



Complex fermatean fuzzy N -soft sets: a new hybrid model with applications

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Abstract

Decision-making methods play an important role in the real-life of human beings and consist of choosing the best options from a set of possible choices. This paper proposes the notion of complex Fermatean fuzzy N -soft set (CFFNS _{f} S) which, by means of ranking parameters, is capable of handling two-dimensional information related to the degree of satisfaction and dissatisfaction implicit in the nature of human decisions. We define the fundamental set-theoretic operations of CFFNS _{f} S and elaborate the CFFNS _{f} S associated with threshold. The algebraic and Yager operations on CFFNS _{f} numbers are also defined. Several algorithms are proposed to demonstrate the applicability of CFFNS _{f} S to multi-attribute decision making. The advanced algorithms are described and accomplished by several numerical examples. Then, a comparative study manifests the validity, feasibility, and reliability of the proposed model. This method is compared with the Fermatean fuzzy Yager weighted geometric (FFY_wG) and the Fermatean fuzzy Yager weighted average (FFY_wA) operators. Further, we developed a remarkable CFFNS _{f} -TOPSIS approach by applying innovative CFFNS _{f} weighted average operator and distance measure. The presented technique is fantastically designed for the classification of the most favorable alternative by examining the closeness of all available choices from particular ideal solutions. Afterward, we demonstrate the amenability of the initiated approach by analyzing its tremendous potential to select the best city in the USA for farming. An integrated comparative analysis with existing Fermatean fuzzy TOPSIS technique is rendered to certify the terrific capability of the established approach. Further, we decisively investigate the rationality and reliability of the presented CFFNS _{f} S and CFFNS _{f} -TOPSIS approach by highlighting its advantages over the existent models and TOPSIS approaches. Finally, we holistically describe the conclusion of the whole work.

Keywords Complex Fermatean fuzzy set · N -soft set · Decision making

1 Introduction

Multi-attribute decision-making (MADM) methods play an important role in the real life of human beings. The process of choosing the best option among a set of possible options is present in all human activities. Decision making in the domain of crisp sets to handle exact and precise data has been a growing field of research for mathematicians.

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1.1 Related work

Given the dubious and erroneous nature of human decisions, the limitations of decision making in the area of crisp set have gained importance over time. Zadeh (1965) was the pioneer who coped with the fuzziness and ambiguity of human decisions in the field of decision making. Fuzzy set (FS) theory refined not only the decision making, but also the related fields like social sciences, production management, etc. (Abdullah et al. 2012; Alcantud and Andrés - Calle 2017; Guiffreda and Nagi 1998). FS theory empowers the experts to use their complacency level (membership/belongingness degree) with attributes whose values are between 0 and 1.

Undoubtedly, FS theory allows to work with unsettling analysis in the field of decision making. Nevertheless, FS theory could not assess the nature of satisfaction and

dissatisfaction with human decisions. To salvage these shortcomings, Atanassov (1986) extended the FS with intuitionistic fuzzy sets (IFS) and added the non-membership function which is limited to the interval $[0,1]$ in order to express the level of discontent with human decisions. In his model the sum of satisfaction and dissatisfaction degrees is in the unit interval.

In 2013, Yager (2013a, 2013b) adapted the conditions of IFS, to present the novel concept of Pythagorean fuzzy set (PFS) with relaxing conditions that the sum of square of belongingness degree and non-membership degree should enclose in unit interval. Due to the constraints in PFS, Yager (2016) introduced the model of q -rung orthopair fuzzy set (q -ROFS) with conditions that sum of q^{th} power of belongingness degree and non-membership degree should not exceed from 1. Later on, Senapati and Yager (2020) developed the theory of Fermatean fuzzy set (FFS) that is more general model than IFS and PFS in which the cubic sum of membership degree and non-membership degree should lie in unit interval. FFS as an extension of IFS and PFS can support more amount of inexactness and vagueness that provide more precise results in decision making framework.

Aforementioned models were not applicable in 2-dimensional problems. Thus, Ramot et al. (2002) introduced the complex fuzzy set (CFS) which was proposed by the emerging relationship of complex and FS theory having complex unit circle as the range of membership function that enables the CFS to handle the 2-dimensional information along with amplitude and phase terms. The amplitude part and phase part both are real-valued functions which can take values from the unit interval to show the vagueness of both dimensions. Later, Alkouri and Salleh (2012) put forward the idea of complex intuitionistic fuzzy set (CIFS), in order to describe the non-membership degree along with membership degree within the complex unit circle, where the sum of phase terms and amplitude terms of belongingness degree and falseness degree should not exceed from 1. Further, Akram and Naz (2019) & Ullah et al. (2020) presented the new model of complex Pythagorean fuzzy set (CPFS), as an extension of CIFS, which has more generalized structure than CFS and CIFS as it possesses more relaxed conditions on the phase and amplitude terms.

The idea of soft sets (S_fS s) theory was proposed by Molodtsov (1999), who also presented its relevancy and remarkable significance in the fields of operational research, probability theory, game theory and smoothness of functions (Molodtsov 1999, 2004). Alcantud and Santos-García (2017) proposed a totally revised approach for S_fS based decision-making issues under imperfect information. Many researchers brought up many models to

enhance the literature of S_fS , inclusive of fuzzy S_fS s (FS_fS s) (Maji et al. 2001b), Intuitionistic FS_fS s (IFS_fS s) (Maji et al. 2001a), Pythagorean FS_fS s (PFS_fS s) (Peng et al. 2015), Fermatean FS_fS s (FFS_fS s) (Sivadas and John 2020), et cetera. The idea of a new perspective for the selection of best alternatives problems based on FS_fS s was given by Alcantud (2016). Fatimah et al. (2019) worked on a new structure of S_fS s, namely, probabilistic S_fS . Alcantud et al. (2017) proposed a new hybrid model named as valuation fuzzy S_fS and used it for real case study that uses data from the Spanish real estate market.

From latest studies of hybrid S_fS models, it can be concluded that primarily work of the researchers was based on real numbers between $[0,1]$ or binary evaluation in S_fS models (Ma et al. 2017). But nowadays, objects are evaluated by non-binary structures such as voting system and rating or ranking objects. Due to that, numerous researchers for instance Alcantud and Laruelle (2014), Chen et al. (2013), and Herawan and Deris (2009) have worked in formal models for non-binary evaluations. Stimulated by these concerns, Fatimah et al. (2018) proposed the model of N -soft set (NS_fS) which is an extension of S_fS and encapsulate the idea of parameterized characterization of the alternatives that depend on the finite number of ordered grades. Fatimah and Alcantud (2021) introduced the idea of multi-fuzzy NS_fS . Later on, Akram et al. (2018, 2021b, 2021d) combined the concept of NS_fS with FS and explored the new hybrid model, namely, fuzzy N -soft set (FNS_fS). This novel concept involves the finite number of ordered grades along with the vagueness in the conception of the attributes that are used for decision making. Another hybrid model called the hesitant N -soft set was introduced by Akram et al. (2019a). Akram et al. (2019b) extended the idea of FNS_fS and presented the hybrid model of intuitionistic fuzzy N -soft set ($IFNS_fS$) that can also capture the non-membership grades. Moreover, Zhang et al. (1965) extended $IFNS_fS$ to Pythagorean fuzzy N -soft set ($PFNS_fS$) that possesses more relaxed conditions than existing models. Recently, Akram and his contributors set forth the hybrid models of bipolar FNS_fS s (Akram et al. 2021a), complex spherical FNS_fS s (Akram et al. 2021c) and complex Pythagorean FNS_fS s ($CPFNS_fS$ s) (Akram et al. 2021e).

The characteristic comparison of proposed and existing models is organized in Table 1 that present a broad view concerning the superiority of the manifested model.

In recent years, a technique for order preference by similarity to the ideal solution (TOPSIS) was proposed by Hwang and Yoon (1981) to solve the MADM problems. The basic idea of TOPSIS technique is to find out the best opt which is closest to the positive ideal solution (PIS) and farthest away from the negative ideal solution (NIS). Chen

Table 1 Characteristic comparison of the proposed and existent models

Models	Capable to address imprecise information	Capable to address 2-D information	Capable to address rating-based parameterized information	Have the features of generalization
FS (Zadeh 1965)	✓	×	×	×
NS _f S (Fatimah et al. 2018)	×	×	✓	×
FNS _f S (Akram et al. 2018)	✓	×	✓	✓
IFNS _f S (Akram et al. 2019b)	✓	×	✓	✓
PFNS _f S (Zhang et al. 1965)	✓	×	✓	✓
CPNS _f S (Akram et al. 2021e)	✓	✓	✓	✓
CFFNS _f S (proposed)	✓	✓	✓	✓

(2000) utilized the TOPSIS technique for multi-attribute group decision-making (MAGDM) under a fuzzy environment. Li et al. (2019) applied the fuzzy TOPSIS approach for the case-study of the Beijing rail transportation system. Boran et al. (2009, 2011, 2012) built up the theory of intuitionistic fuzzy TOPSIS (IF-TOPSIS) and presented various real applications related to technology and business. Akram and his collaborators proposed the methodologies of interval-valued hesitant fuzzy TOPSIS (Akram and Adeel 2019), Pythagorean Fuzzy TOPSIS (PF-TOPSIS) (Akram et al. 2019c), and complex Pythagorean fuzzy TOPSIS (CPF-TOPSIS) (Akram et al. 2020) to address the tricky MAGDM problems. Senapati and Yager (2020) put forward the Fermatean fuzzy TOPSIS (FF-TOPSIS) to capture the MADM problems. Eraslan (2015) redesigned the TOPSIS approach under the environment of S_fS (S_f -TOPSIS) and illustrated the methodology by means of its potential application. Eraslan and Karaaslan (2015) adapted the approach of TOPSIS under the framework of FS_fS s (FS_f -TOPSIS) and demonstrated its cogent applications to select the suitable house. Han et al. (2019) extended the technique of TOPSIS under entropy on PFS_fS s environment and implemented it for the selection of missile position. Salsabeela and John presented the TOPSIS method based on FFS_fS s (FFS_f -TOPSIS) (-Salsabeela and John 2021) and elaborated it with the practical application for the selection of supplier for five-star hotel.

The comparison of proposed and existing techniques based on TOPSIS method, according to their characteristics, is arranged in Table 2 which provide an extensive view about the dominance of the presented methodology.

1.2 Motivation

The motivation of the proposed hybrid model is given by the following facts:

- The idea of NS_fS captures the graded parameterized information but it has no potential to handle the fuzziness and vagueness of the provided data.
- The brilliant models of CIFS and CPFS are competitive frameworks for capturing the 2-dimensional vague data simultaneously. But they also have some restrictions due to the inadequacy of ranking based criteria.
- Moreover, the FFS_fS theory outstandingly renders the binary parameterized mechanism that handles ambiguity and vagueness of information with fantastic universality. But still, it is a 1-dimensional model that cannot present the uncertain periodic information as well as unable to cope with the ordered graded parameters of tricky practical problems.
- The decision-making technique based models FNS_fS, IFNS_fS and PFNS_fS can only deal with 1-dimensional data. None of the described models can handle 2-dimensional problems.
- Further, the novel idea of CPFNS_fS is an efficacious model with splendid characteristics to handle the obscurity of parameterized fuzzy information. Despite that, it has some flaws that spring up due to its restricted space.
- Classical TOPSIS technique is specifically devised to determine the optimal solution based on the assessed closeness of the preferences choices from the ideal solution. But this hypothetical technique must be altered to tackle the ordered graded obscurity and vagueness of inexact information.

Table 2 Characteristic comparison of proposed and existing TOPSIS techniques

Approaches	Have capability to deal uncertain information	Have capability to deal 2-D information	Have capability to deal parameterized information	Have capability to deal rating-based parameterized information
Fuzzy-TOPSIS technique (Chen 2000)	✓	×	×	×
IF-TOPSIS technique (Boran et al. 2009)	✓	×	×	×
PF-TOPSIS technique (Akram et al. 2019c)	✓	×	×	×
FF-TOPSIS technique (Senapati and Yager 2020)	✓	×	×	×
CPF-TOPSIS technique (Akram et al. 2020)	✓	✓	×	×
S_f -TOPSIS technique (Eraslan 2015)	×	×	✓	×
FS_f -TOPSIS technique (Eraslan and Karaaslan 2015)	✓	×	✓	×
FFS_f -TOPSIS technique (Salsabeela and John 2021)	✓	×	✓	×
$CFNS_f$ -TOPSIS technique (proposed)	✓	✓	✓	✓

Because of all these constraints motivated us to put forward the idea of a ground-breaking hybrid model called $CFNS_f$ Ss along with $CFNS_f$ -TOPSIS approach which competently handles two-dimensional information with relaxed conditions that cubic sum of amplitude and phase terms belongs to the interval $[0,1]$. Moreover, $CFNS_f$ S efficiently deals with the finite order grades of the alternatives according to the attributes. Therefore, the proposed model is the extension of FNS_f S (Akram et al. 2018), $IFNS_f$ S (Akram et al. 2019b), $PFNS_f$ S (Zhang et al. 1965), and $CPFNS_f$ S (Akram et al. 2021e) models and in fact dominates overall traditional models of literature as it has comparatively wide range.

1.3 Outline of the article

The essence of the first part of this article is to propose the hybrid model of $CFNS_f$ Ss and the related concepts including score function and accuracy function. Further, we investigate the remarkable properties and basic operations of $CFNS_f$ Ss. We have also constructed the $CFNS_f$ S derived by the threshold. Furthermore, algebraic and Yager operations for $CFNS_f$ numbers ($CFNS_f$ Ns) are also defined. The proposed model is supported by the

construction of three algorithms of decision-making and the applications are presented in contemplation of comparing the results of our algorithms. The comparative results of the model with existing FFY_wA (Garg et al. 2020) and FFY_wG (Garg et al. 2020) operators are given in the paper.

On the other hand, we revamp the TOPSIS approach for the environment of $CFNS_f$ to account for MAGDM problems. The innovative $CFNS_f$ weighted average operator and the distance measure of alternatives from positive and negative ideal solutions are employed to examine the contiguity of optimal variables from ideal solutions. The accountability of the presented technique is illustrated by implementing its magnificent procedure to select the suitable city in the USA for farming. A comparative analysis with the existing FF-TOPSIS (Senapati and Yager 2020) approach has been demonstrated to endorse the phenomenal feasibility and viability of the set forth strategy. The merits of the developed model and TOPSIS approach are also narrated for the appropriate manifestation of its marvelous and incredible feasibility over the existing models and approaches.

We summarize the main contributions of our research work as follows:

1. The article sets up a new theory of CFFNS_fS to deal with imprecise information involving vagueness and periodicity of ordered graded parameterized structure.
2. The algorithms are developed to tackle multi-attribute decision making problems by using numerical examples.
3. The comparative analysis with FFY_wA operator and FFY_wG operator to show the adequacy of the presented method.
4. This research also accomplishes a MAGDM technique, namely CFFNS_f-TOPSIS method.
5. The proposed technique is corroborated by a numerical example related to selecting the most suitable city in the USA for farming.
6. The CFFNS_f-TOPSIS approach is dexterously accomplished by demonstrating a comparative analysis with FF-TOPSIS method.

1.4 Layout of the paper

From this point on, the paper is organized as follows. Section 2 contains some definitions of existing models. In Sect. 3, we introduce the novel concept of CFFNS_fS followed by operations on CFFNS_fSs. Section 4 scrutinizes the algebraic and Yager operations on CFFNS_fNs. Section 5 describes the three proposed algorithms of the decision-making process and also provides some applications of multi-variable decision-making procedures. Section 6 carries out a comparative analysis with existing models and offers experimental results that illustrate the effectiveness of the proposed algorithms. Then, Sect. 7 introduces the CFFNS_f-TOPSIS method for MAGDM problems. A real example and a comparative study of its usefulness is shown in Sects. 8 and 9. Finally, merits of the proposed model and conclusions are drawn in Sects. 10 and 11.

2 Preliminaries

Definition 2.1 (Molodtsov (1999)) Let U be a universe of discourse under consideration and \mathcal{A} be the set of all attributes, $\mathcal{B} \subseteq \mathcal{A}$. A pair (ρ, \mathcal{B}) is called *soft set* over U if $\rho : \mathcal{B} \rightarrow P(U)$ where ρ is a set-valued function.

Definition 2.2 (Fatimah et al. (2018)) Let U be a universe of discourse and \mathcal{A} be the set of all attributes, $\mathcal{B} \subseteq \mathcal{A}$. Consider $\mathcal{R} = \{0, 1, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. A triple $(\mathcal{F}, \mathcal{B}, N)$ is an NS_fS on U if $\mathcal{F} : \mathcal{B} \rightarrow 2^{U \times \mathcal{R}}$, with the property that for each $b_t \in \mathcal{B}$ there exists a unique $(u_g, r_a) \in U \times \mathcal{R}$ such that $(u_g, r_a) \in \mathcal{F}(b_t)$, $u_g \in U$, $r_{a_t} \in \mathcal{R}$.

Definition 2.3 (Senapati and Yager (2020)) Consider U be a universe of discourse. An FFS E on U is defined as an object of the form

$$\mathfrak{F} = (\varrho_E, \varpi_E) = \{(u_g, \varrho_E(u_g), \varpi_E(u_g)) \mid u_g \in U\},$$

where the functions $\varrho_E : U \rightarrow [0, 1]$ and $\varpi_E : U \rightarrow [0, 1]$ denote the degree of membership (namely $\varrho_E(u_g)$) and the degree of non-membership (namely $\varpi_E(u_g)$) of the element $u_g \in U$, respectively, and for all $u_g \in U$, $0 \leq (\varrho_E(u_g))^3 + (\varpi_E(u_g))^3 \leq 1$. The value $\chi_E(u_g) = \sqrt[3]{1 - (\varrho_E(u_g))^3 - (\varpi_E(u_g))^3}$ is called degree of uncertainty of the elements $u_g \in U$ to the FFS E .

Definition 2.4 A complex Fermatean fuzzy set (CFFS, in short) \mathcal{B} , defined on the universal set U , is characterized by the membership and non-membership functions $\mu_{\mathcal{B}}(u_g)$ and $\nu_{\mathcal{B}}(u_g)$, respectively, which assign to each element $u_g \in U$ a complex-valued grade of membership and non-membership functions in \mathcal{B} . The CFFS may be represented as the set of triples:

$$\mathcal{B} = \{(u_g, \mu_{\mathcal{B}}(u_g), \nu_{\mathcal{B}}(u_g)) : u_g \in U\},$$

where $\mu_{\mathcal{B}}(u_g) : U \rightarrow \{u_g \mid u_g \in \mathbb{C}, |u_g| \leq 1\}$, $\nu_{\mathcal{B}}(u_g) : U \rightarrow \{u'_g \mid u'_g \in \mathbb{C}, |u'_g| \leq 1\}$, such that $\mu_{\mathcal{B}}(u_g) = s_{\mathcal{B}}(u_g)e^{i\omega_{\mathcal{B}}(u_g)}$, $\nu_{\mathcal{B}}(u_g) = k_{\mathcal{B}}(u_g)e^{i\psi_{\mathcal{B}}(u_g)}$, where $i = \sqrt{-1}$ and $s_{\mathcal{B}}(u_g), k_{\mathcal{B}}(u_g), \omega_{\mathcal{B}}(u_g), \psi_{\mathcal{B}}(u_g)$ are real-valued functions such that $s_{\mathcal{B}}(u_g), k_{\mathcal{B}}(u_g) \in [0, 1]$, $\omega_{\mathcal{B}}(u_g), \psi_{\mathcal{B}}(u_g) \in [0, 2\pi]$. $s_{\mathcal{B}}(u_g), k_{\mathcal{B}}(u_g)$ are called the amplitude terms and $\omega_{\mathcal{B}}(u_g), \psi_{\mathcal{B}}(u_g)$ are called the phase terms with $0 \leq (s_{\mathcal{B}}(u_g))^3 + (k_{\mathcal{B}}(u_g))^3 \leq 1$, and $0 \leq (\frac{\omega_{\mathcal{B}}(u_g)}{2\pi})^3 + (\frac{\psi_{\mathcal{B}}(u_g)}{2\pi})^3 \leq 1$.

The term $\pi_{\mathcal{B}}(u_g) = j_{\mathcal{B}}(u_g)e^{i2\pi\vartheta_{\mathcal{B}}(u_g)}$ is called degree of indeterminacy, where $j_{\mathcal{B}}(u_g) =$

$$\sqrt[3]{1 - (s_{\mathcal{B}}(u_g))^3 - (k_{\mathcal{B}}(u_g))^3} \quad \text{and} \quad \vartheta_{\mathcal{B}} = \sqrt[3]{1 - (\frac{\omega_{\mathcal{B}}(u_g)}{2\pi})^3 - (\frac{\psi_{\mathcal{B}}(u_g)}{2\pi})^3}.$$

Definition 2.5 Let U be a universal set and \mathcal{A} be the set of all attributes under consideration, $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{P}(U)$ denotes the set of all complex Fermatean fuzzy (CFF) subsets of U . A pair $(\mathcal{P}, \mathcal{B})$ is called a CFFS_fS over U , where \mathcal{P} is a function given by $\mathcal{P} : \mathcal{B} \rightarrow \mathcal{P}(U)$, which is

$$\mathcal{P}_{e_p}(b_t) = \{(u_g, s_{\mathcal{P}}(u_g)e^{i\omega_{\mathcal{P}}(u_g)}, k_{\mathcal{P}}e^{i\psi_{\mathcal{P}}(u_g)}) \mid u_g \in U, b_t \in \mathcal{B}\}.$$

3 Complex Fermatean fuzzy N -soft sets

Definition 3.1 Let U be a universal set and \mathcal{A} be the set of all parameters under consideration, $\mathcal{B} \subseteq \mathcal{A}$. Let $\mathcal{R} = \{0, 1, 2, \dots, N-1\}$ be a set of ordered grades where $N \in \{2, 3, \dots\}$. A triple $(\mathcal{H}, \mathcal{Q}, N)$ is called a *CFF N -soft set* (for short, CFFNS_f S), when $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_f S on U and \mathcal{H} is a mapping $\mathcal{H}: \mathcal{B} \rightarrow \text{CFF}^{(U \times \mathcal{R})}$, where $\text{CFF}^{(U \times \mathcal{R})}$ is the collection of all CFFS s over $U \times \mathcal{R}$, which is

$$\mathcal{H}(b_t) = \{ \langle (u_g, r_{a_{gt}}), s(u_g, r_{a_{gt}})e^{i\omega(u_g, r_{a_{gt}})}, k(u_g, r_{a_{gt}})e^{i\psi(u_g, r_{a_{gt}})} \rangle | b_t \in \mathcal{B}, (u_g, r_{a_{gt}}) \in U \times \mathcal{R} \}.$$

For convenience, $\mathcal{H}(b_t) = \langle (u_g, r_{a_{gt}}), s(u_g, r_{a_{gt}})e^{i\omega(u_g, r_{a_{gt}})}, k(u_g, r_{a_{gt}})e^{i\psi(u_g, r_{a_{gt}})} \rangle$ is denoted by $\alpha_{a_{gt}} = \langle r_{a_{gt}}, (s_{a_{gt}}e^{i\omega_{a_{gt}}}, k_{a_{gt}}e^{i\psi_{a_{gt}}}) \rangle$ which represents CFFNS_f number ($\text{CFFNS}_f N$).

Definition 3.2 Let $\alpha_{a_{gt}} = \langle r_{a_{gt}}, (s_{a_{gt}}e^{i\omega_{a_{gt}}}, k_{a_{gt}}e^{i\psi_{a_{gt}}}) \rangle$ be a $\text{CFFNS}_f N$ then

$$A_{\alpha_{a_{gt}}} = \sqrt[3]{1 - ((s_{a_{gt}})^3 + (k_{a_{gt}})^3)} e^{i2\pi \sqrt[3]{1 - ((\frac{\omega_{a_{gt}}}{2\pi})^3 + (\frac{\psi_{a_{gt}}}{2\pi})^3)}}$$

is called the degree of hesitancy/indeterminacy of $\text{CFFNS}_f N$.

