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ON THE SOLUTIONS OF QUASI-LINEAR INCLUSIONS OF EVOLUTION

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1. INTRODUCTION

The aim of the present paper is to establish two existence theorems based on fixed point techniques and a Filippov type theorem for the mild solutions of quasi-linear differential inclusion

$$\text{(CP)}\qquad \begin{cases} \dfrac{dx(t)}{dt} \in A(t,x(t))x(t) + F(t,x(t)), & \text{a.e. } t \in I = [0,T], T > 0, \\ x(0) = a \end{cases}$$

Here $A(t,w)$ is a linear operator in a Banach space X and it depends on $t \in I$ and $w \in X$, [28].

Other results on quasi-linear differential inclusions are proved in [20-23].

A deep motivation of the usefulness of the differential inclusions in the study of control problems may be found in [10], [2], [11].

If operator A depends on t and w, the differential inclusion in (CP) is said to be *quasi-linear*, if A depends only on t, the differential inclusion is said to be *semi-linear,* and if A depends neither on t nor on w, the differential inclusion is said to be *linear*, [4], [28], [34].

In [11] Frankowska proves, among other results, a set-valued Gronwall lemma (Filippov type theorem), when the differential inclusion is linear, A being the infinitesimal generator of a strongly continuous semigroup $S(t) \in L(X, X)$, $t \geq 0$, of bounded linear operators from X to X and F is a set-valued map from $I \times X$ into the closed nonempty subsets of X.

Tolstonogov, in [37], mainly in [38], studies similar problems to those in [11], when A is the infinitesimal generator of a C_0-semigroup or an m-dissipative operator.

The existence theorems which will be introduced here have been obtained by the first author in [20].

Interesting results are introduced by Qi Ji Zhu in his recent paper [30] in connection with the case where the differential inclusion has the form $dx(t)/d(t) \in F(t, x(t))$. This approach goes back to Filippov's papers [8], [9]. When multifunction F satisfies a Kamke condition similar results may be found in [37], [29].

In [28] Pazy studies also the existence of a mild solution of the following homogeneous Cauchy problem

$$(\mathrm{CP}_0) \qquad \begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + A(t, u(t)) = 0, & t \in I, \\ u(0) = a \end{cases}$$

He shows, using the contraction mapping principle, that under certain conditions inspired by the "hyperbolic" case the initial value problem (CP_0) has a mild solution on an interval $[0, T']$, $0 < T' \le T$.

Sanekata, in [34], proves several results in connection with the following non-homogeneous quasi-linear initial value problem

$$(\mathrm{CP}_1) \qquad \begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} + A(t, u(t)) = f(t, u(t)), & t \in I, \\ u(0) = a \end{cases}$$

using two nonreflexive Banach spaces Y and X, Y being continuously and densely embedded in X. The method used in [34] to establish the main results concerning the existence of mild or strong solutions is based on a difference approximation technique of (CP_1).

In [17] Kobayashi and Sanekata prove similar results to those in [34], but in order to establish the main result the contraction mapping principle is used.

Anguraj and Balachandran, in [1], are concerned with the existence of a solution of (CP), but in case when $X = \boldsymbol{R}^n$. To get the desired result they used the Bohnenblust-Karlin fixed-point theorem, [33] or [36].

Let Z be a linear topological space. We will use the following notation: $P(Z) = \{A \subset Z \mid A \neq \emptyset\}$, $C(Z) = \{A \in P(Z) \mid A \text{ closed}\}$, $CCo(Z) = \{A \in C(Z) \mid A \text{ convex}\}$, $KCo(Z) = \{A \in P(Z) \mid A \text{ compact and convex}\}$.

Let M be a measurable space with a σ-algebra $\mathscr{A}$, and X a separable metrizable space, a multifunction, [6], $F: M \to P(X)$. F is said to be *measurable (weakly measurable)* if $F^{-1}(E) := \{t \in M \mid F(t) \cap M \neq \emptyset\}$ is measurable for each closed (open) subset E of X. If F has closed values and the σ-algebra $\mathscr{A}$ is complete, F is measurable if and only if F is weakly measurable. This result together with other equivalences may be found in [14] or [41]. If $F: Y \to P(X)$ is a multifunction, where Y is a topological space, then the assertion that F is measurable means that F is measurable when Y is assigned with the σ-algebra $\mathscr{B}$ of the Borel subsets of Y. If $F: M \times Y \to P(X)$ and if the measurability of F is defined in terms of the product σ-algebra $\mathscr{A} \otimes \mathscr{B}$ on $M \times Y$ generated by the sets $A \times B$, where $A \in \mathscr{A}$ and $B \in \mathscr{B}$, then F is said to be *product-measurable*. If $F: M \times Y \to P(X)$ and for each multifunction $G: M \to C(Y)$ the multifunction $F_G: M \to P(X)$, defined by $F_G(t) = \bigcup_{y \in G(t)} F(t, y)$ is measurable, then F is said to be *super-positionally measurable*.

Let I be a fixed interval, $I = [0, T]$, $T > 0$, and X be a Banach space. Denote by $C(I, X)$ the Banach space of continuous functions from I to X with the norm given by $\|x\| = \sup_{t \in I} \|x(t)\|$ and by $\mathscr{L}^1(I, X)$ the Banach space of Bochner integrable (classes of) functions from I to X with the norm given by $\|x\|_1 = \int_I \|x(t)\| \, \mathrm{d}t$. Set $\mathscr{L}^1(I) := \mathscr{L}^1(I, \mathbf{R}_+)$, [7].

A set-valued function $G: I \to P(X)$ is said to be *L-Lipschitz* on $K \subset I$ if for all $x, y \in K$, $G(x) \subset G(y) + L\|x - y\|B$, where B denotes the closed unit ball in X.

A set-valued function $G: I \to 2^X$ is said to be *integrable bounded* if there exists $m \in \mathscr{L}^1(I)$ such that $G(t) \subset m(t)B$ a.e. on I.

