Weighted Models for Higher-Order Computation ☆

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Abstract

We study a class of quantitative models for higher-order computation: Lafont categories with (infinite) biproducts. These are models of intuitionistic linear type theory with a canonical enrichment over the modules of a internal complete semiring: we give a semantics of non-deterministic PCF weighted over the elements of this semiring. The fixed points in our model are obtained by constructing a *bifree algebra* for the comonad which gives its exponential structure. By characterizing these fixed points more concretely as infinite sums of finitary approximants indexed over the nested finite multisets, we prove *computational adequacy* with respect to an operational semantics which evaluates a term by taking a weighted sum of the residues of its terminating reduction paths.

We investigate the construction of Lafont categories with biproducts by weighting intensional models such as categories of games over a complete semiring. This transition from a qualitative semantics to a quantitative one is characterized as a "change of base" of enriched categories: it is induced by a monoidal functor from the category of coherence spaces to the category of free modules over a complete semiring. Using the properties of this functor, we characterise some requirements for the change of base to preserve the structure of a Lafont category. As an example, we show that the game semantics of Idealized Algol bears a natural enrichment over the category of coherence spaces, and thus gives rise by change of base to a semiring weighted model, which is fully abstract. We relate this to existing categories of probabilistic games and slot games.

Keywords: Quantitative models, Complete semirings, Game semantics, Fixed points, Linear logic, Computational adequacy

1. Introduction

We study a class of quantitative models of higher-order computation — Lafont categories with (infinite) biproducts: that is, models of intuionistic type theory with a "free" exponential structure, in which the product and coproduct

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functors coincide. The biproducts imply a natural enrichment over the category of $\mathcal{R}$-modules, where $\mathcal{R}$ is the complete "internal semiring" of the category: values of this semiring can be assigned as weights to different outcomes of evaluation, capturing properties such as probability of failure, minimal or maximal cost [1], security level, etc. The free exponentials play a key role in soundly capturing these weights in the denotational semantics of functional programs, by faithfully representing the multiplicity of calls to each argument.

This paper examines these models in an abstract setting: we show that they can be used to give a computationally adequate representation of recursive higher-order computation, weighted over the internal semiring, and give a general recipe for constructing examples from qualitative models such as game semantics. In doing so, we use some basic ideas from axiomatic domain theory — to construct fixed points without any partial order enrichment — and enriched category theory — to change the base category over which our categories are enriched from coherence spaces to free $\mathcal{R}$-modules. As an illustrative example, we apply this change of base to a coherence-enriched category of games and strategies, to give a fully abstract semantics of $\mathcal{R}$-weighted Idealized Algol.

Related Work. Quantitative interpretations of programs as linear functionals — with the direct sum and product as additive and multiplicative connectives, and cofree comonoids as exponentials — played key roles (explicitly and implicitly) in the development of linear logic [2]. Where the direct sum is a biproduct this already implies the existence of a finite sum operation on morphisms (semi-additivity). The construction of the free exponential by forms of Taylor expansion requires certain infinite sums, forcing a choice between models in which these are strictly limited (cf. Ehrhard's finiteness spaces [3]) and those in which all such sums exist. One of our contributions is to show that in the latter case, we can construct fixed points for all endomorphisms. We hope to arrive at a more complete understanding of the relationship between these alternatives.

Our approach generalizes and extends the results presented in [1], which describes a denotational model for nondeterministic PCF with scalar weights from a *continuous* semiring, $\mathcal{R}$ (based on a semantics of linear logic introduced by Lamarche [4]) in the category of sets and matrices over $\mathcal{R}$ (also known as the category of free $\mathcal{R}$-modules and their homomorphisms), which is a particular example of a Lafont category with biproducts. This internal semiring is complete but not necessarily continuous. So to construct fixed points in the absence of any order theoretic structure, we turn to the abstract characterization of such operators provided by *axiomatic domain theory*. Specifically, we use the observation [5, 6] that *uniform* fixed point operators exist (and are unique) for any comonad which is an *algebraically compact* functor [7] — i.e. it has a *bifree algebra* (an initial algebra for which the inverse is a terminal coalgebra). In an example of the utility of this theory, we define such an algebra for the cofree exponential by iterating Lafont's construction of a cofree commutative comonoid as a sum of finite multisets [8, 9], to obtain a bifree algebra which is a biproduct over the *nested finite multisets*. Computationally, this gives a precise representation of the computational resources consumed in evaluating a function in the style of the resource λ-calculus [10] — in this case extended with *nested multiset* resource bounds capturing the call patterns of recursively defined procedures. This suggests further relationships with the rich theory underlying quantitative models, including the differential λ-calculus [11]: on the syntactic side, via their correspondence with the resource calculus [12], and on the semantic side, via the notion of differential category [13], which shares many properties and examples with our notion of categorical model.

Our second objective is to describe further examples of Lafont categories with biproducts: in particular, to show how to derive them from existing qualitative models such as game semantics, which has been used to describe intensional models of a wide variety of programming language features, including state [14]. With some notable exceptions, these models are qualitative rather than quantitative in character, possessing an order-theoretic structure which may be characterized as a categorical *enrichment* over certain categories of domain (such as dI-domains, qualitative domains or prime algebraic lattices). Our aim is to show that this enriched category theory perspective may be used systematically to construct quantitative models (and describe existing ones), by applying the notion of *change of base* [15, 16] to vary the enrichment of the model, independently of its intensional structure. Specifically, we describe a monoidal functor from the category of coherence spaces [17] to the category of free modules over a complete semiring (so we go from using weighted relations directly to interpret programs directly to using them as an enrichment for intensional models).

Change of base thus provides a simple way to identify further examples of our categorical model with richer internal structure than sets and weighted relations. This allows more language features such as side effects to be captured, resolving the *full abstraction* problem for these models (the $\mathcal{R}$-module models for $\mathcal{R}$-weighted PCF are shown not to be fully abstract in [1]). As an illustrative example we study the games model of Idealized Algol introduced by Abramsky and McCusker [14]. It is known that this possesses coherence structure — cf. the projection into a category of (ordered) coherence spaces [18]. We show that a strictly linear (rather than affine) decomposition of this category of games bears a natural enrichment over coherence spaces. Previous quantitative semantics based on the original model include Danos and Harmer's *probabilistic games* [19], in which strategies are defined by attaching probabilities to positions of the game, and Ghica's *slot games* [20], which attach resource weightings to positions in a rather different way — by introducing a class of moves which are persistent when other moves are hidden during composition, allowing the cost of computation to be made explicit. We show that both may be viewed as examples obtained by our change of base construction, and that the corresponding programming languages may be subsumed into a version of Idealized Algol with weights from a complete semiring, for which we describe a denotational semantics. Full abstraction for this model follows from the result in [14] with very little effort.

Change of base allows an approach to constructing Lafont categories described in [21] to be extended to start with any symmetric monoidal category enriched over coherence spaces: we may change its base to free $\mathcal{R}$-modules and build a Lafont category by freely completing with biproducts and idempotents (giving, in particular, symmetric tensor powers) and thus the cofree commutative comonoid on A via Lafont's construction: $!A = \bigoplus_{i \in \mathbb{N}} A^i$. However, we focus on cases (such as the games model of Idealized Algol) in which our coherence-enriched category already has a free exponential given by the generalization of Lafont's construction by Melliès, Tabareau and Tasson [9]. We describe a simple condition under which change of base preserves this structure (and thus the meaning of λ-terms), and show that this is satisfied by our category of games.

Organization of the paper. Our aim is to describe both a general framework for quantitative semantics (Lafont categories with biproducts), and how to apply it to construct examples, based on a change of base functor. Even numbered sections describe the first strand of this approach: biproducts and complete monoid enrichment (Section 2), the free exponential and its construction using biproducts (Section 4), the construction of fixed points from a bifree algebra for the free exponential (Section 6), and a computationally adequate categorical model of $\mathcal{R}$-weighted PCF (Section 8).

The odd-numbered sections describe the second strand: each illustrates change of base and the structure defined in the preceding section using a running example based on game semantics: Section 3 introduces change of base, and applies it to a coherence-space enriched category of games, Sections 5 and 7 show how change of base can preserve the free exponential and fixedpoints (respectively) and Section 9 describes a fully abstract model of $\mathcal{R}$-weighted PCF extended with state (Idealized Algol). Each odd-numbered section depends on the previous even-numbered sections, but not vice-versa.

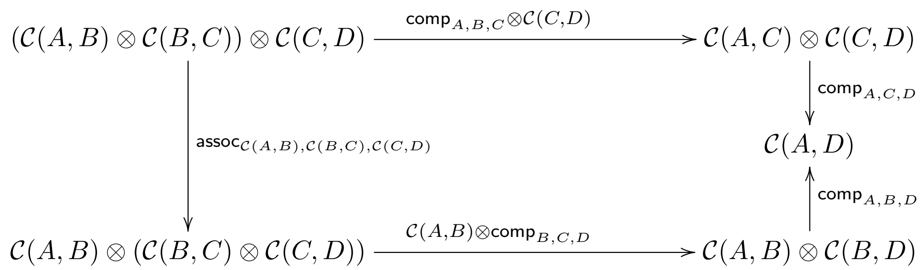
2. Biproducts and Complete Monoid Enrichment

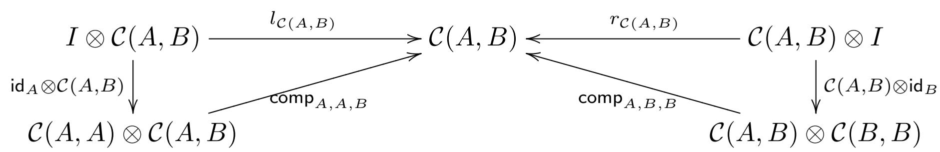
We recall the definition of *biproducts* and describe their relationship to enrichment over the category of complete monoids.

Definition 2.1. *A category $\mathcal{C}$ has (specified, small)* biproducts *if for for any set-indexed family of objects $\{A_i \mid i \in I\}$ there is an object $\bigoplus_{i \in I} A_i$ which is a product and coproduct of the A_i, and for any family of morphisms $\{f_i : A_i \to B_i \mid i \in I\}$, the product $\langle \pi_i; f_i \mid i \in I \rangle$ and coproduct $[f_i; \iota_i \mid i \in I]$ of the f_i are equal (giving a morphism $\bigoplus_{i \in I} f_i : \bigoplus_{i \in I} A_i \to \bigoplus_{i \in I} B_i$).*

The biproduct of the empty family — the *zero object* — is both terminal and initial, and thus for objects A, B there is a unique *zero map* — $0_{A,B} : A \to B$ obtained by composing the terminal map from A with the initial map into B. More generally, it is known that any category with finite biproducts carries a canonical enrichment in the category of commutative monoids. Similarly, categories with all biproducts bear an enrichment over the monoidal category of *complete monoids*, which we now describe.

Recall that if $\mathcal{V}$ is a monoidal category then a $\mathcal{V}$-enriched category (or simply, $\mathcal{V}$-category) $\mathcal{C}$ is given by a set of its objects, a $\mathcal{V}$-object $\mathcal{C}(A, B)$ for each pair of $\mathcal{C}$-objects A and B, and $\mathcal{V}$-morphisms $\mathsf{id}_A : I \to \mathcal{C}(A, A)$ and $\mathsf{comp}_{A,B,C} : \mathcal{V}(A, B) \otimes \mathcal{V}(B, C) \to \mathcal{V}(A, C)$ for each A, B, C, satisfying the following associativity and identity diagrams in $\mathcal{V}$:





Definition 2.2. *A complete monoid A is a pair $(|A|, \Sigma)$ of a set $|A|$ with a sum operation $(\sum)$ on set-indexed families of elements of $|A|$, satisfying the axioms:*

Partition Associativity *For any partitioning of the set I into $\{I_j \mid j \in J\}$:*

$$\sum_{i \in I} a_i = \sum_{j \in J} \sum_{i \in I_j} a_i.$$

Unary Sum *For any singleton family $\{a_i \mid i \in \{j\}\}$, $\sum_{i \in \{j\}} a_i = a_j$.*

We write 0 for the sum of the empty family, which is a neutral element for the sum by the above axioms. Any complete monoid is a commutative monoid in the usual sense (with binary sum $a_1 + a_2 = \sum_{i \in \{1,2\}} a_i$).

A homomorphism of complete monoids from A to B is a function $f : |A| \to |B|$ such that $f(\sum_{i \in I} a_i) = \sum_{i \in I} f(a_i)$ for any family $\{a_i\}_{i \in I}$ over $|A|$. The tensor product of complete monoids A, B is the[1] complete monoid $A \otimes B$ with a natural equivalence (for any C) between the homomorphisms from $A \otimes B$ into C and the functions from $|A| \times |B|$ into $|C|$ which are *bilinear* — i.e. $f(\sum_{i \in I} a_i, \sum_{j \in J} b_j) = \sum_{\langle i,j \rangle \in I \times J} f(a_i, b_j)$. We refer to [22, 23] for details of the construction of this tensor

[1] unique, up to isomorphism

product, which yields symmetric monoidal structure on the category of complete monoids and their homomorphisms. A category bears an enrichment over this category of complete monoids if and only if every hom-set has the structure of a complete monoid, such that composition is a bilinear function — i.e.

$$(\sum_{i\in I} f_i);(\sum_{j\in J} g_j) = \sum_{\langle i,j\rangle\in I\times J} f_i;g_j$$

Proposition 2.3. *Any category with biproducts bears an enrichment over the category of complete monoids.*

Proof. For a family of morphisms $\{f_i : A \to B \mid i \in I\}$, let

$$\sum_{i\in I} f_i = \Delta^I_A; \bigoplus_{i\in I} f_i; \nabla^I_B$$

where $\Delta^A_I : A \to \bigoplus_{i\in I} A = \langle \mathsf{id}_A \mid i \in I\rangle$ and $\nabla^I_B : \bigoplus_{i\in I} B \to A = [\mathsf{id}_B \mid i \in I]$.
This satisfies the unary sum axiom by definition, and partition associativity as follows: for any partitioning of I into $\{I_j \mid j \in J\}$: $\sum_{j\in J}\sum_{i\in I_j} f_i =$

$$\Delta^J_A;\bigoplus_{j\in J}(\Delta^{I_j}_A;(\bigoplus_{i\in I_j} f_i);\nabla^{I_j}_B);\nabla^J_B = \Delta^J_A;(\bigoplus_{j\in J}\Delta^{I_j}_A);(\bigoplus_{j\in J}\bigoplus_{i\in I_j} f_i);(\bigoplus_{j\in J}\nabla^{I_j}_B);\nabla^J_B$$

$= \Delta^I_A;(\bigoplus_{i\in I} f_i);\nabla^I_B = \sum_{i\in I} f_i$.
Bilinearity of composition follows from naturality of the biproduct. □

Conversely:

Lemma 2.4. *In any category which bears a complete monoid enrichment and has specified (small) products or coproducts, these are biproducts.*

Proof. Supposing there are specified products, define coproduct structure on $\prod_{i\in I} A_i$ by $\iota_j : A \to \prod_{i\in I} A_i = \langle \delta_{ij} \mid i \in I\rangle$ where $\delta_{ij} = \begin{cases}\mathsf{id}_A & \text{if } i = j\\ 0_{A,A} & \text{otherwise}\end{cases}$.
The co-pairing of $\{f_i : A_i \to B \mid i \in I\}$ is $\sum_{i\in I} \pi_i; f_i : \prod_{i\in I} A_i \to B$, so that $\iota_j;[f_i \mid i \in I] = \sum_{i\in I}\langle\delta_{ij} \mid i \in I\rangle;\pi_i;f_i = f_i$ and for $g : \prod_{i\in I} \to B$, $[\iota_j;g \mid j \in I] = \sum_{j\in I}(\pi_j;\langle\delta_{ij} \mid i \in I\rangle;g) = \langle\sum_{j\in J}\pi_j;\delta_{ij} \mid i \in I\rangle;g = \langle\pi_i \mid i \in I\rangle;g = g$. □

From this it follows that in any category with biproducts, $\iota_i;\pi_j = \delta_{ij}$, and $\Delta^I_A;\iota_j = \mathsf{id}_A$, and hence that the complete-monoid enrichment is unique:

Proposition 2.5. *If $\mathcal{C}$ is a category with biproducts which bears a complete monoid enrichment then* $\sum_{i\in I}(f_i : A \to B) = \Delta^I_A;\bigoplus_{i\in I} f_i;\nabla^I_B$.

Proof. $\Delta^I_A; \bigoplus_{i\in I} f_i; \nabla^I_B = \Delta^I_A; (\langle \iota_i; f_i \mid i \in I\rangle; (\sum_{j\in I} \pi_j)) = \Delta^I_A; \sum_{j\in I}(\iota_j; f_j)$
$= \sum_{j\in I}(\Delta^I_A; \iota_j; f_j) = \sum_{i\in I} f_i.$ □

Any complete monoid enriched category may be *completed* with biproducts:

Definition 2.6. *For any complete monoid enriched category $\mathcal{C}$, let $\mathcal{C}^\Pi$ be the category in which objects are set-indexed families of objects of $\mathcal{C}$, and morphisms from $\{A_i \mid i \in I\}$ to $\{B_j \mid j \in J\}$ are $I \times J$-indexed sets of morphisms $\{f_{i,j} : A_i \to B_j \mid \langle i, j\rangle \in I \times J\}$, composed by setting $(f; g)_{ik} = \sum_{j\in J}(f_{ij}; g_{jk})$. The biproduct $\bigoplus_{j\in J}\{A_i \mid i \in I_j\}$ is the disjoint sum of families $\{A_k \mid k \in \coprod_{j\in J} I_j\}$.*

2.1. Biproducts, Monoidal Categories and Complete Semirings

The correspondence between complete monoid enrichment and biproducts extends to monoidal categories in which the monoidal product *distributes* over the biproducts — i.e. there is a natural isomorphism $(\bigoplus_{i\in I} A_i) \otimes \bigoplus_{j\in J} B_j \cong \bigoplus_{i\in I, j\in J}(A_i \otimes B_j)$. In monoidal *closed* categories this is always the case because $\otimes$ is a left adjoint and so preserves colimits. (We shall assume that for our specified biproducts this isomorphism is an equality.) The complete monoid enrichment thus extends to the monoidal structure — i.e. $(\sum_{i\in I} f_i) \otimes (\sum_{j\in J} g_j) = \sum_{i\in I, j\in J}(f_i \otimes g_j)$.

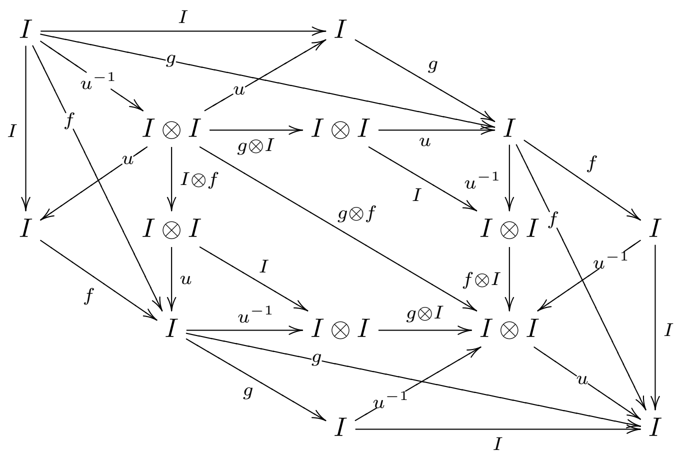
In any complete-monoid-enriched monoidal category, the endomorphisms on the unit I form a *complete semiring.*

Definition 2.7. *A* complete semiring *$\mathcal{R}$ is a tuple $(|\mathcal{R}|, \Sigma, \cdot, 1)$, where $(|\mathcal{R}|, \Sigma)$ is a complete monoid and $(|\mathcal{R}|, \cdot, 1)$ is a monoid which distributes over Σ — i.e. $\sum_{i\in I}(a \cdot b_i) = a \cdot \sum_{i\in I} b_i$ and $\sum_{i\in I}(b_i \cdot a) = (\sum_{i\in I} b_i) \cdot a$.*

Definition 2.8. *For any complete-monoid-enriched monoidal category $\mathcal{C}$, the* internal (complete) semiring *$\mathcal{R}_C$ is the complete monoid $\mathcal{C}(I, I)$, with multiplication being composition in $\mathcal{C}$.*

Lemma 2.9. *The internal semiring is commutative (i.e. for all $f, g \in \mathcal{C}(I, I)$, $f; g = g; f$).*

Proof. In a monoidal category, there is a single isomorphism $u : I \otimes I \to I$ which is both a left and right unitor at I and so the following diagram commutes:



□

From this point, in any reference to complete semirings commutativity should be taken as read. We will also elide the associativity and unital isomorphisms of a monoidal category, as if in a strict monoidal category, where it is not necessary to make them explicit.