Definition 3.3 Let $\alpha_{a_{gt}} = \langle r_{a_{gt}}, (s_{a_{gt}}e^{i\omega_{a_{gt}}}, k_{a_{gt}}e^{i\psi_{a_{gt}}}) \rangle$ be any $\text{CFFNS}_f N$ over U . The score function and accuracy function of $\alpha_{a_{gt}}$ are defined as follows:

$$S(\alpha_{a_{gt}}) = \left(\frac{r_{a_{gt}}}{N-1}\right)^3 + (s_{a_{gt}})^3 - (k_{a_{gt}})^3 + \left(\left(\frac{\omega_{a_{gt}}}{2\pi}\right)^3 - \left(\frac{\psi_{a_{gt}}}{2\pi}\right)^3\right),$$

$$A(\alpha_{a_{gt}}) = \left(\frac{r_{a_{gt}}}{N-1}\right)^3 + (s_{a_{gt}})^3 + (k_{a_{gt}})^3 + \left(\left(\frac{\omega_{a_{gt}}}{2\pi}\right)^3 + \left(\frac{\psi_{a_{gt}}}{2\pi}\right)^3\right),$$

respectively, where $S(\alpha_{a_{gt}}) \in [-2, 3]$ and $A(\alpha_{a_{gt}}) \in [0, 3]$.

Definition 3.4 For any two distinct $\text{CFFNS}_f N$ s $\alpha_{a_{gt}}$ and $\beta_{a_{lt}}$, we have:

1. if $S(\alpha_{a_{gt}}) < S(\beta_{a_{lt}})$, then $\alpha_{a_{gt}} < \beta_{a_{lt}}$,
2. if $S(\alpha_{a_{gt}}) > S(\beta_{a_{lt}})$, then $\alpha_{a_{gt}} > \beta_{a_{lt}}$,
3. if $S(\alpha_{a_{gt}}) = S(\beta_{a_{lt}})$, then
 - a. if $A(\alpha_{a_{gt}}) > A(\beta_{a_{lt}})$, then $\alpha_{a_{gt}} > \beta_{a_{lt}}$,
 - b. if $A(\alpha_{a_{gt}}) < A(\beta_{a_{lt}})$, then $\alpha_{a_{gt}} < \beta_{a_{lt}}$,
 - c. if $A(\alpha_{a_{gt}}) = A(\beta_{a_{lt}})$, then $\alpha_{a_{gt}} = \beta_{a_{lt}}$.

For a better understanding of the concept of our new model, we present the following example:

Example 3.1 Consider that an auto broker decides to purchase the car from auto company. The best car is chosen by spade ratings endowed by an expert. These rankings are on the basis of launched cars in the last 5 years and their performances. Before purchasing the car, auto broker obtained some rating and ranking based information from an expert about four different models of vehicles having different manufacturing dates. Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of vehicles and $\mathcal{B} = \{b_1 = \text{Reliability}, b_2 = \text{Maximum payload}, b_3 = \text{Purchasing cost}\} \subseteq \mathcal{A}$ be the set of

attributes, that are used to set grades for each vehicle with respect to each attribute. The expert assigned the rating of the cars according to the above-mentioned conflicting criteria and the initial review recapped in Table 3, where:

- four spades represent ‘excellent’,
- three spades represent ‘very good’,
- two spades represent ‘good’,
- one spade represents ‘regular’, and
- a bullet represents ‘bad’.

The set of grades $\mathcal{R} = \{0, 1, 2, 3, 4\}$ can be easily associated with rated assessment conducted by bullet and spades as follows:

- 0 stands for ‘•’,
- 1 stands for ‘♠’,
- 2 stands for ‘♠♠’,
- 3 stands for ‘♠♠♠’,
- 4 stands for ‘♠♠♠♠’.

Based on the overall qualities of the cars, the auto broker gives evaluation scores of the cars which is shown as

Table 3 Information extracted from the expert

X/\mathcal{B}	b_1	b_2	b_3
x_1	♠♠	♠♠♠	♠
x_2	♠	♠♠♠	♠♠
x_3	♠♠♠	♠♠♠♠	•
x_4	•	♠♠	♠♠♠♠

Table 3 and the tabular representation of its associated 5-soft set is given in Table 4.

The grade data in the actual information can be easily extracted. However, according to the Definition 3.1 when the data possess fuzzy uncertainty characteristics, we need CFFNS $_f$ N. It provides us information in which the auto brokers evaluate the cars and specify their rankings based on the same multiple fuzzy characteristics from the perspective of the two-dimensional membership degree and non-membership degree. This assessment of cars by auto brokers complies with the guidelines as follows:

- $-2.0 \leq S(X) < -1.2$ when grade 0,
- $-1.2 \leq S(X) < -0.4$ when grade 1,
- $-0.4 \leq S(X) < 0.4$ when grade 2,
- $0.4 \leq S(X) < 1.2$ when grade 3,
- $1.2 \leq S(X) < 2.0$ when grade 4.

According to above criteria, we can obtain Table 5.

Therefore, by Definition 3.1, the CFF5S $_f$ S ($\mathcal{H}, \mathcal{Q}, 5$) can be defined as follows:

$$\begin{aligned} h(b_1) &= \{ \langle (x_1, 2), 0.5e^{i0.9\pi}, 0.6e^{i1.2\pi} \rangle, \langle (x_2, 1), 0.3e^{i0.5\pi}, 0.7e^{i1.5\pi} \rangle, \langle (x_3, 3), 0.7e^{i1.4\pi}, 0.3e^{i0.5\pi} \rangle, \\ &\quad \langle (x_4, 0), 0.1e^{i0.2\pi}, 0.9e^{i1.8\pi} \rangle \}, \\ h(b_2) &= \{ \langle (x_1, 3), 0.7e^{i1.5\pi}, 0.3e^{i0.5\pi} \rangle, \langle (x_2, 3), 0.8e^{i1.6\pi}, 0.4e^{i0.6\pi} \rangle, \langle (x_3, 4), 0.9e^{i1.8\pi}, 0.1e^{i0.3\pi} \rangle, \\ &\quad \langle (x_4, 2), 0.5e^{i0.9\pi}, 0.5e^{i1.0\pi} \rangle \}, \\ h(b_3) &= \{ \langle (x_1, 1), 0.3e^{i0.6\pi}, 0.8e^{i1.6\pi} \rangle, \langle (x_2, 2), 0.6e^{i0.9\pi}, 0.5e^{i1.2\pi} \rangle, \langle (x_3, 0), 0.1e^{i0.2\pi}, 0.9e^{i1.9\pi} \rangle, \\ &\quad \langle (x_4, 4), 0.9e^{i1.9\pi}, 0.2e^{i0.3\pi} \rangle \}. \end{aligned}$$

Table 4 Tabular form of the 5-soft set

$(\mathcal{F}, \mathcal{B}, 5)$	b_1	b_2	b_3
x_1	2	3	1
x_2	1	3	2
x_3	3	4	0
x_4	0	2	4

Table 5 Grading criteria

$r_{a_{gt}}/\mathcal{H}$	Amplitude term		Phase term	
Grades	$s_{a_{gt}}$	$k_{a_{gt}}$	$\omega_{a_{gt}}$	$\psi_{a_{gt}}$
$r_{a_{gt}} = 0$	[0, 0.2)	(0.85, 1]	[0, 0.4 π)	(1.7 π , 2 π]
$r_{a_{gt}} = 1$	[0.2, 0.4)	(0.65, 0.85]	[0.4 π , 0.8 π)	(1.3 π , 1.7 π]
$r_{a_{gt}} = 2$	[0.4, 0.65)	(0.4, 0.65]	[0.8 π , 1.3 π)	(0.8 π , 1.3 π]
$r_{a_{gt}} = 3$	[0.65, 0.85)	(0.2, 0.4]	[1.3 π , 1.7 π)	(0.4 π , 0.8 π]
$r_{a_{gt}} = 4$	[0.85, 1)	(0, 0.2]	[1.7 π , 2 π)	(0, 0.4 π]

The CFF5S $_f$ S ($\mathcal{H}, \mathcal{Q}, 5$) can be represented more clearly in tabular form shown as in Table 6 as follows:

Remark 1 The following observations are in order:

1. In Example 3.1, we consider the five assessment grades, but the assessment grades in practical problems do not necessarily utilize the 5 grades, it can be arbitrary. Generally, the range concerning the score function of CFF numbers can vary with actual grade requirements.
2. Any CFF2S $_f$ S ($\mathcal{H}, \mathcal{Q}, 2$) can be naturally associated with a CFFS $_f$ S. We identify a CFF2S $_f$ S $\mathcal{H} : \mathcal{B} \rightarrow CFF^{(U \times \{0,1\})}$ with a CFFS $_f$ S (\wp, \mathcal{B}) , which is given by:

$$\wp(b_t) = \{ \langle u_g, s_{\wp}(u_g)e^{i\omega_{\wp}(u_g)}, k_{\wp}(u_g)e^{i\psi_{\wp}(u_g)} \rangle | \langle (u_g, 1), s_{\wp}(u_g)e^{i\omega_{\wp}(u_g)}, k_{\wp}(u_g)e^{i\psi_{\wp}(u_g)} \rangle \in \mathcal{H}(b_t) \},$$

for every $b_t \in \mathcal{B}$, where $CFF^{(U \times \{0,1\})}$ is the collection of all CFF subsets of $U \times \{0, 1\}$.

3. An arbitrary CFFNS $_f$ S over a universe U can be identified as a CFF $(N + 1)$ -soft set. For example, from

Table 6, a CFF5S $_f$ S ($\mathcal{H}, \mathcal{Q}, 5$) can be identified as a CFF6S $_f$ S over U . In a CFF6S $_f$ S, we consider that there is a 5 grade, which is never used in Example 3.1.

4. In Definition 3.1, grade 0 describes the lowest score. It does not mean that there is incomplete information or no assessment.

Definition 3.5 A CFFNS $_f$ S ($\mathcal{H}, \mathcal{Q}, N$) over universe of discourse U , where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS $_f$ S, is said to be *efficient* if $\mathcal{H}(b_t) = \langle (u_g, N - 1), 1e^{i2\pi}, 0e^{i0\pi} \rangle$ for some $b_t \in \mathcal{B}, u_g \in U$.

Example 3.2 By inspection, it can be checked that the CFF5S $_f$ S defined in Example 3.1 is not efficient. However, CFF5S $_f$ S ($\mathcal{H}, \mathcal{Q}, 5$) in Table 7 is efficient.

Definition 3.6 Let $(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ be two CFFNS $_f$ Ss over universe of discourse U , where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, N_1)$, $\mathcal{Q}_2 = (\mathcal{F}_2, \mathcal{B}_2, N_2)$ are NS $_f$ Ss, then

Table 6 Tabular form of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}, \mathcal{Q}, 5)$	b_1	b_2	b_3
x_1	$\langle 2, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 3, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 1, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 1, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 3, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 2, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 3, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 2, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$

Table 7 Tabular form of the efficient CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}, \mathcal{Q}, 5)$	b_1	b_2	b_3
x_1	$\langle 2, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 3, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 1, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 1, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 3, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 2, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 3, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 2, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 4, (1.0e^{i2.0\pi}, 0.0e^{i0.0\pi}) \rangle$

$(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ are said to be *equal* if and only if $\mathcal{H}_1 = \mathcal{H}_2$ and $\mathcal{Q}_1 = \mathcal{Q}_2$.

We now define the concept of complementarity CFFNS_fS:

Definition 3.7 Let $(\mathcal{H}, \mathcal{Q}, N)$ be a CFFNS_fS over universe of discourse U , where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS, then $(\mathcal{H}, \mathcal{Q}^c, N)$ is said to be *weak complement* if $\mathcal{Q} = (\mathcal{F}^c, \mathcal{B}, N)$ is a weak complement of $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$. By this mean that $\mathcal{F}^c(b_i) \cap \mathcal{F}(b_i) = \emptyset$ for all $b_i \in \mathcal{B}$.

The term weak complement is used because this complement is not unique.

Definition 3.8 Let $(\mathcal{H}, \mathcal{Q}, N)$ be a CFFNS_fS over universe of discourse U , where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS, then a *CFF complement* is denoted by $(\mathcal{H}^c, \mathcal{Q}, N)$, such that \mathcal{H}^c is defined as $\mathcal{H}^c : \mathcal{B} \rightarrow CFF^{(U \times \mathcal{R})}$, which is given by:

$$\mathcal{H}^c(b_i) = \{ \langle (u_g, r_{a_{gt}}), k(u_g, r_{a_{gt}})e^{i\omega(u_g, r_{a_{gt}})} \rangle, s(u_g, r_{a_{gt}})e^{i\psi(u_g, r_{a_{gt}})} | b_i \in \mathcal{B}, (u_g, r_{a_{gt}}) \in U \times \mathcal{R} \}.$$

Table 8 A weak complement of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(\mathcal{H}, \mathcal{Q}^c, 5)$	b_1	b_2	b_3
x_1	$\langle 3, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 4, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 2, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 0, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 2, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 3, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 2, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 3, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 2, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 4, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 1, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 3, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$

In CFF complement, the grades are same as in the original NS_fS, however all their membership and non-membership degrees are complementary.

Definition 3.9 Let $(\mathcal{H}, \mathcal{Q}, N)$ be a CFFNS_fS over universe of discourse U , where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS, then $(\mathcal{H}^c, \mathcal{Q}^c, N)$ is said to be *weak CFF complement* when $(\mathcal{H}, \mathcal{Q}^c, N)$ is a weak complement and $(\mathcal{H}^c, \mathcal{Q}, N)$ is a CFF complement.

In other words, a weak CFF complement of CFFNS_fS is the CFF complement of any of its weak complement.

Example 3.3 A weak complement $(\mathcal{H}, \mathcal{Q}^c, 5)$ of CFF5S_fS in Example 3.1, Table 6, is represented by Table 8. The CFF complement of this CFF5S_fS is $(\mathcal{H}^c, \mathcal{Q}, 5)$ defined by Table 9. A weak CFF complement of this CFF5S_fS is $(\mathcal{H}^c, \mathcal{Q}^c, 5)$ defined by Table 10.

Definition 3.10 For a CFFNS_fS $(\mathcal{H}, \mathcal{Q}, N)$, where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS, the *top weak complement* of $(\mathcal{H}, \mathcal{Q}, N)$ is $(\mathcal{H}, \mathcal{Q}^>, N)$, and the *top weak CFF complement* of $(\mathcal{H}, \mathcal{Q}, N)$ is $(\mathcal{H}^c, \mathcal{Q}^>, N)$, where $\mathcal{Q}^> = (\mathcal{F}^>, \mathcal{B}, N)$ is the top weak complement of $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ and defined as follows:

Table 9 Tabular representation of the CFF complement of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(H^c, \mathcal{Q}, 5)$	b_1	b_2	b_3
x_1	$\langle 2, (0.6e^{i1.2\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 3, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 1, (0.8e^{i1.6\pi}, 0.3e^{i0.6\pi}) \rangle$
x_2	$\langle 1, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 3, (0.4e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$	$\langle 2, (0.5e^{i1.2\pi}, 0.6e^{i0.9\pi}) \rangle$
x_3	$\langle 3, (0.3e^{i0.5\pi}, 0.7e^{i1.4\pi}) \rangle$	$\langle 4, (0.1e^{i0.3\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 0, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi}) \rangle$
x_4	$\langle 0, (0.9e^{i1.8\pi}, 0.1e^{i0.2\pi}) \rangle$	$\langle 2, (0.5e^{i1.0\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 4, (0.2e^{i0.3\pi}, 0.9e^{i1.9\pi}) \rangle$

Table 10 Tabular representation of a weak CFF complement of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(H^c, \mathcal{Q}^c, 5)$	b_1	b_2	b_3
x_1	$\langle 3, (0.6e^{i1.2\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 4, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 2, (0.8e^{i1.6\pi}, 0.3e^{i0.6\pi}) \rangle$
x_2	$\langle 0, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 2, (0.4e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$	$\langle 3, (0.5e^{i1.2\pi}, 0.6e^{i0.9\pi}) \rangle$
x_3	$\langle 2, (0.3e^{i0.5\pi}, 0.7e^{i1.4\pi}) \rangle$	$\langle 3, (0.1e^{i0.3\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 2, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi}) \rangle$
x_4	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.2\pi}) \rangle$	$\langle 1, (0.5e^{i1.0\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 3, (0.2e^{i0.3\pi}, 0.9e^{i1.9\pi}) \rangle$

$$\begin{aligned}
& \mathcal{F}^>(u_g)(b_t) \text{ (Fatimah et al. 2018)} = (\mathcal{H}, \mathcal{Q}^<, N) = \\
& \begin{cases} N-1, & \text{if } \mathcal{F}(u_g)(b_t) < N-1, \\ 0, & \text{if } \mathcal{F}(u_g)(b_t) = N-1. \end{cases} \\
& (\mathcal{H}, \mathcal{Q}^>, N) = \\
& \begin{cases} \langle N-1, (s_{a_{gt}}e^{i\omega_{a_{gt}}}, k_{a_{gt}}e^{i\psi_{a_{gt}}}) \rangle, & \text{if } r_{a_{gt}} < N-1, \\ \langle 0, (s_{a_{gt}}e^{i\omega_{a_{gt}}}, k_{a_{gt}}e^{i\psi_{a_{gt}}}) \rangle, & \text{if } r_{a_{gt}} = N-1. \end{cases} \\
& (\mathcal{H}^c, \mathcal{Q}^>, N) = \\
& \begin{cases} \langle N-1, (k_{a_{gt}}e^{i\psi_{a_{gt}}}, s_{a_{gt}}e^{i\omega_{a_{gt}}}) \rangle, & \text{if } r_{a_{gt}} < N-1, \\ \langle 0, (k_{a_{gt}}e^{i\psi_{a_{gt}}}, s_{a_{gt}}e^{i\omega_{a_{gt}}}) \rangle, & \text{if } r_{a_{gt}} = N-1. \end{cases}
\end{aligned}$$

Example 3.4 The top weak complement and the top weak CFF complement of the CFF5S_fS Table 6 in Example 3.1 are given by Tables 11 and 12.

Definition 3.11 For a CFFNS_fS ($\mathcal{H}, \mathcal{Q}, N$), where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS, the *bottom weak complement* of $(\mathcal{H}, \mathcal{Q}, N)$ is $(\mathcal{H}, \mathcal{Q}^<, N)$, and the *bottom weak CFF complement* of $(\mathcal{H}, \mathcal{Q}, N)$ is $(\mathcal{H}^c, \mathcal{Q}^<, N)$, where $\mathcal{Q}^< = (\mathcal{F}^<, \mathcal{B}, N)$ is the bottom weak complement of $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ and defined as follows:

$$\begin{aligned}
& \mathcal{F}^<(u_g)(b_t) \text{ (Fatimah et al. 2018)} = \\
& \begin{cases} 0, & \text{if } \mathcal{F}(u_g)(b_t) > 0, \\ N-1, & \text{if } \mathcal{F}(u_g)(b_t) = 0. \end{cases}
\end{aligned}$$

Table 11 Tabular representation of the top weak complement of CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(H, \mathcal{Q}^>, 5)$	b_1	b_2	b_3
x_1	$\langle 4, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 4, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 4, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 4, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 4, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 4, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 0, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 4, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 4, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 4, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 0, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$

Example 3.5 The bottom weak complement and the bottom weak CFF complement of the CFF5S_fS Table 6 in Example 3.1 are given by Tables 13 and 14.

Definition 3.12 Let U be a universe of discourse and $(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ be two CFFNS_fSs over non-empty set U , where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, N_1)$ and $\mathcal{Q}_2 = (\mathcal{F}_2, \mathcal{B}_2, N_2)$ are NS_fSs on U , then their *restricted intersection* is denoted by $(\mathcal{H}_1, \mathcal{Q}_1, N_1) \cap_{\mathfrak{R}} (\mathcal{H}_2, \mathcal{Q}_2, N_2)$ and is defined as $(\sigma, \mathcal{Q}_1 \cap_{\mathfrak{R}} \mathcal{Q}_2, \min(N_1, N_2))$, where $\mathcal{Q}_1 \cap_{\mathfrak{R}} \mathcal{Q}_2 = (\mathcal{F}, \mathcal{B}_1 \cap \mathcal{B}_2, \min(N_1, N_2))$ for all $b_t \in \mathcal{B}_1 \cap \mathcal{B}_2$, $u_g \in U$, $\langle (u_g, r_{a_{gt}}), y, z \rangle \in \sigma(b_t) \Leftrightarrow r_{a_{gt}} = \min(r_{a_{gt}}^1, r_{a_{gt}}^2)$, $y = \min(s_{\mathcal{C}}(u_g, r_{a_{gt}}^1), s_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \min(\omega_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \omega_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}$, $z = \max(k_{\mathcal{C}}(u_g, r_{a_{gt}}^1), k_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \max(\psi_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \psi_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}$, if

Table 12 Tabular representation of the top weak CFF complement of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(\mathcal{H}^c, \mathcal{Q}^>, 5)$	b_1	b_2	b_3
x_1	$\langle 4, (0.6e^{i1.2\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 4, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 4, (0.8e^{i1.6\pi}, 0.3e^{i0.6\pi}) \rangle$
x_2	$\langle 4, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.4e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$	$\langle 4, (0.5e^{i1.2\pi}, 0.6e^{i0.9\pi}) \rangle$
x_3	$\langle 4, (0.3e^{i0.5\pi}, 0.7e^{i1.4\pi}) \rangle$	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi}) \rangle$
x_4	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.2\pi}) \rangle$	$\langle 4, (0.5e^{i1.0\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 0, (0.2e^{i0.3\pi}, 0.9e^{i1.9\pi}) \rangle$

Table 13 Tabular representation of the bottom weak complement of CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(\mathcal{H}, \mathcal{Q}^<, 5)$	b_1	b_2	b_3
x_1	$\langle 0, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 0, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 0, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 0, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 0, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 0, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 0, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 0, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 4, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 4, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 0, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 0, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$

Table 14 Tabulated form of the bottom weak CFF complement of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$) in Example 3.1

$(\mathcal{H}^c, \mathcal{Q}^<, 5)$	b_1	b_2	b_3
x_1	$\langle 0, (0.6e^{i1.2\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 0, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 0, (0.8e^{i1.6\pi}, 0.3e^{i0.6\pi}) \rangle$
x_2	$\langle 0, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 0, (0.4e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$	$\langle 0, (0.5e^{i1.2\pi}, 0.6e^{i0.9\pi}) \rangle$
x_3	$\langle 0, (0.3e^{i0.5\pi}, 0.7e^{i1.4\pi}) \rangle$	$\langle 0, (0.1e^{i0.3\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.1e^{i0.2\pi}) \rangle$
x_4	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.2\pi}) \rangle$	$\langle 0, (0.5e^{i1.0\pi}, 0.5e^{i0.9\pi}) \rangle$	$\langle 0, (0.2e^{i0.3\pi}, 0.9e^{i1.9\pi}) \rangle$

$\langle (u_g, r_{agt}^1), s_{\mathcal{C}}(u_g, r_{agt}^1)e^{i\omega_{\mathcal{C}}(u_g, r_{agt}^1)}, k_{\mathcal{C}}(u_g, r_{agt}^1)e^{i\psi_{\mathcal{C}}(u_g, r_{agt}^1)} \rangle \in \mathcal{B}_1(b_t)$ and $\langle (u_g, r_{agt}^2), s_{\mathcal{D}}(u_g, r_{agt}^2)e^{i\omega_{\mathcal{D}}(u_g, r_{agt}^2)}, k_{\mathcal{D}}(u_g, r_{agt}^2)e^{i\psi_{\mathcal{D}}(u_g, r_{agt}^2)} \rangle \in \mathcal{B}_2(b_t)$, \mathcal{C} and \mathcal{D} are CFFSs on $\mathcal{F}_1(b_t)$ and $\mathcal{F}_2(b_t)$, respectively.