If $F: I \times X \to C(X)$ is a multifunction, then by $S^1_{F_x} := S^1_{F_x(\cdot)} := S^1_{F(\cdot, x(\cdot))} \neq \emptyset$ we denote the set of integrable selections of $F(\cdot, x(\cdot))$, $x: I \to X$. A sufficient condition that $S^1_{F(\cdot, x(\cdot))} \neq \emptyset$ is that F has a measurable selection and that $F(\cdot, x(\cdot))$ is integrable bounded.

A multifunction $F: X \to Y$, X and Y being topological spaces, is said to be *upper semicontinuous on a point* $x_0 \in X$ if for every neighborhood V of $F(x_0)$ there exists a neighborhood U of x_0 such that $F(U) \subset V$. $F: X \to Y$ said to be *upper semicontinuous* (u.sc.) on X if it is upper semicontinuous on every point $x_0 \in X$. A multifunction $F: X \to Y$ is said to be *lower semicontinuous* if $F^{-1}(V)$ is open in X whenever $V \subset Y$ is open.

Let I be the interval $I = [0, T]$, $T > 0$ fixed, and X a Banach space. A family of bounded linear operators $\mathscr{U}(t, s)$, on X, $0 \le s \le t \le T$, depending on two parameters is said to be an *evolution system*, [28], if the following two conditions are fulfilled:

(1) $\mathscr{U}(s, s) = 1$, $\mathscr{U}(t, s)\mathscr{U}(r, s) = \mathscr{U}(t, s)$ for $0 \le s \le r \le t \le T$;

(2) $(t, s) \to \mathscr{U}(t, s)$ is strongly continuous for $0 \le s \le t \le T$, where by strong continuity is meant that $\lim_{t \searrow s} \mathscr{U}(t, s)x = x$ for all $x \in X$.

We use the following assumptions:

(X_1) X is a separable Banach space;

(X_2) X satisfies (X_1) and, moreover, it is reflexive;

(A) For every $u \in C(I,X)$ the family of linear operators $\{A(t,u)|t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}_u(t,s)$, $0 \le s \le t \le T$;

(U_1) If $u \in C(I,X)$, the evolution system $\mathcal{U}_u(t,s)$, $0 \le s \le t \le T$ satisfies

(i) there exists a $c_1 \ge 0$ with $\|\mathcal{U}_u(t,s)\| \le c_1$ for $0 \le s \le t \le T$, uniformly in u;

(ii) there exists a $c_2 \ge 0$ such that for any $u, v \in C(I,X)$ and any $w \in X$ we have

$$\|\mathcal{U}_u(t,s)w - \mathcal{U}_v(t,s)w\| \le c_2 \|w\| \int_s^t \|u(\tau) - v(\tau)\| \mathrm{d}\tau;$$

(U_2) If $u \in C(I,X)$ and $0 \le s \le t \le T$, then $\mathcal{U}_u(t,s)$, is a compact operator, i.e. it transforms bounded sets in relatively compact sets. In this case, (cf. [28] p. 48), $U_u(t,s)$ is continuous in the uniform operatorial topology.

(U_3) If $t, t+\delta \in I$, $\delta > 0$, then $\lim_{\delta \to 0} \mathcal{U}_u(t+\delta, t) = 1$, uniformly in u and t.

Remark. If operator A does not depend on w, but it depends on t, then the assumption (A) reads as follows: $\{A(t)|t \in I\}$ generates a unique strongly continuous evolution system $\mathcal{U}(t,s)$, $0 \le s \le t \le T$. In this case we take $c_2 = 0$ (in (ii) from (U_1)).

In connection with the multifunction F we will use the following assumptions:

(F_1) $F: I \times X \to C(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_2) $F: I \times X \to CCo(X)$ and for any $x \in X$, $F(\cdot, x)$ is measurable;

(F_3) F satisfies (F_1) and for any $t \in I$, $F(t,\cdot): X \to C(X)$ is lower semi-continuous from X in $C(X)$ and it is u.sc. from X in $C(w\text{-}X)$, where $w\text{-}X$ is X endowed with the weak topology;

(F_4) F satisfies (F_1), it is product-measurable and for all $t \in I$, $F(t,\cdot): X \to C(X)$ is u.sc.;

(F_5) F satisfies (F_1) and, moreover, it is $k(t)$-Lipschitz, i.e. exists $k \in \mathcal{L}^1(I, \mathbf{R}_+)$ such that for almost all $t \in I$ and for all $x, y \in X$, $D(F(t,x), F(t,y)) \le k(t)\|x - y\|$, D being the Hausdorff-Pompeiu metric.

(F_6) F is integrable bounded by a function $m \in \mathcal{L}^1(I, \mathbf{R}_+)$, that is for all $x \in C(I,X)$ and $t \in I$ we have $F(t, x(t)) \subset m(t)B$, B is the closed unit ball in X.

(F_7) the function $t \mapsto \mathrm{d}(0, F(t,0))$ is integrable on I.

By an *inclusion of evolution* we mean an inclusion of the following form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \in A(t, x(t))x(t) + F(t, x(t)), \quad \text{a.e. } t \in I.$$

Hereafter we are interested to study the *mild solutions* of (CP), i.e. the continuous functions having the following representation

$$x(t) = \mathcal{U}_x(t,0)a + \int_0^t \mathcal{U}_x(t,s)f(s)\mathrm{d}s, \qquad t \in I, \quad f \in S^1_{F_x}.$$

Remark. The evolution inclusions have been investigated in a series of papers, e.g. [28], [34], [1], [26].

2.1. EXISTENCE RESULTS

From the form of the mild solution it is clear that first of all we have to check that the set of integrable selections is non-empty.

2.1. LEMMA. *If one of the following two conditions is satisfied*

(i) (X_1), (F_4) and (F_6);

(ii) (X_2), (F_3) and (F_6),

then for each $x \in C(I,X)$, $S^1_{F_x} \ne \varnothing$.

Proof. If the condition (i) is satisfied, from (X_1) and (F_4) it follows that F is superpositionally measurable, [40], [42]. Hence for each $x \in C(I,X)$ the multifunction defined by $t \mapsto F(t, x(t))$ is measurable. Now applying the Kuratowski, Ryll-Nardzewski selection theorem, [18], it follows that there exists a measurable selection. Taking into account (F_6) we get that the selection is (Bochner) integrable.