If $\mathcal{C}$ is a complete monoid enriched symmetric monoidal category then we may define the (distributive) tensor product on its biproduct completion $\mathcal{C}^{\Pi}$:

$$\{A_i \mid i \in I\} \otimes \{B_j \mid j \in J\} = \{A_i \otimes B_j \mid \langle i, j\rangle \in I \times J\} \text{ and } (f \otimes g)_{ij,kl} = f_{ik} \otimes g_{jl}.$$

In particular, taking the biproduct completion of a complete semiring $\mathcal{R}$ viewed as a one-object symmetric monoidal (closed) category in which both composition and tensor product are multiplication gives a symmetric monoidal closed category with distributive biproducts. By identifying its objects with their indexing sets it may equivalently be presented as follows [2]

Definition 2.10. *For any complete semiring $\mathcal{R}$, the objects of the category $\mathcal{R}^{\Pi}$ are sets, and morphisms from X to Y are functions from $X \times Y$ into $|\mathcal{R}|$ — i.e. $X \times Y$ matrices with entries in $\mathcal{R}$. They are composed by matrix multiplication: $(f;g)(x,z) = \sum_{y \in Y} f(x,y) \cdot g(y,z)$. Symmetric monoidal structure on $\mathcal{R}$ is given by the cartesian product of sets, with $(f \otimes g)((a,b),(c,d)) = f(a,c) \cdot g(c,d)$. The biproduct of a family of sets is its disjoint union.*

This category plays a key role in our semantics, both as a model itself, and as base over which more intensional models may be enriched. (Observe that if $\mathcal{C}$ is enriched over $\mathcal{R}^{\Pi}$ then the map from $\mathcal{R}$ to $\mathcal{R}_{\mathcal{C}}$ sending a to $a.\mathsf{id}_I$ is

[2] $\mathcal{R}^{\Pi}$ is also equivalent to the category of *free $\mathcal{R}$-modules* and their homomorphisms, and thus the Kleisli category of the monad $\mathcal{R}^{-}$ on **Set**, which sends each set X to the free $\mathcal{R}$-module on X (consisting of functions from X into $|\mathcal{R}|$) — see [1].

a homorphism of complete semirings, since (by Proposition 2.5) the complete monoid enrichment on $\mathcal{C}(I,I)$ is unique, as is the multiplicative structure by the Eckmann-Hilton argument.

2.2. Continuous Semirings and Cpo Enrichment

Important examples of complete semirings include:

- Any complete lattice, with $\sum_{i\in I} a_i = \bigvee\{a_i \mid i \in I\}$, $1 = \top$ and $a \cdot b = a \wedge b$. In particular, the *Boolean semiring* $\mathbb{B} = (\{\top,\bot\}, \bigvee, \wedge, \top)$.
- The *probability semiring*, $\mathbb{R}_+^\infty = (\{a \in \{\mathbb{R} \mid a \geq 0\} \cup \{\infty\}, \Sigma, \cdot, 1)$ — the non-negative real numbers, with arithmetic sum and product, completed with an additively-absorptive infinite element, which is the value assigned to any non-convergent sum.
- The *tropical semiring* $\mathbb{T} = (\mathbb{N}^\infty, \bigvee, +, 0)$ — the natural numbers, completed with an infinite element, with the join operator and arithmetic addition as semiring addition and multiplication, respectively.

These examples are all *continuous semirings*, which were the basis of the weighted relational models studied in [1].

Definition 2.11. *A complete semiring is* ordered *if there is a partial order on $|\mathcal{R}|$ such that multiplication is monotone, and $I \subseteq J$ and $a_i \leq b_i$ for all $i \in I$ implies $\sum_{i\in I} a_i \leq \sum_{j\in J} b_j$. It is* continuous *if $|\mathcal{R}|$ is directed complete, multiplication is continuous and for any set Δ of I-indexed families such that $\{A_i \mid A \in \Delta\}$ is directed for each $i \in I$: $\sum_{i\in I} \bigvee\{A_i \mid A \in \Delta\} = \bigvee\{\sum_{i\in I} A_i \mid A \in \Delta\}$.*[3]

We may obtain further examples of complete semirings by completing a semiring which is positive (i.e. $a+b=0$ implies $a=b=0$) with its *formal sum*, as follows:[4]

Definition 2.12. *Given a positive semiring $(\mathcal{R}, +, 0, \cdot, 1)$, define the complete semiring $\mathcal{R}^\infty = \{|\mathcal{R}| \cup \{\infty\}, \Sigma, \cdot, 1)$, where:*

$$\sum_{i\in I} f(i) \triangleq \begin{cases} \sum_{j\in I\backslash f^{-1}(0)} f(j) & \text{if } I\backslash f^{-1}(0) \text{ is finite, and } f^{-1}(\infty) = \varnothing \\ \infty & \text{otherwise} \end{cases}$$

$$a \cdot \infty = \infty \cdot a \triangleq \begin{cases} 0 & \text{if } a = 0 \\ \infty & \text{otherwise} \end{cases}.$$

[3] Thus it is determined by the binary and empty sum, since $\sum_{i\in I} a_i = \bigvee_{J \subseteq_{fin} I} \sum_{j\in J} a_j$.

[4] Every complete monoid is positive: suppose $a+b=0$ and let $a_i = a$ and $b_i = b$ for all $i \in \mathbb{N}$. Then $a = a + 0 = a_0 + \sum_{i\in\mathbb{N}}(b_i + a_{i+1}) = \sum_{i\in\mathbb{N}}(a_i + b_i) = 0$.

Since we can obtain a positive semiring by adjoining a zero element to any semigroup with a distributive multiplication, formal completion allows any semiring to be used to weight computations. However:

Proposition 2.13. *If $|\mathcal{R}|$ is finite then $\mathcal{R}^\infty$ is not continuous.*

Proof. Let $\underline{n} \triangleq \sum_{1 \leq i \leq n} 1$. If $\mathcal{R}$ is ordered then $\underline{0} \leq \underline{1} \leq \ldots$ and so $\bigvee_{J \subseteq_{fin} \mathbb{N}} \sum_{j \in J} 1 = \bigvee_{m \in \mathbb{N}} \underline{m} = \underline{n}$ for some n. But by definition of the formal sum, $\sum_{i \in \mathbb{N}} 1 = \infty$. □

Any cpo-enrichment on a complete monoid enriched SMC lifts to its biproduct completion — so in particular if $\mathcal{R}$ is a continuous semiring then $\mathcal{R}^\Pi$ bears a cpo-enrichment. Conversely:

Proposition 2.14. *In any cpo-enriched category with biproducts the internal semiring is continuous.*

Proof. For $J \subseteq I$, let $\delta^J_A : A \to \bigoplus_{i \in I} A = \langle g_i \mid i \in I \rangle$, where $g_i = \begin{cases} \mathsf{id}_A & \text{if } i \in J \\ 0_{A,A} & \text{otherwise} \end{cases}$.
By uniqueness, $0_{A,A}$ is the least element in $\mathcal{C}(A, A)$, and so $\delta^I_A = \bigvee_{J \subseteq_{fin} I} \delta^J_A$.
So for any $\{f_i : A \to B \mid i \in I\}$,

$$\sum_{i \in I} f_i = \delta^I_A; [f_i \mid i \in I] = (\bigvee_{J \subseteq_{fin} I} \delta^J_A); [f_i \mid i \in I] = \bigvee_{J \subseteq_{fin} I} \delta^J_A; [f_i \mid i \in I] = \bigvee_{J \subseteq_{fin} I} \sum_{j \in J} f_j$$

□

Hence if $\mathcal{C}$ is a category with biproducts for which the internal semiring $\mathcal{R}_\mathcal{C}$ is not continuous then $\mathcal{C}$ cannot be cpo-enriched.

Remark 2.15. *In Sections 8 and 9 we describe the semantics in categories with biproducts of programming languages (PCF and Idealized Algol) weighted with values from the internal semiring. If this is the formal completion of a finite semiring, Propositions 2.13 and 2.14 imply that the model cannot be cpo-enriched. Standard domain-theoretic techniques for defining fixed points are therefore not available: in Section 6 we shall develop alternatives based on biproducts and complete semiring enrichment.*

3. Change of Base

We now describe a method for obtaining categories enriched over the category of $\mathcal{R}$-weighted relations from intensional semantics such as games models. This is an instance of the notion of *change of base* [15, 16], which uses a monoidal functor between monoidal categories $\mathcal{V}$ to $\mathcal{W}$ to transform any model in a $\mathcal{V}$-category to a model in a $\mathcal{W}$-category satisfying the same equational theory.

Definition 3.1. *Given monoidal categories $\mathcal{V}$ and $\mathcal{W}$ any monoidal functor $(F, m) : \mathcal{V} \to \mathcal{W}$ induces a 2-functor (*change of base*) F_* from the category of $\mathcal{V}$-categories to the category of $\mathcal{W}$-categories: if $\mathcal{C}$ is a $\mathcal{V}$-category, then $F_*(\mathcal{C})$ is the $\mathcal{W}$-category over the same objects, with $F_*(\mathcal{C})(A, B) = F(\mathcal{C}(A, B))$, and composition and identity morphisms $m_{\mathcal{C}(A,B),\mathcal{C}(B,C)}; F(\mathsf{comp}_{A,B,C})$ and $m_I; F(\mathsf{id}_A)$.*

A simple example is the change of base induced by the monoidal hom-functor $\mathcal{V}(I, \_) : \mathcal{V} \to \mathbf{Set}$, which sends each $\mathcal{V}$-category $\mathcal{C}$ to its *underlying category* $\mathcal{C}_0$. The change of base induced by a monoidal functor F comes with an identity-on objects, F-on-morphisms functor $F_0 : \mathcal{C}_0 \to F_*(\mathcal{C})_0$ (which is a natural transformation from $\mathcal{V}(I, \_)_*$ to $\mathcal{W}(I, F\_)_*$).

We will describe a change of base from a class of "qualitative" models — categories enriched over the category of coherence spaces — into a class of quantitative ones — $\mathcal{R}^\Pi$-enriched categories.

Definition 3.2. *A* coherence space *[17] D is a pair $(|D|, \mathrel{\substack{\frown\\\smile}}_D)$ where $\mathrel{\substack{\frown\\\smile}}_D \subseteq |D| \times |D|$ is a symmetric and reflexive relation (coherence) on the set $|D|$ (the web). A clique of D is a set $X \subseteq |D|$ which is pairwise coherent: $d, d' \in X \Longrightarrow d \mathrel{\substack{\frown\\\smile}}_D d'$.*

Definition 3.3. *The category* **CSpace** *has coherence spaces as objects; morphisms from C to D are relations $f \subseteq |C| \times |D|$ such that:*

$$\text{if } (c, d), (c', d') \in f \text{ then } c \mathrel{\substack{\frown\\\smile}}_C c' \text{ implies } d \mathrel{\substack{\frown\\\smile}}_D d' \wedge (d = d' \Longrightarrow c = c')$$

with relational identity and composition — i.e. for morphisms $g : D \to E$,

$$f; g : C \to E = \{(c, e) \in |C| \times |E| \mid \exists d \in D.(c, d) \in f \wedge (d, e) \in g\}$$

The tensor product of coherence spaces is the cartesian product of their webs and coherence relations, giving symmetric monoidal structure on **CSpace**. — i.e. $|D \otimes E| = |D| \times |E|$, with $(d, e) \mathrel{\substack{\frown\\\smile}}_{D\otimes E} (d', e')$ if and only if $d \mathrel{\substack{\frown\\\smile}}_D d'$ and $e \mathrel{\substack{\frown\\\smile}}_E e'$. On morphisms, $f \otimes g = \{((a, b), (c, d) \mid (a, c) \in f \wedge (b, d) \in g\}$. The unit I is the singleton coherence space $\{*\}$.

Typically, models of linear type theory based on sets and relations augmented with some notion of consistency — and thus a forgetful monoidal functor into **CSpace** — may be enriched over coherence spaces: any symmetric monoidal closed category enriches over itself, and thus change of base yields a **CSpace**-category. Examples include categories of hypercoherences [24], event structures [25] and concrete data structures [26].

3.1. From Cliques to Weighted Relations

We define a monoidal functor from coherence spaces into $\mathcal{R}$-weighted relations which sends each coherence space to its web, and each morphism to its characteristic function:

Definition 3.4. *If $\mathcal{R}$ is a complete semiring, define $\Phi^\mathcal{R} : \mathbf{CSpace} \to \mathcal{R}^\Pi$:*

$$\Phi^\mathcal{R}(D) = |D| \text{ and } \Phi^\mathcal{R}(f)(c, d) = \begin{cases} 1 & \text{if } (c, d) \in f \\ 0 & \text{otherwise} \end{cases}$$

To establish functoriality of $\Phi^{\mathcal{R}}$ we use the following property of composition in **CSpace**.

Lemma 3.5. *For any morphisms $f : C \to D$ and $g : D \to E$, if $(c, e) \in f; g$ then there exists a* unique *$d \in D$ such that $(c, d) \in f$ and $(d, e) \in g$.*

Proof. Existence holds by definition. For uniqueness, suppose $(c, d), (c, d') \in f$ and $(d, e), (d', e) \in g$. Then $d \mathrel{\frown\mkern-12mu\smile}_D d'$ by coherence of f and hence $d = d'$ by coherence of g. $\square$

Lemma 3.6. $\Phi^{\mathcal{R}}(f); \Phi^{\mathcal{R}}(g) = \Phi^{\mathcal{R}}(f) : \Phi^{\mathcal{R}}(g)$.

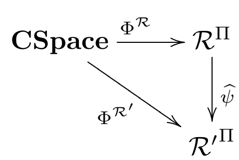
Proof. Suppose $(c, e) \in f; g$. Then by Lemma 3.5 there exists a unique $d' \in |D|$ such that $(c, d') \in f$ and $(d', e) \in g$. Thus for all $d \in |D|$:

$$\Phi^{\mathcal{R}}(f)(c, d).\Phi^{\mathcal{R}}(g)(d, e) = \begin{cases} 1 & \text{if } d = d' \\ 0 & \text{otherwise} \end{cases}$$

Hence $\Phi^{\mathcal{R}}(f); \Phi^{\mathcal{R}}(g)(c, e) = \sum_{d \in D} \Phi^{\mathcal{R}}(f)(c, d).\Phi^{\mathcal{R}}(g)(d, e) = 1 = \Phi^{\mathcal{R}}(f; g)(c, e)$.
If $(c, e) \notin f; g$ then for all $d \in D$, either $(c, d) \notin f$ or $(d, e) \notin g$ and so $\Phi^{\mathcal{R}}(f)(c, d).\Phi^{\mathcal{R}}(g)(d, e) = 0$.
Hence $\Phi^{\mathcal{R}}(f); \Phi^{\mathcal{R}}(g)(c, e) = \sum_{d \in D} \Phi^{\mathcal{R}}(f)(c, d).\Phi^{\mathcal{R}}(g)(d, e) = 0 = \Phi^{\mathcal{R}}(f; g)(c, e)$. $\square$

Evidently, $\Phi^{\mathcal{R}}$ is *strict monoidal* — $\Phi^{\mathcal{R}}(I) = I$ and $\Phi^{\mathcal{R}}(D \otimes E) = \Phi^{\mathcal{R}}(D) \otimes \Phi^{\mathcal{R}}(E)$ — and faithful. So for each complete semiring $\mathcal{R}$ we have a change of base sending each **CSpace**-category $\mathcal{C}$. to a $\mathcal{R}^{\Pi}$-category $\mathcal{C}^{\mathcal{R}}$ with a faithful functor $\Phi^{\mathcal{R}}_0 : \mathcal{C}_0 \to \mathcal{C}^{\mathcal{R}}_0$.

Remark 3.7. *We may also use change of base to move from models enriched over one category of $\mathcal{R}$-weighted relations to another. Any homomorphism of complete semirings $\psi : \mathcal{R} \to \mathcal{R}'$ lifts to an identity-on-objects, strict monoidal functor $\widehat{\psi} : \mathcal{R}^{\Pi} \to \mathcal{R}'^{\Pi}$ by sending an $\mathcal{R}$-weighted relation $f : A \times B \to \mathcal{R}$ to $\psi \cdot f : A \times B \to \mathcal{R}'$. This factorizes $\Phi^{\mathcal{R}'}$ — i.e. the following diagram commutes*



— and thus induces a change of base functor from $\mathcal{R}^{\Pi}$-categories to $\mathcal{R}'^{\Pi}$-categories which factorizes change of base from **CSpace***-categories to $\mathcal{R}^{\Pi}$-categories.*

In particular, if $\mathcal{R}$ is idempotent then the unique homorphism into $\mathcal{R}$ from the Boolean semiring $\mathbb{B} = (\{\top, \bot\}, \bigvee, \wedge, \top)$ induces a monoidal functor from the category **Rel** *of sets and relations (which is isomorphic to $\mathbb{B}^{\Pi}$) into $\mathcal{R}^{\Pi}$. This can be used to change the base of* **Rel***-enriched categories to relations weighted over an idempotent semiring.*

3.2. An Example: Games and History-Sensitive Strategies

We illustrate the change of base induced by $\Phi^{\mathcal{R}} : \mathbf{CSpace} \rightarrow \mathcal{R}^{\Pi}$ by using it to derive a family of quantitative games models from a **CSpace**-enriched category of games and strategies. Its underlying category is essentially the games model of *Idealized Algol* (IA) introduced by Abramsky and McCusker in [14] and obtained by relaxing the *innocence* constraint on strategies in the Hyland-Ong games model of PCF [27] (it is derived from a different "linear decomposition" of this model).

We start with the notions of arena, justified sequence, strategy and composition which are essentially as defined and studied in the literature (e.g. [28, 29]) to which we refer for further details.

Definition 3.8. *An arena A is a labelled, bipartite directed acyclic graph, given as a tuple $(M_A, M_A^I, \vdash_A, \lambda_A)$ — where M_A is a set of moves (nodes), $M_A^I \subseteq M_A$ is a specified set of* initial moves *(source nodes), $\vdash_A \subseteq M_A \times (M_A \backslash M_A^I)$ is the enabling (edge) relation and $\lambda_A : M_A \rightarrow \{O, P\} \times \{Q, A\}$ is a function partitioning the moves between Player and Opponent, and labelling them as either questions or answers, such that initial moves belong to Opponent and each answer is enabled by a question.*

Definition 3.9. *A justified sequence t over an arena A is a sequence over M_A, together with a function (justification) sending each non-empty prefix $s \sqsubseteq t$ to a prefix $j(s) \sqsubset s$.*

The set L_A of legal sequences *over A consists of Opponent/Player alternating justified sequences t on A such that for each non-empty prefix $sb \sqsubseteq t$, $j(sb) = \varepsilon$ if b is initial, and otherwise it is a prefix $ra \sqsubseteq s$ such that $a \vdash b$, and:*

- visibility — *ra is in the set $\mathsf{view}(s)$ of visible prefixes of s, which is defined:* $\mathsf{view}(s') = \begin{cases} \{s'\} & \text{if } s' = \varepsilon \text{ or } j(s') = \varepsilon \\ \mathsf{view}(r') \cup \{s'\} & \text{if } j(s') = r'a' \end{cases}$
- well-bracketing — *if b is an answer then ra is the pending question of s, defined:* $\mathsf{pending}(s'a') = \begin{cases} s'a' = & \text{if } \lambda_A^{QA}(a') = Q \\ \mathsf{pending}(s') & \text{if } \lambda_A^{QA}(a') = A \wedge \mathsf{pending}(s') \downarrow \end{cases}.$

We refer to the literature for further details of these conditions, which may be relaxed or modified in various ways to model different combinations of computational effects: our results do not depend on their particular properties except in that they lead to a well-defined symmetric monoidal category of games, and ultimately a fully abstract model of Idealized Algol. Indeed, the well-bracketing condition may be recovered from the linear structure of our model (cf. *linear continuation passing* [30]: we include it for continuity with [14]).

We require the following constructions on arenas:

- Disjoint union: $A \uplus B \triangleq (M_A + M_B, M_A^I + M_B^I, \vdash_A + \vdash_B, [\lambda_A, \lambda_B])$.
- The *graft* of A onto the root nodes of B: $A \Rightarrow B \triangleq (M_A + M_B, \mathtt{inr}(M_B^I), (\vdash_A + \vdash_B) \cup \mathtt{inr}(M_B^I) \times \mathtt{inl}(M_A^I), [\overline{\lambda_A}, \lambda_B])$.