Example 3.6 Let $(\mathcal{H}_1, \mathcal{Q}_1, 5)$ be a CFF5S_fS and $(\mathcal{H}_2, \mathcal{Q}_2, 4)$ be a CFF4S_fS defined by Tables 15 and 16, respectively, where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, 5)$ and $\mathcal{Q}_2 = (\mathcal{H}_2, \mathcal{B}_2, 4)$ are NS_fSs over U , then their restricted intersection $(\mathcal{H}_1, \mathcal{Q}_1, 5) \cap_{\mathfrak{R}} (\mathcal{H}_2, \mathcal{Q}_2, 4) = (\sigma, \mathcal{Q}_1 \cap_{\mathfrak{r}} \mathcal{Q}_2, 4)$ is defined by Table 17.

Table 15 Tabular representation of CFF5S_fS ($\mathcal{H}_1, \mathcal{Q}_1, 5$) in Example 3.6

$(\mathcal{H}_1, \mathcal{Q}_1, 5)$	b_1	b_2	b_3
x_1	$\langle 4, (0.92e^{i1.88\pi}, 0.08e^{i0.36\pi}) \rangle$	$\langle 3, (0.77e^{i1.58\pi}, 0.37e^{i0.69\pi}) \rangle$	$\langle 2, (0.61e^{i1.14\pi}, 0.42e^{i1.29\pi}) \rangle$
x_2	$\langle 3, (0.73e^{i1.46\pi}, 0.27e^{i0.78\pi}) \rangle$	$\langle 4, (0.92e^{i1.96\pi}, 0.15e^{i0.29\pi}) \rangle$	$\langle 1, (0.37e^{i0.63\pi}, 0.59e^{i1.54\pi}) \rangle$
x_3	$\langle 4, (0.89e^{i1.76\pi}, 0.09e^{i0.36\pi}) \rangle$	$\langle 2, (0.49e^{i1.27\pi}, 0.57e^{i1.18\pi}) \rangle$	$\langle 0, (0.05e^{i0.32\pi}, 0.92e^{i1.94\pi}) \rangle$
x_4	$\langle 1, (0.25e^{i0.45\pi}, 0.75e^{i1.65\pi}) \rangle$	$\langle 0, (0.15e^{i0.25\pi}, 0.95e^{i1.85\pi}) \rangle$	$\langle 2, (0.55e^{i0.95\pi}, 0.61e^{i0.85\pi}) \rangle$

Definition 3.13 Let U be a universe of discourse and $(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ be two CFFNS_fSs over non-empty set U , where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, N_1)$ and $\mathcal{Q}_2 = (\mathcal{F}_2, \mathcal{B}_2, N_2)$ are NS_fSs on U , then their *extended intersection* is denoted by $(\mathcal{H}_1, \mathcal{Q}_1, N_1) \cap_{\mathfrak{E}} (\mathcal{H}_2, \mathcal{Q}_2, N_2)$ and is defined as $(\mathfrak{S}, \mathcal{Q}_1 \cap_{\mathfrak{e}} \mathcal{Q}_2, \max(N_1, N_2))$, where $\mathcal{Q}_1 \cap_{\mathfrak{e}} \mathcal{Q}_2 = (\mathcal{H}, \mathcal{B}_1 \cup \mathcal{B}_2, \max(N_1, N_2))$, and $\mathfrak{S}(b_t)$ is given by:

Table 16 Tabulated form of the CFF4S_fS $(\mathcal{H}_2, \mathcal{Q}_2, 4)$ in Example 3.6

$(\mathcal{H}_2, \mathcal{Q}_2, 4)$	b_1	b_2	v
x_1	$\langle 0, (0.05e^{i0.42\pi}, 0.85e^{i1.76\pi}) \rangle$	$\langle 2, (0.48e^{i1.57\pi}, 0.29e^{i0.67\pi}) \rangle$	$\langle 3, (0.88e^{i1.76\pi}, 0.05e^{i0.19\pi}) \rangle$
x_2	$\langle 1, (0.33e^{i0.57\pi}, 0.69e^{i1.48\pi}) \rangle$	$\langle 3, (0.97e^{i1.96\pi}, 0.19e^{i0.39\pi}) \rangle$	$\langle 2, (0.69e^{i0.97\pi}, 0.38e^{i0.65\pi}) \rangle$
x_3	$\langle 2, (0.76e^{i1.43\pi}, 0.39e^{i0.66\pi}) \rangle$	$\langle 0, (0.19e^{i0.25\pi}, 0.89e^{i1.94\pi}) \rangle$	$\langle 1, (0.29e^{i0.72\pi}, 0.58e^{i1.33\pi}) \rangle$
x_4	$\langle 3, (0.93e^{i1.88\pi}, 0.15e^{i0.25\pi}) \rangle$	$\langle 1, (0.38e^{i0.67\pi}, 0.76e^{i1.29\pi}) \rangle$	$\langle 0, (0.12e^{i0.37\pi}, 0.91e^{i1.87\pi}) \rangle$

Table 17 Tabulated form of $(\mathcal{H}_1, \mathcal{Q}_1, 5) \cap_{\mathcal{R}} (\mathcal{H}_2, \mathcal{Q}_2, 4)$ defined in Example 3.6

$(\sigma, \mathcal{Q}_1 \cap_{\mathcal{R}} \mathcal{Q}_2, 4)$	b_1	b_2
x_1	$\langle 0, (0.05e^{i0.42\pi}, 0.85e^{i1.76\pi}) \rangle$	$\langle 2, (0.48e^{i1.57\pi}, 0.37e^{i0.69\pi}) \rangle$
x_2	$\langle 1, (0.33e^{i0.57\pi}, 0.69e^{i1.48\pi}) \rangle$	$\langle 3, (0.92e^{i1.96\pi}, 0.19e^{i0.39\pi}) \rangle$
x_3	$\langle 2, (0.76e^{i1.43\pi}, 0.39e^{i0.66\pi}) \rangle$	$\langle 0, (0.19e^{i0.25\pi}, 0.89e^{i1.94\pi}) \rangle$
x_4	$\langle 1, (0.25e^{i0.45\pi}, 0.75e^{i1.65\pi}) \rangle$	$\langle 0, (0.15e^{i0.25\pi}, 0.95e^{i1.85\pi}) \rangle$

$$\mathfrak{S}(b_t) = \begin{cases} \mathcal{H}_1(b_t), & \text{if } b_t \in \mathcal{B}_1 - \mathcal{B}_2, \\ \mathcal{H}_2(b_t), & \text{if } b_t \in \mathcal{B}_1 - \mathcal{B}_2, \\ \langle (u_g, r_{a_{gt}}), y, z \rangle, & \text{such that } r_{a_{gt}} = \min(r_{a_{gt}}^1, r_{a_{gt}}^2), \\ & y = \min(s_{\mathcal{C}}(u_g, r_{a_{gt}}^1), s_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \min(\omega_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \omega_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}, \\ & z = \max(k_{\mathcal{C}}(u_g, r_{a_{gt}}^1), k_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \max(\psi_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \psi_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}, \\ & \text{where } \langle (u_g, r_{a_{gt}}^1), s_{\mathcal{C}}(u_g, r_{a_{gt}}^1), k_{\mathcal{C}}(u_g, r_{a_{gt}}^1)e^{i\psi_{\mathcal{C}}(u_g, r_{a_{gt}}^1)} \rangle \in \mathcal{B}_1(b_t) \\ & \text{and } \langle (u_g, r_{a_{gt}}^2), s_{\mathcal{D}}(u_g, r_{a_{gt}}^2), k_{\mathcal{D}}(u_g, r_{a_{gt}}^2)e^{i\psi_{\mathcal{D}}(u_g, r_{a_{gt}}^2)} \rangle \in \mathcal{B}_2(b_t), \\ & \mathcal{C} \text{ and } \mathcal{D} \text{ are CFFSson } \mathcal{F}_1(b_t) \text{ and } \mathcal{F}_2(b_t), \text{ respectively.} \end{cases}$$

Example 3.7 The extended intersection $(\mathfrak{S}, \mathcal{Q}_1 \cap_{\mathcal{C}} \mathcal{Q}_2, 5)$ of $(\mathcal{H}_1, \mathcal{Q}_1, 5)$ (see Table 15) and $(\mathcal{H}_2, \mathcal{Q}_2, 4)$ (see Table 16) is shown by Table 18.

Definition 3.14 Let U be a universe of discourse and $(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ be two CFFNS_fSs over non-empty set U , where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, N_1)$ and $\mathcal{Q}_2 = (\mathcal{F}_2, \mathcal{B}_2, N_2)$ are NS_fSs on U , then their *restricted union* is denoted by $(\mathcal{F}_1, \mathcal{Q}_1, N_1) \cup_{\mathcal{R}} (\mathcal{F}_2, \mathcal{Q}_2, N_2)$ and is defined as

$(\mathfrak{h}, \mathcal{Q}_1 \cup_{\mathcal{R}} \mathcal{Q}_2, \max(N_1, N_2))$, where $\mathcal{Q}_1 \cup_{\mathcal{R}} \mathcal{Q}_2 = (\mathcal{S}, \mathcal{B}_1 \cap \mathcal{B}_2, \max(N_1, N_2))$ for all $b_t \in \mathcal{B}_1 \cap \mathcal{B}_2, u_g \in U, \langle (u_g, r_{a_{gt}}), y, z \rangle \in \mathfrak{h}(b_t) \Leftrightarrow r_{a_{gt}} = \max(r_{a_{gt}}^1, r_{a_{gt}}^2), y = \max(s_{\mathcal{C}}(u_g, r_{a_{gt}}^1), s_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \max(\omega_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \omega_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}, z = \min(k_{\mathcal{C}}(u_g, r_{a_{gt}}^1), k_{\mathcal{D}}(u_g, r_{a_{gt}}^2))e^{i \min(\psi_{\mathcal{C}}(u_g, r_{a_{gt}}^1), \psi_{\mathcal{D}}(u_g, r_{a_{gt}}^2))}$, if $\langle (u_g, r_{a_{gt}}^1), s_{\mathcal{C}}(u_g, r_{a_{gt}}^1), k_{\mathcal{C}}(u_g, r_{a_{gt}}^1)e^{i\psi_{\mathcal{C}}(u_g, r_{a_{gt}}^1)} \rangle \in \mathcal{B}_1(b_t)$ and $\langle (u_g, r_{a_{gt}}^2), s_{\mathcal{D}}(u_g, r_{a_{gt}}^2), k_{\mathcal{D}}(u_g, r_{a_{gt}}^2)e^{i\psi_{\mathcal{D}}(u_g, r_{a_{gt}}^2)} \rangle \in \mathcal{B}_2(b_t)$.

Table 18 Tabulated form of the extended intersection $(\mathcal{H}_1, \mathcal{Q}_1, 5) \cap_{\mathcal{C}} (\mathcal{H}_2, \mathcal{Q}_2, 4)$ in Example 3.7

$(\mathfrak{S}, \mathcal{Q}_1 \cap_{\mathcal{C}} \mathcal{Q}_2, 5)$	b_1	b_2	b_3	v
x_1	$\langle 0, (0.05e^{i0.42\pi}, 0.85e^{i1.76\pi}) \rangle$	$\langle 2, (0.48e^{i1.57\pi}, 0.37e^{i0.69\pi}) \rangle$	$\langle 2, (0.61e^{i1.14\pi}, 0.42e^{i1.29\pi}) \rangle$	$\langle 3, (0.88e^{i1.76\pi}, 0.05e^{i0.19\pi}) \rangle$
x_2	$\langle 1, (0.33e^{i0.57\pi}, 0.69e^{i1.48\pi}) \rangle$	$\langle 3, (0.92e^{i1.96\pi}, 0.19e^{i0.39\pi}) \rangle$	$\langle 1, (0.37e^{i0.63\pi}, 0.59e^{i1.54\pi}) \rangle$	$\langle 2, (0.69e^{i0.97\pi}, 0.38e^{i0.65\pi}) \rangle$
x_3	$\langle 2, (0.76e^{i1.43\pi}, 0.39e^{i0.66\pi}) \rangle$	$\langle 0, (0.19e^{i0.25\pi}, 0.89e^{i1.94\pi}) \rangle$	$\langle 0, (0.05e^{i0.32\pi}, 0.92e^{i1.94\pi}) \rangle$	$\langle 1, (0.29e^{i0.72\pi}, 0.58e^{i1.33\pi}) \rangle$
x_4	$\langle 1, (0.25e^{i0.45\pi}, 0.75e^{i1.65\pi}) \rangle$	$\langle 0, (0.15e^{i0.25\pi}, 0.95e^{i1.85\pi}) \rangle$	$\langle 2, (0.55e^{i0.95\pi}, 0.61e^{i0.85\pi}) \rangle$	$\langle 0, (0.12e^{i0.37\pi}, 0.91e^{i1.87\pi}) \rangle$

$e^{i\psi_D(u_g, r_{agt}^2)} \rangle \in \mathcal{B}_2(b_t), \mathcal{C}$ and \mathcal{D} are CFFSs on $\mathcal{F}_1(b_t)$ and $\mathcal{F}_2(b_t)$, respectively.

Example 3.8 The restricted union $(\bar{h}, \mathcal{Q}_1 \cup_{\mathbb{R}} \mathcal{Q}_2, 5)$ of $(\mathcal{H}_1, \mathcal{Q}_1, 5)$ (see Table 15) and $(\mathcal{H}_2, \mathcal{Q}_2, 4)$ (see Table 16) is defined by Table 19.

$$\mathcal{H}^L(b_t) = \begin{cases} (s(u_g, r_{agt})e^{i\omega(u_g, r_{agt})}, k(u_g, r_{agt})e^{i\psi(u_g, r_{agt})}), & \text{if } \mathcal{H}(b_t) = \langle (u_g, r_{agt}), s(u_g, r_{agt})e^{i\omega(u_g, r_{agt})}, k(u_g, r_{agt})e^{i\psi(u_g, r_{agt})} \rangle, \\ & \text{and } r_{agt} \geq L, \\ (0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}) & \text{otherwise.} \end{cases}$$

Definition 3.15 Let U be a universe of discourse and $(\mathcal{H}_1, \mathcal{Q}_1, N_1)$ and $(\mathcal{H}_2, \mathcal{Q}_2, N_2)$ be two CFFNS_fSs over non-empty set U , where $\mathcal{Q}_1 = (\mathcal{F}_1, \mathcal{B}_1, N_1)$ and $\mathcal{Q}_2 = (\mathcal{F}_2, \mathcal{B}_2, N_2)$ are NS_fSs on U , then their *extended union* is denoted by $(\mathcal{H}_1, \mathcal{Q}_1, N_1) \cup_{\mathbb{E}} (\mathcal{H}_2, \mathcal{Q}_2, N_2)$ and is defined as $(\zeta, \mathcal{Q}_1 \cup_{\mathbb{E}} \mathcal{Q}_2, \max(N_1, N_2))$, where $\mathcal{Q}_1 \cup_{\mathbb{E}} \mathcal{Q}_2 = (\mathfrak{Y}, \mathcal{B}_1 \cup \mathcal{B}_2, \max(N_1, N_2))$, and $\zeta(b_t)$ is given by:

$$\zeta(b_t) = \begin{cases} \mathcal{H}_1(b_t), & \text{if } b_t \in \mathcal{B}_1 - \mathcal{B}_2, \\ \mathcal{H}_2(b_t), & \text{if } b_t \in \mathcal{B}_2 - \mathcal{B}_1, \\ \langle (u_g, r_{agt}), y, z \rangle, & \text{such that } r_{agt} = \max(r_{agt}^1, r_{agt}^2), \\ & y = \max(s_{\mathcal{C}}(u_g, r_{agt}^1), s_{\mathcal{D}}(u_g, r_{agt}^2))e^{i\max(\omega_{\mathcal{C}}(u_g, r_{agt}^1), \omega_{\mathcal{D}}(u_g, r_{agt}^2))}, \\ & z = \min(k_{\mathcal{C}}(u_g, r_{agt}^1), k_{\mathcal{D}}(u_g, r_{agt}^2))e^{i\min(\psi_{\mathcal{C}}(u_g, r_{agt}^1), \psi_{\mathcal{D}}(u_g, r_{agt}^2))}, \\ & \text{where } \langle (u_g, r_{agt}^1), s_{\mathcal{C}}(u_g, r_{agt}^1), k_{\mathcal{C}}(u_g, r_{agt}^1) \rangle \in \mathcal{B}_1(b_t) \\ & \text{and } \langle (u_g, r_{agt}^2), s_{\mathcal{D}}(u_g, r_{agt}^2), k_{\mathcal{D}}(u_g, r_{agt}^2) \rangle \in \mathcal{B}_2(b_t), \\ & \mathcal{C} \text{ and } \mathcal{D} \text{ are CFFSs on } \mathcal{F}_1(b_t) \text{ and } \mathcal{F}_2(b_t), \text{ respectively.} \end{cases}$$

Example 3.9 The extended union $(\zeta, \mathcal{Q}_1 \cup_{\mathbb{E}} \mathcal{Q}_2, 5)$ of $(\mathcal{H}_1, \mathcal{Q}_1, 5)$ (see Table 15) and $(\mathcal{H}_2, \mathcal{Q}_2, 4)$ (see Table 16) is represented by Table 20.

Definition 3.16 Suppose that U be a universe of discourse and $(\mathcal{H}, \mathcal{Q}, N)$ be a CFFNS_fS over non-empty set U , where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ is an NS_fS on U . Let $0 < L < N$ be a threshold. A CFFS_fS related with $(\mathcal{H}, \mathcal{Q}, N)$ and L , denoted by $(\mathcal{H}^L, \mathcal{B})$, is given as follows:

Particularly, $(\mathcal{H}^1, \mathcal{B})$ is called the bottom CFFS_fS linked with $(\mathcal{H}, \mathcal{Q}, N)$ and $(\mathcal{H}^{N-1}, \mathcal{B})$ is called top CFFS_fS associated with $(\mathcal{H}, \mathcal{Q}, N)$.

Definition 3.17 Let $0 < L < N$ and $\partial \in [0, 1]$ be two thresholds. The S_fS over U associated with $(\mathcal{H}, \mathcal{Q}, N)$ and (L, ∂) is $(\mathfrak{f}^{(L, \partial)}, \mathcal{B})$ given by: for each $b_t \in \mathcal{B}$, $\mathfrak{f}^{(L, \partial)}(b_t) = \{u_g \in U : \mathcal{H}^L(b_t) > \partial\}$.

Table 19 Tabulated form of $(\bar{h}, \mathcal{Q}_1 \cup_{\mathbb{R}} \mathcal{Q}_2, 5)$ in Example 3.8

$(\bar{h}, \mathcal{Q}_1 \cup_{\mathbb{R}} \mathcal{Q}_2, 5)$	b_1	b_2
x_1	$\langle 4, (0.92e^{i1.88\pi}, 0.08e^{i0.36\pi}) \rangle$	$\langle 3, (0.77e^{i1.58\pi}, 0.29e^{i0.67\pi}) \rangle$
x_2	$\langle 3, (0.73e^{i1.46\pi}, 0.27e^{i0.78\pi}) \rangle$	$\langle 4, (0.97e^{i1.96\pi}, 0.15e^{i0.29\pi}) \rangle$
x_3	$\langle 4, (0.89e^{i1.76\pi}, 0.09e^{i0.36\pi}) \rangle$	$\langle 2, (0.49e^{i1.27\pi}, 0.57e^{i1.18\pi}) \rangle$
x_4	$\langle 3, (0.93e^{i1.88\pi}, 0.15e^{i0.25\pi}) \rangle$	$\langle 1, (0.38e^{i0.67\pi}, 0.76e^{i1.29\pi}) \rangle$

Example 3.10 Consider the CFF5S_fS in Example 3.1, represented by Table 6. From Definition 3.16, we can find the associated CFFS_fSs with CFF5S_fS. Let $0 < L < 5$ be threshold. Then the possible CFFS_fS associated with thresholds 1, 2, 3 and 4 are shown by Tables 21–24.

Table 20 Tabulated form of $(\mathcal{H}_1, \mathcal{Q}_1, 5) \cup_{\mathbb{E}} (\mathcal{H}_2, \mathcal{Q}_2, 4)$ given in Example 3.9

$(\zeta, \mathcal{Q}_1 \cup_{\mathbb{E}} \mathcal{Q}_2, 5)$	b_1	b_2	b_3	v
x_1	$\langle 4, (0.92e^{i1.88\pi}, 0.08e^{i0.36\pi}) \rangle$	$\langle 3, (0.77e^{i1.58\pi}, 0.29e^{i0.67\pi}) \rangle$	$\langle 2, (0.61e^{i1.14\pi}, 0.42e^{i1.29\pi}) \rangle$	$\langle 3, (0.88e^{i1.76\pi}, 0.05e^{i0.19\pi}) \rangle$
x_2	$\langle 3, (0.73e^{i1.46\pi}, 0.27e^{i0.78\pi}) \rangle$	$\langle 4, (0.97e^{i1.96\pi}, 0.15e^{i0.29\pi}) \rangle$	$\langle 1, (0.37e^{i0.63\pi}, 0.59e^{i1.54\pi}) \rangle$	$\langle 2, (0.69e^{i0.97\pi}, 0.38e^{i0.65\pi}) \rangle$
x_3	$\langle 4, (0.89e^{i1.76\pi}, 0.09e^{i0.36\pi}) \rangle$	$\langle 2, (0.49e^{i1.27\pi}, 0.57e^{i1.18\pi}) \rangle$	$\langle 0, (0.05e^{i0.32\pi}, 0.92e^{i1.94\pi}) \rangle$	$\langle 1, (0.29e^{i0.72\pi}, 0.58e^{i1.33\pi}) \rangle$
x_4	$\langle 3, (0.93e^{i1.88\pi}, 0.15e^{i0.25\pi}) \rangle$	$\langle 1, (0.38e^{i0.67\pi}, 0.76e^{i1.29\pi}) \rangle$	$\langle 2, (0.55e^{i0.95\pi}, 0.61e^{i0.85\pi}) \rangle$	$\langle 0, (0.12e^{i0.37\pi}, 0.91e^{i1.87\pi}) \rangle$

Table 21 CFFS $_f$ S associated with CFF5S $_f$ S and threshold 1

$(\mathcal{H}^1, \mathcal{B})$	b_1	b_2	b_3
x_1	$(0.5e^{i0.9\pi}, 0.6e^{i1.2\pi})$	$(0.7e^{i1.5\pi}, 0.3e^{i0.5\pi})$	$(0.3e^{i0.6\pi}, 0.8e^{i1.6\pi})$
x_2	$(0.3e^{i0.5\pi}, 0.7e^{i1.5\pi})$	$(0.8e^{i1.6\pi}, 0.4e^{i0.6\pi})$	$(0.6e^{i0.9\pi}, 0.5e^{i1.2\pi})$
x_3	$(0.7e^{i1.4\pi}, 0.3e^{i0.5\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_4	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.5e^{i0.9\pi}, 0.5e^{i1.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$

Table 22 CFFS $_f$ S associated with CFF5S $_f$ S and threshold 2

$(\mathcal{H}^2, \mathcal{B})$	b_1	b_2	b_3
x_1	$(0.5e^{i0.9\pi}, 0.6e^{i1.2\pi})$	$(0.7e^{i1.5\pi}, 0.3e^{i0.5\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_2	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.8e^{i1.6\pi}, 0.4e^{i0.6\pi})$	$(0.6e^{i0.9\pi}, 0.5e^{i1.2\pi})$
x_3	$(0.7e^{i1.4\pi}, 0.3e^{i0.5\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_4	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.5e^{i0.9\pi}, 0.5e^{i1.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$

Table 23 CFFS $_f$ S associated with CFF5S $_f$ S and threshold 3

$(\mathcal{H}^3, \mathcal{B})$	b_1	b_2	b_3
x_1	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.7e^{i1.5\pi}, 0.3e^{i0.5\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_2	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.8e^{i1.6\pi}, 0.4e^{i0.6\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_3	$(0.7e^{i1.4\pi}, 0.3e^{i0.5\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_4	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$