If we consider (ii), from (X_2) and (F_3), using [25] theorem 3.4, we get that F is super-positionally measurable. From here we continue as above. ■

Remark. In [30] the problem of non-emptyness of the set of integrable selections is solved by considering it as an assumption, (A_3), p.218.

We use two fixed point theorems. One is called as the Bochnenblust-Karlin fixed point theorem in [1], [33] p.74, [15] p.160 or as the Himmelberg fixed point theorem in [36]. The other is a multivalued version of the Banach fixed point theorem.

2.2. THEOREM. *Let K be a nonempty, closed and convex subset of a locally convex space X. Let $\psi: K \to CCo(K)$ be an upper semi-continuous multifunction such that $\psi(K)$ is a compact set. Then ψ has a fixed point.*

2.3. THEOREM [3]. *Let Y be a non-empty and closed subset of a Banach space X and let $F: Y \to C(Y)$ be a multifunction with the property that there exists a constant $c \in (0,1)$ such that for any $x, y \in Y$ and any $u \in F(x)$ there exists $v \in F(y)$ which satisfies the following inequality*

$$\|u - v\| \le c\|x - y\|.$$

Then F has a fixed point in Y.

Admit (X_1) and (A). Let M be defined by $M = \{x \in C(I,X) | x(0) = a\}$. Obviously, M is a non-empty convex and closed subset of the Banach space $C(I, X)$. Consider the multifunction $\psi: M \to P(M)$ defined by

$$\text{(2.1)} \qquad \psi(x)=\left\{y \in M \middle| y(t)=\mathcal{U}_x(t,0)a+\int_0^t \mathcal{U}_x(t,s)f(s)\mathrm{d}s,\ t \in I, f \in S^1_{F_x}\right\},\ x \in M.$$

If the assumptions of lemma 2.1 are satisfied, the multifunction ψ is properly defined, that is $\psi(x) \neq \varnothing$, for each $x \in M$. Denote by $b = (\|a\| + \|m\|_1) c_1$ (m in (F_6)) and let M_b be the set defined by

$$M_b = M \cap \{x \in C(I,X) | \|x\| \le b\}.$$

Let M_b (t) be the *t-section* of the set $\psi(M_b)$

$$M_b(t) = \{y(t) |\ y \in \psi(x),\ x \in M_b\}.$$

We consider one more assumption

(M_1) If A depends on w we suppose that for each $t \in I$, $M_b(t)$ is relatively compact in X.

Remark. If A does not depend on w, then we prove by lemma 2.6 that $M_b(t)$ is relatively compact, hence the assumption (M_1) is unnecessary.

In what follows we study some properties of the multifunction ψ, useful in applying theorems 2.2 and 2.3.

2.4. LEMMA. *Suppose the following assumptions are satisfied:* (X_2), (A), (U_1), (F_2), (F_6), (F_3) *or* (F_4). *Then for each* $x \in M$, $\psi(x) \in CCo(C(I, X))$.

Proof. Under the above assumptions, taking into account lemma 2.1, for each $x \in M$, $S^1_{F_x} \neq \varnothing$ and hence $\psi(x) \neq \varnothing$. The convexity of the set $\psi(x)$ follows from the assumptions (X_2), (A) and from the linearity of the Bochner integral. All we have to do is to show that $\psi(x)$ is a closed set in $C(I, X)$, that is $\psi(x) \in C(C(I, X))$. For it we consider a sequence $(y_n)_{n\in\mathrm{N}} \subset \psi(x)$ convergent in the uniform topology to an element $y \in C(I, X)$. We show that $y \in \psi(x)$, it means that there exists an element $f \in S^1_{F_x}$ such that

$$y(t) = \mathcal{U}_x(t,0)a + \int_0^t \mathcal{U}_x(t,s)f(s)\mathrm{d}s, \quad t \in I.$$

Now, if $y_n \in \psi(x)$, there exists $f_n \in S^1_{F_x}$ such that

$$y_n(t) = \mathcal{U}_x(t,0)\,a + \int_0^t \mathcal{U}_x(t,s)f_n(s)\mathrm{d}s, \quad t \in I,\ n \in \mathrm{N}.$$

Since F is integrable bounded, $\{f_n | n \in \mathrm{N}\}$ is a bounded set in $\mathcal{L}^1(I, X)$. From Pettis theorem, theorem 2.11.2 [13], and (X_2) it follows that the set $\{f_n(t) | n \in \mathrm{N}\}$ is sequentially weak compact, $t \in I$. From proposition 1.2 [38] we have that the set $\{f_n(t) | n \in \mathrm{N}\}$ is metrizable relatively weak compact in $\mathcal{L}^1(I, X)$. It means that (passing to a subsequence and keeping the notation, if necessary) $(f_n)_{n\in\mathrm{N}}$ converges weakly to an element $f \in \mathcal{L}^1(I, X)$. It remained to see if $f \in S^1_{F_x}$. From Mazur lemma, [39] p.199, or [32] there exists a sequence $(g_n)_{n\in\mathrm{N}}$, as convex combinations of elements from $\{f_n\}_{n\in\mathrm{N}}$, which converges strongly to $f \in \mathcal{L}^1(I, X)$. It is clear that $g_n(t) \in F(t, x(t))$, $t \in I$ and, moreover, $g_n \in S^1_{F_x}$, $n \in \mathrm{N}$. Since $(g_n)_{n\in\mathrm{N}}$ converges strongly to $f \in \mathcal{L}^1(I, X)$ and F has closed values, we have that $f(t) \in F(t, x(t))$ a.e. I and hence $f \in S^1_{F_x}$.