Definition 3.10. *A deterministic strategy on an arena A is set of even-length legal sequences $\sigma \subseteq L_A$ which is* even-branching *— i.e. if $s, t \in \sigma$ then $s \sqcap t$ (their greatest common prefix) is even-length, so they first differ (if at all) on an Opponent move.*

The only point of significant difference from the definitions of [14] is that strategies are not required to be non-empty and even-prefix-closed. This does not affect the proofs of key results nor change the denotations of programs in the model, which possess both of these properties.

Strategies are composed by "parallel composition plus hiding":

Definition 3.11. *For $S \subseteq L_{A \Rightarrow B}$ and $T \subseteq L_{B \Rightarrow C}$, let $S|T$ be the set of "interaction sequences", which are justified sequences u on $M_A + M_B + M_B$ such that $u\restriction(A \Rightarrow B) \in S$ and $u\restriction(B \Rightarrow C) \in T$. We define:*

$$\rho; \sigma = \{t \in L_{A \Rightarrow B} \mid \exists u \in \sigma|\tau.t = u\restriction A \Rightarrow C\}.$$

The identity $\mathsf{id}_A : A \Rightarrow A$ is the strategy consisting of the sequences on $A \Rightarrow A$ for which each even prefix projects to the same (legal) sequence on both components. Proof that composition is well-defined and associative, and id is its unit on both sides follows similar results in the literature (e.g. [28, 29]) which do not depend fundamentally on even-prefix-closure and non-emptiness of strategies.

3.3. Coherence Space Enrichment of Games

We will now define a **CSpace**-category of games based on the constructions already described. The fact that the inclusion order on strategies provides an enrichment over the category of cpos and continuous functions was already used in [14], as in other games models, to construct fixed points. Our results amount to showing that in fact composition is a bilinear and stable function. This is also implicit in the definition in [18] of a monoidal functor from a similar category of games into a category of ordered coherence spaces. However, this depends on a number of particular features — notably the reconstruction of a strategy on $G_1 \multimap G_2$ from its projections on G_1 and G_2, which is not always possible, so **CSpace**-enrichment is in this sense a more general property.

Definition 3.12. *A* game *is a pair (A, P) of an arena A and a set of justified sequences $P \subseteq L_A$.*

For any game $G = (A, P)$ let $\mathsf{Coh}(G)$ be the coherence space $(P^E, \frown)$, where P^E is the set of even-length sequences in P and $s \frown t$ if $s \sqcap t$ is even length. Thus the morphisms from I to $\mathsf{Coh}(G)$ in **CSpace** — i.e. the cliques of $\mathsf{Coh}(G)$ — correspond precisely to the strategies on A which are subsets of P.

Definition 3.13. *Let* **G** *be the* **CSpace***-category in which objects are games, and the coherence space of morphisms from G_1 to G_2 is $\mathsf{Coh}(G_1 \multimap G_2)$, where $G_1 \multimap G_2$ is the game $(A_1 \Rightarrow A_2, \{s \in L_{A_1 \Rightarrow A_2} \mid s\restriction A_1 \in P_1 \wedge s\restriction A_2 \in P_2\})$.*

- *For each object $G = (A, P)$, the identity on G is the set of $(*, s) \in I \rightarrow$ $\mathbf{CSpace}(G \multimap G)$ such that $s \in \mathsf{id}_A$.*
- $\mathsf{comp}_{G_1,G_2,G_3} : \mathsf{Coh}(G_1 \multimap G_2) \otimes \mathsf{Coh}(G_2 \multimap G_3) \rightarrow \mathsf{Coh}(G_1 \multimap G_3)$ *is the set of* $((r, s), t) \in |\mathsf{Coh}(G_1 \multimap G_2)| \times |\mathsf{Coh}(G_2 \multimap G_3)| \times |\mathsf{Coh}(G_1 \multimap G_3)|$ *such that* $\exists u \in \{r\}|\{s\}.u{\restriction}G_1 \multimap G_3 = t$.

To prove that this is well-defined we use two observations. First, that the parallel composition with hiding of $\rho : I \rightarrow \mathsf{Coh}(G_1 \multimap G_1)$ and $\sigma : I \rightarrow \mathsf{Coh}(G_2 \multimap G_3)$ as strategies on $A_1 \Rightarrow A_2$ and $A_2 \Rightarrow A_3$ (Definition 3.11) is equal to the relational composition of $\rho \otimes \sigma$ with $\mathsf{comp}_{G_1,G_2,G_3}$. The second is essentially a version of the "zipping lemma" established in [31] for AJM games. First, we prove:

Lemma 3.14 (Switching Lemma)**.** *Suppose $sm \sqsubseteq u \in \sigma|\tau$, then:*

- *If m is a move in B then $s{\restriction}A \Rightarrow C$ is odd-length.*
- *If m is a move in C (resp. A) then $s \restriction A \Rightarrow B$ (resp. $s \restriction B \Rightarrow C$) is even-length.*

Proof. By induction on length — e.g. suppose m is in B but the last move in s is (w.l.o.g.) in A. Then by hypothesis $s{\restriction}B \Rightarrow C$ is even-length, so m must be an Opponent move in $B \Rightarrow C$ — and thus a Player move in $A \Rightarrow B$. So the preceding move is an Opponent move in $A \Rightarrow C$ and thus $s{\restriction}A \Rightarrow C$ is odd-length as required. □

Note that this property holds independently of any "switching condition" for play in $\Rightarrow$, so it does not depend on the visibility condition.

Lemma 3.15 (Zipping Lemma)**.** *For any $t \in \sigma;\tau$ there exists a* unique *$u \in \sigma|\tau$ such that $t = u{\restriction}A \Rightarrow C$.*

Proof. Suppose $u, u' \in \sigma|\tau$ such that $u \restriction A \Rightarrow C = u' \restriction A \Rightarrow C = t$. We show by induction on length that every prefix of u is a prefix of u' and vice-versa: suppose $rm \sqsubseteq u$, where $r \sqsubseteq u'$. If m is a move in $A \Rightarrow C$ then $r \neq u'$ (since m appears in u') and if m is a move in B then by the switching lemma, $r{\restriction}A \Rightarrow C$ is odd-length and so $r \neq u'$ (since $u{\restriction}A \Rightarrow C$ is even-length). So suppose $rm' \sqsubseteq u'$.

If m, m' are both moves in $A \Rightarrow C$ then $m = m'$, since $u{\restriction}A \Rightarrow C = u'{\restriction}A \Rightarrow C$. Otherwise, suppose m is a move in B, and thus a Player move in either $B \Rightarrow C$ or $A \Rightarrow B$. Suppose, without loss of generality, the former. Then if m' is a move in A then it must be an Opponent move in $A \Rightarrow B$ (since m is an Opponent move in $A \Rightarrow B$) and a Player move in $A \Rightarrow C$ (by Lemma 3.14, since $r{\restriction}A \Rightarrow C$ is odd-length). This is impossible and so m is also a Player move in $A \Rightarrow B$. But $(u{\restriction}A \Rightarrow B) \sqcap (u{\restriction}A \Rightarrow B)$ is even-length and therefore $m = m'$. □

Proposition 3.16. $\mathbf{G}$ *is a well-defined* $\mathbf{CSpace}$*-enriched category.*

Proof. To show that $\mathsf{comp}_{G_1,G_2,G_3}$ is a well-defined morphism from $\mathsf{Coh}(G_1 \multimap G_2) \otimes \mathsf{Coh}(G_2 \multimap G_3) \to \mathsf{Coh}(G_1 \multimap G_3)$ in **CSpace**, suppose $((r,s),t),((r,s'),t') \in \mathsf{comp}_{G_1,G_2,G_3}$, and $(r,s) \frown (r',s')$ — i.e. $r \frown r'$ and $s \frown s'$. Let $\rho : G_1 \multimap G_2 = \{r, r'\}$ and $\sigma = \{s, s'\}$. Then:

- By definition of composition, $t, t' \in \rho; \sigma$ and hence $t \frown t'$.
- If $t = t'$ then there exist $u \in \{r\}|\{s\}$ and $u' \in \{r'\}|\{s'\}$ such that $u{\upharpoonright}G_1 \multimap G_3 = u'{\upharpoonright}G_1 \multimap G_3 = t$. Then $u, u' \in \rho|\sigma$ and hence by Lemma 3.15, $u = u'$ and hence $r = r'$ and $s = s'$ as required.

CSpace is well-pointed with respect to morphisms from the tensor unit I (i.e. $f : X \to Y = g : X \to Y$ if $x; f = x; g$ for all $x : I \to X$). Hence by the associativity and unitality properties of parallel composition with hiding, comp and id satisfy the corresponding diagrams in **CSpace**, making **G** a well-defined **CSpace**-category. □

We define **CSpace**-enriched symmetric monoidal (closed) structure on **G**, given by the enriched functor $\odot$ with the action on objects:

$$G_1 \odot G_2 = (A_1 \uplus A_2, \{s \in L_{A_1 \uplus A_2} \mid s{\upharpoonright}A_1 \in P_1 \wedge s{\upharpoonright}A_2 \in P_2\})$$

and the **CSpace**-morphism from $\mathsf{Coh}(G_1 \multimap G_3) \otimes \mathsf{Coh}(G_2 \multimap G_4)|$ to $\mathsf{Coh}(G_1 \odot G_2 \multimap G_3 \odot G_4)$ consisting of triples $((r,s),t) \in |\mathsf{Coh}(G_1 \multimap G_3) \times \mathsf{Coh}(G_2 \multimap G_4)| \times |\mathsf{Coh}(G_1 \odot G_2 \multimap G_3 \odot G_4)|$ such that $r = t{\upharpoonright}G_1 \multimap G_3$ and $s = t{\upharpoonright}G_2 \multimap G_4$. The unit for $\odot$ is the game over the arena with no moves, with ε as its only play.

Remark 3.17. *Unlike [14] (and similar games models) the underlying symetric monoidal category of games is not* affine *— the unit for the tensor is not a terminal object. This is a necessary requirement for non-trivial* **CSpace***-enrichment — in any* **CSpace***-category $\mathcal{C}$, each coherence space $\mathcal{C}(A,B)$ must contain the empty clique $\bot_{A,B}$ but since $\bot_{I,I} \otimes f = \bot_{A,B}$ for any $f : A \to B$, the identity on I is not equal to $\bot_{I,I}$ unless $\mathcal{C}(A,B)$ is empty for every A, B. In* **G** *there are two morphisms from the tensor unit to itself, one empty ($\bot_{I,I}$) and the other containing the empty sequence (id_I).*

In other respects, this corresponds to symmetric monoidal structure on the underlying category of games already identified in [14], giving associator, unitor and twist maps making the relevant diagrams commute. The (natural) isomorphism $\mathsf{Coh}(G_1 \otimes G_2, G_3) \cong \mathsf{Coh}(G_1, G_2 \multimap G_3)$ in **CSpace** yields symmetric monoidal closure.

3.4. Weighted Strategies

For any complete semiring $\mathcal{R}$, change of base yields a symmetric monoidal closed category $\mathbf{G}^{\mathcal{R}} \triangleq \Phi^{\mathcal{R}}_*(\mathbf{G})$ enriched in $\mathcal{R}^{\Pi}$. Concretely, a morphism $\phi : G_1 \to G_2$ in the underlying category $\mathbf{G}^{\mathcal{R}}_0$ is a $\mathcal{R}$*-weighted strategy* — a map from the set of even-length plays $P^E_{G_1 \multimap G_2}$ into $\mathcal{R}$. Composition of ϕ with $\psi : G_2 \to G_3$ may be defined directly — $(\phi;\psi)(t) =$

$$\sum \{\phi(u{\upharpoonright}A_1 \Rightarrow A_2) \cdot \psi(u{\upharpoonright}A_2 \Rightarrow A_3) \mid u \in P^E_{A_1 \multimap A_2} | P^E_{A_2 \multimap A_3} \wedge u{\upharpoonright}A_1 \Rightarrow A_3 = t\}$$

The tensor product of $\mathcal{R}$-weighted strategies $\phi : G_1 \to G_2, \psi : G_3 \to G_4$ is given:

$$(\phi \odot \psi)(s) = \phi(s{\upharpoonright}A_1 \Rightarrow A_2) \cdot \psi(s{\upharpoonright}A_3 \Rightarrow A_4)$$

The induced (identity-on-objects) monoidal functor between the underlying categories $\Phi^{\mathcal{R}}_0 : \mathbf{G}_0 \to \mathbf{G}^{\mathcal{R}}_0$ sends each deterministic strategy $\sigma : G_1 \to G_2$ to the $\mathcal{R}$-weighted strategy $\sigma_{\mathcal{R}}$ with $\sigma_{\mathcal{R}}(s) = \begin{cases} 1 & \text{if } s \in \sigma \\ 0 & \text{otherwise} \end{cases}$.

By choosing particular semirings we may relate our categories of weighted strategies to examples in the literature. For instance, if $\mathcal{R}$ is the *Boolean semiring* then morphisms in $\mathbf{G}^{\mathcal{R}}_0$ correspond to sets of even-length legal sequences — i.e. non-deterministic strategies, as in the model of may-testing studied in [32] and [21].

If $\mathcal{R}$ is the *probability semiring*, $\mathbb{R}^{\infty}_{+}$ then $\mathcal{R}$-weighted strategies correspond precisely to the "probabilistic pre-strategies", introduced by Danos and Harmer [19]. These may be refined further by imposing constraints requiring the weights to correspond to probabilities, although the precursor model of prestrategies is already fully abstract for Probabilistic Algol.

If $\mathcal{R}$ is the *tropical semiring* $\mathbb{T}$ then $\mathbf{G}^{\mathcal{R}}_0$ corresponds to a sequential version of Ghica's category of *slot games* [33]. This was introduced as a model quantifying resources used in stateful and concurrent computation, in a presentation rather different to weighted strategies. Assuming a distinguished token Ⓢ, which does not occur in the set of moves of any arena, we may define a *sequence with costs* on the game A to be an interleaving of a sequence $s \in P_A$ of A with a sequence of Ⓢ moves: a *strategy-with-costs* on a A is a set of such sequences, closed under permutation of Ⓢ and non-Ⓢ moves, and erasure of Ⓢ moves. Composition of strategies with costs does not hide Ⓢ moves, so that all Ⓢ moves of σ and τ propagate to $\sigma;\tau$. In other words, if we take the weight of a sequence-with-costs to be the number of Ⓢ moves it contains, strategies-with-costs correspond to $\mathbb{T}$-weighted strategies on the underlying arena, and their composition to composition in the category $\mathbf{G}^{\mathbb{T}}_0$.

Remark 3.18. *We are able to enrich our category of games over coherence spaces because the alternation condition on sequences leads to a simple notion of coherence (branching on Opponent moves), with cliques corresponding to deterministic strategies. These can be used to interpret sequential programs with various side-effects, but in games models for concurrent languages, including the original category of slot games [33], the alternation condition is lifted, and so this enrichment is not available. However, we may enrich over the category of relations, using similar definitions for composition of strategies but no longer requiring stability. As we have noted, we may change the base of any* **Rel**-*enriched model to the category of free semimodules over an* idempotent *semiring — of which the tropical semiring is an instance.*

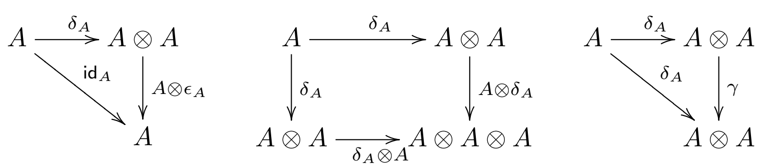
4. Lafont Categories

We now return to our general notion of categorical model (symmetric monoidal categories with biproducts). To allow controlled duplication and discarding of resources in a weighted setting, we interpret the exponential $!A$ of intuitionistic linear type theory as the *cofree commutative comonoid* on A. Such a model is known as a Lafont category.

Definition 4.1. *A* Lafont category *[8] is a symmetric monoidal* closed *category $\mathcal{C}$ with finite products such that the forgetful functor into $\mathcal{C}$ from its category of commutative comonoids and comonoid morphisms has a right adjoint.*

To spell this out: in the category $\mathsf{comon}(\mathcal{C})$ of commutative comonoids on $\mathcal{C}$:

- Objects are commutative comonoids in $\mathcal{C}$ — triples $(A, \delta_A, \epsilon_A)$ consisting of an object A of $\mathcal{C}$, and morphisms $\delta_A : A \otimes A \to A$ and $\epsilon_A : A \to I$ in $\mathcal{C}$ such that $\delta_A; (A \otimes \epsilon_A) = \mathsf{id}_A$, $\delta_A; \delta_A \otimes A = \delta_A; (A \otimes \delta_A)$ (up to associativity and unity isomorphisms of $\mathcal{C}$) and $\delta_A; \gamma_A = \delta_A$, where $\gamma_A : A \otimes A \to A \otimes A$ is the symmetry isomorphism:



- Morphisms from $(A, \delta_A, \epsilon_A)$ to $(B, \delta_B, \epsilon_B)$ are morphisms $f \in \mathcal{C}(A, B)$ which satisfy $\delta_A; (f \otimes f) = f; \delta_B$ and $\epsilon_A = f; \epsilon_B$.

A cofree commutative comonoid on an object B in $\mathcal{C}$ is a commutative comonoid $!B$ with a natural equivalence between the morphisms into B in $\mathcal{C}$, and the comonoid morphisms into $!B$. To be more precise:

Definition 4.2. *A commutative comonoid $(!B, \delta_{!B}, \epsilon_{!B})$ is a* cofree commutative comonoid *on B if there is a morphism $\mathsf{der}_B \in \mathcal{C}(!B, B)$ and for each commutative comonoid $(A, \delta_A, \epsilon_A)$ an operation $(\_)^\dagger$ sending each morphism $f \in \mathcal{C}(A, B)$ to a comonoid morphism $f^\dagger$ from $(A, \delta_A, \epsilon_A)$ to $(!B, \delta_{!B}, \epsilon_{!B})$ such that:*

$f^\dagger; \mathsf{der}_B = f$ *for all* $f \in \mathcal{C}(A, B)$ *and* $(g; \mathsf{der}_B)^\dagger = g$ *for all* $g \in \mathsf{comon}(\mathcal{C})(A, !B)$

$(\_)^\dagger$ *should be natural in A — i.e. $h; f^\dagger = (h; f)^\dagger$ for any comonoid morphism $h : C \to A$,*

Thus $\mathcal{C}$ is a Lafont category if and only if there are cofree commutative comonoids on all objects. In this case the (monoidal) adjunction between the forgetful and cofree functors resolves a monoidal comonad $! : \mathcal{C} \to \mathcal{C}$, with Kleisli triple $(!, \mathsf{der}, (\_)^\dagger)$. Its co-Kleisli category $\mathcal{C}_!$ is therefore Cartesian closed. We will refer to such a comonad as the *free exponential.*

4.1. The Lafont Construction

In fact, distributive biproducts are a key element of a general construction of the cofree commutative comonoids from symmetric tensor powers.

Definition 4.3. *A family of objects* $\{B^i \mid i \in \mathbb{N}\}$ *of a symmetric monoidal category are* symmetric tensor powers *of the object* B *if:*

- *For each* n *there is a morphism* $\mathsf{eq}_n : B^n \to B^{\otimes n}$ *such that* (B^n, eq_n) *is an* equalizer *for the group* $\mathsf{perm}(B^{\otimes n})$ *of automorphisms on* $B^{\otimes n}$ *derived from the permutations on* $\{1, \ldots, n\}$.
- *These equalizers are preserved by the monoidal product — i.e.* $(B^m \otimes B^n, \mathsf{eq}_m \otimes \mathsf{eq}_n)$ *is an equalizer for the monoidal products of pairs of permutation automorphisms on* $\{1, \ldots m\}$ *and* $\{1, \ldots, n\}$.

An object $!B$ *in a SMC with biproducts is a* Lafont exponential *of* B *if it is the biproduct of all symmetric tensor powers of* B *— i.e. if* $!B = \bigoplus_{n \in \mathbb{N}} B^n$.