Table 24 CFFS $_f$ S associated with CFF5S $_f$ S and threshold 4

$(\mathcal{H}^4, \mathcal{B})$	b_1	b_2	b_3
x_1	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_2	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_3	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$
x_4	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$

4 Operations

Definition 4.1 Let $\alpha_{a_{1t}} = \langle r_{a_{1t}}, (s_{a_{1t}}e^{i\omega_{a_{1t}}}, k_{a_{1t}}e^{i\psi_{a_{1t}}}) \rangle$ ($t = 1, 2$) and $\alpha = \langle r, (se^{i\omega}, ke^{i\psi}) \rangle$ be three CFFNS $_f$ Ns over U and $\xi > 0$. Then, some operations for CFFNS $_f$ Ns are:

- $\alpha_{a_{11}} \cup \alpha_{a_{12}} = \langle \max(r_{a_{11}}, r_{a_{12}}), (\max(s_{a_{11}}, s_{a_{12}})e^{i\max(\omega_{a_{11}}, \omega_{a_{12}})}, \min(k_{a_{11}}, k_{a_{12}})e^{i\min(\psi_{a_{11}}, \psi_{a_{12}})}) \rangle$
- $\alpha_{a_{11}} \cap \alpha_{a_{12}} = \langle \min(r_{a_{11}}, r_{a_{12}}), (\min(s_{a_{11}}, s_{a_{12}})e^{i\min(\omega_{a_{11}}, \omega_{a_{12}})}, \max(k_{a_{11}}, k_{a_{12}})e^{i\max(\psi_{a_{11}}, \psi_{a_{12}})}) \rangle$
- $\alpha^c = \langle r, (ke^{i\psi}, se^{i\omega}) \rangle$
- $\alpha_{a_{11}} \oplus \alpha_{a_{12}} = \langle \max(r_{a_{11}}, r_{a_{12}}), \sqrt[3]{(s_{a_{11}})^3 + (s_{a_{12}})^3 - (s_{a_{11}})^3(s_{a_{12}})^3} e^{i2\pi\sqrt[3]{(\frac{\omega_{a_{11}}}{2\pi})^3 + (\frac{\omega_{a_{12}}}{2\pi})^3 - (\frac{\omega_{a_{11}}}{2\pi})^3(\frac{\omega_{a_{12}}}{2\pi})^3}}, k_{a_{11}}k_{a_{12}}e^{i2\pi(\frac{\psi_{a_{11}}}{2\pi})(\frac{\psi_{a_{12}}}{2\pi})} \rangle$

5. $\alpha_{a_{11}} \otimes \alpha_{a_{12}} = \langle \min(r_{a_{11}}, r_{a_{12}}), (s_{a_{11}} s_{a_{12}} e^{i2\pi(\frac{\omega_{a_{11}}}{2\pi})(\frac{\omega_{a_{12}}}{2\pi})}, \sqrt[3]{(k_{a_{11}})^3 + (k_{a_{12}})^3 - (k_{a_{11}})^3(k_{a_{12}})^3} e^{i2\pi\sqrt[3]{(\frac{\psi_{a_{11}}}{2\pi})^3 + (\frac{\psi_{a_{12}}}{2\pi})^3 - (\frac{\psi_{a_{11}}}{2\pi})^3(\frac{\psi_{a_{12}}}{2\pi})^3}} \rangle$
6. $\zeta\alpha = \langle r, (\sqrt[3]{1 - (1 - s^3)^\zeta} e^{i2\pi\sqrt[3]{1 - (1 - (\frac{\omega}{2\pi})^3)^\zeta}}, k^\zeta e^{i2\pi(\frac{\psi}{2\pi})^\zeta} \rangle$
7. $\alpha^\zeta = \langle r, (s^\zeta e^{i2\pi(\frac{\omega}{2\pi})^\zeta}, \sqrt[3]{1 - (1 - k^3)^\zeta} e^{i2\pi\sqrt[3]{1 - (1 - (\frac{\psi}{2\pi})^3)^\zeta}} \rangle$
3. $\zeta\alpha = \langle r, (\sqrt[3]{\min(1, (\zeta s^{3\varphi})^{\frac{1}{\varphi}})} e^{i2\pi\sqrt[3]{\min(1, (\zeta(\frac{\omega}{2\pi})^{3\varphi})^{\frac{1}{\varphi}}}}, \sqrt[3]{1 - \min(1, (\zeta(1 - k^3)^\varphi)^{\frac{1}{\varphi}}} e^{i2\pi\sqrt[3]{1 - \min(1, (\zeta(1 - (\frac{\psi}{2\pi})^3)^\varphi)^{\frac{1}{\varphi}}}} \rangle$
4. $\alpha^\zeta = \langle r, (\sqrt[3]{1 - \min(1, (\zeta(1 - s^3)^\varphi)^{\frac{1}{\varphi}})} e^{i2\pi\sqrt[3]{1 - \min(1, (\zeta(1 - (\frac{\omega}{2\pi})^3)^\varphi)^{\frac{1}{\varphi}}}}, \sqrt[3]{\min(1, (\zeta k^{3\varphi})^{\frac{1}{\varphi}}} e^{i2\pi\sqrt[3]{\min(1, (\zeta(\frac{\psi}{2\pi})^{3\varphi})^{\frac{1}{\varphi}}}} \rangle$

where φ is the parameter and $\varphi \in (0, \infty)$.

Definition 4.2 Let $\alpha_{a_{1t}} = \langle r_{a_{1t}}, (s_{a_{1t}} e^{i\omega_{a_{1t}}}, k_{a_{1t}} e^{i\psi_{a_{1t}}}) \rangle (t = 1, 2)$ and $\alpha = \langle r, (s e^{i\omega}, k e^{i\psi}) \rangle$ be three CFFNS_fNs over U and $\zeta > 0$. Then, Yager operations for CFFNS_fNs are:

1. $\alpha_{a_{11}} \oplus \alpha_{a_{12}} = \langle \max(r_{a_{11}}, r_{a_{12}}), (\sqrt[3]{\min(1, (s_{a_{11}}^{3\varphi} + s_{a_{12}}^{3\varphi})^{\frac{1}{\varphi}})} e^{i2\pi\sqrt[3]{\min(1, ((\frac{\omega_{a_{11}}}{2\pi})^{3\varphi} + (\frac{\omega_{a_{12}}}{2\pi})^{3\varphi})^{\frac{1}{\varphi}}}}, \sqrt[3]{1 - \min(1, ((1 - k_{a_{11}}^3)^\varphi + (1 - k_{a_{12}}^3)^\varphi)^{\frac{1}{\varphi}}}} \rangle$

Remark 2 We have used the name “Yager operations” because these operations are derived from the use of the theoretical foundations of Yager s-norm and t-norm in the CFFNS_fS environment. These operations carry the accuracy feature and aggregation skills of the Yager norm for flexible model of CFFNS_fNs.

Example 4.1 Let $\alpha_1 = \langle 3, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$ and $\alpha_2 = \langle 2, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$ be two CFFNS_fNs, and then by using Definition 4.2 for $\zeta = 2$, $\varphi = 4$ they are:

$$\begin{aligned}
 \alpha_1 \oplus \alpha_2 &= \left\langle \max(3, 2), \left(\sqrt[3]{\min(1, (0.8^{12} + 0.6^{12})^{\frac{1}{4}})} e^{i2\pi\sqrt[3]{\min(1, ((\frac{1.6\pi}{2\pi})^{12} + (\frac{0.9\pi}{2\pi})^{12})^{\frac{1}{4}}}}, \sqrt[3]{1 - \min(1, ((1 - 0.4^3)^4 + (1 - 0.5^3)^4)^{\frac{1}{4}}} e^{i2\pi\sqrt[3]{1 - \min(1, ((1 - (\frac{0.6\pi}{2\pi})^3)^4 + (1 - (\frac{1.2\pi}{2\pi})^3)^4)^{\frac{1}{4}}}} \right) \right\rangle \\
 &= \langle 3, (0.8e^{i1.6\pi}, 0e^{i0}) \rangle \\
 \alpha_1 \otimes \alpha_2 &= \left\langle \min(3, 2), \left(\sqrt[3]{1 - \min(1, ((1 - 0.8^3)^4 + (1 - 0.6^3)^4)^{\frac{1}{4}}} e^{i2\pi\sqrt[3]{1 - \min(1, ((1 - (\frac{1.6\pi}{2\pi})^3)^4 + (1 - (\frac{0.9\pi}{2\pi})^3)^4)^{\frac{1}{4}}}}, \sqrt[3]{\min(1, (0.4^{12} + 0.5^{12})^{\frac{1}{4}})} e^{i2\pi\sqrt[3]{\min(1, ((\frac{0.6\pi}{2\pi})^{12} + (\frac{1.2\pi}{2\pi})^{12})^{\frac{1}{4}}}} \right) \right\rangle \\
 &= \langle 2, (0.57e^{i0.83\pi}, 0.5e^{i1.2\pi}) \rangle \\
 2\alpha_1 &= \langle 3, (\sqrt[3]{\min(1, (2(0.8)^{12})^{\frac{1}{4}})} e^{i2\pi\sqrt[3]{\min(1, (2(\frac{1.6\pi}{2\pi})^{12})^{\frac{1}{4}}}}, \sqrt[3]{1 - \min(1, (2(1 - 0.4^3)^4)^{\frac{1}{4}}} e^{i2\pi\sqrt[3]{1 - \min(1, (2(1 - (\frac{0.6\pi}{2\pi})^3)^4)^{\frac{1}{4}}}} \rangle \\
 &= \langle 3, (0.85e^{i1.7\pi}, 0e^{i0}) \rangle \\
 \alpha_1^2 &= \langle 3, (\sqrt[3]{1 - \min(1, (2(1 - 0.8^3)^4)^{\frac{1}{4}}} e^{i2\pi\sqrt[3]{1 - \min(1, (2(1 - (\frac{1.6\pi}{2\pi})^3)^4)^{\frac{1}{4}}}}, \sqrt[3]{\min(1, (2(0.4)^{12})^{\frac{1}{4}})} e^{i2\pi\sqrt[3]{\min(1, (2(\frac{0.6\pi}{2\pi})^{12})^{\frac{1}{4}}}} \rangle \\
 &= \langle 3, (0.75e^{i1.5\pi}, 0.42e^{i0.64\pi}) \rangle
 \end{aligned}$$

2. $\alpha_{a_{11}} \otimes \alpha_{a_{12}} = \langle \min(r_{a_{11}}, r_{a_{12}}), (\sqrt[3]{1 - \min(1, ((1 - s_{a_{11}}^3)^\varphi + (1 - s_{a_{12}}^3)^\varphi)^{\frac{1}{\varphi}}}} e^{i2\pi\sqrt[3]{1 - \min(1, ((1 - (\frac{\omega_{a_{11}}}{2\pi})^3)^\varphi + (1 - (\frac{\omega_{a_{12}}}{2\pi})^3)^\varphi)^{\frac{1}{\varphi}}}}, \sqrt[3]{\min(1, (k_{a_{11}}^{3\varphi} + k_{a_{12}}^{3\varphi})^{\frac{1}{\varphi}}} e^{i2\pi\sqrt[3]{\min(1, ((\frac{\psi_{a_{11}}}{2\pi})^{3\varphi} + (\frac{\psi_{a_{12}}}{2\pi})^{3\varphi})^{\frac{1}{\varphi}}}} \rangle$

5 Algorithms and applications

In this section, we clarify the decision-making (DM) process for the constructed model. Firstly, we construct the procedures as shown in Algorithms 1–3 for problems that are described by CFFNS_fSs. Then, we apply them to real circumstances to get the particular results.

Algorithm 1: The algorithm of choice values of CFFNS_fSs

- 1 **Input** $U = \{u_1, u_2, u_3, \dots, u_n\}$ as a universal set.
- 2 **Input** $\mathcal{B} = \{b_1, b_2, b_3, \dots, b_m\}$ as a set of attributes.
- 3 **Input** $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ with $\mathcal{R} = \{0, 1, 2, 3, \dots, N-1\}$, $N \in \{2, 3, \dots\}$, for each $u_g \in U$, $b_t \in \mathcal{B}$, there exists $r_{agt} \in \mathcal{R}$.
- 4 **Construct** CFFNS_fNs $(\mathcal{H}, \mathcal{Q}, N)$, where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$, corresponding to each attribute for alternative u_g .
- 5 **Compute** $\mathcal{W}_i = \bigoplus_{t=1}^m \alpha_{agt}$, where $(u_g, r_{agt}) \in \mathcal{F}(b_t)$, and the addition of two CFFNS_fNs is interpreted as, if $\alpha_{a_{11}} = \langle r_{a_{11}}, (s_{a_{11}} e^{i\omega_{a_{11}}}, k_{a_{11}} e^{i\psi_{a_{11}}}) \rangle$ and $\alpha_{a_{12}} = \langle r_{a_{12}}, (s_{a_{12}} e^{i\omega_{a_{12}}}, k_{a_{12}} e^{i\psi_{a_{12}}}) \rangle$ are two CFFNS_fNs then $\alpha_{a_{11}} \oplus \alpha_{a_{12}} = \left\langle \max(r_{a_{11}}, r_{a_{12}}), \left(\sqrt[3]{(s_{a_{11}})^3 + (s_{a_{12}})^3 - (s_{a_{11}})^3 (s_{a_{12}})^3} e^{i2\pi \sqrt[3]{(\frac{\omega_{a_{11}}}{2\pi})^3 + (\frac{\omega_{a_{12}}}{2\pi})^3 - (\frac{\omega_{a_{11}}}{2\pi})^3 (\frac{\omega_{a_{12}}}{2\pi})^3}}, k_{a_{11}} k_{a_{12}} e^{i2\pi (\frac{\psi_{a_{11}}}{2\pi})(\frac{\psi_{a_{12}}}{2\pi})} \right) \right\rangle$.
- 6 **Compute** score function $S(\mathcal{W}_i)$ as defined in Definition 3.3 and identify all indices i for which $S(\mathcal{W}_i) = \max_{i=1,2,\dots,n} S(\mathcal{W}_i)$.
- 7 **Output** Any of the alternative for which $S(\mathcal{W}_i) = \max_{i=1,2,\dots,n} S(\mathcal{W}_i)$ will be picked.

Algorithm 2: The algorithm of L -choice values of CFFNS_fSs

- 1 **Input** $U = \{u_1, u_2, u_3, \dots, u_n\}$ as a universal set.
- 2 **Input** $\mathcal{B} = \{b_1, b_2, b_3, \dots, b_m\}$ as a set of attributes.
- 3 **Input** $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$ with $\mathcal{R} = \{0, 1, 2, 3, \dots, N-1\}$, $N \in \{2, 3, \dots\}$, for each $u_g \in U$, $b_t \in \mathcal{B}$, there exists $r_{agt} \in \mathcal{R}$.
- 4 **Construct** CFFNS_fS $(\mathcal{H}, \mathcal{Q}, N)$, where $\mathcal{Q} = (\mathcal{F}, \mathcal{B}, N)$.
- 5 **Input** L threshold.
- 6 **Determine** $\mathcal{H}^L(b_t) = \begin{cases} (s_{agt} e^{i\omega_{agt}}, k_{agt} e^{i\psi_{agt}}), & \text{if } \mathcal{H}_{agt} = \langle \langle r_{agt}, (s_{agt} e^{i\omega_{agt}}, k_{agt} e^{i\psi_{agt}}) \rangle \rangle, \\ & \text{and } r_{agt} \geq L, \\ (0.0e^{i0.0\pi}, 1.0e^{i2.0\pi}) & \text{otherwise.} \end{cases}$
- 7 **Compute** $\mathcal{W}_i^L = \bigoplus_{t=1}^m \beta_{agt}$, where $(u_g, r_{agt}) \in \mathcal{F}(b_t)$ and $\beta_{agt} = (s_{agt} e^{i\omega_{agt}}, k_{agt} e^{i\psi_{agt}})$ are CFF threshold numbers.
- 8 **Compute** score function $S(\mathcal{W}_i^L) = (s_{agt})^3 - (k_{agt})^3 + ((\frac{\omega_{agt}}{2\pi})^3 - (\frac{\psi_{agt}}{2\pi})^3)$
- 9 **Compute** all indices i for which $S(\mathcal{W}_i^L) = \max_{i=1,2,\dots,n} S(\mathcal{W}_i^L)$.
- 10 **Output** Any of the alternative for which $S(\mathcal{W}_i^L) = \max_{i=1,2,\dots,n} S(\mathcal{W}_i^L)$ will be decision.

Algorithm 3: The algorithm of comparison table of CFFNS_fSs

- 1 **Input** $U = \{u_1, u_2, u_3, \dots, u_n\}$ as a universal set.
- 2 **Input** $\mathcal{B} = \{b_1, b_2, b_3, \dots, b_m\}$ as a set of attributes.
- 3 **Consider** a CFFNS_fS in tabulated form.
- 4 **Determine** the comparison table of information table.
- 5 **Determine** the information ranking order from comparison table.
- 6 **Output** Any of the alternative for which maximum outcome and maximum grade will be decision.

5.1 Selection of buy new car

Selection of a car is a difficult task for an auto broker. Productive selection is possible only when there is an essential matching. By choosing the best car, the auto broker will get quality performance. In Example 3.1, different CFFNS_fNs for the cars have been defined on the basis of their qualities, by the auto broker. Tabulated form of CFF5S_fS is represented by Table 25.

• Choice value (CV) of CFF5S_fS

We can calculate the CV of CFF5S_fS of the car's selection by using Algorithm 1 and calculated results are given in Table 26, where

$$S(\mathcal{W}_i) = \left(\frac{r_{agt}}{N-1}\right)^3 + (s_{agt})^3 - (k_{agt})^3 + \left(\left(\frac{\omega_{agt}}{2\pi}\right)^3 - \left(\frac{\psi_{agt}}{2\pi}\right)^3\right).$$

From Table 26, it is concluded that according to $S(\mathcal{W}_i)$ values, $x_3 > x_4 > x_2 > x_1$ and hence x_3 has maximum value. So, the auto broker will choose the car x_3 .

• L-Choice value (L-CV) of CFF5S_fS

Now, we will choose the threshold L and will calculate the CV by using Algorithm 2, where

$$S(\mathcal{W}_i^L) = (s_{agt})^3 - (k_{agt})^3 + \left(\left(\frac{\omega_{agt}}{2\pi}\right)^3 - \left(\frac{\psi_{agt}}{2\pi}\right)^3\right).$$

The result is shown by Table 27.

Table 25 Tabular form of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}, \mathcal{Q}, 5)$	b_1	b_2	b_3
x_1	$\langle 2, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 3, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 1, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$
x_2	$\langle 1, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 3, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 2, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$
x_3	$\langle 3, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$
x_4	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 2, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$

Table 26 Tabular form of CV of CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}, \mathcal{Q}, 5)$	b_1	b_2	b_3	\mathcal{W}_i	$S(\mathcal{W}_i)$
x_1	$\langle 2, (0.5e^{i0.9\pi}, 0.6e^{i1.2\pi}) \rangle$	$\langle 3, (0.7e^{i1.5\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 1, (0.3e^{i0.6\pi}, 0.8e^{i1.6\pi}) \rangle$	$\langle 3, (0.76e^{i1.58\pi}, 0.14e^{i0.24\pi}) \rangle$	1.35
x_2	$\langle 1, (0.3e^{i0.5\pi}, 0.7e^{i1.5\pi}) \rangle$	$\langle 3, (0.8e^{i1.6\pi}, 0.4e^{i0.6\pi}) \rangle$	$\langle 2, (0.6e^{i0.9\pi}, 0.5e^{i1.2\pi}) \rangle$	$\langle 3, (0.86e^{i1.65\pi}, 0.14e^{i0.27\pi}) \rangle$	1.61
x_3	$\langle 3, (0.7e^{i1.4\pi}, 0.3e^{i0.5\pi}) \rangle$	$\langle 4, (0.9e^{i1.8\pi}, 0.1e^{i0.3\pi}) \rangle$	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.9\pi}) \rangle$	$\langle 4, (0.94e^{i1.87\pi}, 0.03e^{i0.07\pi}) \rangle$	2.65
x_4	$\langle 0, (0.1e^{i0.2\pi}, 0.9e^{i1.8\pi}) \rangle$	$\langle 2, (0.5e^{i0.9\pi}, 0.5e^{i1.0\pi}) \rangle$	$\langle 4, (0.9e^{i1.9\pi}, 0.2e^{i0.3\pi}) \rangle$	$\langle 4, (0.91e^{i1.91\pi}, 0.09e^{i0.14\pi}) \rangle$	2.62

Table 27 Tabular form of 2-CV of the CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}^2, \mathcal{B})$	b_1	b_2	b_3	\mathcal{W}_i	$S(\mathcal{W}_i^L)$
x_1	$(0.5e^{i0.9\pi}, 0.6e^{i1.2\pi})$	$(0.7e^{i1.5\pi}, 0.3e^{i0.5\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.75e^{i1.56\pi}, 0.18e^{i0.30\pi})$	0.89
x_2	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.8e^{i1.6\pi}, 0.4e^{i0.6\pi})$	$(0.6e^{i0.9\pi}, 0.5e^{i1.2\pi})$	$(0.85e^{i1.65\pi}, 0.20e^{i0.36\pi})$	1.16
x_3	$(0.7e^{i1.4\pi}, 0.3e^{i0.5\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.94e^{i1.87\pi}, 0.03e^{i0.08\pi})$	1.65
x_4	$(0.0e^{i0.0\pi}, 1.0e^{i2.0\pi})$	$(0.5e^{i0.9\pi}, 0.5e^{i1.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$	$(0.91e^{i1.91\pi}, 0.10e^{i0.15\pi})$	1.62

Table 28 Tabular form of the membership and non-membership values of CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

$(\mathcal{H}, \mathcal{Q}, 5)$	b_1	b_2	b_3
x_1	$(0.5e^{i0.9\pi}, 0.6e^{i1.2\pi})$	$(0.7e^{i1.5\pi}, 0.3e^{i0.5\pi})$	$(0.3e^{i0.6\pi}, 0.8e^{i1.6\pi})$
x_2	$(0.3e^{i0.5\pi}, 0.7e^{i1.5\pi})$	$(0.8e^{i1.6\pi}, 0.4e^{i0.6\pi})$	$(0.6e^{i0.9\pi}, 0.5e^{i1.2\pi})$
x_3	$(0.7e^{i1.4\pi}, 0.3e^{i0.5\pi})$	$(0.9e^{i1.8\pi}, 0.1e^{i0.3\pi})$	$(0.1e^{i0.2\pi}, 0.9e^{i1.9\pi})$
x_4	$(0.1e^{i0.2\pi}, 0.9e^{i1.8\pi})$	$(0.5e^{i0.9\pi}, 0.5e^{i1.0\pi})$	$(0.9e^{i1.9\pi}, 0.2e^{i0.3\pi})$

In Table 27, we took $L = 2$ for DM and get the 2-CV of CFF5S_fS. It can observe from Table 27 that the car x_3 has highest output value. So, x_3 will be selected by the auto broker.

• Comparison table of CFF5 S_fS

Comparison table is a square table in which rows and columns are represented by the name of objects of universe such as $u_1, u_2, u_3, \dots, u_n$ and $q_{gt} =$ the CFFNS_f values of the attributes for which the value of score function of $u_d \geq u_j$.

Membership and non-membership values of Table 25 are given in tabular form in Table 28.

The comparison table of Table 28 is given by Table 29.

The result will be derived by subtracting the row and column sum of Table 29.

From Table 30, it is concluded that the highest rank and grade sum is 3 and 7, respectively, which is obtained by x_3 . So, x_3 car is selected by the auto broker.