For each $t \in I$ the map $h \mapsto \int_0^t \mathcal{U}_x(t,s)h(s)\mathrm{d}s$ from, $\mathcal{L}^1(I, X)$ in X is linear and continuous (in fact Lipschitz, from (U_1)) and, from theorem IV.7.4 in [35], it remains continuous as a map from $w - \mathcal{L}^1(I, X)$ in w - X. Hence, for each $t \in I$, the sequence $(y_n(t))_{n\in\mathrm{N}}$ converges to $y(t)$ in $w - X$. From hypothesis we have that $y_n \to y$ uniformly, which implies that $y \in \psi(x)$. ∎

2.5. LEMMA. *If the assumptions from lemma* 2.1 *and* (U_1) *are satisfied, then* $\psi(M) \subset M_b$.

Proof. We show that for each $x \in M$ and each $y \in \psi(x)$ the estimation $\|y\| \le b$ holds. Indeed

$$\|y(t)\| \le \|\mathcal{U}_x(t,0)\|\|a\| + \int_0^t \|\mathcal{U}_x(t,s)\|\|f(s)\|\,\mathrm{d}s \le \|a\|c_1 + \|m\|_1 c_1 = b, \quad t \in I.$$

∎

2.6. LEMMA. *Under the assumptions of lemma* 2.1 *and* (U_1) *and* ($F_{5\text{-}6}$), *the map* $x \mapsto \psi(x)$ *from* M *to* $CC_0(M_b)$ *is uniformly upper semi-continuous in respect to Hausdorff-Pompeiu metric.*

Proof. Choose an arbitrary $\varepsilon > 0$. We want to find an $\eta > 0$ such that if $u, v \in M$ with $\|u - v\| < \eta$, then $\mathrm{d}(\psi(u), \psi(v)) < \varepsilon$, that is, for each $y \in \psi(u)$ there exists $z \in \psi(v)$ with $\|y - z\| < \varepsilon$.

If $u = v$ then $\psi(u) = \psi(v)$ and $\mathrm{d}(\psi(u), \psi(v)) = 0 < \varepsilon$.

Suppose $u \neq v$. Let $\mu > 0$ be arbitrary. If $d(\psi(u), \psi(v)) < \varepsilon$, then for each $y \in \psi(u)$ there exists $z \in \psi(v)$ such that $\|y - z\| < \varepsilon$. If $y \in \psi(u)$ there exists $f \in S^1_{F_u}$ such that

$$y(t) = \mathcal{U}_u(t,0)a + \int_0^t \mathcal{U}_u(t,s)f(s)\,ds, \quad t \in I.$$

Write $k_\mu(t) = k(t) + \mu$. We have the assumption (F_5)

$$D(F(t,u), F(t,v)) \leq k(t)\|u - v\| < k_\mu(t)\|u - v\|.$$

It follows that there exists $g \in S^1_{F_v}$ such that

$$\|f(t) - g(t)\| \leq k_\mu(t)\|u(t) - v(t)\|, \text{ a.e. on } I.$$

Let us take

$$z(t) = \mathcal{U}_v(t,0)a + \int_0^t \mathcal{U}_v(t,s)g(s)\,ds, \quad t \in I.$$

We have

$$\|y(t) - z(t)\| \leq \|\mathcal{U}_u(t,0)a - \mathcal{U}_v(t,0)a\| + \int_0^t \|\mathcal{U}_u(t,s)f(s) - \mathcal{U}_v(t,s)g(s)\|\,ds, \quad t \in I.$$

Taking into account (U_1) we have the estimation

$$\|\mathcal{U}_u(t,0)a - \mathcal{U}_v(t,0)a\| \leq c_2\|a\| \int_0^t \|u(s) - v(s)\|\,ds \leq c_2\|a\|T\|u - v\|.$$

On the other side we have

$$\|\mathcal{U}_u(t,s)f(s) - \mathcal{U}_v(t,s)g(s)\| \leq \|\mathcal{U}_u(t,s)f(s) - \mathcal{U}_v(t,s)f(s)\| +$$

$$+\|\mathcal{U}_v(t,s)f(s) - \mathcal{U}_v(t,s)g(s)\| \leq c_1\|f(s) - g(s)\| + c_2\|f(s)\|\int_s^t \|u(\tau) - v(\tau)\|\,d\tau \leq$$

$$\leq c_1\|f(s) - g(s)\| + c_2\|f(s)\|t\|u - v\|.$$

Hence

$$\|y(t) - z(t)\| \leq c_2\|a\|T\|u - v\| + c_1\int_0^t \|f(s) - g(s)\|ds + c_2T\|m\|_1\|u - v\| \leq$$

$$\leq \left[c_2T(\|a\| + \|m\|_1) + c_1\|k_\mu\|_1\right]\|u - v\|, \ t \in I.$$

Let us write $c_3 = c_2T(\|a\| + \|m\|_1) + c_1\|k_\mu\|_1$. Then

$$\|y - z\| \leq c_3\|u - v\|.$$

If $c_3 = 0$, then $\eta > 0$ is arbitrary. If $c_3 > 0$, then we may consider $0 < \eta < \varepsilon / c_3$. Then if $\|u - v\| < \eta$, it follows that $\|y - z\| < \varepsilon$. It is clear that η does not depend on u or v. From the method of finding of y and z we have that $d(\psi(y), \psi(v)) \leq \leq c_3\|u - v\|$. ■

2.7. THEOREM. *If the following assumptions (X_2), (A), (U_1), (F_2), $(F_{5\text{-}6})$ are satisfied and if $0 < c_3 < 1$, then there exists a mild solution of the problem (CP) in M_b.*

Proof. We consider the map $\psi_b: M_b \to C(M_b)$ defined by $\psi_b = \psi|_{M_b}$ and then we use theorem 2.3. ■

2.8. LEMMA. *Under the assumptions (X_1), (A), $(F_{5\text{-}6})$, (U_1) and (U_3), $\psi(M)$ is a family of equicontinuous maps.*