The Lafont exponential is given the structure of a commutative comonoid by defining $\epsilon_{!B} : !B \to I = \pi_0$ and $\delta_{!B} : !B \to !B \otimes !B = \langle \pi_{m+n}; \delta_{m,n} \mid m, n \in \mathbb{N} \rangle$, where $\delta_{m,n} : B^{m+n} \to B^m \otimes B^n$ is the unique morphism such that $\mathsf{eq}_{m+n} = \delta_{m,n}; (\mathsf{eq}_m \otimes \mathsf{eq}_n)$.

Proposition 4.4. *If* $!B$ *is the Lafont exponential of* B *then* $(!B, \delta_{!B}, \epsilon_{!B})$ *is the cofree commutative comonoid on* B.

Proof. For details, see e.g. [9]: $\mathsf{der}_B : !B \to B = \pi_1$ and for any $f : A \to B$, $f^\dagger : A \to !B = \langle f^i \mid i \in \mathbb{N} \rangle$, where $f^n : A \to B^n$ is the unique morphism such that $f^n; \mathsf{eq}_n = \delta(n); f^{\otimes n}$ (where $\delta(n) : A \to A^{\otimes n}$ is the unique n-ary co-multiplication for A). □

4.2. Constructing Lafont Categories with Biproducts

Thanks to the Lafont construction, any symmetric monoidal closed category with biproducts is a Lafont category if it has symmetric tensor powers. In [21], it was observed that if $\mathcal{C}$ is a symmetric monoidal category enriched over the category of complete lattices, then its *Karoubi envelope* or idempotent completion has symmetric tensor powers. We refer to *loc. cit.* for more detailed discussion, but observe here that this construction may be carried through for any category $\mathcal{C}$ in which the internal semiring $\mathcal{R}_\mathcal{C}$ has *fractions.*

Definition 4.5. *A semiring* $\mathcal{R}$ *has fractions if* $\underline{n} = \sum_{1 \leq i \leq n} 1$ *has a multiplicative inverse* $\frac{1}{n}$ *for each* $n > 0$.

Definition 4.6. *The Karoubi envelope* $\mathcal{K}(\mathcal{C})$ *of a category* $\mathcal{C}$ *is the category in which:*

- *Objects are idempotent morphisms of* $\mathcal{C}$ *(i.e. pairs* $(A, d : A \to A)$ *such that* $d; d = d$*).*

- *Morphisms $g : (A, d) \to (B, e)$ are $\mathcal{C}$-morphisms $g : A \to B$ such that $g = d; g; e$. They are composed as in $\mathcal{C}$, and the identity on (A, d) is d.*

The fully faithful (identity-on-morphisms) functor from $\mathcal{C}$ to $\mathcal{K}(\mathcal{C})$ sending A to (A, id_A) preserves monoidal (closed) structure on-the-nose. If $\mathcal{C}$ has biproducts the so does $\mathcal{K}(\mathcal{C})$ — we specify $\bigoplus_{i\in I}(A_i, d_i) = (\bigoplus_{i\in I} A_i, \bigoplus_{i\in I} d_i)$.

Proposition 4.7. *If $\mathcal{R}_\mathcal{C}$ has fractions then $\mathcal{K}(\mathcal{C})$ has symmetric tensor powers.*

Proof. Let $p : A^{\otimes n} \to A^{\otimes n}$ be the idempotent $\frac{1}{n!} \otimes \sum_{\theta\in\mathsf{perm}(A^{\otimes n})} \theta$.
Then $(A, f)^n \triangleq (A^{\otimes n}, f^{\otimes n}; p)$ is the nth symmetric tensor power for (A, f), as $((A, f^n), f^{\otimes n})$ is the equalizer for the automorphisms in $\mathsf{perm}(A^{\otimes n})$. □

Thus we have a recipe for constructing a Lafont category with biproducts, starting with any **CSpace**-enriched symmetric monoidal closed category $\mathcal{C}$ and complete semiring $\mathcal{R}$ with fractions (such as any idempotent semiring, the completed reals, or any semifield extended with the formal sum):

1. Change the base of $\mathcal{C}$ from **CSpace** to $\mathcal{R}^\Pi$, giving a category $\mathcal{C}_\mathcal{R}$ enriched over $\mathcal{R}$-modules.
2. Complete with biproducts.
3. Complete with idempotents.

(Since idempotent completion preserves biproducts, and vice-versa, steps 2 and 3 may be performed in either order.)

Remark 4.8. *In [21], instead of Step 1 (change-of-base), the starting category $\mathcal{C}$ is augmented with semi-additive structure by taking its* sup-lattice *completion (in which morphisms from A to B are subsets of $\mathcal{C}(A, B)$, composed pointwise). We can see this as a particular case of change-of-base: we may freely construct a* **CSpace***-category $\mathcal{C}_\bullet$ over the objects of $\mathcal{C}$ by taking the web $|\mathcal{C}_\bullet(A, B)|$ to be the set $\mathcal{C}(A, B)$ with the discrete coherence, for which the underlying category is the free pointed category on $\mathcal{C}$ (i.e. we adjoin an additional morphism $\perp_{A,B}$ to each hom-set, which is preserved by composition on both sides). Changing the basis of $\mathcal{C}_\bullet$ to $\mathcal{R}^\mathbb{B}$ (i.e.* **Rel***) corresponds to sup-lattice completion of the original category $\mathcal{C}$. More generally, changing the base of $\mathcal{C}_\bullet$ to $\mathcal{R}^\Pi$ corresponds to a free $\mathcal{R}$-module completion of $\mathcal{C}$.*

For the main examples of categories $\mathcal{C}$ considered in [21] — the terminal SMCC (one-object, with one map); and a category of games corresponding to the free SMCC generated by one object — there is no available enrichment over coherence spaces other than the discrete one, and thus sup-lattice enrichment gives the minimal $\mathcal{R}$-enriched category with a faithful functor from our underlying category of games — as is shown by universality results for a corresponding resource calculus. However, as we have seen, our category of history sensitive games does bear a non-trivial enrichment over coherence spaces, giving rise to a $\mathcal{R}$-enriched category of games which is not equivalent to $\mathcal{R}$-module completion.

Composing steps $1-3$ gives a Lafont category with biproducts — $\mathcal{K}(\mathcal{C}_{\mathcal{R}}^{\Pi})$ — with a faithful (strong monoidal) functor from $\mathcal{C}$, so they represent a methodology for building a quantitative model which retains the intensional character of the base category $\mathcal{C}$: e.g. in [21], it was shown that the full structure of Hyland-Ong style games may be reconstructed by applying these steps to a basic SMCC of tree traversals. However, a typical situation (as in the game semantics of Idealized Algol) is that the underlying category of our coherence-enriched SMCC has already been used to give a qualitative interpretation of the λ-calculus (and possibly other features such as side-effects), and we want to *preserve* this structure through the change of base rather than having to reconstruct it using the Karoubi envelope. In the next section, we consider conditions under which this is possible.

5. Change of Base and the Free Exponential

Any functor interpreting the exponential of linear logic cannot be **CSpace**-enriched.[5] So we assume only that the *underlying* category of our **CSpace**-category has cofree commutative comonoids. Since this is not an enriched notion we do not show directly that these are preserved by change of base. However, we can instead show that (under certain conditions, which are satisfied in our **CSpace**-category of games) change of base preserves the *construction* of a cofree commutative comonoid due to Melliès, Tabareau and Tasson [9]. This subsumes Lafont's construction, and characterizes the cofree exponential in many other models — we will show that the cofree commutative comonoids in our coherence category of games are an instance.

First, to outline the MTT-construction. If a symmetric monoidal category $(\mathcal{C}, \otimes, I)$ has the cartesian product $B_\bullet = B \times I$ (the "free pointed object" on B) and the symmetric tensor power $B^{\leq i} \triangleq (B_\bullet)^i$ exists for each $i \in \omega$, then the morphisms $\mathsf{eq}_{i+1}; (B_\bullet^{\otimes i} \otimes \pi_r); \theta : B^{\leq i+1} \to B^{\otimes n}$ are equal for each permutation θ on $B^{\otimes i}$ and hence by the universal property of the symmetric tensor power, there is a unique morphism $p_i : (B_\bullet^{i+1}) \to (B_\bullet)^i$ such that $p_i; \mathsf{eq}_i : B_\bullet^{i+1} \to \mathbb{B}_\bullet^{\otimes i} = eq_{i+1}; (B_\bullet^{\otimes i} \otimes \pi_r)$.

In [9] it is shown that the cofree comonoid on B is given by the limit (where it exists) of the diagram:

$$I \xleftarrow{p_0} B^{\leq 1} \xleftarrow{p_1} B^{\leq 2} \ldots B^{\leq i} \xleftarrow{p_i} B^{\leq i+1} \ldots$$

This is a refinement of Lafont's construction: in a category with distributive biproducts, $B^{\leq n+1} = \bigoplus_{i \leq n+1} B^i$ and $p_n : B^{\leq n+1} \to B^{\leq n}$ is its left projection. Hence the limit of the above diagram is $\bigoplus_{i \in \mathbb{N}} B^i$.

[5] For instance, $!1 \cong I$ implies that ! sends the identity on the terminal object to the identity on the monoidal unit. The former must be the empty clique and so the latter is also the empty clique if ! is **CSpace** enriched. But in this case $1 \cong I$ and the model is degenerate.

The definition of comonoid structure on $!B$, and the natural equivalence between morphisms into B and comonoid morphisms into $!B$ are defined in [9]. They all derive from the universal properties of the free pointed object, its symmetric tensor powers, and the limit of the above diagram. So if we can show that change of base preserves these limits in the underlying category, then it preserves the free exponential. To do so, we relate them to limits in the enriching category in the following sense.

Definition 5.1. *Let $\mathcal{C}$ be a $\mathcal{V}$-category. A representable limit for a diagram $D : K \to \mathcal{C}_0$ is a cone $(L, \vec{\phi})$ to D in $\mathcal{C}_0$ such that for each object A of $\mathcal{V}$, $(h^A(L), h^A(\vec{\phi}))$ is a limit of $h^A(D) : K \to \mathcal{V}$ in $\mathcal{V}$ (where $h^A : \mathcal{C}_0 \to \mathcal{V}$ is the hom-functor sending an object B to $\mathcal{C}(A, B)$ and $f : I \to \mathcal{C}(B, C)$ to $(\mathcal{C}(A, B) \otimes f); \mathsf{comp}_{A,B,C} : \mathcal{C}(A, B) \to \mathcal{C}(A, C)$).*

In particular, any representable limit is already a limit for D in $\mathcal{C}_0$ (in other words, a limit for D in $\mathcal{C}$, weighted over the constant I functor): if $(N, \vec{\psi})$ is a cone into D in $\mathcal{C}_0$ then $(I, \vec{\psi})$ is a cone into $h^N(D)$ in $\mathcal{V}$ and so has a unique mediating morphism $u : I \to h^N(L)$ — which is also a unique mediating morphism from $(N, \vec{\psi})$ into $(L, \vec{\phi})$ in $\mathcal{C}_0$.

This gives us a recipe for establishing that change of base via $F : \mathcal{V} \to \mathcal{W}$ preserves limits in the underlying category: show that they are representable limits and that the corresponding limits in $\mathcal{V}$ are preserved by F.

Lemma 5.2. *Given a diagram $D : K \to \mathcal{C}_0$, if F preserves the limit of each $h^A(D) : K \to \mathcal{V}$ (where they exist) then F_0 preserves the representable limit of D, where it exists.*

So, for example, change of base via $\Phi^{\mathcal{R}}$ preserves all products (and thus the free pointed object, since is strict monoidal) because h^A sends products in $\mathcal{C}$ to products in **CSpace**, which are preserved by $\Phi^{\mathcal{R}}$.

5.1. Preservation of Symmetric Tensor Powers

Given an object D in a category $\mathcal{C}$, we write $G : D \Rightarrow D$ to mean that G is a finite group of $\mathcal{C}$-automorphisms on D under composition. By Lemma 5.2, change of base via $\Phi^{\mathcal{R}}$ preserves the symmetric tensor power B^n — the equalizer of the the group $\mathsf{perm}(B^{\otimes n}) : B^{\otimes n} \Rightarrow B^{\otimes n}$ of permutation isomorphisms — provided that $h^A(B^n)$ is the equalizer of the group $h^A(\mathsf{perm}(B^{\otimes n}) : h^A(B^{\otimes n}) \Rightarrow h^A(B^{\otimes n})$ for each object A and that $\Phi^{\mathcal{R}}$ preserves these equalizers.

We give a necessary and sufficient condition on a **CSpace**-enriched category for this to hold. Given an automorphism group $G : D \Rightarrow D$ in **CSpace**, let $\sim_G$ be the equivalence relation on $|D|$ induced by G — i.e. $d \sim_G d'$ if there exists $g \in G$ with $(d, d') \in g$. Say that G is *coherent* if $\sim_G \subseteq \bigcirc_G$.

Proposition 5.3. CSpace *has equalizers for any coherent automorphism group $G : D \Rightarrow D$.*

Proof. The equalizer for G in **CSpace** is the coherence space $D_{/G}$ for which the web is the set of equivalence classes of $\sim_G$ — with coherence $X \frown_{D_{/G}} Y$ if $x \frown_D y$ for all $x \in X, y \in Y$.
This comes with the stable map $\mathsf{eq} : D_{/G} \to D = \{([d]_G, d) \mid d \in |D_{/G}\}$, which evidently satisfies $\mathsf{eq}; g = g$ for all $g \in G$, and for any $f : A \to D$ such that $f; g = f$ for all $g \in G$ we have a unique mediating morphism $u_f : A \to D_{/G} = \{(a, [d]_G) \mid (a, d) \in f\}$. $\square$

The equalizer of $\Phi^{\mathcal{R}}(G) : |D| \Rightarrow |D|$ in $\mathcal{R}^{\Pi}$ is the quotient of $|D|$ by $\sim_G$ — i.e. $(|D|_{/G}, \Phi^{\mathcal{R}}(\mathsf{eq}))$. In other words:

Proposition 5.4. *$\Phi^{\mathcal{R}}$ preserves the equalizer of $G : D \Rightarrow D$ if G is coherent.*

Remark 5.5. *If G is not coherent it may have an equalizer in* **CSpace***, but this is not $D_{/G}$ (note that* eq *is not a well-defined clique of $D_{/G}$). It is therefore not preserved by $\Phi^{\mathcal{R}}$. E.g. if $\mathbb{B}$ is the coherence space of Booleans (with two incoherent atoms* $\mathtt{tt}$ *and* $\mathtt{ff}$*) then the symmetric tensor power $\mathbb{B}^2$ (equalizer for the two permutations on $\mathbb{B} \otimes \mathbb{B}$) has two incoherent atoms,* $\{(\mathtt{tt}, \mathtt{tt}), (\mathtt{ff}, \mathtt{ff})\}$*, whereas $\Phi^{\mathcal{R}}(\mathbb{B})^2$ also has an element* $[(\mathtt{tt}, \mathtt{ff})]$*.*

Motivated by Proposition 5.4 and Lemma 5.2, we define:

Definition 5.6. *A symmetric tensor power B^n in a* **CSpace***-category is* coherent *if $h^A(G) : \mathcal{C}(A, B^{\otimes n}) \Rightarrow \mathcal{C}(A, B^{\otimes n})$ is coherent for every object A, where G is the group of permutation isomorphisms on $B^{\otimes n}$.*

Proposition 5.7. *The change of base induced by $\Phi^{\mathcal{R}}$ preserves coherent symmetric tensor powers.*

5.2. Preservation of the MTT-exponential

Assuming that the symetric tensor powers are coherent we may establish that change of base induced by $\Phi^{\mathcal{R}}$ preserves the limit of the chain of projections $\langle p_i : B_\bullet^{i+1} \to B_\bullet^i \mid i \in \omega\rangle$ (if it exists) by showing that for all objects A, the limit of $\langle h^A(p_i) : \mathcal{C}(A, B_\bullet^{i+1}) \to \mathcal{C}(A, B_\bullet^i) \mid i \in \omega\rangle$ in **CSpace** is preserved by $\Phi^{\mathcal{R}}$. We establish this by characterising this limit in **CSpace**, using the fact that each of the $h^A(p_i)$ are *projections* in the following sense.

Definition 5.8. *In a partial-order enriched category, $p : D \to C$ is a* projection *if there exists $e : C \to D$ such that $e; p = \mathsf{id}_C$ and $p; e \leq \mathsf{id}_D$ (i.e. (e, p) form an* embedding-projection pair *from C to D).*

A proof of the following fact is given in Appendix A.

Lemma 5.9. *For each object A in a* **CSpace***-enriched SMC with coherent symmetric tensor powers of $B_\bullet$, $h^A(p_n) : \mathcal{C}(A, B_\bullet^{n+1}) \to \mathcal{C}(A, B_\bullet^n)$ is a projection.*

So it suffices to show that $\Phi^{\mathcal{R}}$ preserves limits of chains of projections.

Proposition 5.10. **CSpace** *has limits for any ω-chain of projections.*

Proof. Given a chain of $e-p$ pairs, $D_0 \overset{e_0,p_0}{\trianglelefteq} D_1 \overset{e_1,p_1}{\trianglelefteq} \ldots$, the web of $\bigsqcup D$ is the quotient of $\coprod_{i\in\omega}|D_i|$ by the reflexive, symmetric and transitive closure of the relation $\{(c,i+1),(d,i) \mid i \in \omega \wedge (c,d) \in p_i\}$, with the coherence $[(c,i)] \frown_{\bigsqcup D} [(d,i)]$ iff $c \frown_{D_i} d$.
This comes with projections $p^i : \bigsqcup D \to D_i = \{(d,[(d,i)]) \mid d \in |D_i|\}$, and for any cone $(A, \langle f_i : A \to D_i \mid i \in \omega\rangle$, a mediating morphism $u : A \to \bigsqcup D = \bigcup_{i\in\omega}\{(a,[(d,i)]) \mid (a,d) \in f_i\}$. □

Proposition 5.11. Φ^R *preserves the limit of any* ω*-chain of projections.*

Proof. $(|\bigsqcup D|, \langle \Phi^{\mathcal{R}}(p^i) \mid i \in \omega\rangle$ is the limit for $|D_0| \overset{\Phi^{\mathcal{R}}(p_0)}{\longleftarrow} |D_1| \overset{\Phi^{\mathcal{R}}(p_1)}{\longleftarrow} \ldots$ in $\mathcal{R}^\Pi$ — for any cone $(A, \langle f_i : A \to |D_i| \mid i \in \omega\rangle)$ we may define a mediating morphism $f : A \to \bigsqcup|D|$ by $f(a,[(d,i)]) = f_i(a,d)$. □

Thus by Lemma 5.2, change of base preserves the construction of Melliès, Tabareau and Tasson:

Proposition 5.12. *In any* **CSpace***-category* $\mathcal{C}$ *with coherent symmetric tensor powers, if* $!B$ *is a limit for the chain* $B^0_\bullet \overset{p_0}{\longleftarrow} B^1_\bullet \overset{p_1}{\longleftarrow} B^2_\bullet \ldots$ *in* $\mathcal{C}_0$ *then it is a limit for* $I \overset{\Phi^{\mathcal{R}}_0(p_0)}{\longleftarrow} B_\bullet \overset{\Phi^{\mathcal{R}}_0(p_1)}{\longleftarrow} B^2_\bullet \ldots$ *in* $\mathcal{C}^{\mathcal{R}}_0$.

5.3. A CCC of Weighted Strategies

We now return to our example of a **CSpace**-enriched category of games **G**. Its underlying category has cofree commutative comonoids for a collection of "well-opened" games (essentially, structure which was used to construct a cartesian closed category in [14]). We establish that these comonoids may be constructed from (representable) limits in **G** à la Melliès, Tabareau and Tasson, and are therefore preserved by change of base.

Lemma 5.13. *For any games* A *and* B*, the group of permutations* $h^A(\mathsf{perm}(B^{\otimes n})) : \mathsf{Coh}(A \multimap B^{\otimes n}) \Rightarrow \mathsf{Coh}(A \multimap B^{\otimes n})$ *is coherent.*

Proof. For any permutation π on n, the action of π on the justified sequences over $M_A \uplus (M_B \times \{1,\ldots,n\})$ by composition with the corresponding isomorphism $\theta_\pi : B^{\otimes n} \to B^{\otimes n}$ satisfies:

- $\pi(s(b,i)) = \pi(s)(b,\pi(i))$ if (b,i) is a move from $M_{B^{\otimes n}}$,
- $\pi(sa) = \pi(s)a$ if a is a move from M_A.