5.2 Selection of the best cellular telecommunication company in Pakistan

Since due to COVID-19, everything has shifted to the online mode. So, the usage of the internet has increased. Due to that, in market, the competition among different network provider companies has been tough day by day. All companies are presenting different internet packages according to the needs

Table 29 Comparison table of CFF5S_fS ($\mathcal{H}, \mathcal{Q}, 5$)

	x_1	x_2	x_3	x_4
x_1	3	1	1	2
x_2	2	3	1	2
x_3	2	2	3	2
x_4	1	1	1	3

Table 30 Ranking Table

	Grade sum (G_s)	Row sum (R_s)	Column sum (C_s)	Final Rank ($R_s - C_s$)
x_1	6	7	8	-1
x_2	6	8	7	1
x_3	7	9	6	3
x_4	6	6	9	-3

of the customers. So, it's a difficult task for customers to choose the sim card of the best telecommunication company. Suppose that a student, in Pakistan, decides to purchase a new sim card to attend online classes. Before buying the sim, the student has collected some relevant rating based information related to the internet packages and internet speed about five different network companies such as Zong, Ufone, Telenor, Jazz, and Warid. Each network company has different prices of internet packages depending on time as well. Let $Y = \{y_1 = \text{zong}, y_2 = \text{SCOM}, y_3 = \text{telenor}, y_4 = \text{jazz}, y_5 = \text{ufone}\}$ be a universal set and $\mathcal{O} = \{o_1 = \text{3G/4Gspeed}, o_2 = \text{Packageestimateduration}, o_3 = \text{Monthlycost}, o_4 = \text{Signalstrength}\} \subseteq \mathcal{A}$ be a set of attributes, which are used to assign grades to network companies. The ratings are on the basis of internet packages and speed provided in last year and users reviews. It may be noted that the ranking of alternatives with respect to parameters may get affected and altered if the time and location is different for a particular network company. The initial survey is organized in Table 31, where:

- five ♦ represents 'outstanding',
- four ♦ represents 'very good',
- three ♦ represents 'good',
- two ♦ represents 'average',
- one ♦ represents 'subpar',
- ◇ represents 'poor'.

The set of grades $\mathcal{R} = \{0, 1, 2, 3, 4, 5\}$ can be easily associated with ♦ and ◇ as follows:

- 0 stands for '◇',
- 1 stands for '♦',
- 2 stands for '♦♦',
- 3 stands for '♦♦♦',
- 4 stands for '♦♦♦♦',
- 5 stands for '♦♦♦♦♦'.

Based on the overall qualities of the network companies, the student gives evaluation scores to the sim cards which is shown as Table 31 and the tabular representation of its associated 6-soft set is given in Table 32.

Although it is easy to extract the grade data in actual information, the data possess the fuzzy uncertainty characteristics. In order to address the ambiguity of data, we construct CFFNS $_f$ S by using a certain grade. This evaluation of sim cards by students complies with the guidelines as follows:

- $-2.0 \leq S(Y) < -1.4$ when grade 0,
- $-1.4 \leq S(Y) < -0.5$ when grade 1,
- $-0.5 \leq S(Y) < 0.0$ when grade 2,
- $0.0 \leq S(Y) < 0.5$ when grade 3,
- $0.5 \leq S(Y) < 1.4$ when grade 4,
- $1.4 \leq S(Y) \leq 2.0$ when grade 5.

By Definition 3.1, the CFF6S $_f$ S ($\mathcal{H}, \mathcal{Q}, 6$) can be defined as follows:

The CFF6S $_f$ S ($\mathcal{H}, \mathcal{Q}, 6$) can be represented more clearly in tabular form shown as in Table 33.

• Choice value (CV) of CFF6 S $_f$ S

We can calculate the CV of CFF6S $_f$ S of the sim card's selection by using Algorithm 1 as given by Table 34, where

Table 31 Information extracted from the related data

Y/\mathcal{O}	o_1	o_2	o_3	o_4
y_1	♦♦♦♦♦	♦♦♦♦	♦♦♦	♦♦♦♦♦
y_2	◇	♦♦	♦♦	♦
y_3	♦♦♦	♦♦♦♦	♦♦♦	♦♦
y_4	♦	♦♦	♦♦♦	♦♦
y_5	♦♦	♦	◇	♦♦♦♦

Table 32 Tabular form of the 6-soft set ($\mathcal{F}, \mathcal{B}, 6$)

$(\mathcal{F}, \mathcal{B}, 6)$	o_1	o_2	o_3	o_4
y_1	5	4	3	5
y_2	1	2	2	1
y_3	3	4	3	2
y_4	0	2	3	2
y_5	2	1	0	4

$$\begin{aligned}
h(o_1) &= \{ \langle (y_1, 5), 0.95e^{i1.78\pi}, 0.05e^{i0.24\pi} \rangle, \langle (y_2, 1), 0.32e^{i0.54\pi}, 0.89e^{i1.74\pi} \rangle, \langle (y_3, 3), 0.48e^{i1.06\pi}, 0.17e^{i0.30\pi} \rangle, \\
&\quad \langle (y_4, 0), 0.21e^{i0.04\pi}, 0.83e^{i1.94\pi} \rangle, \langle (y_5, 2), 0.29e^{i0.84\pi}, 0.57e^{i1.16\pi} \rangle \}, \\
h(o_2) &= \{ \langle (y_1, 4), 0.76e^{i1.69\pi}, 0.39e^{i0.29\pi} \rangle, \langle (y_2, 2), 0.24e^{i0.64\pi}, 0.43e^{i1.22\pi} \rangle, \langle (y_3, 4), 0.7e^{i1.71\pi}, 0.32e^{i0.21\pi} \rangle, \\
&\quad \langle (y_4, 2), 0.26e^{i0.56\pi}, 0.43e^{i1.21\pi} \rangle, \langle (y_5, 1), 0.34e^{i0.84\pi}, 0.95e^{i1.71\pi} \rangle \}, \\
h(o_3) &= \{ \langle (y_1, 3), 0.58e^{i1.38\pi}, 0.23e^{i0.52\pi} \rangle, \langle (y_2, 2), 0.29e^{i0.54\pi}, 0.49e^{i0.98\pi} \rangle, \langle (y_3, 3), 0.52e^{i1.44\pi}, 0.32e^{i0.72\pi} \rangle, \\
&\quad \langle (y_4, 3), 0.69e^{i1.28\pi}, 0.45e^{i0.86\pi} \rangle, \langle (y_5, 0), 0.07e^{i0.24\pi}, 0.99e^{i1.88\pi} \rangle \}, \\
h(o_4) &= \{ \langle (y_1, 5), 0.91e^{i1.93\pi}, 0.02e^{i0.05\pi} \rangle, \langle (y_2, 1), 0.43e^{i0.66\pi}, 0.85e^{i1.46\pi} \rangle, \langle (y_3, 2), 0.44e^{i0.44\pi}, 0.35e^{i1.08\pi} \rangle, \\
&\quad \langle (y_4, 2), 0.19e^{i0.38\pi}, 0.37e^{i1.06\pi} \rangle, \langle (y_5, 4), 0.81e^{i1.53\pi}, 0.12e^{i0.34\pi} \rangle \}.
\end{aligned}$$

$$S(\mathcal{W}_i) = \left(\frac{r_{a_{gt}}}{N-1}\right)^3 + (s_{a_{gt}})^3 - (k_{a_{gt}})^3 + \left(\left(\frac{\omega_{a_{gt}}}{2\pi}\right)^3 - \left(\frac{\psi_{a_{gt}}}{2\pi}\right)^3\right).$$

From Table 34, it is concluded that according to $S(\mathcal{W}_i)$, $y_1 > y_3 > y_5 > y_4 > y_2$ and hence y_1 has maximum value. So, the student will choose the sim card of zong.

• L-Choice value (L-CV) of CFF6 S_fS

Now, by using the second procedure as given by Algorithm 2, L value will be chosen and the results are given by Table 35, where

$$S(\mathcal{W}_i^L) = (s_{a_{gt}})^3 - (k_{a_{gt}})^3 + \left(\left(\frac{\omega_{a_{gt}}}{2\pi}\right)^3 - \left(\frac{\psi_{a_{gt}}}{2\pi}\right)^3\right).$$

In Table 35, we have chosen $L = 3$ for DM and get the 3-CV of CFF6 S_fS . It can observe from Table 35 that the telecommunication company $y_1 = \text{zong}$ has highest output value. So, y_1 will be selected by the student.

• Comparison table of CFF6 S_fS

Now to apply the third procedure as shown in Algorithm 3, membership and non-membership values of Table 33 are given by Table 36.

The comparison table of Table 36 is given by Table 37.

The result will be derived by subtracting the row and column sum of Table 37.

From Table 38, it is concluded that the highest rank and grade sum is 17 and 16, respectively, which is obtained by $y_1 = \text{zong}$. So, sim card y_1 is selected by the student.

6 Comparison analysis

To certify the rationality of our proposed model, we solve the same example “Selection of the best cellular telecommunication company in Pakistan” using FFY_wA (Garg et al. 2020) and FFY_wG (Garg et al. 2020) operators:

- **Step 1.** The membership and non-membership terms of amplitude part are the same as given in Table 33, but their grades have neglected and phase terms in all CFFNS_jNs have taken to be zero given by Table 39.
- **Step 2.** Let $\tau_k = (0.2, 0.4, 0.1, 0.3)^t$ is the weight vector on alternatives.
- **Step 3.** The entries of aggregated values W_i of the alternatives by using FFY_wA (Garg et al. 2020) operator defined as follows:

Table 33 Tabular form of the CFF6 S_fS ($\mathcal{H}, \mathcal{Q}, 6$)

$(\mathcal{H}, \mathcal{Q}, 6)$	o_1	o_2	o_3	o_4
y_1	$\langle 5, (0.95e^{i1.78\pi}, 0.05e^{i0.24\pi}) \rangle$	$\langle 4, (0.76e^{i1.69\pi}, 0.39e^{i0.29\pi}) \rangle$	$\langle 3, (0.58e^{i1.38\pi}, 0.23e^{i0.52\pi}) \rangle$	$\langle 5, (0.91e^{i1.93\pi}, 0.02e^{i0.05\pi}) \rangle$
y_2	$\langle 1, (0.32e^{i0.54\pi}, 0.89e^{i1.74\pi}) \rangle$	$\langle 2, (0.24e^{i0.64\pi}, 0.43e^{i1.22\pi}) \rangle$	$\langle 2, (0.29e^{i0.54\pi}, 0.49e^{i0.98\pi}) \rangle$	$\langle 1, (0.43e^{i0.66\pi}, 0.85e^{i1.46\pi}) \rangle$
y_3	$\langle 3, (0.48e^{i1.06\pi}, 0.17e^{i0.30\pi}) \rangle$	$\langle 4, (0.70e^{i1.71\pi}, 0.32e^{i0.21\pi}) \rangle$	$\langle 3, (0.52e^{i1.44\pi}, 0.32e^{i0.72\pi}) \rangle$	$\langle 2, (0.44e^{i0.44\pi}, 0.35e^{i1.08\pi}) \rangle$
y_4	$\langle 0, (0.21e^{i0.04\pi}, 0.83e^{i1.94\pi}) \rangle$	$\langle 2, (0.26e^{i0.56\pi}, 0.43e^{i1.21\pi}) \rangle$	$\langle 3, (0.69e^{i1.28\pi}, 0.45e^{i0.86\pi}) \rangle$	$\langle 2, (0.19e^{i0.38\pi}, 0.37e^{i1.06\pi}) \rangle$
y_5	$\langle 2, (0.29e^{i0.84\pi}, 0.57e^{i1.16\pi}) \rangle$	$\langle 1, (0.34e^{i0.84\pi}, 0.95e^{i1.71\pi}) \rangle$	$\langle 0, (0.07e^{i0.24\pi}, 0.99e^{i1.88\pi}) \rangle$	$\langle 4, (0.81e^{i1.53\pi}, 0.12e^{i0.34\pi}) \rangle$

Table 34 Tabular form of the CV of CFF6S $_j$ S ($\mathcal{H}, \mathcal{Q}, 6$)

$(\mathcal{H}, \mathcal{Q}, 6)$	\mathcal{O}_1	\mathcal{O}_2	\mathcal{O}_3	\mathcal{O}_4	\mathcal{W}_i	$S(\mathcal{W}_i)$
γ_1	$\langle 5, (0.95e^{i1.78\pi}, 0.05e^{i0.24\pi}) \rangle$	$\langle 4, (0.76e^{i1.69\pi}, 0.39e^{i0.29\pi}) \rangle$	$\langle 3, (0.58e^{i1.38\pi}, 0.23e^{i0.52\pi}) \rangle$	$\langle 5, (0.91e^{i1.93\pi}, 0.02e^{i0.05\pi}) \rangle$	$\langle 5, (0.99e^{i1.99\pi}, 8.97 \times 10^{-5}e^{i2.26 \times 10^{-4}\pi}) \rangle$	2.96
γ_2	$\langle 1, (0.32e^{i0.54\pi}, 0.89e^{i1.74\pi}) \rangle$	$\langle 2, (0.24e^{i0.64\pi}, 0.43e^{i1.22\pi}) \rangle$	$\langle 2, (0.29e^{i0.54\pi}, 0.49e^{i0.98\pi}) \rangle$	$\langle 1, (0.43e^{i0.66\pi}, 0.85e^{i1.46\pi}) \rangle$	$\langle 2, (0.52e^{i0.94\pi}, 0.16e^{i0.38\pi}) \rangle$	0.30
γ_3	$\langle 3, (0.48e^{i1.06\pi}, 0.17e^{i0.30\pi}) \rangle$	$\langle 4, (0.70e^{i1.71\pi}, 0.32e^{i0.21\pi}) \rangle$	$\langle 3, (0.52e^{i1.44\pi}, 0.32e^{i0.72\pi}) \rangle$	$\langle 2, (0.44e^{i0.44\pi}, 0.35e^{i1.08\pi}) \rangle$	$\langle 4, (0.82e^{i1.86\pi}, 6.09 \times 10^{-3}e^{i6.12 \times 10^{-3}\pi}) \rangle$	1.87
γ_4	$\langle 0, (0.21e^{i0.04\pi}, 0.83e^{i1.94\pi}) \rangle$	$\langle 2, (0.26e^{i0.56\pi}, 0.43e^{i1.21\pi}) \rangle$	$\langle 3, (0.69e^{i1.28\pi}, 0.45e^{i0.86\pi}) \rangle$	$\langle 2, (0.19e^{i0.38\pi}, 0.37e^{i1.06\pi}) \rangle$	$\langle 3, (0.70e^{i1.32\pi}, 0.06e^{i0.27\pi}) \rangle$	0.84
γ_5	$\langle 2, (0.29e^{i0.84\pi}, 0.57e^{i1.16\pi}) \rangle$	$\langle 1, (0.34e^{i0.84\pi}, 0.95e^{i1.71\pi}) \rangle$	$\langle 0, (0.07e^{i0.24\pi}, 0.99e^{i1.88\pi}) \rangle$	$\langle 4, (0.81e^{i1.53\pi}, 0.12e^{i0.34\pi}) \rangle$	$\langle 4, (0.82e^{i1.62\pi}, 0.06e^{i0.16\pi}) \rangle$	1.59

Table 35 Tabular form of the 3-CV of CFF6S $_j$ S ($\mathcal{H}, \mathcal{Q}, 6$)

$(\mathcal{H}^3, \mathcal{B})$	\mathcal{O}_1	\mathcal{O}_2	\mathcal{O}_3	\mathcal{O}_4	\mathcal{W}_i^3	$S(\mathcal{W}_i^3)$
γ_1	$\langle 0.95e^{i1.78\pi}, 0.05e^{i0.24\pi} \rangle$	$\langle 0.76e^{i1.69\pi}, 0.39e^{i0.29\pi} \rangle$	$\langle 0.58e^{i1.38\pi}, 0.23e^{i0.52\pi} \rangle$	$\langle 0.91e^{i1.93\pi}, 0.02e^{i0.05\pi} \rangle$	$\langle 0.99e^{i1.99\pi}, 8.97 \times 10^{-5}e^{i2.26 \times 10^{-4}\pi} \rangle$	1.96
γ_2	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	-2.00
γ_3	$\langle 0.48e^{i1.06\pi}, 0.17e^{i0.30\pi} \rangle$	$\langle 0.70e^{i1.71\pi}, 0.32e^{i0.21\pi} \rangle$	$\langle 0.52e^{i1.44\pi}, 0.32e^{i0.72\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.79e^{i1.86\pi}, 0.02e^{i0.01\pi} \rangle$	1.30
γ_4	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.69e^{i1.28\pi}, 0.45e^{i0.86\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.69e^{i1.28\pi}, 0.45e^{i0.86\pi} \rangle$	0.42
γ_5	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.00e^{i0.00\pi}, 1.00e^{i2.00\pi} \rangle$	$\langle 0.81e^{i1.53\pi}, 0.12e^{i0.34\pi} \rangle$	$\langle 0.81e^{i1.53\pi}, 0.12e^{i0.34\pi} \rangle$	0.97

6.1 Discussion

1. We present a comparative study with existing MADM techniques, namely, FFY_wA and FFY_wG operators

$$\mathcal{W}_i = \left\langle \sqrt[3]{\min(1, (\sum_{t=1}^n (\tau_t s_{gt}^{3\varsigma}))^{\frac{1}{\varsigma}})}, \sqrt[3]{1 - \min(1, (\sum_{t=1}^n (\tau_t (1 - k_{gt}^{3\varsigma}))^{\frac{1}{\varsigma}})} \right\rangle$$

For $\varsigma = 3$, the values are:

$$\mathcal{W}_1 = (0.87, 0.29)$$

$$\mathcal{W}_2 = (0.38, 0.64)$$

$$\mathcal{W}_3 = (0.64, 0.31)$$

$$\mathcal{W}_4 = (0.53, 0.51)$$

$$\mathcal{W}_5 = (0.71, 0.64)$$

- **Step 4.** The entries of aggregated values \mathcal{W}_i of the alternatives by using FFY_wG (Garg et al. 2020) operator defined as follows:

which manifest the proficiency and adeptness of proposed methods. The ranking of alternatives by applying the proposed and compared techniques are given in Table 41.

2. According to Table 41, Zong (y_1) is the best alternative obtained from the extant and proposed model which shows the validity and authenticity of proposed MADM methods.
3. Figure 1 skillfully depicts the comparison between the outcomes of proposed and existing decision-making methodologies by displaying an illustrated bar

$$\mathcal{W}_i = \left\langle \sqrt[3]{1 - \min(1, (\sum_{t=1}^n (\tau_t (1 - s_{gt}^3)^\phi))^{\frac{1}{\phi}})}, \sqrt[3]{\min(1, (\sum_{t=1}^n (\tau_t k_{gt}^{3\phi})^{\frac{1}{\phi}})} \right\rangle$$

For $\phi = 3$, the values are:

$$\mathcal{W}_1 = (0.79, 0.35)$$

$$\mathcal{W}_2 = (0.33, 0.80)$$

$$\mathcal{W}_3 = (0.57, 0.33)$$

$$\mathcal{W}_4 = (0.33, 0.69)$$

$$\mathcal{W}_5 = (0.50, 0.89)$$

- **Step 5.** The score of each executed value from FFY_wA and FFY_wG operators are assembled in Table 40.

chart among network companies and their order of ranking, demonstrating the consistency and competency of the presented MADM techniques.

Table 37 Comparison table of CFF6S_fS ($\mathcal{H}, \mathcal{Q}, 6$)

	y_1	y_2	y_3	y_4	y_5
y_1	4	4	4	4	4
y_2	0	4	0	2	2
y_3	0	4	4	4	3
y_4	0	2	0	4	2
y_5	0	2	1	2	4

Table 36 Tabular form of the membership and non-membership values of CFF6S_fS ($\mathcal{H}, \mathcal{Q}, 6$)

$(\mathcal{H}, \mathcal{Q}, 6)$	o_1	o_2	o_3	o_4
y_1	$(0.95e^{i1.78\pi}, 0.05e^{i0.24\pi})$	$(0.76e^{i1.69\pi}, 0.39e^{i0.29\pi})$	$(0.58e^{i1.38\pi}, 0.23e^{i0.52\pi})$	$(0.91e^{i1.93\pi}, 0.02e^{i0.05\pi})$
y_2	$(0.32e^{i0.54\pi}, 0.89e^{i1.74\pi})$	$(0.24e^{i0.64\pi}, 0.43e^{i1.22\pi})$	$(0.29e^{i0.54\pi}, 0.49e^{i0.98\pi})$	$(0.43e^{i0.66\pi}, 0.85e^{i1.46\pi})$
y_3	$(0.48e^{i1.06\pi}, 0.17e^{i0.30\pi})$	$(0.70e^{i1.71\pi}, 0.32e^{i0.21\pi})$	$(0.52e^{i1.44\pi}, 0.32e^{i0.72\pi})$	$(0.44e^{i0.44\pi}, 0.35e^{i1.08\pi})$
y_4	$(0.21e^{i0.04\pi}, 0.83e^{i1.94\pi})$	$(0.26e^{i0.56\pi}, 0.43e^{i1.21\pi})$	$(0.69e^{i1.28\pi}, 0.45e^{i0.86\pi})$	$(0.19e^{i0.38\pi}, 0.37e^{i1.06\pi})$
y_5	$(0.29e^{i0.84\pi}, 0.57e^{i1.16\pi})$	$(0.34e^{i0.84\pi}, 0.95e^{i1.71\pi})$	$(0.07e^{i0.24\pi}, 0.99e^{i1.88\pi})$	$(0.81e^{i1.53\pi}, 0.12e^{i0.34\pi})$

Table 38 Ranking Table

	Grade sum (G_s)	Row sum (R_s)	Column sum (C_s)	Final rank ($R_s - C_s$)
y_1	17	20	4	16
y_2	6	8	16	-8
y_3	12	15	9	6
y_4	7	8	16	-8
y_5	7	9	15	-6

Table 39 Fermatean fuzzy decision matrix

Y/O	o_1	o_2	o_3	o_4
y_1	(0.95, 0.05)	(0.76, 0.39)	(0.58, 0.23)	(0.91, 0.02)
y_2	(0.32, 0.89)	(0.24, 0.43)	(0.29, 0.49)	(0.43, 0.85)
y_3	(0.48, 0.17)	(0.70, 0.32)	(0.52, 0.32)	(0.44, 0.35)
y_4	(0.21, 0.83)	(0.26, 0.43)	(0.69, 0.45)	(0.19, 0.37)
y_5	(0.29, 0.57)	(0.34, 0.95)	(0.07, 0.99)	(0.81, 0.12)

4. Our proposed approaches provide the most comprehensive and adaptable methodologies because they effectively incorporate the existing proficient MADM methods, namely, FFY_wA operators and FFY_wG operators, by taking phase term equal to zero and neglecting the parameterized grading of alternatives. On the contrary, existing techniques cannot handle the two-dimensional parameterized fuzzy information. They are designed to deal with one-dimensional information only.