Proof. To prove that $\psi(M)$ is a family of equicontinuous maps it is enough to show that for any $\varepsilon > 0$ there exists $\mu > 0$ such that for every t, $t + \delta \in I$, $0 < \delta < \mu$, $x \in M$ and $y \in \psi(x)$ the following inequality takes place

$$\|y(t + \delta) - y(t)\| < \varepsilon.$$

We have

$$\|y(t + \delta) - y(t)\| \leq \|\mathcal{U}_x(t + \delta,0)a - \mathcal{U}_x(t,0)a\| +$$

$$+\left\|\int_0^{t+\delta} \mathcal{U}_x(t + \delta, s)f(s)\,ds - \int_0^t \mathcal{U}_x(t + s)f(s)\,ds\right\| \leq$$

$$\leq \|[\mathcal{U}_x(t + \delta, t) - 1_X]\mathcal{U}_x(t,0)a\| +$$

$$+\left\|\int_0^t [\mathcal{U}_x(t + \delta, t) - 1_X]\,\mathcal{U}_x(t,s)f(s)ds\right\| + \int_t^{t+\delta} \|\mathcal{U}_x(t + \delta, s)f(s)\|\,ds,$$

and since a linear and continuous operator commutes with the integral we have further

$$\leq \left[\|\mathcal{U}_x(t + \delta, t) - 1_X\|\right]\left[\|\mathcal{U}_x(t,0)a\| + \int_0^t \|\mathcal{U}_x(t,s)f(s)\|\,ds\right] + \int_t^{t+\delta} \|\mathcal{U}_x(t + \delta, s)f(s)\|\,ds.$$

From the hypothesis and from theorem 9 p. 49 [7], (by which the last integral is as small as we like provided δ is sufficiently small) there results our lemma. ■

Remark. Similar evaluations appear in [11] and [38].

2.9. LEMMA [11]. *Let $U: I \to C(X)$ be a measurable set-valued map and $g: I \to X$, $k: I \to \mathbf{R}_+$ be measurable single-valued maps. Assume that*

$$W(t) = U(t) \bigcap \{g(t) + k(t)B\} \neq \emptyset, \text{ a.e. on } I.$$

Then there exists a measurable selection from W on I.

Proof. $g(t) + k(t)B = B(g(t), k(t))$, i.e., the closed ball centered in $g(t)$ with radius $k(t)$. By theorem III.41 and proposition III.13 in [5] the multifunction $t \mapsto B(g(t), k(t))$ has measurable graph. Now, by theorem III.40 in [5] the multifunction $t \to W(t)$ has measurable graph, and by theorem 3.5 in [14], it is a measurable closed valued multifunction. Then it has a measurable selection. ■

2.10. LEMMA. *If the assumptions (X_1), (A), (U_1), (F_1), (F_6) and (M_1)* or *(X_2), then the set $M_b(t)$ is relatively compact in X, for each $t \in I$.*

Proof. If A depends on w, then by the assumption (M_1) it follows that the lemma is true. Suppose, further, that A does not depend on w and we follow the way in [27]. Then

$$M_b(t) \subset \mathcal{U}(t,0)a + \int_0^t \mathcal{U}(t,s)P(s)\,ds,$$

where $P(t) = \{x \in X \mid \|x\| \leq \sup\{|F(t,z)|: \|z\| \leq b\}$. By (F_6) we have that $P(s) = m(s)B$, with $m \in \mathcal{L}^1(I, \mathbf{R}_+)$, iar and by (U_2) that $\overline{\mathcal{U}(t,s)P(s)}$ is convex and compact in X. Since $P(s)$ is measurable, (lemma 2.9 with $g = 0$, $k = m$), it follows that the map $s \mapsto \mathcal{U}(t,s)P(s)$, $s \in [0,t]$, is measurable, too. By the embedding theorem of Rådström, [31], [16] or [12] theorem 3.6 (2°) and theorem 4.5 (2°) we get that $\overline{\int_0^t \mathcal{U}(t,s)P(s)\,ds}$ is convex and compact in X. From here it follows that $\overline{M_b(t)}$ is compact in X. ■

2.11. THEOREM. *Suppose the following assumptions are satisfied*

(i) (X_2), (A), (U_1), (U_3), (F_2), $(F_{5\text{-}6})$;
(ii) (M_1) *or* (U_2) ;
(iii) (F_3) *or* (F_4) .

Then there exists a mild solution in M_b of the (CP) problem.

Proof. Consider in theorem 2.2. $K = M_b \in CCo(C(I,X))$ and ψ defined by (2.1).From lemma 2.1 we have that $\psi(x) \neq \emptyset$, for each $x \in M_b$, and from

lemmas 2.4 and 2.5 it follows that $\psi(M_b) \subset \psi(M) \subset M_b$. Moreover, it is valid that $\psi(x) \in CCo(M_b)$, for each $x \in M_b$. So, we have checked that $\psi: M_b \to CCo(M_b)$.

By lemma 2.8 it follows that $\psi(M_b)$ is a family of equicontinuous maps, and by lemma 2.10 it follows that $M_b(t)$ (t-section) is relatively compact in X, for each $t \in I$. Hence, based on the Ascoli-Arzelà, [2] p.13, we have that $\overline{\psi(M_b)}$ is compact in $C(I,X)$.

From $\psi(x) \subset \psi(M_b)$ it follows that $\overline{\psi(x)} = \psi(x) \subset \overline{\psi(M_b)}$ for each $x \in M_b$, such that $\psi(x) \in KCo(M_b)$. Taking into account that ψ is upper semi-continuous in respect to the Hausdoff-Pompeiu metric and it has compact values, based on an observation in [2] p. 45, we have that ψ is upper semi-continuous. In this way we have verified all the requirements of theorem 2.2. Hence the multifunction ψ has a fixed point in M_b. This fixed point is a mild solution of the problem (CP). ■

2.12. THEOREM. *Consider $f, g \in \mathcal{L}^1(I,X)$, $\chi = \|f - g\|_1$ and $\delta = \|x_0 - y_0\|$ such that the assumptions of theorem [2.11] are fulfilled with f and g instead of F. Denote by x and y two mild solutions of the quasi-linear equations corresponding to the cases f, x_0, respectively g, y_0. Then the estimation holds.*

$$\|x(t) - y(t)\| \leq c_1(\chi + \delta)\exp\left[c_2\left(\min\{\|x_0\|, \|y_0\|\} + \min\{\|f\|_1, \|g\|_1\}\right)t\right], \quad t \in I.$$