For any $t \in \mathsf{Coh}(A \multimap B^{\otimes n})$ we claim that if $sa \sqsubseteq t$ and $s \sqsubseteq \pi(t)$, where a is a Player move, then $sa \sqsubseteq \pi(t)$ and so t is coherent with $\pi(t)$. Observe that $\pi(s) = s$. Either a is a move in A — in which case $\pi(sa) = sa$ and so $sa \sqsubseteq \pi(t)$ — or else $a = (b,i)$ for some move b in B. In the latter case, a is justified by an Opponent move of the form (b',i) which occurs in s. Then $\pi(i) = i$ since $\pi(s) = s$, and so $sa \sqsubseteq \pi(t)$ as required. □

Thus cofree commutative comonoids in $\mathbf{G}_0$ given by the MTT-construction are preserved by change of base to $\mathbf{G}_0^{\mathcal{R}}$. Not all games have a MTT-exponential — the free pointed object $B_\bullet = B \times I$ required does not always exist in $\mathbf{G}_0$, which does not have all finite products — e.g. there is no object $I \times I$ such that $\mathbf{G}(I, I \times I) \cong \mathbf{G}(I, I) \times \mathbf{G}(I, I)$. However we may identify a class of objects of $\mathbf{G}$ for which the free pointed objects, their symmetric tensor powers, and the cofree commutative comonoid all exist and have a simple characterization. These are essentially the well-opened games used to construct a cartesian closed category in [14].

Definition 5.14. *For an arena B, the set L_B^1 of* well-opened *sequences on B consists of sequences in L_B which contain precisely one initial move. Let $\underline{B}$ be the game (B, L_B^1).*

For $n \leq \omega$, let $L_B^{\leq n}$ be the subset of L_B consisting of those legal sequences which contain at most n initial moves, and let $B_{\leq n} = (B, L_B^{\leq n})$.

Lemma 5.15. *For any arena B, $B_{\leq n}$ is the symmetric tensor power $\underline{B}^{\leq n}$.*

Proof. For any game G, $\mathsf{Coh}(G \multimap B_{\leq 1})$ is the free pointed object on $\mathsf{Coh}(G \multimap \underline{B})$, since it includes an additional element (ε) which is coherent with every element of $\mathsf{Coh}(G \multimap \underline{B})$. Hence $B_{\leq 1}$ is the free pointed object on $\underline{B}$.

Thus it suffices to show that for any game G,

$$\mathsf{Coh}(G \multimap (B_{\leq 1})^{\otimes n})_{/\mathsf{perm}(\underline{B}^{\otimes n})} \cong \mathsf{Coh}(G \multimap B_{\leq n})$$

. The map from $\mathsf{Coh}(G \multimap (B_{\leq 1})^{\otimes n})$ to $\mathsf{Coh}(G \multimap B_{\leq n})$ which simply erases the tags on moves in B identifies two plays precisely when they are permutation equivalent. It preserves (and reflects) coherence by Lemma 5.13. It is surjective: we may define a pre-image for any play s on $\mathsf{Coh}(G \multimap B_{\leq n})$ by choosing a fresh tag for each initial move. Thus it is the required isomorphism. □

Proposition 5.16. *For any arena B, there is a cofree commutative comonoid on $\underline{B}$ in $\mathbf{G}_0$ given by the MTT-construction.*

Proof. The projection $p_n : \underline{B}^{\leq n+1} \to \underline{B}^{\leq n}$ is the restriction of the identity strategy to $B_{\leq n+1} \multimap B_{\leq n}$. Therefore the limit for the diagram:

$$I \overset{h_A(p_0)}{\leftarrow} h_A(\underline{B})^{\leq 1} \overset{h_A(p_1)}{\leftarrow} h_A(\underline{B})^{\leq 2} \ldots$$

is given by $\mathsf{Coh}(A \multimap !\underline{B})$, where $!\underline{B} = (B, L_B)$. □

Concretely, the cofree commutative comonoid on $\underline{B}$ in $\mathbf{G}_0$ is therefore $(!\underline{B}, \delta_B, \epsilon_B)$, where $\epsilon_B = \{\varepsilon\}$ and $\delta_B : !\underline{B} \to !\underline{B} \otimes !\underline{B}$ consists of sequences on $B \Rightarrow (B \uplus B)$ for which the left restriction of each even prefix is equal to the right restriction with tags removed.

Thus we may define a "co-Kleisli" category $\mathbf{G}_!$ in which objects are arenas, morphisms from A to B are morphisms from $!\underline{A}$ to $\underline{B}$ in $\mathbf{G}_0$, the identity on A is $\mathsf{der}_A : !A \to A$ and the composition of $f : A \to B$ with $g : B \to C$ is $f^\dagger; g$.

Proposition 5.17. $\mathbf{G}_!$ *is cartesian closed:*

Proof. The cartesian product of B with C is given by the disjoint sum $B \uplus C$ (note that $\mathsf{Coh}(A, \underline{B \uplus C}) \cong \mathsf{Coh}(A, \underline{B}) \times \mathsf{Coh}(A, \underline{C})$), while $A \Rightarrow B$ is the exponential of B by A since $!\underline{A} \multimap \underline{B} = \underline{A \Rightarrow B}$. □

$\mathbf{G}_!$ is the cartesian closed category of games constructed in [14], minus the non-emptiness and even-prefix-closure condition on strategies. The change of base functor $\Phi^{\mathcal{R}} : \mathbf{G}_0 \to \mathbf{G}_0^{\mathcal{R}}$ preserves all of this structure, yielding a cartesian closed category $\mathbf{G}_!^{\mathcal{R}}$ of well-opened games and $\mathcal{R}$-weighted strategies with a cartesian closed functor $\Phi_!^{\mathcal{R}} : \mathbf{G}_! \to \mathbf{G}_!^{\mathcal{R}}$. Concretely, composition of weighted strategies $\phi : A \to B$ and $\psi : B \to C$ in $\mathbf{G}_!^{\mathcal{R}}$ is weighted composition of $\phi^\dagger$ with ψ, where $\phi^\dagger(s) = \phi(s_1) \cdot \ldots \cdot \phi(s_n)$ if s is the interleaving of the well-opened sequences $s_1, \ldots, s_n$.

6. Uniform Fixed Points

We now return to the general setting of Lafont categories with biproducts. The co-Kleisli category of the free exponential furnishes us with a model of the simply-typed λ-calculus (a cartesian closed category). Recursively defined functions should therefore correspond to fixed points in this category. We will show that it has a unique uniform fixed point operator.

Definition 6.1. *A* fixed point operator *for a category $\mathcal{C}$ with a terminal object* 1 *is a map taking each endomorphism $f \in \mathcal{C}(A, A)$ to a morphism $\mathsf{fix}(f) \in \mathcal{C}(1, A)$ satisfying $\mathsf{fix}(f) = \mathsf{fix}(f); f$.*

Let $L : \mathcal{C} \to \mathcal{C}$ be a comonad with co-Kleisli triple $(L, (\_)^\dagger, \mathsf{der})$. A fixed point operator for the co-Kleisli category $\mathcal{C}_L$ is uniform *if for any morphisms (in $\mathcal{C}$) $f : LA \to A$, $g : LB \to B$ and $h : A \to B$ which satisfy $f; h = L(h); g$, we have $\mathsf{fix}(g) = \mathsf{fix}(f); h$.*

Proposition 6.2. *If $\mathcal{C}$ and L are cpo-enriched then $\mathcal{C}_L$ has a uniform fixed point operator.*

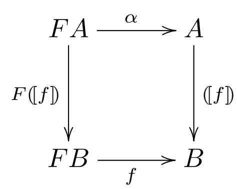
Proof. Taking $\mathsf{fix}(f)$ to be the least fixed point of f — the supremum of the ω-chain $f_0 \leq f_1 \leq \ldots$ defined $f_0 = \bot_{L1,B}$ and $f_{i+1} = f_i^\dagger; f$, we may show by induction that if $f; h = Lh; g$ then $f_i; h = g_i$ for all i and hence

$$\mathsf{fix}(f); h = (\bigvee_{i \in \omega} f_i); h = \bigvee_{i \in \omega} (f_i; h) = \bigvee_{i \in \omega} g_i = \mathsf{fix}(g).$$

□

However, by Proposition 2.14, categories with biproducts in which the induced sum is *not* continuous cannot be cpo-enriched, and therefore the least fixed point construction does not apply. Instead, we shall use the observation of [5], further developed in [6], that any comonad which has a *bifree algebra* has a unique uniform fixed point operator on its co-Kleisli category.

Definition 6.3. *Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. An* initial algebra *for F is an initial object in the category of F-algebras. In other words, an F-algebra — a pair (A, α) of an object A of $\mathcal{C}$ and a morphism $\alpha : FA \to A$ — such that for any F-algebra $(B, f : FB \to B)$ there is a unique morphism of F-algebras from (A, α) to (B, f) — that is, a unique morphism $(\!|f|\!) : A \to B$ in $\mathcal{C}$ (the* catamorphism *of f) for which the following diagram commutes:*



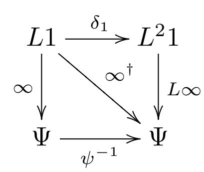
Dually, a final coalgebra is a terminal object in the category of F-coalgebras — i.e. a F-coalgebra $(C, \gamma : C \to FC)$ such that for any F-coalgebra $(D, g : D \to FD)$ there is a unique coalgebra morphism from (C, γ) into (D, g) (the anamorphism *of g).*

Observe that if (A, α) is an initial algebra then $\alpha : FA \to A$ is an isomorphism in $\mathcal{C}$ (Lambek's lemma). Thus we may define:

Definition 6.4. *A* bifree algebra *for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is an initial algebra (A, α) for F such that (A, α^{-1}) is a final coalgebra for F. In other words, F is* algebraically compact *[7].*

Proposition 6.5. *If a comonad $L : \mathcal{C} \to \mathcal{C}$ has a bifree algebra $(\Psi, \psi : L\Psi \to \Psi)$ then $\mathcal{C}_L$ has a unique uniform fixed point operator.*

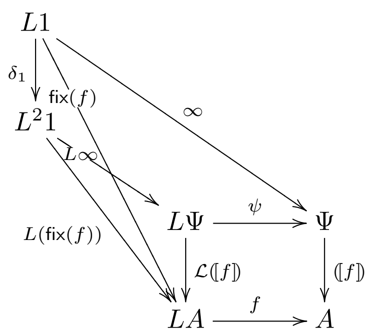
Proof. The L-comultiplication $\delta_1 : L1 \to L^2 1$ is a L-coalgebra and therefore has a unique *anamorphism* $\infty : L1 \to \Psi$ into the coalgebra (Ψ, ψ^{-1}), satisfying:



i.e. ∞ is the unique morphism such that $\infty = \infty^\dagger; \psi$.

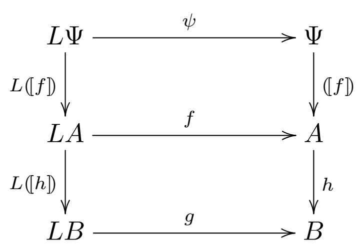
For any endomorphism $f \in \mathcal{C}_L(A, A)$, $(A, f : LA \to A)$ is an L-algebra in $\mathcal{C}$ and therefore has a unique catamorphism $(\!|f|\!) : \Psi \to A$ such that $\psi; (\!|f|\!) = L(\!|f|\!); f$. Define the *algebraic fixed point* $\mathsf{fix}(f) \triangleq \infty; (\!|f|\!)$. This satisfies the following conditions:

- Fixed Point Property: The following diagram commutes:



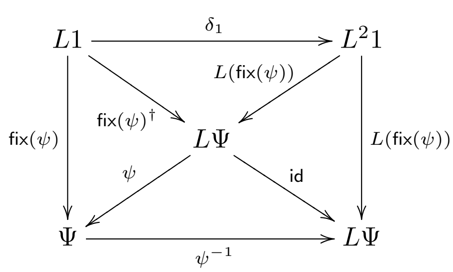
and so $\mathsf{fix}(f) = \mathsf{fix}(f)^\dagger; f$.

- Uniformity: Suppose $g : LB \to B$ and $h : A \to B$ satisfy $f; h = Lh; g$. Then $\llparenthesis f \rrparenthesis; h$ is a L-algebra morphism from ψ into g — i.e. the following diagram commutes:



— and so by uniqueness of catamorphisms, $\llparenthesis g \rrparenthesis = \llparenthesis f \rrparenthesis; h$ and $\mathsf{fix}(g) = \infty; \llparenthesis g \rrparenthesis = \mathsf{fix}(f); h$.

- Uniqueness: Suppose $\mathsf{fix}(\_)$ is a uniform fixed point operator. Then $\mathsf{fix}(\psi) : L1 \to \Psi$ is a coalgebra morphism from $(\delta_1, L1)$ into (Ψ, ψ), since the following diagram commutes:



and so $\mathsf{fix}(\psi) : L1 \to \Psi = \infty$ by uniqueness of anamorphisms.

For any $f : LA \to A$, $\psi; \llparenthesis f \rrparenthesis = L\llparenthesis f \rrparenthesis; f$, by definition of $\llparenthesis f \rrparenthesis$ as an algebra morphism from ψ into f.
So by uniformity $\mathsf{fix}(\psi); \llparenthesis f \rrparenthesis = \mathsf{fix}(f)$, and so $\mathsf{fix}(f) = \infty; \llparenthesis f \rrparenthesis$ as required.
□

6.1. The Bifree Algebra of Nested Finite Multisets

We now show that in any symmetric monoidal category with distributive biproducts, if the cofree exponential exists then it has a bifree algebra (and thus a uniform fixed point operator). We can prove that $! : \mathcal{C} \to \mathcal{C}$ has an initial algebra using Adámek's theorem [34]: if the diagram

$$0 \xrightarrow{0_{!0}} !0 \xrightarrow{!0_{!0}} !!(0) \xrightarrow{!!0_{!0}} \dots$$

has a colimit A, which is preserved by ! — i.e. $!A$ is also a colimit for this diagram, yielding an isomorphism $\alpha : A \to !A$ — then (A, α) is an initial algebra for the cofree exponential. We establish existence of this colimit by showing that it is the image under a colimit-preserving functor of a diagram in **Set** for which the colimit is easy to construct.

Definition 6.6. *For a set S, let $\mathcal{M}_*(S)$ be the set of finite multisets with* support *in S: i.e. its elements are permutation-equivalence classes of finite sequences of elements of S, which we can write in the form $[a_1, \dots, a_n]$ for $a_1, \dots, a_n \in S$.*

Let $M : \mathbf{Set} \to \mathbf{Set}$ be the finite multiset functor — i.e. $M(S) = \mathcal{M}_*(S)$ and $M(f)[a_1, \dots, a_n] = [f(a_1), \dots, f(a_n)]$. M restricts to a functor on the category of sets and inclusions which is cocontinuous (i.e. preserves inclusions and directed unions). Thus the initial ω-chain

$$\varnothing \overset{i_{M(\varnothing)}}{\to} M(\varnothing) \overset{M(i)}{\to} M^2(\varnothing) \overset{M^2 i}{\to} \dots$$

consists of inclusions, and its colimit $\mathbb{M} = \bigcup_{i \in \omega} M^i(\varnothing)$ in **Set** (the set of *nested finite multisets*) is preserved by M.

Proposition 6.7. *The functor $h^I : \mathcal{C} \to \mathbf{Set}$ has a left adjoint $\widetilde{(\_)} : \mathbf{Set} \to \mathcal{C}$.*

Proof. For each set S, let $\widetilde{S} = \bigoplus_{s \in S} I$, and define $\eta_S : S \to \mathcal{C}(I, \widetilde{S})$ by $\eta_S(s) = \iota_s$.

Given $f : S \to \mathcal{C}(I, B)$, define $\widetilde{f} : \widetilde{S} \to B = [f(s) \mid s \in S]$. Then for all $s \in S$, $(\eta_S; h^I(\widetilde{f}))(s) = h^I(\widetilde{f})(\iota_s) = \iota_s; \widetilde{f} = \iota_s; [; f(s') \mid s' \in S] = f(s)$. □

We now show that $\widetilde{\_}$ sends the finite multiset functor on **Set** to the cofree exponential functor on $\mathcal{C}$ — i.e. $\widetilde{M} : \mathbf{Set} \to \mathcal{C}$ is naturally isomorphic to $!\widetilde{(\_)}$ — by establishing that for any set S, $\widetilde{\mathcal{M}_*(S)}$ is the cofree commutative comonoid on $\widetilde{S}$. This generalizes the proof from [1], that $\mathcal{M}_*(S)$ is the Lafont exponential of S in the category of $\mathcal{R}$-weighted relations $\mathcal{R}^{\Pi}$.

For each set S, let $\mathcal{M}_k(S)$ denote the set of finite multisets over S of cardinality k.

Lemma 6.8. *The objects $\{\widetilde{\mathcal{M}_k(S)} \mid k \in \mathbb{N}\}$ are symmetric tensor powers of $\widetilde{S}$.*

Proof. $\mathcal{M}_k(S)$ corresponds to the set of permutation equivalence classes of elements of $\Pi_{i\leq k}S$ and so for each $X \in \mathcal{M}_k(S)$ there exists $\widehat{X} \in \Pi_{i\leq k}S$ such that $[\widehat{X}] = X$.

$\widetilde{S}^{\otimes k} = \widetilde{\Pi_{i\leq k}S}$ by distributivity of $\otimes$ over biproducts. Thus for each permutation θ on $\{1,\ldots,k\}$, the corresponding automorphism $\theta_{\widetilde{S}} : \widetilde{S}^{\otimes k} \to \widetilde{S}^{\otimes k}$ is the map $\langle \pi_{\theta^{-1}(s)} \mid s \in \Pi_{i\leq k}S\rangle$. (Observe that every permutation automorphism on $I^{\otimes k}$ is the identity — i.e. I is a k-ary tensor power of itself by equality of the left and right unitors $r_I, l_I : I \otimes I \to I$.)

Let $\mathsf{eq}_k : \widetilde{\mathcal{M}_k(S)} \to \widetilde{S}^{\otimes k} = \langle \pi_{[x]} \mid x \in \Pi_{i\leq k}S\rangle$. Then $(\widetilde{\mathcal{M}_k(S)}, \mathsf{eq}_k)$ is an equalizer for the permutation automorphisms on $\widetilde{S}^{\otimes k}$ (and these equalizers are preserved by the tensor product):

- $\mathsf{eq}_n;\theta_{\widetilde{S}} = \langle \pi_{[x]} \mid x \in \Pi_{i\leq k}S\rangle; \langle \pi_{\theta^{-1}(s)} \mid s \in S^k\rangle = \langle \pi_{[\theta^{-1}(s)]} \mid s \in S^k\rangle = \mathsf{eq}_n$.
- For any $f : B \to \widetilde{\Pi_{i\leq k}S}$ such that $f;\theta_{\widetilde{S}} = f$ for all θ, let $u : B \to \widetilde{\mathcal{M}_k(S)} = \langle f;\pi_{\widehat{X}} \mid X \in \mathcal{M}_k(S)\rangle$, so that $g;\mathsf{eq}_n = f$ if and only if $g = u$. □

For any set S, $\widetilde{\mathcal{M}_*(S)} = \bigoplus_{k\in\mathbb{N}} \widetilde{\mathcal{M}_k(S)}$ and hence $\widetilde{\mathcal{M}_*(S)}$ is a Lafont exponential for $\widetilde{S}$. (The action of M on morphisms is also preserved by the Lafont construction.) Thus we have a natural isomorphism $\alpha_S : !\widetilde{S} \cong \widetilde{M(S)}$ and hence:

Proposition 6.9. *$(\widetilde{\mathbb{M}}, \alpha_{\widetilde{\mathbb{M}}})$ is an initial algebra for the functor $! : \mathcal{C} \to \mathcal{C}$.*

The proof that $(\widetilde{\mathbb{M}}, \alpha^{-1}_{\widetilde{\mathbb{M}}})$ is a final coalgebra for the cofree exponential functor uses a symmetric argument. The functor from **Set** to $\mathcal{C}^{op}$ sending S to $\widetilde{S}$ and $f : S \to S'$ to $\sum_{s\in S} \pi_{f(s)};\iota_S$ is right adjoint to the contravariant hom functor $\overline{h^I} : \mathcal{C}^{op} \to \mathbf{Set} = \mathcal{C}(\_, I)$ and therefore sends the colimit of the diagram

$$\varnothing \xrightarrow{i_{M(\varnothing)}} M(\varnothing) \xrightarrow{M(i)} M^2(\varnothing) \xrightarrow{M^2 i} \ldots$$

(the set of nested finite multisets) to the limit of the diagram

$$0 \xleftarrow{0_{!0}} !0 \xleftarrow{!0_{!0}} !!0 \xleftarrow{!!0_{!0}} \ldots$$

in $\mathcal{C}$, which must therefore be $\widetilde{\mathbb{M}}$. By Adàmek's theorem, $(\widetilde{\mathbb{M}}, \alpha^{-1}_{\mathbb{M}})$ is therefore a final coalgebra in $\mathcal{C}$.