Table 40 Score values

Methods	$S(W_1)$	$S(W_2)$	$S(W_3)$	$S(W_4)$	$S(W_5)$	Best alternative
FFY _w A operator	0.63	-0.21	0.23	0.02	0.10	y_1
FFY _w G operator	0.45	-0.48	0.15	-0.29	-0.58	y_1

Table 41 Comparison analysis with existing methods

Methods	$S(W_1)$	$S(W_2)$	$S(W_3)$	$S(W_4)$	$S(W_5)$	Ranking order	Best alternative
CV of CFF6S _f S (proposed)	2.96	0.30	1.87	0.84	1.59	$y_1 > y_3 > y_5 > y_4 > y_2$	y_1
3-CV of CFF6S _f S (proposed)	1.96	-2.00	1.30	0.42	0.97	$y_1 > y_3 > y_5 > y_4 > y_2$	y_1
FFY _w A operator (Garg et al. 2020)	0.63	-0.21	0.23	0.02	0.10	$y_1 > y_3 > y_5 > y_4 > y_2$	y_1
FFY _w G operator (Garg et al. 2020)	0.45	-0.48	0.15	-0.29	-0.58	$y_1 > y_3 > y_4 > y_2 > y_5$	y_1

7 Development of CFFNS_f-TOPSIS method for MAGDM problems

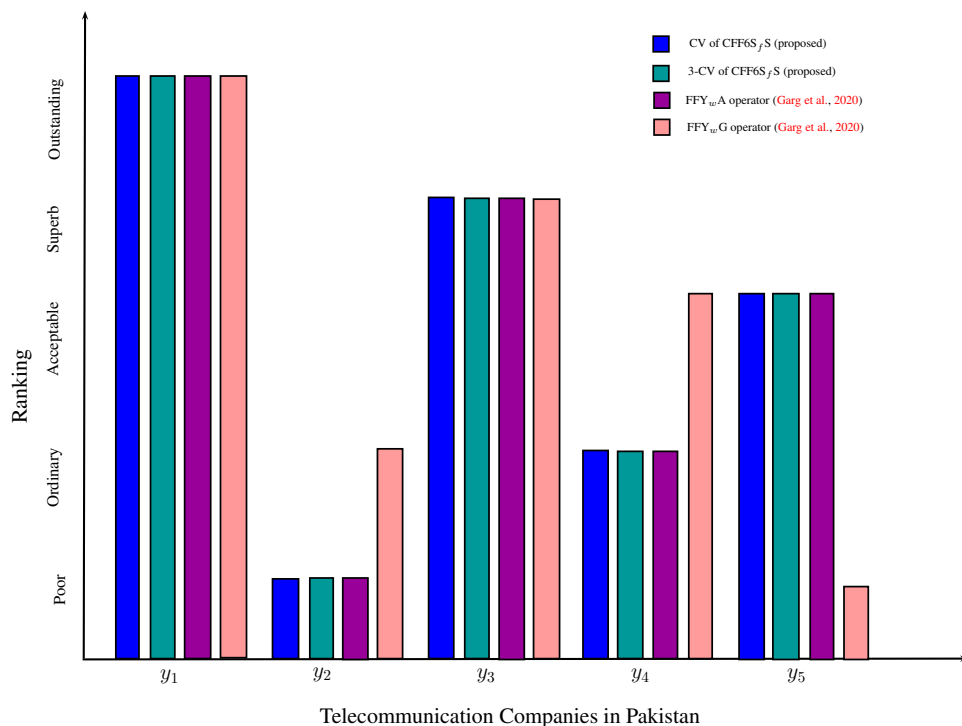
In this section, we aim to renovate the TOPSIS approach for the environment of CFFNS_f to account for MAGDM problems. The chief idea of the proposed CFFNS_f-TOPSIS technique is to find the most appropriate alternative having maximum distance from negative ideal solution (NIS) and shortest distance from positive ideal solution (PIS). Mathematical steps of MAGDM are as follows:

Let $\mathfrak{D} = \{\mathfrak{D}_1, \mathfrak{D}_2, \dots, \mathfrak{D}_l\}$ be the set of l distinct experts which are appointed for the selection of best alternative from $\mathfrak{I} = \{\mathfrak{I}_1, \mathfrak{I}_2, \mathfrak{I}_3, \dots, \mathfrak{I}_n\}$ relating to some specific attributes. $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \dots, \mathfrak{B}_m\}$ represents the collection of attributes which is selected by the experts according to the necessities of decision-making problem and $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_l)^T$ be the weight vector, represents the weightage of experts such that $\sum_{c=1}^l \zeta_c = 1$.

The step-by-step procedure of CFFNS_f-TOPSIS method is as follows:

- **Step 1.** Firstly, decision-makers will give grades corresponding to the linguistic terms according to the importance of alternatives on the basis of attributes. Then each expert \mathfrak{D}_c will assign CFFNS_fN,

Fig. 1 Comparison chart



corresponding to each grade in NS_fS ($\mathcal{H}^{(c)}, \mathcal{Q}, N$), according to the defined criteria for ranking. The CFFNS_fNs allocated by the decision-maker \mathfrak{D}_c are adapted in complex Fermatean fuzzy NS_f decision matrix (CFFNS_fDM) $\mathfrak{G}^{(c)} = (\mathfrak{G}_{gt}^{(c)})_{n \times m}$. Hence, CFFNS_fDMs $\mathfrak{G}^{(1)}, \mathfrak{G}^{(2)}, \dots, \mathfrak{G}^{(l)}$ are arranged by l decision-experts as follows:

$$\mathfrak{G}^{(c)} = \begin{pmatrix} \langle r_{a11}^{(c)}, (\mu_{a11}^{(c)}, v_{a11}^{(c)}) \rangle & \langle r_{a12}^{(c)}, (\mu_{a12}^{(c)}, v_{a12}^{(c)}) \rangle & \dots & \langle r_{a1m}^{(c)}, (\mu_{a1m}^{(c)}, v_{a1m}^{(c)}) \rangle \\ \langle r_{a21}^{(c)}, (\mu_{a21}^{(c)}, v_{a21}^{(c)}) \rangle & \langle r_{a22}^{(c)}, (\mu_{a22}^{(c)}, v_{a22}^{(c)}) \rangle & \dots & \langle r_{a2m}^{(c)}, (\mu_{a2m}^{(c)}, v_{a2m}^{(c)}) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle r_{an1}^{(c)}, (\mu_{an1}^{(c)}, v_{an1}^{(c)}) \rangle & \langle r_{an2}^{(c)}, (\mu_{an2}^{(c)}, v_{an2}^{(c)}) \rangle & \dots & \langle r_{anm}^{(c)}, (\mu_{anm}^{(c)}, v_{anm}^{(c)}) \rangle \end{pmatrix} \quad (1)$$

- **Step 2.** For MAGDM, the individual opinions of the decision-makers are organized to have a generic opinion of all experts about an alternative related to the attributes. This directs to the formation of aggregated CFFNS_fDM (ACFFNS_fDM) $\mathfrak{G} = (\mathfrak{G})_{n \times m}$. The CFFNS_fDM of all experts are assembled with the help of CFFNS_f weighted average (CFFNS_fWA) operator as follows:

$$\begin{aligned} \mathfrak{G}_{gt} &= \text{CFFNS}_f \text{WA}_{\zeta}(\mathfrak{G}_{gt}^{(1)}, \mathfrak{G}_{gt}^{(2)}, \dots, \mathfrak{G}_{gt}^{(l)}) \\ &= \zeta_1 \mathfrak{G}_{gt}^{(1)} \oplus \zeta_2 \mathfrak{G}_{gt}^{(2)} \oplus \dots \oplus \zeta_l \mathfrak{G}_{gt}^{(l)} \\ &= \left\langle \max_{c=1}^l r_{a_{gt}}^{(c)}, \left(\sqrt[3]{1 - \prod_{c=1}^l (1 - (s_{a_{gt}}^{(c)})^3)^{\zeta_c}} e^{i2\pi \sqrt[3]{1 - \prod_{c=1}^l (1 - (\frac{\omega_{a_{gt}}^{(c)}}{2\pi})^3)^{\zeta_c}}} \right), \prod_{c=1}^l (k_{a_{gt}}^{(c)})^{\zeta_c} e^{i2\pi \prod_{c=1}^l (\frac{\psi_{a_{gt}}^{(c)}}{2\pi})^{\zeta_c}} \right\rangle, \end{aligned} \quad (2)$$

where $c = \{1, 2, \dots, l\}$.

Each entry of the CFFNS_fDM has the form $\mathfrak{G}_{gt}^{(c)} = \langle r_{a_{gt}}^{(c)}, (\mu_{a_{gt}}^{(c)}, v_{a_{gt}}^{(c)}) \rangle = \langle r_{a_{gt}}^{(c)}, (s_{a_{gt}}^{(c)} e^{i\omega_{a_{gt}}^{(c)}}, k_{a_{gt}}^{(c)} e^{i\psi_{a_{gt}}^{(c)}}) \rangle$.

where $\mathfrak{G}_{gt} = \langle r_{a_{gt}}, (\mu_{a_{gt}}, v_{a_{gt}}) \rangle = \langle r_{a_{gt}}, (s_{a_{gt}} e^{i\omega_{a_{gt}}}, k_{a_{gt}} e^{i\psi_{a_{gt}}}) \rangle, g = 1, 2, 3, \dots, n$, and $t = 1, 2, 3, \dots, m$. The ACFFNS_fDM can be form as follows:

$\mathfrak{G} =$

$$\begin{pmatrix} \langle r_{a_{11}}, (\mu_{a_{11}}, v_{a_{11}}) \rangle & \langle r_{a_{12}}, (\mu_{a_{12}}, v_{a_{12}}) \rangle & \cdots & \langle r_{a_{1m}}, (\mu_{a_{1m}}, v_{a_{1m}}) \rangle \\ \langle r_{a_{21}}, (\mu_{a_{21}}, v_{a_{21}}) \rangle & \langle r_{a_{22}}, (\mu_{a_{22}}, v_{a_{22}}) \rangle & \cdots & \langle r_{a_{2m}}, (\mu_{a_{2m}}, v_{a_{2m}}) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle r_{a_{n1}}, (\mu_{a_{n1}}, v_{a_{n1}}) \rangle & \langle r_{a_{n2}}, (\mu_{a_{n2}}, v_{a_{n2}}) \rangle & \cdots & \langle r_{a_{nm}}, (\mu_{a_{nm}}, v_{a_{nm}}) \rangle \end{pmatrix}$$

- **Step 3.** The attributes nominated by experts may not be equally important and valuable in a MAGDM problem. Therefore, each decision-maker ranks these attributes and assigns a CFFNS_f weightage according to the grading criteria defined by the experts. Let $\kappa_t^{(c)} = \langle r_{a_t}^{(c)}, (\mu_{a_t}^{(c)}, v_{a_t}^{(c)}) \rangle = \langle r_{a_t}^{(c)}, (s_{a_t}^{(c)} e^{i\omega_{a_t}^{(c)}}, k_{a_t}^{(c)} e^{i\psi_{a_t}^{(c)}}) \rangle$ be the CFFNS_f weight assigned by decision-maker \mathfrak{D}_c to the attribute \mathfrak{B}_t . To compute the CFFNS_f weight vector $\kappa = (\kappa_1, \kappa_2, \kappa_3, \dots, \kappa_m)^T$ of attributes, the CFFNS_fNs corresponding to the grade values, assigned by decision-makers are aggregated as follows:

$$\begin{aligned} \kappa_t &= \text{CFFNS}_f \text{WA}_\zeta(\kappa_t^{(1)}, \kappa_t^{(2)}, \dots, \kappa_t^{(l)}) \\ &= \zeta_1 \kappa_t^{(1)} \oplus \zeta_2 \kappa_t^{(2)} \oplus \dots \oplus \zeta_l \kappa_t^{(l)} \\ &= \left\langle \max_{c=1}^l r_{a_t}^{(c)}, \left(\sqrt[3]{1 - \prod_{c=1}^l (1 - (s_{a_t}^{(c)})^3)^{\zeta_c}} e^{i2\pi \sqrt[3]{1 - \prod_{c=1}^l (1 - (\frac{\omega_{a_t}^{(c)}}{2\pi})^3)^{\zeta_c}}} \right), \prod_{c=1}^l (k_{a_t}^{(c)})^{\zeta_c} e^{i2\pi \prod_{c=1}^l (\frac{\psi_{a_t}^{(c)}}{2\pi})^{\zeta_c}} \right\rangle, \\ &= \langle r_{a_t}, (\mu_{a_t}, v_{a_t}) \rangle, \\ &= \langle r_{a_t}, (s_{a_t} e^{i\omega_{a_t}}, k_{a_t} e^{i\psi_{a_t}}) \rangle, \end{aligned} \quad (3)$$

where $t = 1, 2, 3, \dots, m$.

- **Step 4.** Construct the aggregated weighted CFFNS_fDM (AWCFFNS_fDM) $\widehat{\mathfrak{G}} = (\widehat{\mathfrak{G}}_{gt})_{n \times m}$ by using CFFNS_fDM \mathfrak{G} and the weight vector κ_t of attributes, as follows:

$$\begin{aligned} \widehat{\mathfrak{G}}_{gt} &= \mathfrak{G}_{gt} \otimes \kappa_t \\ &= \left\langle \min(r_{a_{gt}}, r_{a_t}), \left(s_{a_{gt}} s_{a_t} e^{i2\pi (\frac{\omega_{a_{gt}}}{2\pi}) (\frac{\omega_{a_t}}{2\pi})}, \sqrt[3]{(k_{a_{gt}})^3 + (k_{a_t})^3 - (k_{a_{gt}})^3 (k_{a_t})^3} \right. \right. \\ &\quad \left. \left. e^{i2\pi \sqrt[3]{(\frac{\psi_{a_{gt}}}{2\pi})^3 + (\frac{\psi_{a_t}}{2\pi})^3 - (\frac{\psi_{a_{gt}}}{2\pi})^3 (\frac{\psi_{a_t}}{2\pi})^3}} \right) \right\rangle, \\ &= \langle \hat{r}_{a_{gt}}, (\hat{\mu}_{a_{gt}}, \hat{v}_{a_{gt}}) \rangle, \\ &= \langle \hat{r}_{a_{gt}}, (\hat{s}_{a_{gt}} e^{i\hat{\omega}_{a_{gt}}}, \hat{k}_{a_{gt}} e^{i\hat{\psi}_{a_{gt}}}) \rangle. \end{aligned} \quad (4)$$

The AWCFFNS_fDM is constructed as follows:

$$\widehat{\mathfrak{G}} = \begin{pmatrix} \langle \hat{r}_{a_{11}}, (\hat{\mu}_{a_{11}}, \hat{v}_{a_{11}}) \rangle & \langle \hat{r}_{a_{12}}, (\hat{\mu}_{a_{12}}, \hat{v}_{a_{12}}) \rangle & \cdots & \langle \hat{r}_{a_{1m}}, (\hat{\mu}_{a_{1m}}, \hat{v}_{a_{1m}}) \rangle \\ \langle \hat{r}_{a_{21}}, (\hat{\mu}_{a_{21}}, \hat{v}_{a_{21}}) \rangle & \langle \hat{r}_{a_{22}}, (\hat{\mu}_{a_{22}}, \hat{v}_{a_{22}}) \rangle & \cdots & \langle \hat{r}_{a_{2m}}, (\hat{\mu}_{a_{2m}}, \hat{v}_{a_{2m}}) \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle \hat{r}_{a_{n1}}, (\hat{\mu}_{a_{n1}}, \hat{v}_{a_{n1}}) \rangle & \langle \hat{r}_{a_{n2}}, (\hat{\mu}_{a_{n2}}, \hat{v}_{a_{n2}}) \rangle & \cdots & \langle \hat{r}_{a_{nm}}, (\hat{\mu}_{a_{nm}}, \hat{v}_{a_{nm}}) \rangle \end{pmatrix}$$

- **Step 5.** Let \mathfrak{B}^- and \mathfrak{B}^+ represent the collection of cost-type and benefit-type attributes, respectively. Then CFFNS_f positive ideal solution (CFFNS_f-PIS) $\widetilde{\mathfrak{G}}_t = \langle \tilde{r}_{a_t}, (\tilde{\mu}_{a_t}, \tilde{v}_{a_t}) \rangle$ related to attribute \mathfrak{B}_t can be chosen as follows:

$$\widetilde{\mathfrak{G}}_t = \begin{cases} \max_{1 \leq g \leq n} \widehat{\mathfrak{G}}_{gt}, & \text{if } \mathfrak{B}_t \in \mathfrak{B}^+, \\ \min_{1 \leq g \leq n} \widehat{\mathfrak{G}}_{gt}, & \text{if } \mathfrak{B}_t \in \mathfrak{B}^-. \end{cases} \quad (5)$$

The CFFNS_f negative ideal solution (CFFNS_f-NIS) $\check{\mathfrak{G}}_t = \langle \check{r}_{a_t}, (\check{\mu}_{a_t}, \check{v}_{a_t}) \rangle$ with respect to the attribute \mathfrak{B}_t can be determined as follows:

$$\check{\mathfrak{G}}_t = \begin{cases} \min_{1 \leq g \leq n} \hat{\mathfrak{G}}_{gt}, & \text{if } \mathfrak{B}_t \in \mathfrak{B}^+, \\ \max_{1 \leq g \leq n} \hat{\mathfrak{G}}_{gt}, & \text{if } \mathfrak{B}_t \in \mathfrak{B}^-. \end{cases} \quad (6)$$

The CFFNS_fNs are compared on the basis of accuracy function and score function to obtain CFFNS_f-PIS and CFFNS_f-NIS.

- **Step 6.** Now, to find the optimal alternative which is away from CFFNS_f-NIS and closest to CFFNS_f-PIS, we evaluate the distance of each alternative \mathfrak{Z}_g from CFFNS_f-PIS and CFFNS_f-NIS. The distance between any of the alternative and CFFNS_f-PIS can be calculated as follows:

$$d(\mathfrak{Z}_g, \check{\mathfrak{G}}_t) = \sum_{i=1}^m \sqrt{\frac{1}{3} \left\{ \left(\frac{\hat{r}_{a_{gt}}}{N-1} - \frac{\check{r}_{a_t}}{N-1} \right)^2 + (\hat{s}_{a_{gt}}^3 - \check{s}_{a_t}^3)^2 + (\hat{k}_{a_{gt}}^3 - \check{k}_{a_t}^3)^2 + \frac{1}{64\pi^6} \sqrt{\left((\hat{\omega}_{a_{gt}}^3 - \check{\omega}_{a_t}^3)^2 + (\hat{\psi}_{a_{gt}}^3 - \check{\psi}_{a_t}^3)^2 \right)} \right\}} \quad (7)$$

Similarly, the distance between any of the alternative and CFFNS_f-NIS can be calculated as follows:

$$d(\mathfrak{Z}_g, \check{\mathfrak{G}}_t) = \sum_{i=1}^m \sqrt{\frac{1}{3} \left\{ \left(\frac{\hat{r}_{a_{gt}}}{N-1} - \frac{\check{r}_{a_t}}{N-1} \right)^2 + (\hat{s}_{a_{gt}}^3 - \check{s}_{a_t}^3)^2 + (\hat{k}_{a_{gt}}^3 - \check{k}_{a_t}^3)^2 + \frac{1}{64\pi^6} \sqrt{\left((\hat{\omega}_{a_{gt}}^3 - \check{\omega}_{a_t}^3)^2 + (\hat{\psi}_{a_{gt}}^3 - \check{\psi}_{a_t}^3)^2 \right)} \right\}} \quad (8)$$

- **Step 7.** To find the most suitable alternative, we compare the alternative by some ranking index. The revised closeness index (Vencheh and Mirjaberi 2014) corresponding to the alternative \mathfrak{Z}_g can be evaluated by utilizing the formula:

$$\psi(\mathfrak{Z}_g) = \frac{d(\mathfrak{Z}_g, \check{\mathfrak{G}}_t)}{d_{\max}(\mathfrak{Z}_g, \check{\mathfrak{G}}_t)} - \frac{d(\mathfrak{Z}_g, \check{\mathfrak{G}}_t)}{d_{\min}(\mathfrak{Z}_g, \check{\mathfrak{G}}_t)}, \quad (9)$$

where $g = 1, 2, 3, \dots, n$, and

$$d_{\max}(\mathfrak{Z}_g, \check{\mathfrak{G}}_t) = \max_{1 \leq g \leq n} (\mathfrak{Z}_g, \check{\mathfrak{G}}_t),$$

$$d_{\min}(\mathfrak{Z}_g, \check{\mathfrak{G}}_t) = \min_{1 \leq g \leq n} (\mathfrak{Z}_g, \check{\mathfrak{G}}_t).$$

- **Step 8.** After the evaluated results of closeness index, the alternatives are arranged in an ascending order with respect to revised closeness index. The alternative having maximum value of closeness index will be the optimal solution of MAGDM problem.

The general steps of CFFNS_f-TOPSIS method are summarized in Figure 2.

8 Selection of the most suitable city in the USA for farming

Suppose that an investor X wants to purchase land for agriculture farming in a favorable city of the USA. For this purpose, he arranged a panel of four decision-makers $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$ and \mathfrak{D}_4 to thoroughly judge the essential needs of the best land for farming. Since, each decision-maker has his own importance and opinions, so, $\zeta = (0.240.350.230.18)^T$ represents the weightage of experts in the decision-making panel. Clearly, $\sum_{c=1}^4 \zeta_c = 1$. The following cities are under consideration as alternatives for this problem:

- \mathfrak{Z}_1 : Boston, Massachusetts
- \mathfrak{Z}_2 : Portland, Oregon
- \mathfrak{Z}_3 : New York city, New York
- \mathfrak{Z}_4 : Minneapolis, Minnesota

After discussion, all decision-makers identify the following factors as the attributes for this MAGDM problem:

- \mathfrak{B}_1 : Initial cost
- \mathfrak{B}_2 : Environmental destruction
- \mathfrak{B}_3 : Topography
- \mathfrak{B}_4 : Climate of the Area
- \mathfrak{B}_5 : Maintenance cost
- \mathfrak{B}_6 : Soil
- \mathfrak{B}_7 : Water quality & availability

The stepwise solution of this MAGDM problem by following CFFNS_f-TOPSIS method is given as follows:

- **Step 1.** According to the above-mentioned attributes, each expert assesses the alternatives regarding all attributes using 5-soft, given in Table 42, where:

- ♡♡♡♡ represent ‘Outstanding’,
- ♡♡♡ represent ‘Good’,
- ♡♡ represent ‘Average’,
- ♡ represents ‘Satisfactory’,
- ◇ represents ‘Below average’.

The experts $\mathfrak{D}_1, \mathfrak{D}_2, \mathfrak{D}_3$ and \mathfrak{D}_4 will use the grading criteria given by Table 5, to assign the CFFNS_fN corresponding to each rank. The individual CFFNS_fDMs of the experts are arranged by Tables 43–46.