Proof. Let us denote by

$$x(t) = \mathcal{U}_x(t,0)x_0 + \int_0^t \mathcal{U}_x(t,s)f(s)\,ds, \quad t \in I,$$

$$y(t) = \mathcal{U}_y(t,0)y_0 + \int_0^t \mathcal{U}_y(t,s)g(s)\,ds, \quad t \in I$$

the two mild solutions. Then

$$\|x(t) - y(t)\| \leq \|\mathcal{U}_x(t,0)x_0 - \mathcal{U}_y(t,0)y_0\| + \int_0^t \|\mathcal{U}_x(t,s)f(s) - \mathcal{U}_x(t,s)g(s)\|\,ds +$$

$$+ \int_0^t \|\mathcal{U}_x(t,s)g(s) - \mathcal{U}_y(t,s)g(s)\|\,ds \leq$$

$$\leq c_1\delta + c_2\|y_0\| \int_0^t \|x(s) - y(s)\|\,ds + c_1 \int_0^t \|f(s) - g(s)\|\,ds +$$

$$+c_2\int_0^t \|g(s)\| \int_s^t \|x(\tau) - y(\tau)\| d\tau ds \le$$

$$\le c_1(\delta + \chi) + c_2\left(\|y_0\| + \|g\|_1\right)\int_0^t \|x(s) - y(s)\| ds.$$

Using Bellman lemma, [19] p.353, we get the inequality

$$\|x(t) - y(t)\| \le c_1(\delta + \chi)\exp\left[c_2\left(\|y_0\| + \|g\|_1\right)t\right], \quad t \in I.$$

From here there results the desired estimation. ■

COROLLARY. *Under the assumptions of the above theorem we have*

$$\|x - y\|_{C(I,X)} \le c_1(\delta + \chi)\exp\left[c_2\left(\min\{\|x_0\|, \|y_0\|\} + \min\{\|f\|_1, \|g\|_1\}\right)T\right].$$

COROLLARY. *Under the assumptions of the above theorem and if, moreover,* $\delta = \chi = 0$, *then* $x = y$, *hence the mild solution of an initial value quasi-linear equation is unique.*

2.2. FILIPPOV-TYPE THEOREM

In this section we consider two Cauchy problems:

$$\text{(2.2)} \quad \begin{cases} \dfrac{dx(t)}{dt} \in A(t, x(t))x(t) + F(t, x(t)), \ \text{a.e. } t \in I, \\ x(0) = a = x_0 \end{cases}$$

and

$$\text{(2.3)} \quad \begin{cases} \dfrac{dy(t)}{dt} = A(t, y(t))y(t) + g(t), \ g \in \mathscr{L}^1(I, X), \ \text{a.e. } t \in I, \\ y(0) = y_0. \end{cases}$$

(S_1) Suppose that problem (2.3) has a mild solution

$$y(t) = \mathscr{U}_y(t,0)y_0 + \int_0^t \mathscr{U}_y(t,s)g(s)ds, \quad t \in I.$$

We show that if the initial values and the non-linear parts are sufficiently close, then problem (2.2) has a mild solution x whose distance to y does not exceed a certain value.

2.13. LEMMA [11]. *Let* $G: I \times X \to C(X)$ *be measurable in the first variable and continuous in the second variable and* $z \in C(I, X)$, *then the set-valued map* $t \to G(t, z(t))$ *is measurable.*

Proof. From proposition 2.3 [24] we have that G is product-measurable. By theorem 1 in [40] there results that G is super-positionally measurable.

The above result may be obtained from theorem 2.2 in [30], too.

2.14. LEMMA [11]. *Let* $U: I \to C(X)$ *be a measurable multifunction and* $u: I \to X$ *be a measurable map. Then the function* $t \to d(u(t), U(t))$ *is measurable.*

2.15. LEMMA [11]. *Let* G *and* z *be as in lemma* 2.10 *and* $h \in \mathscr{L}^1(I, X)$. *Then, if* G *satisfies* (F_7) *and a* $k(t)$*-Lipschitz condition, the function* $t \to d(h(t), G(t, z(t)))$, *is integrable on* I.

Proof. From lemmas 2.13 and 2.14 we have that our function is measurable. It is also bounded by an integrable one since

$$d(h(t), G(t, z(t))) \le \|h(t)\| + d(0, G(t,0)) + d(G(t,0), G(t, z(t))) \le$$

$$\le \|h(t)\| + d(0, G(t,0)) + D(G(t,0), G(t, z(t))) \le$$

$$\le \|h(t)\| + d(0, G(t,0)) + k(t)\|z(t)\|. \quad ■$$

2.16. LEMMA. *Suppose that under the assumptions* (X_1), (A) and (U_1) *each quasi-linear Cauchy problem*

$$\begin{cases} x_n'(t) = A(t, x_n(t))x_n(t) + f_n(t), \quad a.e. \ on \ I \\ x_n(0) = a, \end{cases}$$

$n \in \mathbf{N}$, *has a mild solution*

$$x_n(t) = \mathscr{U}_{x_n}(t,0)a + \int_0^t \mathscr{U}_{x_n}(t,s)f(s)ds, \quad t \in I.$$

Suppose, also, that there exist $x \in C(I, X)$ *and* $f \in \mathscr{L}^1(I, X)$ *such that* $x_n \xrightarrow{C(I,X)} x$ *and* $f_n \to f$ *in* $\mathscr{L}^1(I, X)$ *and that the set* $\{f\} \cup \{f_n\}_{n \in \mathbf{N}}$ *is integrable bounded by a function* $m \in \mathscr{L}^1(I, X)$. *Then*

$$x(t) = \mathscr{U}_x(t,0)a + \int_0^t \mathscr{U}_x(t,s)f(s) \ ds, \quad t \in I.$$