Proposition 6.10. *$(\widetilde{\mathbb{M}}, \alpha_{\widetilde{\mathbb{M}}})$ is a bifree algebra for the free exponential on $\mathcal{C}$.*

So by Proposition 6.5, the free the co-Kleisli category of the free exponential on $\mathcal{C}$ has a unique uniform fixed point operator. If $\mathcal{C}$ is cpo-enriched (e.g. $\mathcal{R}^{\Pi}$ for a continuous semiring $\mathcal{R}$), by Proposition 6.2, this operator sends each endomorphism to its least fixed point (note that the free exponential is continuous, that is, it is a cpo-functor).

6.2. Parameterised Fixed Points

To interpret fixed points of terms with free variables we will require a parameterised fixed point operator.

Definition 6.11. *A* parameterised fixed point operator *for a category $\mathcal{C}$ with Cartesian products is a family of operators* $\mathsf{fix}_A : \mathcal{C}(A \times B, B) \to \mathcal{C}(A, B)$ *(indexed over the objects of $\mathcal{C}$) which satisfy:*

- *Fixed Point Property: For each A, $\langle A, \mathsf{fix}_A(f)\rangle ; f = \mathsf{fix}_A(f)$.*
- *Naturality: If $g : C \to A$ then $g; \mathsf{fix}_A(f) = \mathsf{fix}_C((g \times B); f)$.*

The following parametrization of our algebraic fixed points will be used in our denotational model. For each object A we have a comonad $!A \otimes \_$ on $\mathcal{C}$ with a *distributive law* [35] $l :!A\otimes !\_ \to !(!A\otimes \_)$, yielding a comonad $!_A$ on the category $\mathcal{C}_{!A\otimes\_}$: it is sufficient to show that $\widetilde{\mathbb{M}}$ is a bifree algebra for each of these.

Theorem 6.12. *If $\mathcal{C}$ is a symmetric monoidal category with distributive biproducts and cofree exponential then $\mathcal{C}_!$ has a uniform parameterised fixed point operator.*

Proof. For any object A, the monoidal comonad $!A \otimes \_$ preserves biproducts and thus its co-Kleisli category $\mathcal{C}_{!A\otimes\_}$ is a symmetric monoidal category with distributive biproducts and cofree commutative comonoids. So by Proposition 6.10 the comonad $!_A : \mathcal{C}_{!A\otimes\_} \to \mathcal{C}_{!A\otimes\_}$ has $\epsilon_{!A} \otimes \alpha_{\widetilde{\mathbb{M}}} :!A\otimes !\widetilde{\mathbb{M}} \to \widetilde{\mathbb{M}}$ as its bifree algebra. Hence by Proposition 6.5, for each A there is a uniform fixed point operator fix_A on the co-Kleisli category of $!_A$.

This family is natural in A: given $f \in \mathcal{C}_{!A\otimes\_}(!B, B)$ with a catamorphism $(\!|f|\!)_A \in \mathcal{C}_{!A\otimes\_}(\widetilde{\mathbb{M}}, B)$, for any $g :!C \to A$, $g^\dagger \otimes \widetilde{\mathbb{M}}); (\!|f|\!)_A$ is a morphism of $!_C$ algebras from $\epsilon_{!C} \otimes \mathsf{id}_{\widetilde{\mathbb{M}}}$ to $(g^\dagger \otimes B); f$ and is therefore equal to $(\!|(g^\dagger \otimes B); f|\!)_C$. Hence $\mathsf{fix}_C(g^\dagger \otimes B); f) = (!C \otimes \infty); (\!|(g^\dagger \otimes \widetilde{\mathbb{M}}); f|\!)_C = (!C \otimes \infty); (g^\dagger \otimes \widetilde{\mathbb{M}}); (\!|f|\!)_A = g^\dagger; (!A \otimes \infty); (\!|f|\!)_A = g^\dagger; \mathsf{fix}_A(f)$.

By the natural isomorphism $!A\otimes !\_ \cong !(A \times \_)$ the co-Kleisli category of $!_A : \mathcal{C}_{!A\otimes\_} \to \mathcal{C}_{!A\otimes\_}$ is isomorphic to the co-Kleisli category of the comonad $A \times \_ : \mathcal{C}_! \to \mathcal{C}_!$, Thus we have a family of uniform fixed point operators for the latter which is natural in A and is therefore a parameterised fixed point operator for $\mathcal{C}_!$. □

Note that uniformity in $!_A$ implies *parametric uniformity* in ! — i.e. for any $f :!(A \times B) \to B$, $g :!(A \times C) \to C$, and $h : B \to C$ such that $f; h =!(A \times h); g$, $\mathsf{fix}_A(g) = \mathsf{fix}_A(f); h$.

6.3. Nested Multiset Approximants

By unpicking the definition of the fixed point a little we may describe the action of this fixed point operator more directly as a sum of approximants indexed over the nested finite multisets, relating it more closely to the resource λ-calculus [10] and differential λ-calculus [11], and providing the basis for our proof of computational adequacy for $\mathcal{R}$-weighted PCF.

The following identities relating the biproduct structure of the Lafont exponential to its comonoid structure follow directly from its definition ($X \subseteq Y$ denotes multiset inclusion of X in Y, $X + Y$ denotes their multiset union and $X - Y$ their multiset difference).

Lemma 6.13. *For any set S, the Lafont exponential $!\widetilde{S}$ satisfies:*

1. $\iota_{[\_]}; \epsilon_{!\widetilde{S}} = \mathsf{id}_I$ *and* $\pi_{[\_]} = \epsilon_{!\widetilde{S}}$, *and if* $X \neq [\_]$ *then* $\iota_X; \epsilon_{!\widetilde{S}} = 0_{I,I}$.
2. *For all* $X \in \mathcal{M}_{*(S)}$, $\iota_X; \delta_{!\widetilde{S}} = \sum_{Y \subseteq X} (\iota_Y \otimes \iota_{X-Y})$ *and for all* $Y \subseteq X$, $\pi_X = \delta_{!\widetilde{S}}; (\pi_Y \otimes \pi_{X-Y})$.
3. *For all* $x \in S$, $\iota_{[x]}; \mathsf{der}_{\widetilde{S}} = \iota_x$ *and* $\pi_{[x]} = \mathsf{der}_{\widetilde{S}}; \pi_x$, *and if* $X \notin \mathcal{M}_1(S)$ *then* $\iota_X; \mathsf{der}_{\widetilde{S}} = 0_{I,\widetilde{S}}$.

We may thus give a direct characterization of the anamorphism $\infty : I \to \widetilde{\mathbb{M}}$.

Lemma 6.14. $\infty : I \to \widetilde{\mathbb{M}} = \sum_{X \in \mathbb{M}} \iota_X$.

Proof. Observe that:

- $\sum_{X \in \mathbb{M}} \iota_X; \epsilon_{\widetilde{\mathbb{M}}} = \iota_{[\_]}; \epsilon_{\widetilde{\mathbb{M}}} + \sum_{X \in \mathbb{M} \setminus \{[\_]\}} (\iota_X; \epsilon_{\widetilde{\mathbb{M}}}) = \epsilon_{\widetilde{\mathbb{M}}} + 0_{\widetilde{\mathbb{M}},\widetilde{\mathbb{M}}} = \epsilon_{\widetilde{\mathbb{M}}}$ by (1).
- $(\sum_{X \in \mathbb{M}} \iota_X); \delta_{\widetilde{\mathbb{M}}} = \sum_{X \in \mathbb{M}} \sum_{Y \subseteq X} (\delta_{\widetilde{\mathbb{M}}}; (\phi_Y \otimes \phi_{X-Y}))$ by (2)
  $= \delta_{\widetilde{\mathbb{M}}}; \sum_{X \in \mathbb{M}_{n+1}} \sum_{Y \subseteq X} (\iota_Y \otimes \iota_{X-Y}) = \delta_{\widetilde{\mathbb{M}}}; ((\sum_{Y \in \mathbb{M}} \iota_Y) \otimes (\sum_{Z \in \mathbb{M}} \iota_Z))$.

Therefore $\sum_{X \in \mathbb{M}} \iota_X$ is a comonoid morphism. Moreover,
$(\sum_{X \in \mathbb{M}} \iota_X); \mathsf{der}_{\mathbb{M}} = \sum_{X \in \mathbb{M}} (\iota_X; \mathsf{der}_{\widetilde{\mathbb{M}}}) = \sum_{X \in \mathbb{M}} (\iota_{[X]}; \mathsf{der}_{\widetilde{\mathbb{M}}}) + \sum_{X \in \mathbb{M} \setminus \mathbb{M}_1(\mathbb{M})} (\iota_X; \mathsf{der}_{\widetilde{\mathbb{M}}})$
$= \sum_{X \in \mathbb{M}} \iota_X + 0_{I,\widetilde{\mathbb{M}}} = \sum_{X \in \mathbb{M}} \iota_X$ by (3).
Therefore $(\sum_{X \in \mathbb{M}} \iota_X)^\dagger = \sum_{X \in \mathbb{M}} \iota_X$. By uniqueness of anamorphisms, $\infty = \sum_{X \in \mathbb{M}} \iota_X$. ☐

Guided by Theorem 6.12, we obtain *nested finite multiset approximants* to parameterized fixed points as follows:

Definition 6.15. *Given* $f : !A \otimes !B \to B$ *and* $X \in \mathbb{M}$, *let* $f^X : !A \to !B = (!A \otimes \iota_X); (\!|f|\!)^\dagger_A$ *so that* $\mathsf{fix}_A(f)^\dagger =$:

$$(!A \otimes \infty); (\!|f|\!)^\dagger_A = (!A \otimes \infty^\dagger); (\!|f|\!)^\dagger_A = (!A \otimes (\sum_{X \in \mathbb{M}} \iota_X)); (\!|f|\!)^\dagger_A = \sum_{X \in \mathbb{M}} f^X.$$

We may think of each nested finite multiset X as representing a unique forest of nested calls to f, which compute the approximant f^X — i.e. $f^{[X_1,\ldots,X_k]}$ corresponds to k calls to f at top level, each of which makes nested calls to f with call-patterns $X_1, \ldots, X_k$, and so on. These approximants, and their properties, are used directly in the semantics of the abstract machine with nested multiset resource bounds defined in Section 8, which may be considered as a form of the resource λ-calculus [10], in which functions are supplied with multisets of arguments. The following identities derive from the corresponding properties of the Lafont exponential (Lemma 6.13) and its bifree algebra.

Lemma 6.16. *For any $f :!B \to B$:*

$$f^X; \epsilon_{!B} = \begin{cases} \epsilon_{!A} & \text{if } X = [\_] \\ 0_{I,I} & \text{otherwise} \end{cases}. \tag{1}$$

$$f^X; \delta_{!B} = \delta_{!A}; \sum_{Y \subseteq X} (f^Y \otimes f^{X-Y}). \tag{2}$$

$$f^X; \mathsf{der}_B = \begin{cases} f^Y; f & \text{if } X = [Y] \\ 0_{!A,B} & \text{if } X \neq [Y] \text{ for all } Y \in \mathbb{M} \end{cases}. \tag{3}$$

As a special case, "promotion" to a comonoid morphism may also be expressed as a sum of approximants (subsuming the approximation of promotion as a sum of integer approximants implicit in the Lafont exponential). Given $f :!A \to B$, we have $\mathsf{fix}_A((!A \otimes \epsilon_{!B}); f) = f$ and so $f^\dagger :!A \to !B = \sum_{X \in \mathbb{M}} \iota_X; (!A \otimes \epsilon_{!B}); f$.

7. Fixed Points and Change of Base

In this short section, we will show that for any **CSpace**-category with a cofree exponential, change of base sends the least fixed points of the cpo-enriched category $\mathcal{C}_!$ to the algebraic fixed points of the complete monoid enriched category $\mathcal{C}_!^{\mathcal{R}}$ — i.e. for any endomorphism $f :!A \to A$, in $\mathcal{C}_0$, $\Phi_0^{\mathcal{R}}(\mathsf{fix}(f)) = \mathsf{fix}(\Phi_0^{\mathcal{R}}(f))$, where fix is the unique uniform fixed point operator in each category.

This provides another way to understand algebraic fixed points more concretely. Note in particular that if change of base preserves fixed points in this sense then it preserves the fixed point *combinator* $\mathbf{Y} :!1 \to !(!A \multimap A) \multimap A$ given by the fixed point of the morphism

$$F :!(!A \multimap A) \multimap A \vdash \lambda f.f(F\ f) :!(!A \multimap A) \multimap A$$

which satisfies $\Phi_0^{\mathcal{R}}(\mathbf{Y})(f) = f(\Phi_0^{\mathcal{R}}(\mathbf{Y})(f))$, allowing us to derive fixed points for any endomorphism in $\mathcal{C}_!^{\mathcal{R}}$.

If $\mathcal{R}$ is continuous, then by Lemma 6.2 the least fixed point operator is uniform and so equal to the algebraic fixed point. Thus change of base sends least fixed points to algebraic fixed points. If $\mathcal{R}$ is not continuous, we may use the fact that we can factor the change of base $\Phi_*^{\mathcal{R}}$ into one which preserves least fixed points followed by one which preserves algebraic fixed points.

Definition 7.1. *Given a complete semiring $\mathcal{R}$ let $\mathbb{N}^\infty[\mathcal{R}]$ be the "free complete semiring" over the multiplicative structure of $\mathcal{R}$ — elements are functions from $|\mathcal{R}|$ into $\mathbb{N}^\infty$ with the following operations:*

- $(\sum_{i\in I} f_i)(a) = \sum_{i\in I} f_i(a)$
- $(f.g)(a) = \sum_{b\cdot c=a} (f(b).g(c))$, *with unit* $u(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$.

Then:

1. $\mathbb{N}^\infty[\mathcal{R}]$ is a continuous semiring, with $f \leq g$ if $f(a) \leq g(a)$ for all $a \in |\mathcal{R}|$.
2. We may define a homomorphism of complete semirings $\beta : \mathbb{N}^\infty[\mathcal{R}] \to \mathcal{R}$:

$$\beta(f) \triangleq \sum_{a\in|\mathcal{R}|} \underline{f(a)} \cdot a.$$

3. By Remark 3.7 this lifts to a functor $\widehat{\beta} : \mathbb{N}^\infty[\mathcal{R}]^\Pi \to \mathcal{R}^\Pi$ which factorizes $\Phi^{\mathcal{R}}$ — i.e. $\Phi^{\mathcal{R}} : \mathbf{CSpace} \to \mathcal{R}^\Pi = \Phi^{\mathbb{N}[\mathcal{R}]}; \widehat{\beta}$.

Proposition 7.2. *For any* **CSpace***-category* $\mathcal{C}$*, the uniform fixed operator on* $\mathcal{C}_!$ *is preserved by the change-of-base functor* $\Phi_0^{\mathcal{R}}$*.*

Proof. By (1), $\mathcal{C}_!^{\mathbb{N}^\infty[\mathcal{R}]}$ is cpo-enriched, and since change of base from $\mathcal{C}_!$ to $\mathcal{C}_!^{\mathbb{N}^\infty[\mathcal{R}]}$ preserves cpo-enrichment, it preserves the least fixed point operator, which is uniform (Lemma 6.2) and thus equal to the algebraic fixed point operator on $\mathcal{C}_!^{\mathbb{N}^\infty[\mathcal{R}]}$ by uniqueness of the latter.

Change of base from $\mathcal{C}_!^{\mathbb{N}^\infty[\mathcal{R}]}$ to $\mathcal{C}_!^{\mathcal{R}}$ via the functor $\widehat{\beta}$ preserves all of the symmetric monoidal and biproduct structure on which the algebraic fixed point depends, and thus preserves algebraic fixed points themselves.

Thus change of base sends least fixed points in $\mathcal{C}_!$ to algebraic fixed points in $\mathcal{C}_!^{\mathcal{R}}$ as required. □

Finally, we note that our category $\mathbf{G}_0^{\mathcal{R}}$ of games and $\mathcal{R}$-weighted strategies does not have all biproducts. In particular it does not have a bifree algebra for the cofree exponential (which is only given for well-opened games). However, we may obtain free pointed objects, and thus Lafont exponentials, by biproduct completion.

Proposition 7.3. *If* $\mathcal{C}$ *has symmetric tensor powers, then its biproduct completion* $\mathcal{C}^\Pi$ *has symmetric tensor powers.*

Proof. For $A = \{A_i \mid i \in I\}$, $A^n = \{A_X \mid X \in \mathcal{M}_n(I)\}$, where if X has *support* $i_1, \ldots, i_k$ then $A_X = A_{i_1}^{X(1)} \otimes \ldots \otimes A_{i_k}^{X(i_k)}$. □

Thus $\mathbf{G}_0^{\mathcal{R}\Pi}$ has all cofree commutative comonoids by the Lafont construction, and hence a uniform fixed point operator on $\mathbf{G}_!^{\mathcal{R}\Pi}$. Since the "inclusion" of $\mathbf{G}_0^{\mathcal{R}}$ into $\mathbf{G}_0^{\mathcal{R}\Pi}$ is fully faithful, this restricts to a fixed point operator on $\mathbf{G}_!^{\mathcal{R}}$. Change of base from $\mathbf{G}_!$ to $\mathbf{G}_!^{\mathcal{R}}$ preserves fixed points as above.

8. Weighted PCF and its Semantics

To illustrate our quantitative semantics, we recall from [1] the syntax and operational semantics of an extension of Scott's prototypical functional programming language PCF with bounded non-deterministic choice and "scalar" weights from a given complete semiring. We will show that any monoid in a Lafont category with biproducts gives rise to a model of this language in which the weights come from its internal semiring. This generalizes to an abstract setting the model presented in [1], in which each program denotes a matrix of values in a continuous semiring.

Typing is Church-style — i.e. every variable has a fixed PCF type (generated with the constructor $\to$ from a single ground type `nat`). The well-typed terms are defined with respect to contexts (finite sequences) of variables according to the rules in Table 1. So, in particular, every typable term has a unique type. The operational semantics for $\mathrm{PCF}^{\mathcal{R}}$ determines a weight in $\mathcal{R}$ for each (normal

$$\frac{}{\Gamma, x{:}T, \Gamma' \vdash x{:}T} \qquad \frac{\Gamma \vdash M{:}S \to T \quad \Gamma \vdash N{:}S}{\Gamma \vdash M\, N{:}T} \qquad \frac{\Gamma \vdash M{:}\mathtt{nat}}{\Gamma \vdash a.M{:}\mathtt{nat}}\, a \in \mathcal{R}$$

$$\frac{\Gamma, x{:}T \vdash M{:}T}{\Gamma \vdash \mu x.M{:}T} \qquad \frac{\Gamma, x{:}S \vdash M{:}T}{\Gamma \vdash \lambda x.M{:}S \to T} \qquad \frac{\Gamma \vdash M{:}\mathtt{nat} \quad \Gamma \vdash N{:}\mathtt{nat}}{\Gamma \vdash M \;\mathtt{or}\; N{:}\mathtt{nat}}$$

$$\frac{}{\Gamma \vdash \mathtt{0}{:}\mathtt{nat}} \qquad \frac{\Gamma \vdash M{:}\mathtt{nat}}{\Gamma \vdash \mathtt{succ}(M){:}\mathtt{nat}} \qquad \frac{\Gamma \vdash M{:}\mathtt{nat}}{\Gamma \vdash \mathtt{pred}(M){:}\mathtt{nat}}$$

$$\frac{\Gamma \vdash M{:}\mathtt{nat}}{\Gamma \vdash \mathtt{Ifz}(M){:}\mathtt{nat} \to \mathtt{nat} \to \mathtt{nat}}$$

Table 1: Typing Judgments for $\mathrm{PCF}^{\mathcal{R}}$

order) *reduction path* from a program (closed term of type `nat`) to a terminal value (numeral), by assigning a weight to each reduction step and multiplying together the weights from each step in the path. Since the only computation step at which there is a choice of reduction rules to apply is the reduction of explicit choice, each reduction path for a given program is uniquely determined by the finite sequence of branching decisions (whether to choose left or right) made at these points — i.e. by a unique element of the free monoid $\{l, r\}^*$ (for which the elements are sequences over $\{l, r\}$, composed by concatenation). Thus we define a reduction relation labelled with actions from the monoid $\{l, r\}^* \times (|\mathcal{R}|, \cdot, 1)$.