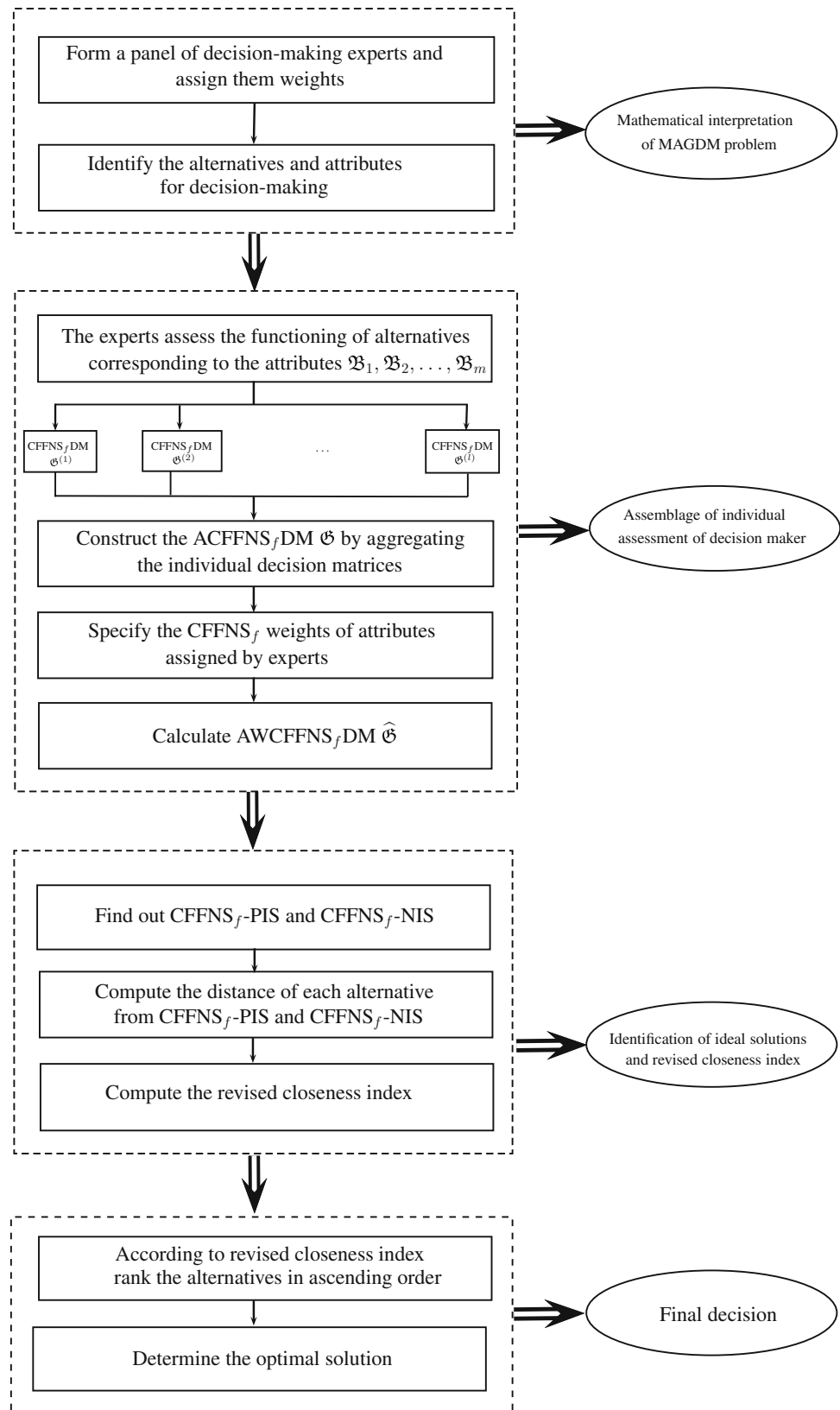
Fig. 2 Flow chart of CFFNS_{*f*}-TOPSIS method

Table 42 Expert's assessment of alternatives corresponding to each attribute

Attributes	Alternatives	\mathfrak{D}_1	\mathfrak{D}_2	\mathfrak{D}_3	\mathfrak{D}_4
\mathfrak{B}_1	\mathfrak{I}_1	$\heartsuit = 1$	$\heartsuit = 1$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$
	\mathfrak{I}_3	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
	\mathfrak{I}_4	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
\mathfrak{B}_2	\mathfrak{I}_1	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit = 1$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_3	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$
	\mathfrak{I}_4	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$
\mathfrak{B}_3	\mathfrak{I}_1	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_3	$\heartsuit = 1$	$\heartsuit = 1$	$\diamond = 0$	$\diamond = 0$
	\mathfrak{I}_4	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
\mathfrak{B}_4	\mathfrak{I}_1	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit = 1$	$\heartsuit = 1$
	\mathfrak{I}_2	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
	\mathfrak{I}_3	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit = 3$
	\mathfrak{I}_4	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
\mathfrak{B}_5	\mathfrak{I}_1	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$
	\mathfrak{I}_3	$\diamond = 0$	$\diamond = 0$	$\diamond = 0$	$\diamond = 0$
	\mathfrak{I}_4	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit = 2$
\mathfrak{B}_6	\mathfrak{I}_1	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit = 3$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit = 1$	$\heartsuit\heartsuit = 2$	$\heartsuit = 1$
	\mathfrak{I}_3	$\heartsuit\heartsuit = 2$	$\heartsuit = 1$	$\heartsuit = 1$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_4	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$
\mathfrak{B}_7	\mathfrak{I}_1	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$
	\mathfrak{I}_2	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit = 2$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit = 2$
	\mathfrak{I}_3	$\diamond = 0$	$\diamond = 0$	$\heartsuit = 1$	$\heartsuit = 1$
	\mathfrak{I}_4	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit = 3$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$	$\heartsuit\heartsuit\heartsuit\heartsuit = 4$

Table 43 Tabulated form of CFFNS_JDM $\mathfrak{G}^{(1)}$ of expert \mathfrak{D}_1

$(\mathcal{H}^{(1)}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3
\mathfrak{I}_1	$\langle 1, (0.38e^{i0.78\pi}, 0.73e^{i1.37\pi}) \rangle$	$\langle 2, (0.42e^{i1.23\pi}, 0.51e^{i0.96\pi}) \rangle$	$\langle 3, (0.68e^{i1.61\pi}, 0.31e^{i0.67\pi}) \rangle$
\mathfrak{I}_2	$\langle 2, (0.41e^{i0.98\pi}, 0.52e^{i1.12\pi}) \rangle$	$\langle 2, (0.56e^{i0.87\pi}, 0.45e^{i1.11\pi}) \rangle$	$\langle 2, (0.64e^{i1.21\pi}, 0.56e^{i0.92\pi}) \rangle$
\mathfrak{I}_3	$\langle 3, (0.81e^{i1.51\pi}, 0.38e^{i0.68\pi}) \rangle$	$\langle 2, (0.44e^{i1.25\pi}, 0.53e^{i0.94\pi}) \rangle$	$\langle 1, (0.33e^{i0.55\pi}, 0.81e^{i1.35\pi}) \rangle$
\mathfrak{I}_4	$\langle 2, (0.53e^{i1.01\pi}, 0.44e^{i0.82\pi}) \rangle$	$\langle 2, (0.52e^{i0.93\pi}, 0.41e^{i1.12\pi}) \rangle$	$\langle 2, (0.44e^{i1.19\pi}, 0.61e^{i0.90\pi}) \rangle$
	\mathfrak{B}_4	\mathfrak{B}_5	\mathfrak{B}_6
\mathfrak{I}_1	$\langle 2, (0.45e^{i1.14\pi}, 0.59e^{i1.23\pi}) \rangle$	$\langle 3, (0.74e^{i1.36\pi}, 0.26e^{i0.61\pi}) \rangle$	$\langle 3, (0.80e^{i1.34\pi}, 0.33e^{i0.59\pi}) \rangle$
\mathfrak{I}_2	$\langle 3, (0.67e^{i1.51\pi}, 0.35e^{i0.42\pi}) \rangle$	$\langle 2, (0.59e^{i1.12\pi}, 0.54e^{i1.21\pi}) \rangle$	$\langle 2, (0.61e^{i1.13\pi}, 0.48e^{i0.87\pi}) \rangle$
\mathfrak{I}_3	$\langle 3, (0.78e^{i1.37\pi}, 0.24e^{i0.54\pi}) \rangle$	$\langle 0, (0.07e^{i0.18\pi}, 0.92e^{i1.77\pi}) \rangle$	$\langle 2, (0.50e^{i0.95\pi}, 0.54e^{i1.25\pi}) \rangle$
\mathfrak{I}_4	$\langle 3, (0.68e^{i1.49\pi}, 0.32e^{i0.47\pi}) \rangle$	$\langle 3, (0.77e^{i1.49\pi}, 0.34e^{i0.52\pi}) \rangle$	$\langle 3, (0.71e^{i1.43\pi}, 0.24e^{i0.76\pi}) \rangle$
	\mathfrak{B}_7		
\mathfrak{I}_1	$\langle 4, (0.93e^{i1.93\pi}, 0.07e^{i0.32\pi}) \rangle$		
\mathfrak{I}_2	$\langle 2, (0.47e^{i1.23\pi}, 0.62e^{i0.98\pi}) \rangle$		
\mathfrak{I}_3	$\langle 0, (0.16e^{i0.32\pi}, 0.86e^{i1.91\pi}) \rangle$		
\mathfrak{I}_4	$\langle 3, (0.69e^{i1.33\pi}, 0.31e^{i0.58\pi}) \rangle$		

Table 44 Tabulated form of CFFNS _{f} DM $\mathfrak{G}^{(2)}$ of expert \mathfrak{D}_2

$(\mathcal{H}^{(2)}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3
\mathfrak{J}_1	$\langle 1, (0.29e^{i0.45\pi}, 0.69e^{i1.67\pi}) \rangle$	$\langle 2, (0.54e^{i0.88\pi}, 0.43e^{i1.24\pi}) \rangle$	$\langle 3, (0.78e^{i1.54\pi}, 0.34e^{i0.57\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.69e^{i1.62\pi}, 0.33e^{i0.57\pi}) \rangle$	$\langle 1, (0.32e^{i0.51\pi}, 0.82e^{i1.59\pi}) \rangle$	$\langle 2, (0.63e^{i1.17\pi}, 0.57e^{i0.88\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.97e^{i1.93\pi}, 0.03e^{i0.07\pi}) \rangle$	$\langle 2, (0.58e^{i1.25\pi}, 0.47e^{i0.98\pi}) \rangle$	$\langle 1, (0.23e^{i0.61\pi}, 0.72e^{i1.47\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.51e^{i1.11\pi}, 0.42e^{i1.21\pi}) \rangle$	$\langle 2, (0.46e^{i0.96\pi}, 0.55e^{i1.23\pi}) \rangle$	$\langle 2, (0.43e^{i1.15\pi}, 0.62e^{i0.86\pi}) \rangle$
\mathfrak{B}_4	\mathfrak{B}_5	\mathfrak{B}_6	
\mathfrak{J}_1	$\langle 2, (0.60e^{i1.16\pi}, 0.55e^{i1.25\pi}) \rangle$	$\langle 4, (0.97e^{i1.89\pi}, 0.03e^{i0.09\pi}) \rangle$	$\langle 3, (0.82e^{i1.39\pi}, 0.26e^{i0.73\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.79e^{i1.39\pi}, 0.38e^{i0.45\pi}) \rangle$	$\langle 3, (0.76e^{i1.37\pi}, 0.31e^{i0.72\pi}) \rangle$	$\langle 1, (0.37e^{i0.73\pi}, 0.79e^{i1.39\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.86e^{i1.85\pi}, 0.03e^{i0.17\pi}) \rangle$	$\langle 0, (0.13e^{i0.29\pi}, 0.88e^{i1.85\pi}) \rangle$	$\langle 1, (0.25e^{i0.67\pi}, 0.71e^{i1.47\pi}) \rangle$
\mathfrak{J}_4	$\langle 3, (0.69e^{i1.48\pi}, 0.22e^{i0.56\pi}) \rangle$	$\langle 3, (0.75e^{i1.47\pi}, 0.29e^{i0.48\pi}) \rangle$	$\langle 2, (0.49e^{i0.85\pi}, 0.55e^{i0.96\pi}) \rangle$
\mathfrak{B}_7			
\mathfrak{J}_1	$\langle 4, (0.92e^{i1.95\pi}, 0.09e^{i0.31\pi}) \rangle$		
\mathfrak{J}_2	$\langle 2, (0.63e^{i0.83\pi}, 0.55e^{i0.94\pi}) \rangle$		
\mathfrak{J}_3	$\langle 0, (0.02e^{i0.11\pi}, 0.91e^{i1.88\pi}) \rangle$		
\mathfrak{J}_4	$\langle 3, (0.78e^{i1.41\pi}, 0.31e^{i0.47\pi}) \rangle$		

Table 45 Tabulated form of CFFNS _{f} DM $\mathfrak{G}^{(3)}$ of expert \mathfrak{D}_3

$(\mathcal{H}^{(3)}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3
\mathfrak{J}_1	$\langle 2, (0.43e^{i1.07\pi}, 0.54e^{i0.83\pi}) \rangle$	$\langle 2, (0.60e^{i1.27\pi}, 0.49e^{i1.04\pi}) \rangle$	$\langle 3, (0.81e^{i1.38\pi}, 0.38e^{i0.61\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.75e^{i1.48\pi}, 0.24e^{i0.49\pi}) \rangle$	$\langle 2, (0.48e^{i0.94\pi}, 0.57e^{i1.09\pi}) \rangle$	$\langle 2, (0.62e^{i1.13\pi}, 0.58e^{i0.84\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.89e^{i1.85\pi}, 0.15e^{i0.28\pi}) \rangle$	$\langle 2, (0.50e^{i1.29\pi}, 0.59e^{i1.06\pi}) \rangle$	$\langle 0, (0.02e^{i0.04\pi}, 0.87e^{i1.75\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.55e^{i1.13\pi}, 0.46e^{i0.89\pi}) \rangle$	$\langle 2, (0.62e^{i0.84\pi}, 0.61e^{i1.24\pi}) \rangle$	$\langle 2, (0.42e^{i1.11\pi}, 0.63e^{i0.82\pi}) \rangle$
\mathfrak{B}_4	\mathfrak{B}_5	\mathfrak{B}_6	
\mathfrak{J}_1	$\langle 1, (0.24e^{i0.42\pi}, 0.67e^{i1.39\pi}) \rangle$	$\langle 4, (0.93e^{i1.77\pi}, 0.04e^{i0.35\pi}) \rangle$	$\langle 4, (0.88e^{i1.83\pi}, 0.02e^{i0.31\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.80e^{i1.37\pi}, 0.28e^{i0.48\pi}) \rangle$	$\langle 2, (0.46e^{i1.15\pi}, 0.58e^{i0.82\pi}) \rangle$	$\langle 2, (0.53e^{i0.82\pi}, 0.47e^{i1.11\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.92e^{i1.76\pi}, 0.15e^{i0.28\pi}) \rangle$	$\langle 0, (0.09e^{i0.12\pi}, 0.89e^{i1.86\pi}) \rangle$	$\langle 1, (0.34e^{i0.77\pi}, 0.84e^{i1.58\pi}) \rangle$
\mathfrak{J}_4	$\langle 3, (0.70e^{i1.45\pi}, 0.37e^{i0.49\pi}) \rangle$	$\langle 3, (0.84e^{i1.45\pi}, 0.25e^{i0.63\pi}) \rangle$	$\langle 2, (0.48e^{i0.97\pi}, 0.56e^{i1.22\pi}) \rangle$
\mathfrak{B}_7			
\mathfrak{J}_1	$\langle 4, (0.86e^{i1.91\pi}, 0.15e^{i0.29\pi}) \rangle$		
\mathfrak{J}_2	$\langle 3, (0.67e^{i1.32\pi}, 0.23e^{i0.59\pi}) \rangle$		
\mathfrak{J}_3	$\langle 1, (0.27e^{i0.48\pi}, 0.78e^{i1.51\pi}) \rangle$		
\mathfrak{J}_4	$\langle 4, (0.88e^{i1.89\pi}, 0.13e^{i0.27\pi}) \rangle$		

- **Step 2.** The individual opinions of all decision-makers are assembled by employing the CFFNS _{f} WA operator, as defined in Equation 2 and the results are accumulated in the ACFFNS _{f} DM as shown in Table 47.
- **Step 3.** Experts associate CFFNS _{f} N to each attribute indicating the importance of that attribute in MAGDM problems which are summarized in Table 48. The CFFNS _{f} weight of each attribute is accumulated by CFFNS _{f} WA operator defined in Equation 3 to form a CFFNS _{f} weight vector κ , given by:

$$\kappa = \begin{pmatrix} 3, (0.731811e^{i1.527149\pi}, 0.304928e^{i0.582725\pi}) \\ 4, (0.945072e^{i1.861182\pi}, 0.054923e^{i0.236347\pi}) \\ 4, (0.860568e^{i1.651356\pi}, 0.189314e^{i0.416769\pi}) \\ 3, (0.545443e^{i1.019597\pi}, 0.613887e^{i1.185280\pi}) \\ 4, (0.826558e^{i1.645141\pi}, 0.185730e^{i0.402961\pi}) \\ 3, (0.739877e^{i1.434762\pi}, 0.312240e^{i0.694595\pi}) \\ 3, (0.670899e^{i1.289629\pi}, 0.423652e^{i0.802981\pi}) \end{pmatrix} \quad (10)$$

Table 46 Tabulated form of CFFNS_fDM $\mathfrak{G}^{(4)}$ of expert \mathfrak{D}_4

$(\mathcal{H}^{(4)}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3
\mathfrak{J}_1	$\langle 2, (0.45e^{i1.14\pi}, 0.56e^{i0.91\pi}) \rangle$	$\langle 2, (0.49e^{i0.92\pi}, 0.50e^{i1.21\pi}) \rangle$	$\langle 4, (0.94e^{i1.92\pi}, 0.16e^{i0.31\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.79e^{i1.65\pi}, 0.29e^{i0.43\pi}) \rangle$	$\langle 2, (0.47e^{i0.82\pi}, 0.58e^{i1.08\pi}) \rangle$	$\langle 2, (0.61e^{i1.20\pi}, 0.59e^{i1.29\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.88e^{i1.78\pi}, 0.11e^{i0.36\pi}) \rangle$	$\langle 3, (0.72e^{i1.58\pi}, 0.31e^{i0.53\pi}) \rangle$	$\langle 0, (0.08e^{i0.22\pi}, 0.87e^{i1.79\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.57e^{i1.25\pi}, 0.48e^{i0.83\pi}) \rangle$	$\langle 3, (0.67e^{i1.67\pi}, 0.23e^{i0.69\pi}) \rangle$	$\langle 2, (0.41e^{i1.18\pi}, 0.64e^{i1.27\pi}) \rangle$
	\mathfrak{B}_4	\mathfrak{B}_5	\mathfrak{B}_6
\mathfrak{J}_1	$\langle 1, (0.22e^{i0.47\pi}, 0.69e^{i1.43\pi}) \rangle$	$\langle 4, (0.89e^{i1.72\pi}, 0.17e^{i0.32\pi}) \rangle$	$\langle 3, (0.73e^{i1.35\pi}, 0.31e^{i0.59\pi}) \rangle$
\mathfrak{J}_2	$\langle 4, (0.91e^{i1.89\pi}, 0.05e^{i0.08\pi}) \rangle$	$\langle 3, (0.76e^{i1.46\pi}, 0.27e^{i0.43\pi}) \rangle$	$\langle 1, (0.31e^{i0.63\pi}, 0.77e^{i1.61\pi}) \rangle$
\mathfrak{J}_3	$\langle 3, (0.81e^{i1.34\pi}, 0.21e^{i0.57\pi}) \rangle$	$\langle 0, (0.12e^{i0.25\pi}, 0.91e^{i1.88\pi}) \rangle$	$\langle 2, (0.52e^{i0.87\pi}, 0.48e^{i0.98\pi}) \rangle$
\mathfrak{J}_4	$\langle 4, (0.87e^{i1.72\pi}, 0.13e^{i0.12\pi}) \rangle$	$\langle 2, (0.51e^{i0.89\pi}, 0.49e^{i1.02\pi}) \rangle$	$\langle 2, (0.47e^{i0.98\pi}, 0.57e^{i1.12\pi}) \rangle$
	\mathfrak{B}_7		
\mathfrak{J}_1	$\langle 4, (0.91e^{i1.81\pi}, 0.01e^{i0.24\pi}) \rangle$		
\mathfrak{J}_2	$\langle 2, (0.44e^{i0.81\pi}, 0.59e^{i0.92\pi}) \rangle$		
\mathfrak{J}_3	$\langle 1, (0.21e^{i0.59\pi}, 0.69e^{i1.34\pi}) \rangle$		
\mathfrak{J}_4	$\langle 4, (0.87e^{i1.72\pi}, 0.12e^{i0.21\pi}) \rangle$		

Table 47 Tabulated form of ACFNS_fDM \mathfrak{G}

$(\mathcal{H}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2
\mathfrak{J}_1	$\langle 2, (0.384320e^{i0.89467\pi}, 0.636677e^{i1.215568\pi}) \rangle$	$\langle 2, (0.525656e^{i1.098516\pi}, 0.474341e^{i1.114966\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.693071e^{i1.510336\pi}, 0.334196e^{i0.615372\pi}) \rangle$	$\langle 2, (0.463232e^{i0.793682\pi}, 0.613584e^{i1.247357\pi}) \rangle$
\mathfrak{J}_3	$\langle 4, (0.921191e^{i1.84144\pi}, 0.100947e^{i0.223135\pi}) \rangle$	$\langle 3, (0.57605e^{i1.341568\pi}, 0.472935e^{i0.884443\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.535991e^{i1.122995\pi}, 0.444246e^{i0.959586\pi}) \rangle$	$\langle 3, (0.565281e^{i1.196937\pi}, 0.448675e^{i1.085823\pi}) \rangle$
\mathfrak{B}_3		\mathfrak{B}_4
\mathfrak{J}_1	$\langle 4, (0.821814e^{i1.669277\pi}, 0.297877e^{i0.539368\pi}) \rangle$	$\langle 2, (0.474829e^{i0.990753\pi}, 0.609707e^{i1.307223\pi}) \rangle$
\mathfrak{J}_2	$\langle 2, (0.626753e^{i1.176826\pi}, 0.573407e^{i0.942689\pi}) \rangle$	$\langle 4, (0.805816e^{i1.591906\pi}, 0.24108e^{i0.329189\pi}) \rangle$
\mathfrak{J}_3	$\langle 1, (0.235591e^{i0.495882\pi}, 0.800397e^{i1.553293\pi}) \rangle$	$\langle 4, (0.856807e^{i1.705982\pi}, 0.101565e^{i0.312849\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.426781e^{i1.156826\pi}, 0.623415e^{i0.922475\pi}) \rangle$	$\langle 4, (0.740156e^{i1.535327\pi}, 0.246763e^{i0.394609\pi}) \rangle$
\mathfrak{B}_5		\mathfrak{B}_6
\mathfrak{J}_1	$\langle 4, (0.925437e^{i1.773344\pi}, 0.073543e^{i0.24464\pi}) \rangle$	$\langle 4, (0.820664e^{i1.551716\pi}, 0.157531e^{i0.548194\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.683699e^{i1.297906\pi}, 0.399011e^{i0.765837\pi}) \rangle$	$\langle 2, (0.48671e^{i0.874574\pi}, 0.619182e^{i1.211136\pi}) \rangle$
\mathfrak{J}_3	$\langle 0, (0.10998e^{i0.236042\pi}, 0.89715e^{i1.838062\pi}) \rangle$	$\langle 2, (0.414822e^{i0.813233\pi}, 0.644054e^{i1.336384\pi}) \rangle$
\mathfrak{J}_4	$\langle 3, (0.757506e^{i1.4106\pi}, 0.320004e^{i0.596584\pi}) \rangle$	$\langle 4, (0.56322e^{i1.110101\pi}, 0.455531e^{i0.98608\pi}) \rangle$
\mathfrak{B}_7		
\mathfrak{J}_1	$\langle 4, (0.91033e^{i1.921529\pi}, 0.064167e^{i0.293766\pi}) \rangle$	
\mathfrak{J}_2	$\langle 3, (0.586502e^{i1.096052\pi}, 0.469085e^{i0.849702\pi}) \rangle$	
\mathfrak{J}_3	$\langle 1, (0.193176e^{i0.414263\pi}, 0.824363e^{i1.688289\pi}) \rangle$	
\mathfrak{J}_4	$\langle 4, (0.814511e^{i1.658152\pi}, 0.213975e^{i0.376418\pi}) \rangle$	

- Step 4.** The entries of AWCFFNS_fDM $\widehat{\mathfrak{G}}$ are obtained by Equation 4 by utilizing ACFNS_fDM, given by Table 47, and the weight vector κ of attributes in Equation 10. These entries are tabulated, as shown in Table 49.
- Step 5.** In the proposed MAGDM problem, the attributes topography, soil and water quality & availability are benefit-type attributes whereas initial cost, environmental destruction, climate of the area and maintenance cost are cost-type attributes. CFFNS_f-PIS and CFFNS_f-NIS relative to each attribute, opted by Equations 5 and 6, are arranged in Table 50.
- Step 6.** Distance of each alternative from CFFNS_f-PIS and CFFNS_f-NIS is calculated by employing Equations 7 and 8, respectively. These distance measures are tabulated in Table 51.
- Step 7.** Table 52 represents the revised closeness index corresponding to each alternative, evaluated by using Equation 9.
- Step 8.** The ranking of cities on the basis of revised closeness index is shown by Table 53. Since \mathfrak{J}_1 has maximum index value. Hence the experts will give suggestions to the investor to select Boston, Massachusetts for farming.