Proof. We show that $\left\| x_n(t) - \mathscr{U}_x(t,0)a - \int_0^t \mathscr{U}_x(t,s)f(s)ds \right\|$ converges to 0 provided $n \to +\infty$, uniformly in t. We have

$$\left\| x_n(t) - \mathcal{U}_x(t,0)a - \int_0^t \mathcal{U}_x(t,s)f(s)\mathrm{d}s \right\| \le$$

$$\le \left\| \mathcal{U}_{x_n}(t,0) - \mathcal{U}_x(t,0) \right\| + \int_0^t \left\| \mathcal{U}_{x_n}(t,s)f_n(s) - \mathcal{U}_x(t,s)f(s) \right\| \mathrm{d}s \le$$

$$\le c_2\|a\| \int_0^t \|x_n(s) - x(s)\| \mathrm{d}s + \int_0^t \left\| \mathcal{U}_{x_n}(t,s)f_n(s) - \mathcal{U}_{x_n}(t,s)f(s) \right\| \mathrm{d}s +$$

$$+ \int_0^t \left\| \mathcal{U}_{x_n}(t,s)f(s) - \mathcal{U}_x(t,s)f(s) \right\| \mathrm{d}s \le$$

$$\le c_2\|a\| \int_0^t \|x_n(s) - x(s)\| \mathrm{d}s + c_1 \int_0^t \|f_n(s) - f(s)\| \mathrm{d}s +$$

$$+ c_1 \int_0^t \| f(s) \| \int_s^t \|x_n(\tau) - x(\tau)\| \mathrm{d}\tau \mathrm{d}s \le$$

$$\le \left(c_2\|a\| + \|m\|_1\right) \int_0^t \|x_n(s) - x(s)\| \mathrm{d}s + c_1 \int_0^t \| f_n(s) - f(s) \| \mathrm{d}s \le$$

$$\le \left(c_2\|a\| + \|m\|_1\right) T \|x_n - x\| + c_1 \|f_n - f\|_1 .$$

Remark. Convergence results as stated above may be found in [38]. ∎

We consider problems (2.2) and (2.3) under the assumptions (X_1), (A), (F_5) and (F_6). Write $\delta = \|x_0 - y_0\|$, $p = c_2(\|x_0\| + \|m\|_1)$, $k_\varepsilon(t) = k(t) + \varepsilon$, $\varepsilon > 0$, $K(t) = \int_0^t [p + k_\varepsilon(s)]\mathrm{d}s$, $E(t) = \exp(K(t))$, $t \in I$. Moreover, we admit the assumption (S_1) and let $\gamma(t) = \mathrm{d}(g(t), F(t, y(t)))$, $t \in I$. Based on lemma 2.15 we have that $\gamma \in \mathcal{L}^1$ and then consider $n(t) = c_1\left[\delta + \int_0^t (\gamma(s) + \varepsilon)\mathrm{d}s\right]$, $t \in I$.

2.17. THEOREM. *Suppose the following assumptions are satisfied:* (X_1), (A), (U_1), (F_5), (F_6), (F_7) *and* (S_1). *Then problem* (2.2) *has a mild solution* $x \in C(I, X)$ *such that*

(2.4) $$\|x(t) - y(t)\| \le n(t)E(t) = c_1\left[\delta E(t) + E(t)\int_0^t (\gamma(s) + \varepsilon)\mathrm{d}s\right], \quad t \in I,$$

and for almost every $t \in I$

(2.5) $$\|f(t) - g(t)\| \le \gamma(t) + \varepsilon + n(t)k_\varepsilon(t)E(t).$$

Proof. The method (as in [8], [10], [11] [38]) consists in constructing two convergent sequences $(x_n)_{n\ge1} \subset C(I,X)$ and $(f_n)_{n\ge1} \subset \mathcal{L}^1(I,X)$ such that x, the limit of $(x_n)_{n\ge1}$ in the uniform topology from $C(I, X)$, is the mild solution of

the problem (2.2) and it satisfies (2.4). f, the limit of the sequence $(f_n)_{n\ge1}$ in $\mathcal{L}^1(I, X)$, satisfies (2.5) and appears in the formula of x.

Let us see the first two steps of this inductive procedure. Consider the multifunction given by

$$t \mapsto W_1(t) := F(t, y(t)) \cap \{g(t) + (\gamma(t) + \varepsilon)B\} \ne \varnothing, \quad t \in I.$$

From the definition of W_1 it results that it is measurable, integrable bounded and closed valued. Hence it has an integrable selection $f_1 \in \mathcal{L}^1(I, X)$. Then $\|f_1(t) - g(t)\| \le \gamma(t) + \varepsilon$ .Define x_1 as

$$x_1(t) = \mathcal{U}_y(t,0)x_0 + \int_0^t \mathcal{U}_y(t,s)f_1(s)\,\mathrm{d}s, \quad t \in I,$$

and we get that

$$\|x_1(t) - y(t)\| \le c_1\left[\delta + \int_0^t (\gamma(s) + \varepsilon)\mathrm{d}s\right] = n(t).$$

Consider the multifunction given by

$$t \mapsto W_2(t) := F(t, x_1(t)) \cap \{f_1(t) + k_\varepsilon(t)\|x_1(t) - y(t)\|B\} \ne \varnothing, \quad t \in I.$$

We show that $W_2(t) \ne \varnothing$ for each $t \in I$. If $x_1(t) = y(t)$, then $F(t,x_1) = F(t, y(t))$, hence $f_1(t) \in F(t,x_1(t))$. Suppose $x_1(t) \ne y(t)$. From the following inequalities

$$\mathrm{d}(f_1(t), F(t,x_1)) \le \mathrm{d}(F(t, y(t)), \; F(t,x_1(t)) \le D(F(t, y(t)), \; F(t,x_1(t)) \le$$
$$\le k(t)\|x_1(t) - y(t)\| < k_\varepsilon(t)\|x_1(t) - y(t)\|$$

we infer that there exists $z \in F(t, x_1(t))$ such that

$$z \in f_1(t) + k_\varepsilon(t)\|x_1(t) - y(t)\|B,$$

and so $W_2(t) \ne \varnothing$. Hence for each $t \in I$, $W_2(t) \ne \varnothing$. By lemma 2.13 the multifunction $t \mapsto F(t, x_1(t))$ is measurable, $t \in I$; the function $\mapsto k_\varepsilon(t)\|x_1(t) - y(t)\|$ is summable on I. By lemma 2.9 we have that multifunction $t \mapsto W_2(t)$, $t \in I$ admits a measurable selection which is also summable, hence $f_2 \in \mathcal{L}^1(I, X)$. Then there holds the estimation

$$\|f_2(t) - f_1(t)\| \le k_\varepsilon(t)n(t).$$

Define x_2 as

$$x_2(t) = \mathcal{U}_{x_1}(t,0)x_0 + \int_0^t \mathcal{U}_{x_1}(t,s)f_2(s)\mathrm{d}s, \quad t \in I.$$