Definition 8.1. *The operational semantics of* $\mathrm{PCF}^{\mathcal{R}}$ *is the* labelled transition system *(LTS) in which the* states *are the programs (closed terms of type* `nat`*) of* $\mathrm{PCF}^{\mathcal{R}}$*, the set of* actions *is* $\{l, r\}^* \times |\mathcal{R}|$ *and the* transitions *are instances of* $E[M] \xrightarrow{u,a} E[M']$ *where* $M \xrightarrow{u,a} M'$ *is an instance of the rules defined in Table 2 and the* evaluation context $E[\_]$ *is an element of the grammar:*

$$E[\_] ::= [\_] \mid E\, N \mid \mathtt{Ifz}(E) \mid \mathtt{succ}(E) \mid \mathtt{pred}(E)$$

2(a) — Arithmetic, choice, control and weights

$\mathtt{Ifz}(0)$	$\xrightarrow{\varepsilon,1}$	$\lambda x.\lambda y.x$	$M \;\mathtt{or}\; N$	$\xrightarrow{l,1}$	M
$\mathtt{Ifz}(\mathtt{n}+1)$	$\xrightarrow{\varepsilon,1}$	$\lambda x.\lambda y.y$	$M \;\mathtt{or}\; N$	$\xrightarrow{r,1}$	N
$\mathtt{pred}(\mathtt{n}+1)$	$\xrightarrow{\varepsilon,1}$	$\mathtt{n}$	$a.M$	$\xrightarrow{\varepsilon,a}$	M

2(b) — β-reduction and fixed point unfolding

$(\lambda x.M)\, N$	$\xrightarrow{\varepsilon,1}$	$M[N/x]$	$\mu x.M$	$\xrightarrow{\varepsilon,1}$	$M[\mu x.M/x]$

Table 2: Reduction Rules for PCF$^{\mathcal{R}}$

Definition 8.2. *$P \overset{u,a}{\Longrightarrow} Q$ if there exists $i \in \mathbb{N}$ such that $P \overset{u,a}{\Longrightarrow}_i Q$, where:*

$$\frac{}{P \overset{\varepsilon,1}{\Longrightarrow}_i P} \qquad \frac{P \overset{u,a}{\longrightarrow} P' \quad P' \overset{v,b}{\Longrightarrow}_i P''}{P \overset{u\cdot v,a\cdot b}{\Longrightarrow}_j P''}\, i < j$$

Lemma 8.3. *If $P \overset{u,a}{\Longrightarrow} \mathtt{m}$ and $P \overset{u,b}{\Longrightarrow} \mathtt{n}$ then $a = a'$ and $m = n$.*

Proof. By induction on i that if $P \overset{u,a}{\Longrightarrow}_i \mathtt{m}$ and $P \overset{u,a'}{\Longrightarrow} \mathtt{n}$ then $a = a'$ and $m = n$. If $i = 0$ then $a = a' = 1$ and $m = n$. If $i > 0$ then:
$P \overset{v,b}{\longrightarrow} P'$ for some P' such that $P' \overset{w,c}{\Longrightarrow}_{i-1} \mathtt{m}$, where $u = v \cdot w$ and $a = b \cdot c$, and $P \overset{v',b'}{\longrightarrow} P''$ for some P'' such that $P'' \overset{w',c'}{\Longrightarrow} \mathtt{n}$ where $u = v'.w'$ and $a' = b' \cdot c'$. Observe that $v = v'$ (since either $P = E[Q_1 + Q_2]$, and so $v, v' \in \{l, r\}$ and $v.w = v'.w'$ implies $v = v'$, or $v = v' = \varepsilon$), $b = b'$ and $P' = P''$. So we may apply the inductive hypothesis to P' to get $c = c'$ (and thus $a = a'$) and $m = n$. ☐

This allows us to make the following definition of a path-weighting function for each sequence of branching choices.

Definition 8.4. *For each $u \in \{l, r\}^*$, the path weight from P to n along u is:*

$$w_u(P, n) \triangleq \begin{cases} a & \text{if } P \overset{u,a}{\Longrightarrow} \mathtt{n} \\ 0 & \text{if } \neg\exists a \in |\mathcal{R}|.P \overset{u,a}{\Longrightarrow} \mathtt{n}. \end{cases}$$

P is evaluated at n by taking the sum of the weights of all paths from P to n:

$$w(P, n) \triangleq \sum_{u \in \{l,r\}^*} w_u(P, n)$$

As discussed in [1, 36] this value can represent aspects of the evaluation behaviour of P, such as the number of distinct paths or length of the shortest or longest path to a given value or the probability of reaching it, depending on the choice of weighting semiring. From this notion of testing we derive a notion of observational equivalence:

Definition 8.5. *For terms $\Gamma \vdash P, Q : T$, $P \approx_T^\Gamma Q$ if for any context $C[\_ : T]$: $\mathtt{nat}$ binding the variables in Γ, and $n \in \mathbb{N}$, $w(C[P], n) = w(C[Q], n)$.*

If $\mathcal{R}$ is ordered then we have a notion of approximation: $P \lesssim_T^\Gamma Q$ if for any closing context $C[\_ : T] : \mathtt{nat}$ and $n \in \mathbb{N}$, $w(C[P], n) \leq_{\mathcal{R}} w(C[Q], n)$.

8.1. Denotational Semantics of $\mathrm{PCF}^{\mathcal{R}}$

Let $\mathcal{C}$ be a Lafont category with biproducts, and $\mathcal{R} \subseteq \mathcal{R}_{\mathcal{C}}$ a complete semiring contained in its internal semiring. Following [1] we may interpret $\mathrm{PCF}^{\mathcal{R}}$ in $\mathcal{C}$ by fixing an interpretation of `nat` as an "object of numerals" N in $\mathcal{C}$ with morphisms $z : I \to \mathsf{N}$, $s, p : \mathsf{N} \to \mathsf{N}$ and $c : \mathsf{N} \to !\mathsf{N} \multimap !\mathsf{N} \multimap \mathsf{N}$ which satisfy

$$s; p = \mathsf{id}_{\mathsf{N}},\ z; c = \lambda x.\lambda y.x \text{ and } z; s^{n+1}; c = \lambda x.\lambda y.y.$$

For any monoid M in $\mathcal{C}$, the biproduct $\bigoplus_{i \in \mathbb{N}} M$ gives an object of numerals in which the numeral `n` denotes the map $\eta; \iota_n : I \to \mathsf{N}_M$.

Definition 8.6. *Let* $(M, \mu : M \otimes M \to M, \eta : I \to M)$ *be a monoid in* $\mathcal{C}$ *such that* $f; \eta = g; \eta$ *implies* $f = g$ *for all* $f, g \in \mathcal{R}_{\mathcal{C}}$*. The* monoidal object of numerals $\mathsf{N}_M = \bigoplus_{i \in \mathbb{N}} M$ *satisfies* $\mathsf{N}_M = M \oplus \mathsf{N}_M$ *and so we may define* $z = \eta; \iota_l$*,* $s = \iota_r$*,* $p = \pi_r$ *and* c *to be the currying of:*

$$((\pi_0 \otimes \mathsf{der} \otimes \eta) + \sum_{1 \leq i < \omega} (\pi_i \otimes \eta \otimes \mathsf{der})); \bigoplus_{i \in \omega} \mu : \mathsf{N_M} \otimes !\mathsf{N_M} \otimes !\mathsf{N_M} \to \mathsf{N}_M$$

For example:

- Taking M to be the monoid given by the isomorphism $I \cong I \otimes I$ (i.e. $\mathsf{N}_M = \widetilde{\mathbb{N}}$) gives an interpretation of $\mathrm{PCF}^{\mathcal{R}}$ in $\mathcal{C}$ equivalent to the weighted relational model — i.e. the denotation of each term in $\mathcal{C}$ is the image of its denotation in $\mathcal{R}^{\Pi}$ under the functor $\widetilde{(\_)}$.
- For an object A of $\mathcal{C}$, taking M to be the monoid on $!A \multimap A$ given by function composition in $\mathcal{C}_!$ yields (up to isomorphism) a call-by-name CPS interpretation of PCF with answer object A. (Similarly, taking M to be the monoid on $A \multimap A$ given by composition in $\mathcal{C}$ yields a linear CPS model.)
- In Section 9 we take M to be the monoid denoted by sequential composition in the games model of Idealized Algol (under change of base). This yields an object of numerals equivalent to the interpretation of the natural numbers as the game with a single, initial Opponent question with answers for each natural number (see [37, 27, 14] etc.)

Given an object of numerals N in a Lafont category $\mathcal{C}$ with biproducts, we interpret

- A *type* as an object of $\mathcal{C}$ by setting $[\![\mathtt{nat}]\!] = \mathsf{N}$ and $[\![S \to T]\!] = ![\![S]\!] \multimap [\![T]\!]$.
- A *context* $\Gamma = x_1 : S_1, \ldots, x_n : S_n$ as $![\![S_1]\!] \otimes \ldots \otimes ![\![S_n]\!] \cong !([\![S_1]\!] \times \ldots \times [\![S_n]\!])$.
- A *term-in-context* $\Gamma \vdash P : T$ as a morphism $[\![P]\!]_{\Gamma} : [\![\Gamma]\!] \to [\![T]\!]$ in $\mathcal{C}$.

The Cartesian closed structure of $\mathcal{C}_!$ yields interpretations of the operations of the λ-calculus and:

- μ-abstraction denotes a parameterised fixpoint — $[\![\mu x.M]\!]_\Gamma = \mathsf{fix}_{[\![\Gamma]\!]}([\![M]\!]_\Gamma)$.
- Choice and scalar weighting denote the corresponding operations on the $\mathcal{R}_\mathcal{C}$-modules of morphisms in $\mathcal{C}$ — $[\![M \texttt{ or } N]\!]_\Gamma = [\![M]\!]_\Gamma + [\![N]\!]_\Gamma$ and $[\![a.M]\!]_\Gamma = a \otimes [\![M]\!]_\Gamma$.
- Other operations denote composition (in $\mathcal{C}$) with the corresponding morphisms given by the object of numerals — $[\![\texttt{0}]\!]_\Gamma = \epsilon_\Gamma; z$, $[\![\texttt{succ}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; s$, $[\![\texttt{pred}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; p$ and $[\![\texttt{Ifz}(M)]\!]_\Gamma = [\![M]\!]_\Gamma; c$.

8.2. Computational Adequacy

The key result relating our operational and denotational semantics is a form of *computational adequacy* — the denotation of a program P is the weighted sum of terminals of its reduction paths — i.e. $[\![P]\!] = \sum_{n \in \mathbb{N}} (w(P,n) \otimes [\![\texttt{n}]\!])$. By Lemma 8.3 we may define:

Definition 8.7. *The* path interpretation *of a program P with respect to $u \in \{l,r\}^*$ is the morphism $\langle\!|P|\!\rangle_u : I \to N$ such that:*

$$\langle\!|P|\!\rangle_u = \begin{cases} a.[\![\texttt{n}]\!] & \text{if } P \stackrel{u,a}{\Longrightarrow} \texttt{n} \\ 0 & \text{if } P \stackrel{u,a}{\not\Longrightarrow} \texttt{n} \text{ for all } a,n \end{cases}$$

Then $\langle\!|P|\!\rangle_u = \sum_{n \in \mathbb{N}} w_u(P,n).[\![\texttt{n}]\!]$ and so it suffices to show that $\sum_{u \in \{l,r\}^*} \langle\!|P|\!\rangle_u = [\![P]\!]$.

The proof of adequacy via logical relations for the weighted relational model in [1] depends on continuity of the weighting semiring. Our proof for general Lafont categories with biproducts requires a different approach. We define a new operational semantics, more directly related to our interpretation of fixed points — an *abstract machine* in which the environment is instrumented with bounds characterizing a particular call-pattern for each variable, and show that (a) this gives an equivalent notion of weighted reduction path to the original operational semantics, and (b) its denotational semantics is computationally adequate in the above sense.

An *environment* $\mathcal{E}$ is a finite sequence of triples, $(x_1, M_1, r_1), \ldots, (x_n, M_n, r_n)$, where each x_i is a (distinct) variable, M_i is a term, and $r_i \in \mathbb{M}$ is a nested finite multiset *resource bound*. Note that the latter are upper *and* lower bounds — precise specifications of how many times a procedure may be and must be called or a fixed point unfolded. We write $|\mathcal{E}|$ for the sequence of variables $x_1 : T_1, \ldots, x_n : T_n$ and define typing judgements $\Gamma \vdash \mathcal{E}$ for well formed environments as follows:

$$\frac{}{\Gamma \vdash \varepsilon} \qquad\qquad \frac{\Gamma \vdash \mathcal{E} \quad \Gamma, |\mathcal{E}|, x{:}T \vdash M{:}T}{\Gamma \vdash \mathcal{E}, (x^T, M, X)} X \in \mathbb{M}$$

$$\begin{array}{lcl}\mathcal{E}; (\lambda x.M)\, N & \xrightarrow{\varepsilon,1} & \mathcal{E}, (y, N, X); M[y/x] \\ \mathcal{E}; \mu x.M & \xrightarrow{\varepsilon,1} & \mathcal{E}, (y, M, X); M[y/x] \\ \mathcal{E}, (x, M, [Y] + X), \mathcal{E}'; x & \xrightarrow{\varepsilon,1} & \mathcal{E}, (y, M[y/x], Y), (x, M, X), \mathcal{E}'; M[y/x]\end{array}$$

Table 3: Labelled Transitions for Configurations

(Variables occuring only on the right of a rule are assumed fresh.)

A *configuration* is a pair $\mathcal{E}; P$ of an environment $\_ \vdash \mathcal{E}$ and a term P such that $|\mathcal{E}| \vdash P : \mathtt{nat}$.

Definition 8.8. *The* bounded abstract machine *for* $\mathrm{PCF}^{\mathcal{R}}$ *is the LTS in which the* states *are configurations (up to α-equivalence),* actions *are elements of* $\{l, r\}^* \times |\mathcal{R}|$ *and* transitions *are instances of* $\mathcal{E}; E[M] \xrightarrow{u,a} \mathcal{E}'; E[M']$*, where* either*:*

- $\mathcal{E} = \mathcal{E}'$ *and* $M \xrightarrow{u,a} M'$ *is a rule from Table 2a,* or
- $\mathcal{E}; M \xrightarrow{u,a} \mathcal{E}'; M'$ *is a rule from Table 3.*

Remark 8.9. *The bounded abstract machine may be refined by assigning natural number bounds to non-recursive bindings (i.e. those arising from function application). It is straightforward to show that any configuration* $\mathcal{E}; P$ *cannot terminate successfully if* $\mathcal{E}$ *contains a binding* (x, M, X) *where* x *does not occur in* M *and* X *is not a natural number (i.e. in* $\mathbb{M}_1$*), and so all such paths can safely be avoided. Observe also that where the bound is a natural number the rule for calling* x *may be simplified to the expected form:*

$$\mathcal{E}, (x, M, k+1), \mathcal{E}'; x \xrightarrow{\varepsilon,1} \mathcal{E}, (x, M, k), \mathcal{E}'; M$$

Every reduction path of this LTS is, in fact, terminating (Lemma 8.22) — we say that a reduction path terminates *successfully* if it both reaches a value (a numeral) *and* consumes all of the resources in the environment — i.e. all bounds in its final configuration are the empty multiset.

Definition 8.10. *Let* Env_0 *be the set of all environments in which all bounds are empty (i.e.* $\mathcal{E}, (x, M, X) \in Env_0$ *if and only* $X = [\_]$ *and* $\mathcal{E} \in Env_0$*).*

We define the "many-step" evaluation relation $\mathcal{E}; P \overset{u,a}{\Longrightarrow} n$ *(*$\mathcal{E}; P$ *successfully reduces to* $\mathtt{n}$ *along the path* u *with weight* a*) if* $\mathcal{E}; P \overset{u,a}{\Longrightarrow}_i n$ *for some* $i \in \mathbb{N}$*, where:*

$$\frac{}{\mathcal{E}; \mathtt{n} \overset{\varepsilon,1}{\Longrightarrow}_i n}\ \mathcal{E} \in Env_0 \qquad \frac{\mathcal{E}; P \xrightarrow{u,a} \mathcal{E}'; P' \quad \mathcal{E}'; P' \overset{v,b}{\Longrightarrow}_i n}{\mathcal{E}; P \overset{u \cdot v, a \cdot b}{\Longrightarrow}_j n}\ i < j$$

Although the abstract machine semantics is nondeterministic in the sense that a state may have one-step reductions with the same label to (countably many) different states, only at most one of those states is on a successfully terminating reduction path.

Proposition 8.11. *If* $|\mathcal{E}| = |\mathcal{E}'|$, $\mathcal{E}; P \stackrel{u,a}{\Longrightarrow}_i m$ *and* $\mathcal{E}'; P \stackrel{u,b}{\Longrightarrow} n$ *then* $\mathcal{E} = \mathcal{E}'$, $m = n$ *and* $a = b$.

Proof. By induction on i:
If $i = 0$ then $P = \mathtt{m} = \mathtt{n}$ and $\mathcal{E}, \mathcal{E}' \in Env_0$ and hence $\mathcal{E} = \mathcal{E}'$.
Otherwise $\mathcal{E}; P \stackrel{v,b}{\longrightarrow} \mathcal{E}''; Q$ and $\mathcal{E}'; P \stackrel{v,b}{\longrightarrow} \mathcal{E}'''; Q'$ such that $|\mathcal{E}''| = |\mathcal{E}'''|$, and $\mathcal{E}'; Q \stackrel{w,c}{\Longrightarrow}_{i-1} m$ and $\mathcal{E}'''; Q \stackrel{w,c}{\Longrightarrow} n$ where $u = v.w$ and $a = b \cdot c$ and hence by inductive hypothesis, $m = n$ and $\mathcal{E}'' = \mathcal{E}'''$ and so $\mathcal{E} = \mathcal{E}'$ as required. ☐

Thus we may define the path interpretation of configurations:

$$\langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \begin{cases} a.[\![\mathtt{n}]\!] & \text{if } \mathcal{E}; P \Longrightarrow^{u,a} n \\ 0 & \text{if } \mathcal{E}; P \not\Longrightarrow^{u,a} n \text{ for all } a, n \end{cases}$$

We now show that this agrees with the path interpretation of programs according to the original operational semantics.

Lemma 8.12. *For any* $(x, N, X), \mathcal{E}; P$*, where* $X \in \mathbb{M}$*:*

- *If* $(x, N, X), \mathcal{E}; P \stackrel{u,a}{\Longrightarrow}_i n$ *then* $(\mathcal{E}; P)[\mu x.M/x] \stackrel{u,a}{\Longrightarrow}_i n$.
- *If* $(\mathcal{E}; P)[\mu x.N/x] \stackrel{u,a}{\Longrightarrow}_i n$ *then* $\exists X' \in \mathbb{M}$ *such that* $(x, N, X'), \mathcal{E}; P \stackrel{u,a}{\Longrightarrow} n$.

Proof. By induction on i. The key cases are where $P = E[x]$ — for example:

- If $(x, N, X), \mathcal{E}; P \stackrel{u,a}{\Longrightarrow}_{i+1} n$ then for some Y in the support of X,
  $(x, N, X - [Y]), (y, N[y/x], Y), \mathcal{E}; E[N[y/x]] \stackrel{u,a}{\Longrightarrow}_i n$.
  By hypothesis, $((y, N[y/x], Y), \mathcal{E}; E[N[y/x]])[\mu x.N/x] \stackrel{u,a}{\Longrightarrow}_i n$, and so
  $(\mathcal{E}; E[\mu x.N])[\mu x.N/x] \stackrel{u,a}{\Longrightarrow}_{i+1} n$ as required.
- If $(\mathcal{E}; P)[\mu x.N/x] = (\mathcal{E}; E[\mu x.N])[\mu x.N/x] \stackrel{u,a}{\Longrightarrow}_{i+1} n$ then
  $\mathcal{E}; E[\mu x.N])[\mu x.N/x] \stackrel{\varepsilon,1}{\longrightarrow} \mathcal{E}, ((y, N[y/x], Y); E[N[y/x]])[\mu x.N/x]$ for some $Y \in \mathbb{M}$ such that $\mathcal{E}; ((y, N[y/x], Y); E[N[y/x]])[\mu x.N/x] \stackrel{u,a}{\Longrightarrow}_i n$.
  So by induction hypothesis there exists $X' \in \mathbb{M}$ such that
  $(x, N, X'), \mathcal{E}, (y, N[y/x], Y); E[N[y/x]] \stackrel{u,a}{\Longrightarrow} n$ and so
  $(x, N, X'+[Y]), \mathcal{E}; E[x] \stackrel{\varepsilon,1}{\longrightarrow} (x, N, X'), \mathcal{E}, (y, N[y/x], Y); E[N[y/x]] \stackrel{u,a}{\Longrightarrow} n$.
  ☐

Lemma 8.13. $P \stackrel{u,a}{\Longrightarrow} \mathtt{n}$ *if and only if* $\_; P \stackrel{u,a}{\Longrightarrow} n$.