Table 48 Importance weights of each attributes

$(\mathcal{H}, \mathcal{Q}, 5)$	\mathfrak{D}_1	\mathfrak{D}_2
\mathfrak{B}_1	$\langle 3, (0.67e^{i1.38\pi}, 0.23e^{i0.43\pi}) \rangle$	$\langle 3, (0.74e^{i1.65\pi}, 0.37e^{i0.73\pi}) \rangle$
\mathfrak{B}_2	$\langle 4, (0.96e^{i1.79\pi}, 0.09e^{i0.17\pi}) \rangle$	$\langle 4, (0.97e^{i1.92\pi}, 0.07e^{i0.29\pi}) \rangle$
\mathfrak{B}_3	$\langle 3, (0.71e^{i1.49\pi}, 0.26e^{i0.52\pi}) \rangle$	$\langle 3, (0.82e^{i1.53\pi}, 0.24e^{i0.49\pi}) \rangle$
\mathfrak{B}_4	$\langle 1, (0.31e^{i0.49\pi}, 0.67e^{i1.62\pi}) \rangle$	$\langle 0, (0.02e^{i0.21\pi}, 0.89e^{i1.82\pi}) \rangle$
\mathfrak{B}_5	$\langle 3, (0.68e^{i1.57\pi}, 0.31e^{i0.67\pi}) \rangle$	$\langle 3, (0.69e^{i1.46\pi}, 0.33e^{i0.76\pi}) \rangle$
\mathfrak{B}_6	$\langle 3, (0.75e^{i1.63\pi}, 0.27e^{i0.71\pi}) \rangle$	$\langle 3, (0.73e^{i1.33\pi}, 0.25e^{i0.55\pi}) \rangle$
\mathfrak{B}_7	$\langle 3, (0.81e^{i1.35\pi}, 0.35e^{i0.47\pi}) \rangle$	$\langle 2, (0.43e^{i1.21\pi}, 0.52e^{i0.92\pi}) \rangle$
	\mathfrak{D}_3	\mathfrak{D}_4
\mathfrak{B}_1	$\langle 3, (0.72e^{i1.37\pi}, 0.28e^{i0.57\pi}) \rangle$	$\langle 3, (0.79e^{i1.56\pi}, 0.34e^{i0.58\pi}) \rangle$
\mathfrak{B}_2	$\langle 4, (0.89e^{i1.85\pi}, 0.05e^{i0.22\pi}) \rangle$	$\langle 4, (0.88e^{i1.78\pi}, 0.02e^{i0.27\pi}) \rangle$
\mathfrak{B}_3	$\langle 4, (0.93e^{i1.73\pi}, 0.12e^{i0.29\pi}) \rangle$	$\langle 4, (0.91e^{i1.82\pi}, 0.14e^{i0.36\pi}) \rangle$
\mathfrak{B}_4	$\langle 3, (0.76e^{i1.47\pi}, 0.32e^{i0.45\pi}) \rangle$	$\langle 2, (0.57e^{i0.97\pi}, 0.61e^{i1.17\pi}) \rangle$
\mathfrak{B}_5	$\langle 4, (0.94e^{i1.75\pi}, 0.08e^{i0.14\pi}) \rangle$	$\langle 4, (0.87e^{i1.79\pi}, 0.09e^{i0.23\pi}) \rangle$
\mathfrak{B}_6	$\langle 2, (0.62e^{i0.86\pi}, 0.54e^{i1.02\pi}) \rangle$	$\langle 3, (0.83e^{i1.61\pi}, 0.29e^{i0.65\pi}) \rangle$
\mathfrak{B}_7	$\langle 2, (0.51e^{i0.92\pi}, 0.43e^{i1.23\pi}) \rangle$	$\langle 3, (0.78e^{i1.55\pi}, 0.36e^{i0.73\pi}) \rangle$

9 Comparative analysis of CFFNS_f-TOPSIS technique

In this section, we solve the MAGDM problem “Selection of the most suitable city in the USA for farming” by Fermatean fuzzy TOPSIS (FF-TOPSIS) method, proposed by Senapati and Yager (2020), to authenticate the importance and validity of proposed model. The step wise solution of MAGDM problem following the Fermatean fuzzy TOPSIS method is given as follows:

- **Step 1.** The linguistic terms along with grades are same as given in Table 42. Since the existing technique only deals with multi-attribute decision-making (MADM) problems. Hence, the aggregated opinion of all experts, given in Table 47 is used by the investor but the grading part is excluded and CFFNs have taken to be zero to apply FF-TOPSIS method. Fermatean fuzzy decision matrix (FFDM) is arranged in Table 54. Moreover, to determine the role of each criteria, the decision-maker sets the weights of attributes as follows:

$$\lambda = (0.150.20.170.10.160.130.09)^T$$

Step 2. The score of all FF numbers (FFNs) are determined to identify the Fermatean fuzzy positive and negative ideal solutions. The score of a FFN can be calculated by the following formula (Senapati and Yager 2020):

$$S_c(\mathfrak{G}_{a_{gt}}) = s_{a_{gt}}^3 - k_{a_{gt}}^3. \quad (11)$$

The score values of all entries of FFDM are assembled in Table 55. Table 56 represents the FF-PIS and FF-NIS relative to each attribute.

- **Step 3.** Distance of each alternative \mathfrak{I}_g from FF-PIS \mathfrak{B}^+ and FF-NIS \mathfrak{B}^- is computed by employing the equations as follows (Senapati and Yager 2020):

$$d(\mathfrak{I}_g, \mathfrak{G}^+) = \frac{1}{2} \sum_{i=1}^m \lambda_i \sqrt{\frac{1}{2} \left\{ (s_{a_{gt}}^3 - (s_{a_i}^+)^3)^2 + (k_{a_{gt}}^3 - (k_{a_i}^+)^3)^2 + ((\pi_{a_{gt}}^3 - (\pi_{a_i}^+)^3)^2) \right\}}$$

$$d(\mathfrak{I}_g, \mathfrak{G}^-) = \frac{1}{2} \sum_{i=1}^m \lambda_i \sqrt{\frac{1}{2} \left\{ (s_{a_{gt}}^3 - (s_{a_i}^-)^3)^2 + (k_{a_{gt}}^3 - (k_{a_i}^-)^3)^2 + ((\pi_{a_{gt}}^3 - (\pi_{a_i}^-)^3)^2) \right\}}$$

The results are tabulated in Table 57.

- **Step 4.** To find out the most suitable alternative, Table 58 represents the closeness index corresponding to each alternative which is evaluated by utilizing the Equation 9.
- **Step 5.** The increasing order ranking of cities is shown by Table 59, where 1 is for minimum closeness index value and 4 is for highest index value. Since \mathfrak{I}_1 has the maximum index value. Hence the investor will select Boston, Massachusetts for farming.

9.1 Results

1. Now, we present a comparison of the proposed technique with the existing FF-TOPSIS method (Senapati and Yager 2020) to assess the accuracy of CFFNS_f-

Table 49 Tabular form of AWCFNS_fDM $\widehat{\mathfrak{G}}$

$(\mathcal{H}, \mathcal{Q}, 5)$	\mathfrak{B}_1	\mathfrak{B}_2
\mathfrak{J}_1	$\langle 2, (0.281249e^{i0.683147\pi}, 0.653525e^{i1.249243\pi}) \rangle$	$\langle 2, (0.496783e^{i1.022269\pi}, 0.474560e^{i1.117885\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.507197e^{i1.153254\pi}, 0.401287e^{i0.751931\pi}) \rangle$	$\langle 2, (0.437787e^{i0.738593\pi}, 0.613697e^{i1.249496\pi}) \rangle$
\mathfrak{J}_3	$\langle 3, (0.674138e^{i1.406076\pi}, 0.308470e^{i0.593173\pi}) \rangle$	$\langle 3, (0.544408e^{i1.248451\pi}, 0.473156e^{i0.889553\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.392244e^{i0.857490\pi}, 0.484229e^{i1.019488\pi}) \rangle$	$\langle 3, (0.534231e^{i1.113859\pi}, 0.448924e^{i1.088949\pi}) \rangle$
	\mathfrak{B}_3	\mathfrak{B}_4
\mathfrak{J}_1	$\langle 4, (0.707227e^{i1.378285\pi}, 0.320871e^{i0.610807\pi}) \rangle$	$\langle 2, (0.258992e^{i0.505084\pi}, 0.740208e^{i1.508697\pi}) \rangle$
\mathfrak{J}_2	$\langle 2, (0.539364e^{i0.971679\pi}, 0.578935e^{i0.966398\pi}) \rangle$	$\langle 3, (0.439527e^{i0.811552\pi}, 0.623269e^{i1.191945\pi}) \rangle$
\mathfrak{J}_3	$\langle 1, (0.202742e^{i0.409439\pi}, 0.802114e^{i1.558591\pi}) \rangle$	$\langle 3, (0.467339e^{i0.869707\pi}, 0.614598e^{i1.191005\pi}) \rangle$
\mathfrak{J}_4	$\langle 2, (0.367274e^{i0.955166\pi}, 0.627793e^{i0.947371\pi}) \rangle$	$\langle 3, (0.403713e^{i0.782708\pi}, 0.623937e^{i1.196714\pi}) \rangle$
	\mathfrak{B}_5	\mathfrak{B}_6
\mathfrak{J}_1	$\langle 4(0.764928e^{i1.458701\pi}, 0.189473e^{i0.430803\pi}) \rangle$	$\langle 3, (0.607190e^{i1.113172\pi}, 0.324696e^{i0.789955\pi}) \rangle$
\mathfrak{J}_2	$\langle 3, (0.565117e^{i1.067620\pi}, 0.411197e^{i0.799440\pi}) \rangle$	$\langle 2, (0.360106e^{i0.627403\pi}, 0.638742e^{i1.267695\pi}) \rangle$
\mathfrak{J}_3	$\langle 0, (0.090905e^{i0.194161\pi}, 0.897887e^{i1.839505\pi}) \rangle$	$\langle 2, (0.306917e^{i0.583398\pi}, 0.661504e^{i1.378904\pi}) \rangle$
\mathfrak{J}_4	$\langle 3, (0.626123e^{i1.160318\pi}, 0.339023e^{i0.651104\pi}) \rangle$	$\langle 3, (0.416714e^{i0.796365\pi}, 0.496091e^{i1.078299\pi}) \rangle$
	\mathfrak{B}_7	
\mathfrak{J}_1	$\langle 3, (0.610740e^{i1.239030\pi}, 0.424105e^{i0.815057\pi}) \rangle$	
\mathfrak{J}_2	$\langle 3, (0.393484e^{i0.706751\pi}, 0.555490e^{i1.029621\pi}) \rangle$	
\mathfrak{J}_3	$\langle 1, (0.129602e^{i0.267123\pi}, 0.840450e^{i1.712080\pi}) \rangle$	
\mathfrak{J}_4	$\langle 3, (0.546455e^{i1.069201\pi}, 0.439838e^{i0.827983\pi}) \rangle$	

Table 50 CFFNS_f-PIS and CFFNS_f-NIS

Attributes	CFFNS _f -PIS ($\widetilde{\mathfrak{G}}_t$)	CFFNS _f -NIS ($\check{\mathfrak{G}}_t$)
\mathfrak{B}_1	$\langle 2, (0.281249e^{i0.683147\pi}, 0.653525e^{i1.249243\pi}) \rangle$	$\langle 3, (0.674138e^{i1.406076\pi}, 0.308470e^{i0.593173\pi}) \rangle$
\mathfrak{B}_2	$\langle 2, (0.437787e^{i0.738593\pi}, 0.613697e^{i1.249496\pi}) \rangle$	$\langle 3, (0.544408e^{i1.248451\pi}, 0.473156e^{i0.889553\pi}) \rangle$
\mathfrak{B}_3	$\langle 4, (0.707227e^{i1.378285\pi}, 0.320871e^{i0.610807\pi}) \rangle$	$\langle 1, (0.202742e^{i0.409439\pi}, 0.802114e^{i1.558591\pi}) \rangle$
\mathfrak{B}_4	$\langle 2, (0.258992e^{i0.505084\pi}, 0.740208e^{i1.508697\pi}) \rangle$	$\langle 3, (0.467339e^{i0.869707\pi}, 0.614598e^{i1.191005\pi}) \rangle$
\mathfrak{B}_5	$\langle 0, (0.090905e^{i0.194161\pi}, 0.897887e^{i1.839505\pi}) \rangle$	$\langle 4, (0.764928e^{i1.458701\pi}, 0.189473e^{i0.430803\pi}) \rangle$
\mathfrak{B}_6	$\langle 3, (0.607190e^{i1.113172\pi}, 0.324696e^{i0.789955\pi}) \rangle$	$\langle 2, (0.360106e^{i0.627403\pi}, 0.638742e^{i1.267695\pi}) \rangle$
\mathfrak{B}_7	$\langle 3, (0.610740e^{i1.239030\pi}, 0.424105e^{i0.815057\pi}) \rangle$	$\langle 1, (0.129602e^{i0.267123\pi}, 0.840450e^{i1.712080\pi}) \rangle$

Table 51 Distance of each alternative from ideal solution

Alternatives	$d(\mathfrak{J}_g, \widehat{\mathfrak{G}}_t)$	$d(\mathfrak{J}_g, \check{\mathfrak{G}}_t)$
\mathfrak{J}_1	1.002077	2.187073
\mathfrak{J}_2	1.955608	1.434256
\mathfrak{J}_3	2.274187	0.950996
\mathfrak{J}_4	1.851251	1.589221

Table 52 Revised closeness index of each alternative

Alternatives	$\Psi(\mathfrak{J}_g)$
\mathfrak{J}_1	0
\mathfrak{J}_2	-1.29577
\mathfrak{J}_3	-1.83465
\mathfrak{J}_4	-1.12077

TOPSIS method. Despite the difference in revised closeness index calculated by both techniques, the final ranking of cities is the same. Thus, the same city is proclaimed as the most suitable one for farming in both methods. The results of the proposed and existing methods, including the final ranking and best alternative, are summarized in Table 60 as follows:

Table 53 Ranking of each alternative

Alternatives	\mathfrak{J}_1	\mathfrak{J}_2	\mathfrak{J}_3	\mathfrak{J}_4
Ranking	4	2	1	3

Table 54 Tabular representation of FFDM $\vec{\mathfrak{G}}$

	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3	\mathfrak{B}_4
\mathfrak{J}_1	(0.384320, 0.636677)	(0.525656, 0.474341)	(0.821814, 0.297877)	(0.474829, 0.609707)
\mathfrak{J}_2	(0.693071, 0.334196)	(0.463232, 0.613584)	(0.626753, 0.573407)	(0.805816, 0.241080)
\mathfrak{J}_3	(0.921191, 0.100947)	(0.576050, 0.472935)	(0.235591, 0.800397)	(0.856807, 0.101565)
\mathfrak{J}_4	(0.535991, 0.444246)	(0.565281, 0.448675)	(0.426781, 0.623415)	(0.740156, 0.246763)
	\mathfrak{B}_5	\mathfrak{B}_6	\mathfrak{B}_7	
\mathfrak{J}_1	(0.925437, 0.073543)	(0.820664, 0.157531)	(0.910330, 0.064167)	
\mathfrak{J}_2	(0.683699, 0.399011)	(0.486710, 0.619182)	(0.586502, 0.469085)	
\mathfrak{J}_3	(0.109980, 0.897150)	(0.414822, 0.644054)	(0.193176, 0.824363)	
\mathfrak{J}_4	(0.757506, 0.320004)	(0.563220, 0.455531)	(0.814511, 0.213975)	

Table 55 Score values of FFNs to opt ideal solutions

	\mathfrak{B}_1	\mathfrak{B}_2	\mathfrak{B}_3	\mathfrak{B}_4	\mathfrak{B}_5	\mathfrak{B}_6	\mathfrak{B}_7
\mathfrak{J}_1	-0.20132	0.038520	0.528605	-0.11960	0.792178	0.548798	0.754126
\mathfrak{J}_2	0.295590	-0.131603	0.057666	0.509237	0.256065	-0.12209	0.098530
\mathfrak{J}_3	0.780688	0.085372	-0.49969	0.627949	-0.72077	-0.19578	-0.55301
\mathfrak{J}_4	0.066309	0.090309	-0.16455	0.390455	0.401899	0.084136	0.530573

Table 56 FF-PIS and FF-NIS

Attributes	FF-PIS (\mathfrak{G}^+)	B _p FNS _f -NIS (\mathfrak{G}^-)
\mathfrak{B}_1	(0.384320, 0.636677)	(0.921191, 0.100947)
\mathfrak{B}_2	(0.463232, 0.613584)	(0.565281, 0.448675)
\mathfrak{B}_3	(0.821814, 0.297877)	(0.235591, 0.800397)
\mathfrak{B}_4	(0.474829, 0.609707)	(0.856807, 0.101565)
\mathfrak{B}_5	(0.109980, 0.897150)	(0.925437, 0.073543)
\mathfrak{B}_6	(0.820664, 0.157531)	(0.414822, 0.644054)
\mathfrak{B}_7	(0.910330, 0.064167)	(0.193176, 0.824363)

Table 57 Distance of each alternative from ideal solution

Alternatives	$d(\mathfrak{J}_g, \mathfrak{G}^+)$	$d(\mathfrak{J}_g, \mathfrak{G}^-)$
\mathfrak{J}_1	0.071595	0.174813
\mathfrak{J}_2	0.152928	0.130212
\mathfrak{J}_3	0.182976	0.062972
\mathfrak{J}_4	0.151493	0.137461

Table 58 Revised closeness index of each alternative

Alternatives	$\psi(\mathfrak{J}_g)$
\mathfrak{J}_1	0
\mathfrak{J}_2	-1.39116
\mathfrak{J}_3	-2.19548
\mathfrak{J}_4	-1.32965

Table 59 Ranking of each alternative

Alternatives	\mathfrak{J}_1	\mathfrak{J}_2	\mathfrak{J}_3	\mathfrak{J}_4
Ranking	4	2	1	3

2. A comparison chart is designed in Figure 3 to envision the conformity of final results of compared and proposed MAGDM approaches which shows the effectuality and accountability of our proposed technique.
3. It is clear from the figure that both techniques elucidate the same outcome and ranking order that indicates the feasibility and sustain-ability of the presented technique.

4. Our proposed CFFNS_f-TOPSIS technique has capability to handle the vagueness and periodicity involve in the data simultaneously, but the compared FF-TOPSIS technique is limited to capture the ambiguity of non-periodic data that may cause to the inconsistency and specious outcomes. This extraordinary trait of the proposed strategy depicts that it is the more effective and generalized MAGDM strategy.
5. Due to the inadequacy of multi-valued grades and periodic terms, FF-TOPSIS cannot deal with CFFNS_f information. On the other hand, CFFNS_f-TOPSIS method has potential to handle the FF information by taking phase terms equal to zero and neglecting the grades. Since the results are same in both cases which

depicts the proposed method more adaptable than existing methods.

10 Merits of CFFNS_fS model and CFFNS_f-TOPSIS approach

1. In the modern era, the performance appraisal system is commonly used for the rating of restaurant management, schools, candidates for job, online services, online applications, products and websites, etc. The proposed model is designed to handle the rating-based assessment framework along with imprecise and vague two-dimensional information.
2. In this article, the robust technique of CFFNS_f-TOPSIS is developed for determining the best solution obtaining the closest distance from PIS and far away from NIS. The framework of the presented MAGDM strategy has remarkable aspects: it merges the fascinating advantages of TOPSIS with the hybrid model of CFFNS_fS. The advantage of the hybrid model is that it has the potency to handle vagueness and periodicity of parameterized graded information simultaneously.
3. The proposed model shows the same accuracy when applied to the existing models inclusive of FF, CIF, CPF, FFS_f, CIFS_f, CPFS_f, FFNS_f, CIFNS_f, and CPFNS_f by taking either $N = 2$ or substituting phase terms equal to zero or by applying both strategies. Hence, the developed technique deprives a adaptable tool that skillfully and efficiently accomplishes its decision-making chores with preciseness under traditional as well as two-dimensional vague information along with finely-graded parameters.

11 Conclusion

Decision-making methods play an important role in the real life of human beings. The process of choosing the best option among a set of possible options is present in all human activities. In this paper, a new theory has been developed that serves as a mathematical tool which deals

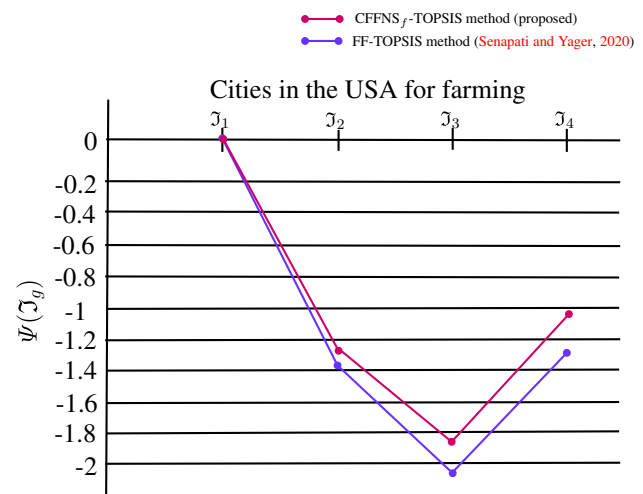


Fig. 3 Comparative analysis

with the two-dimensional vague information, and which is a generalization of the fuzzy N -soft set. We have advanced a model, CFFNS_fS, that assesses the uncertain and vague data which has complex membership and non-membership values, parameterized information, and ordinal ranking systems. To establish a comparison between two CFFNS_fNs, we have developed score and accuracy functions in a CFFNS_f environment. We have defined some basic operations for the CFFNS_fS model that include: complement (weak complement, CFFNS_fS complement, and weak CFFNS_fS complement), union (extended union and restricted union), intersection (extended intersection and restricted intersection). We have also included relevant examples for these operations. In addition, we have presented algebraic and Yager operations for CFFNS_fNs.

Moreover, we have accomplished three algorithms to resolve multi-attribute decision-making problems. These algorithms have been validated by two real-life examples related to the selection of cars and the selection of the best telecommunication company in Pakistan.

Furthermore, in order to analyze the validity, feasibility, and reliability of the proposed model, we have conducted a comparative study of our approach with two operators: the FFY_wG operator and the FFY_wA operator.

With respect to the proposed CFFNS_f-TOPSIS method, our method possesses the MAGDM potential of TOPSIS along with the adequacy of the proposed CFFNS_f model to

Table 60 Comparative analysis

Methods	Ranking of the most suitable city for farming	Best city
CFFNS _f -TOPSIS method (proposed)	$\mathfrak{J}_3 \prec \mathfrak{J}_2 \prec \mathfrak{J}_4 \prec \mathfrak{J}_1$	\mathfrak{J}_1
FF-TOPSIS method (Senapati and Yager 2020)	$\mathfrak{J}_3 \prec \mathfrak{J}_2 \prec \mathfrak{J}_4 \prec \mathfrak{J}_1$	\mathfrak{J}_1

improve the exactness of decision-making results. The proposed method's primary dominance is due to its capability to tackle two-dimensional imprecise information along with level of attributes based on alternative with the help of N -soft grading values as well as complex membership and non-membership values. The basic principle of the CFFNS $_f$ -TOPSIS method is to find out the best solution possessing the proximity distance from the ideal solutions.

In the presented approach, the primary information has been equipped by ordered grades and their corresponding CFFNS $_f$ Ns. The individual opinions have been aggregated by employing CFFNS $_f$ WA operator. Further, the AWCFFNS $_f$ DM has been acquired by the multiplication of CFFNS $_f$ weight vector of criteria and CFFNS $_f$ DM. After examining the CFFNS $_f$ -PIS and CFFNS $_f$ -NIS, distance of each alternative from ideal solutions have been computed. Further, the revised closeness index of each alternative has been calculated by evolving the discrepancy of these variables from the ideal solution. After the evaluated results of the closeness index, the alternatives are arranged in an ascending order. The alternative having maximum value of closeness index will be the optimal solution of the MAGDM problem. The proposed approach has been endorsed by a numerical example related to the selection of the suitable city in the USA for farming.

Along with beneficial characteristics of the proposed technique based on TOPSIS method for MAGDM problems in two-dimensional data, it ensures the same level of authenticity under Fermatean fuzzy environment by eliminating the grades and substituting phase terms equal to zero. On the contrary, the adeptness of the FF-TOPSIS method is restricted to handle one dimensional phenomena, also it is unable to deal with MAGDM problems.

Moreover, the proposed CFFNS $_f$ -TOPSIS method has an edge over the extant decision-making approaches as the CFFNS $_f$ S model can effectively apply in the environments of FFS, CIFS, CPFS, FFS $_f$ S, FNS $_f$, IFNS $_f$ S, PFNS $_f$ S, CPFNS $_f$ and so forth by taking either $N = 2$ or substituting phase terms equal to zero or by applying both strategies.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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