We have

$$\|x_2(t)-x_1(t)\| \le \|\mathscr{U}_{x_1}(t,0)a-\mathscr{U}_y(t,0)a\| + \int_0^t \|\mathscr{U}_{x_1}(t,s)f_2(s)-\mathscr{U}_y(t,s)f_1(s)\|\mathrm{d}s \le$$

$$\le c_2\|a\|\int_0^t \|x_1(s)-y(s)\|\,\mathrm{d}s + \int_0^t \|\mathscr{U}_{x_1}(t,s)f_2(s)-\mathscr{U}_{x_1}(t,s)f_1(s)\|\,\mathrm{d}s +$$

$$+\int_0^t \|\mathscr{U}_{x_1}(t,s)f_1(s)-\mathscr{U}_y(t,s)f_1(s)\|\,\mathrm{d}s \le$$

$$\le c_2\|a\|\int_0^t \|x_1(s)-y(s)\|\,\mathrm{d}s + c_1\int_0^t \|f_2(s)-f_1(s)\|\,\mathrm{d}s +$$

$$+c_2\int_0^t \|f_1(\tau)\|\int_\tau^t \|x_1(s)-y(s)\|\,\mathrm{d}s\mathrm{d}\tau \le$$

$$\le \int_0^t [p+c_1k_\varepsilon(s)]\|x_1(s)-y(s)\|\,\mathrm{d}s \le \int_0^t [p+c_1k_\varepsilon(s)]n(s)\mathrm{d}s.$$

If the last integral is increased to $n(t)K(t)$ (this is allowed since the function $n(t)$ is positive and increasing), then we get

$$\|x_2(t)-x_1(t)\| \le n(t)K(t)$$

and

$$\|x_2-y\|_{C(I,X)} \le n(T)[1+K(T)].$$

Now, let us take $n\in\mathbb{N}$, $n\ge 2$ and suppose we have determined the sequences $(x_i)_{1\le i\le n}\subset C(I,X)$ and $(f_i)_{1\le i\le n}\subset \mathscr{L}^1(I,X)$ such that

$$W_i(t) := F(t,x_{i-1}(t))\cap\{f_{i-1}(t)+k_\varepsilon(t)\|x_{i-1}(t)-x_{i-2}(t)\|B\},\quad f_i\in S^1_{W_i},$$

$$x_i(t)=\mathscr{U}_{x_{i-1}}(t,0)x_0+\int_0^t \mathscr{U}_{x_{i-1}}(t,s)f_i(s)\mathrm{d}s,$$

$$\|x_i(t)-x_{i-1}(t)\| \le n(t)\frac{[K(t)]^{i-1}}{(i-1)!},$$

$$\|x_i(t)-y(t)\| \le n(t)\sum_{j=0}^{i-1}\frac{[K(t)]^j}{j!},$$

$$\|f_i(t)-f_{i-1}(t)\| \le k_\varepsilon(t)\|x_{i-1}(t)-x_{i-2}(t)\|,$$

$$\|f_i(t)-g(t)\| \le \gamma(t)+\varepsilon+k_\varepsilon(t)n(t)\sum_{j=0}^{i-2}\frac{[K(t)]^j}{j!}$$

$t\in I$ and $i\ge 2$. We accept that $x_0(t)=y(t)$, $t\in I$. Consider the following multifunction

$$t\mapsto W_{n+1}(t) := F(t,x_n(t))\cap\{f_n(t)+k_\varepsilon(t)\|x_n(t)-x_{n-1}(t)\|B\},$$

which admits an integrable selection f_{n+1}. Thus we have

$$\|f_{n+1}(t)-f_n(t)\| \le k_\varepsilon(t)\|x_n(t)-x_{n-1}(t)\|, \tag{2.6}$$

$$\|f_{n+1}(t)-g(t)\| \le \gamma(t)+\varepsilon+k_\varepsilon(t)n(t)\sum_{j=0}^{i-2}\frac{[K(t)]^j}{j!}. \tag{2.7}$$

Using the selection f_{n+1} we construct x_{n+1} as

$$x_{n+1}(t)=\mathscr{U}_{x_n}(t,0)x_0+\int_0^t \mathscr{U}_{x_n}(t,s)f_{n+1}(s)\,\mathrm{d}s \tag{2.8}$$

and it results that

$$\|x_{n+1}(t)-x_n(t)\| \le n(t)\frac{[K(t)]^n}{n!}, \tag{2.9}$$

$$\|x_{n+1}(t)-y(t)\| \le n(t)\sum_{i=0}^{n}\frac{[K(t)]^i}{i!}. \tag{2.10}$$

From (2.9) (2.5) for all $n\in\mathbb{N}$ we have

$$\|x_{n+1}-x_n\|_{C(I,X)} \le n(T)\frac{[K(T)]^n}{n!},$$

$$\|f_{n+1}-f_n\|_1 \le n(T)\|k\|_1\|x_{n+1}-x_n\|_{C(I,X)}.$$

These inequalities imply that $(x_n)_{n\ge 1}$ and $(f_n)_{n\ge 1}$ are convergent sequences. Let $x\in C(I,X)$, respectively $f\in\mathscr{L}^1(I,X)$, be their limits. Then by lemma 2.16 we have that $f\in S^1_{F_x}$, more exactly that x is a mild solution of problem (2.2) which corresponds to the selection f. From (2.10) we get estimation (2.4), and from (2.7) it follows (2.5). ■

Remark. If the differential inclusion in (2.2) is semi-linear, then $p=0$ and thus we get a result in [11].

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