, \; z(0) = 1, \; z'(0) = 0, \; z''(0) = 1.$$

The method is tested using this example in the interval [0, 1] with step size $h = 0.1$ where $r = 0$.

The analytical solution is:

$$y(x) = e^x - x,$$

$$z(x) = e^{-x} + x.$$

The tabulated results, appearing in the following table, are evaluated at the point $x = 0.25$.

Table

	analytical value	numerical value	absolute error
y	1.03403	1.03392	1.05417E–04
z	1.028800783	1.028698843	1.0194007E–04
y'	0.2840254167	0.28258275	0.0014426667
z'	0.2211992169	0.2199325875	0.001266629424
y''	1.284025417	1.27159	0.012435417
z''	0.7788007831	0.7685835	0.0102172831
y'''	1.284025417	1.22140	0.06262541668
z'''	–0.7788007831	–0.81873	0.0399292169

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