Proof. We prove that $\_; P \stackrel{u,a}{\Longrightarrow}_i n$ implies $P \stackrel{u,a}{\Longrightarrow} \mathtt{n}$ by induction on i.
E.g. suppose $P \equiv E[\mu x.N]$. If $\_; P \stackrel{u,a}{\Longrightarrow}_{i+1} n$ then $\_; P \stackrel{\varepsilon,1}{\longrightarrow} (y, N, X); E[N[y/x]]$ for some $X \in \mathbb{M}$ such that $(y, N, X); E[N[y/x]] \stackrel{u,a}{\Longrightarrow}_i n$. By Lemma 8.12, $\_; E[N[\mu x.N/x]] \stackrel{u,a}{\Longrightarrow}_i n$ and so by induction hypothesis, $E[N[\mu x.N/x]] \stackrel{u,a}{\Longrightarrow} \mathtt{n}$. Hence $E[\mu x.N] \stackrel{u,a}{\Longrightarrow} \mathtt{n}$ by definition of the operational semantics.

We prove that if $P \stackrel{u,a}{\Longrightarrow}_i \mathtt{n}$ then $\_; P \stackrel{u,a}{\Longrightarrow} n$ by induction on i. Again giving the case $P \equiv E[\mu x.N]$, if $P \stackrel{u,a}{\Longrightarrow}_{i+1} \mathtt{n}$, then $E[N[\mu x.N/x]] \stackrel{u,a}{\Longrightarrow}_i \mathtt{n}$.

By induction hypothesis $\_; E[N[\mu x.N/x]] \overset{u,a}{\Longrightarrow} n$, and so by Lemma 8.12, there exists $X \in \mathbb{M}$ such that $(x, N, X); E[N] \overset{u,a}{\Longrightarrow} n$ and hence $\_; E[\mu x.N] \overset{\varepsilon,a}{\longrightarrow} (y, N, X); E[N[y/x]] \overset{u,a}{\Longrightarrow} n$ as required. □

Hence by the definitions of $\langle\!| \_ |\!\rangle$ for programs and configurations:

Proposition 8.14. *For any $u \in \{l, r\}^*$, $\langle\!| P |\!\rangle_u = \langle\!| \_; P |\!\rangle_u$.*

8.3. Denotational Semantics for Configurations

We now extend the denotational semantics of programs to configurations, and show that it is computationally adequate. Environments are interpreted using nested finite multiset indexed approximants derived from the construction of the uniform fixed point operator (Definition 6.15).

Definition 8.15. *An* environment $\Gamma \vdash \mathcal{E}$ *is interpreted as a morphism $[\![\mathcal{E}]\!]_\Gamma : [\![\Gamma]\!] \to [\![|\mathcal{E}|]\!]$. defined by induction on its length:*

- $[\varepsilon]\!]_\Gamma : [\![\Gamma]\!] \to I = \epsilon_{[\![\Gamma]\!]}$,
- $[\![(x, M : T, X), \mathcal{E}]\!]_\Gamma = \delta_{[\![\Gamma]\!]}; ([\![[\![M]\!]^X_{|\mathcal{E}|, x:T} \otimes [\![\Gamma]\!]); [\![\mathcal{E}]\!]_{x:T,\Gamma}$.

For $\Gamma \vdash P$ and $\Gamma, |\mathcal{E}| \vdash P : \mathtt{nat}$ define $[\![\mathcal{E}; P]\!]_\Gamma = \delta_{[\![\Gamma}; ([\![\Gamma]\!] \otimes [\![\mathcal{E}]\!]_\Gamma); [\![P]\!]_{|\mathcal{E}|} : [\![\Gamma]\!] \to [\![\mathtt{nat}]\!]$ so that (up to coherence isomorphisms) $[\![\mathcal{E}, \mathcal{E}'; P]\!]_\_ = [\![\mathcal{E}]\!]_\_; [\![\mathcal{E}'; P]\!]_{|\mathcal{E}'|}$.

Lemma 8.16. *If $\mathcal{E} \in Env_0$ then $[\![\mathcal{E}; \mathtt{n}]\!] = z; s^n$. Otherwise, $[\![\mathcal{E}; \mathtt{n}]\!] = 0$.*

Proof. $[\![\mathcal{E}; \mathtt{n}]\!] = [\![\mathcal{E}]\!]; \epsilon_{|\mathcal{E}|}; z; s^n$. So it suffices to observe that $[\![\mathcal{E}]\!]; \epsilon_{|\mathcal{E}|} : I \to I = \mathsf{id}_I$ if $\mathcal{E} \in Env_0$, and $[\![\mathcal{E}]\!]; \epsilon_{|\mathcal{E}|} = 0$ otherwise, by Lemma 6.16 (1). □

To prove soundness for the reduction rules we require an interpretation of evaluation contexts as morphisms in $\mathcal{C}$ (rather than the $\mathcal{C}_!$), which is derived as follows.

Proposition 8.17. *If $\Gamma, \bullet : S \vdash E[\bullet] : T$ there is a morphism $[\![E[\bullet]]\!]_\Gamma : [\![\Gamma]\!] \otimes [\![S]\!] \to [\![T]\!]$ such that for all $\Gamma \vdash M : S$, $[\![E[M]]\!]_\Gamma = \delta_\Gamma; ([\![\Gamma]\!] \otimes [\![M]\!]_\Gamma); [\![E[\bullet]]\!]_\Gamma$.*

Proof. $[\![E[\bullet]]\!]_\Gamma$ is defined inductively as follows:

- $[\![[\bullet]]\!]_\Gamma = (\epsilon_{[\![\Gamma]\!]} \otimes [\![T]\!])$
- $[\![E[\bullet]]\!]_\Gamma = ((\delta_{[\![\Gamma]\!]} \otimes [\![S]\!]); ([\![N : T']\!]_\Gamma \otimes [\![E'[\bullet]_\Gamma]\!]); \mathsf{app}_{[\![T]\!], [\![T']\!]}$.
- $[\![\mathtt{Ifz}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; c$, $[\![\mathtt{succ}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; s$, $[\![\mathtt{pred}(E[\bullet])]\!]_\Gamma = [\![E[\bullet]]\!]_\Gamma; p$. □

We use Proposition 8.17 (together with the properties of Lafont categories with biproducts and fixpoint approximants) to establish a key soundness property. Say that a configuration $\mathcal{E}; P$ is *silent* if P is neither a value $\mathtt{n}$ for $n \in \mathbb{N}$ nor of the form $E[M\ \mathtt{or}\ N]$. Any silent configuration denotes the weighted sum of the denotations of configurations which are reachable in one reduction step.

Lemma 8.18. *If $\mathcal{E}; P$ is silent then $[\![\mathcal{E}; P]\!] = \sum\{a.[\![\mathcal{E}'; P']\!] \mid (\mathcal{E}; P) \overset{(\varepsilon,a)}{\longrightarrow} (\mathcal{E}'; P')\}$.*

Proof. See Appendix B. □

To give an inductive proof of the adequacy of the resource-bounded semantics, we show that reduction is strictly decreasing with respect to a measure on terms based on the *nested multiset order* [38].

Definition 8.19. *For each $i \in \mathbb{N}$, $(\mathbb{M}_{i+1}, \ll_{i+1})$ is the* multiset order *generated by $(\mathbb{M}_i, \ll_i)$ — i.e. $X \ll_{i+1} Y$ if for all $x \in \sup(X - Y)$ there exists $y \in \sup(Y - X)$ with $x \ll_i y$.*
This is a well-founded partial order [38], and $\ll_i \subseteq \ll_{i+1}$. Hence we may define a well-founded order $\ll^* = \bigcup_{i \in \mathbb{N}} \ll_i$ on $\mathbb{M}$. We write $<^*$ for the corresponding strict inequality.

Note that if $X \ll^* X'$ and $Y \ll^* Y'$ then $X + Y \ll^* X' + Y'$. Let $k.X$ denote the k-fold multiset union of X with itself.

Lemma 8.20. *$k.X <^* [X]$ for all $k \in \mathbb{N}$ and $X \in \mathbb{M}$.*

Proof. Define $X \prec Y$ if for all $x \in \sup(X)$ there exists $y \in \sup(Y - X)$ such that $x \in \sup(y)$, so that $k.X \prec [X]$. Note that $x \in \sup(y)$ implies $x \prec y$, since if $a \in \sup x$, there exists $b \in \sup(y - x)$ such that $a \in b$ by taking $b = x$ (since $x \notin \sup(x)$).

We prove that $X, Y \in \mathbb{M}_i$ and $X \prec Y$ implies $X \ll_i Y$ by induction on i: for the induction case, suppose $X, Y \in \mathbb{M}_i$ and $X \prec Y$. Then if $x \in \sup(X - Y)$ there exists $y \in \sup(Y - X)$ such that $x \in \sup(y)$. Then $x \prec y$ and so by induction hypothesis $x \ll_i y$. Thus $X \ll_{i+1} Y$ as required. □

Definition 8.21. *Let ℓ be the map from $\mathrm{PCF}^{\mathcal{R}}$ terms into $\mathbb{M}$ defined by:*

$$\begin{gathered}\ell(x) = \ell(\mathtt{0}) = 1 \qquad \ell(M \;\mathtt{or}\; N) = \ell(N) + \ell(N) \\ \ell(\mu x.M) = [\ell(M)] \qquad \ell(M\,N) = \ell(M) + [\ell(N)] \\ \ell(\lambda x.M) = \ell(\mathtt{Ifz}(M)) = \ell(\mathtt{succ}(M)) = \ell(\mathtt{pred}(M)) = \ell(a.M) = \ell(M) + 1\end{gathered}$$

ℓ is extended to environments, and thus configurations, by setting:

$$\ell\langle (x_i, M_i, X_i) \mid i \le n\rangle = \sum_{i \le n} |X_i|.\ell(M_i) \text{ and } \ell(\mathcal{E}; P) = \ell(\mathcal{E}) + \ell(P)$$

where $|X| \in \mathbb{N}$ is defined inductively by $|[X_1, \ldots, X_k]| = 1 + \sum_{i \le k} |X_i|$.

Lemma 8.22. *$\mathcal{E}; P \xrightarrow{u,a} \mathcal{E}'; P'$ implies $\ell(\mathcal{E}'; P') <^* \ell(\mathcal{E}; P)$.*

Proof. ℓ is extended to evaluation contexts by setting $\ell(\bullet) = \varnothing$, so $\ell(E[M]) = \ell(E[\bullet]) + \ell(M)$. Then e.g.

- Suppose $P = E[\mu x.N]$, so $\mathcal{E}'; P' = (\mathcal{E}, (x, N, X); E[N]$ for some $X \in \mathbb{M}$. Then $\ell(\mathcal{E}'; P') = \ell(\mathcal{E}) + |X|.\ell(N) + \ell(E[\bullet]) + \ell(N)$
$<^* \ell(\mathcal{E}) + \ell(E[\bullet]) + [\ell(N)]$ (since $(|X|+1).\ell(N) <^* [\ell(N)]$ by Lemma 8.20)
$= \ell(\mathcal{E}; P)$.

- Suppose $\mathcal{E} = \mathcal{E}'', (x, N, X + [Y]), \mathcal{E}'''$ and $P = E[x]$,
so $\mathcal{E}' = \mathcal{E}'', (x, N, X), \mathcal{E}''', (y, N[y/x], Y), \mathcal{E}'''$ and $P = E[N[y/x]]$. Then:
$\ell(\mathcal{E}'; P') = \ell(\mathcal{E}'') + |X|.\ell(N) + |Y|.\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet]) + \ell(N[y/x])$
$= \ell(\mathcal{E}'') + (|X| + |Y| + 1).\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet])$
$<^* \ell(\mathcal{E}'') + |X + [Y]|.\ell(N) + \ell(\mathcal{E}''') + \ell(E[\bullet]) + \ell(x)$
$= \ell(\mathcal{E}; P)$.

The remaining cases are similar, or simpler. □

We can now establish adequacy for the bounded abstract machine.

Proposition 8.23. *For any configuration* $\mathcal{E}; P$*:* $\sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = [\![\mathcal{E}; P]\!]$.

Proof. By nested multiset induction on $\ell(\mathcal{E}; P)$:

- Suppose $P = \mathtt{n}$. Then by Lemma 8.16:
If $\mathcal{E} \in Env_0$ then $\sum_{s \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon = z; s^n = [\![\mathcal{E}; P]\!]$.
Otherwise $\sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon = 0 = [\![\mathcal{E}; P]\!]$.

- Suppose $P = E[N_l \;\mathtt{or}\; N_r]$. Observe that $\{l, r\}^* = \{\varepsilon\} \cup \{lu, ru \mid u \in \{l, r\}^*\}$, so
$\sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle = \langle\!\langle \mathcal{E}; P \rangle\!\rangle_\varepsilon + \sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_{lu} + \sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_{ru}$
$= 0 + \sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; E[N_l] \rangle\!\rangle_u + \sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; E[N_r] \rangle\!\rangle_u$
$= [\![\mathcal{E}; E[N_l]]\!] + [\![\mathcal{E}; E[N_r]]\!]$ (by the induction hypothesis)
$= [\![\mathcal{E}; P]\!]$ (by Proposition 8.17 and bilinearity of composition).

- Otherwise, $\mathcal{E}; P$ reduces silently and so
$\sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}; P \rangle\!\rangle_u = \sum_{u \in \{l,r\}^*} \sum\{\langle\!\langle \mathcal{E}'; P' \rangle\!\rangle_u \mid (\mathcal{E}; P) \xrightarrow{(\varepsilon,1)} (\mathcal{E}'; P')\}$ (by Lemma 8.18)
$= \sum\{\sum_{u \in \{l,r\}^*} \langle\!\langle \mathcal{E}'; P' \rangle\!\rangle_u \mid (\mathcal{E}; P) \xrightarrow{(\varepsilon,1)} (\mathcal{E}'; P')\}$ (by partition associativity)
$= \sum\{[\![\mathcal{E}'; P']\!] \mid (\mathcal{E}; P) \xrightarrow{(\varepsilon,1)} (\mathcal{E}'; P')\}$ by induction hypothesis
$= [\![\mathcal{E}; P]\!]$ by Lemma 8.18. □

Theorem 8.24. *The semantics of* $\mathrm{PCF}^{\mathcal{R}}$ *in a Lafont category with biproducts, an object of numerals and internal semiring* $\mathcal{R}_\mathcal{C} \supseteq \mathcal{R}$ *is computationally adequate.*

Proof. For any program P, $[\![P]\!] = [\![\_; P]\!] = \sum_{u \in \{l,r\}^*} \langle\!\langle \_; P \rangle\!\rangle_u = \sum_{u \in \{l,r\}^*} \langle\!\langle P \rangle\!\rangle_u$ by Propositions 8.14 and 8.23. □

Corollary 8.25. *The semantics of* $\mathrm{PCF}^{\mathcal{R}}$ *in a Lafont category with biproducts, a monoidal object of numerals and internal semiring* $\mathcal{R}_\mathcal{C} \supseteq \mathcal{R}$ *is equationally sound — i.e. if* $[\![M : T]\!]_\Gamma = [\![N : T]\!]_\Gamma$ *then* $M \approx^\Gamma_T N$.

Proof. If $[\![M : T]\!]_\Gamma = [\![N]\!]_\Gamma$ then for any compatible, closing context $C[\_] : \mathtt{nat}$, $[\![C[M]]\!] = [\![C[N]]\!]$ and so $\sum_{k\in\mathbb{N}} w(C[M], k).[\![\mathtt{k}]\!] = \sum_{\mathtt{k}\in\mathbb{N}} \mathtt{w(C[M],k)}.[\![\mathtt{k}]\!]$ by Theorem 8.24. For each $k \in \mathbb{N}$, $w(C[M], k); \eta = [\![C[M]]\!]; \pi_k = [\![C[N]]\!]; \pi_k = w(C[N], k); \eta$ and so $w(C[M], k) = w(C[N], k)$. □

If $\mathcal{C}$ is order enriched (thus $\mathcal{R}_\mathcal{C}$ is ordered) and $f; \eta \leq g; \eta$ implies $f \leq g$ for all $f, g \in \mathcal{R}_\mathcal{C}$ then we also have inequational soundness:
$[\![M]\!] \leq [\![N]\!]$ implies $w(C[M], k); \eta = C[M]; \pi_k \leq [\![C[N]]\!]; \pi_k = w(C[N], k); \eta$ and thus $w(C[M], k) \leq w(C[N], k)$.

9. Full Abstraction: $\mathcal{R}$-Weighted Idealized Algol

As noted in [1], the weighted relational model of $\mathrm{PCF}^\mathcal{R}$ is not *fully abstract* (although the weighted relational model of probabilistic PCF, which is a sublanguage of PCF weighted over $\mathcal{R}_+^\infty$, inherits full abstraction from the probabilistic coherence spaces model [39]): for example, the terms $\lambda x.\mu y.\mathtt{0} \;\mathtt{or}\; y$ and $\lambda x.(\mu y.\mathtt{0} \;\mathtt{or}\; y) \;\mathtt{or}\; (\mathtt{Ifz}(x)\; \mathtt{0}\; \mu y.y)$ are observationally equivalent but denote distinct weighted relations.

Arguably, this is due to the limited expressiveness of PCF, which allows only indirect observation of the call-patterns for variables which are recorded in the model. However, in the change of base we have a recipe for constructing quantitative interpretations from game semantics, which has furnished fully abstract qualitative models of a wide range of functional languages with side-effects. In this section, we show how change of base can preserve full abstraction results by considering a basic example, Reynolds' *Idealized Algol* [40]. By the results in [14] we know that our CCC of games $\mathbf{G}_!$ furnishes a semantics of Idealized Algol — which may be considered as an extension of PCF with integer state (conservative with respect to the operational semantics). So applying the change-of-base induced functor $\Phi_!^\mathcal{R} : \mathbf{G}_! \to \mathbf{G}_!^\mathcal{R}$ gives us a semantics of Idealized Algol in $\mathbf{G}_!^\mathcal{R}$. The latter is also an instance of our categorical model of $\mathcal{R}$-weighted PCF which agrees with the semantics of Idealized Algol on their common part (PCF itself). So by combining both models we obtain an interpretation of $\mathrm{IA}^\mathcal{R}$ — erratic Idealized Algol with scalar weights from $\mathcal{R}$. Moreover, this semantics inherits the full abstraction property from the qualitative model, as we will now show.

Types of Idealized Algol are formed from ground types `nat`, `com` (commands) and `var` (integer references). Terms of $\mathrm{IA}^\mathcal{R}$ are formed by extending the simply typed λ-calculus with the operations of $\mathrm{PCF}^\mathcal{R}$ and the following constants for imperative programming[6]:

- Sequential composition — $\mathtt{seq} : \mathtt{com} \to B \to B$ where $B \in \{\mathtt{com}, \mathtt{nat}\}$,
- New variable declaration — $\mathtt{new} : \mathtt{nat} \to (\mathtt{var} \to B) \to B$,
- Read and write — $\mathtt{set} : \mathtt{nat} \to \mathtt{var} \to \mathtt{com}$, $\mathtt{deref} : \mathtt{var} \to \mathtt{nat}$,

[6]We consider the variant of IA with *active expressions* and *bad variables* as in [14].

$\mathcal{S}; \texttt{seq}\,\texttt{skip}$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}; \lambda x.x$	$\mathcal{S}; (\texttt{set}\,\texttt{n})\,(\texttt{mkvr}\,M\,N)$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}; N\,\texttt{n}$
$\mathcal{S}; (\texttt{new}\,\texttt{n})\,P$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}, (a,n); P\,a$	$\mathcal{S}; \texttt{deref}(\texttt{mkvr}\,M\,N)$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}; M$
$\mathcal{S}; (\texttt{set}\,\texttt{n})\,a$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}[a \mapsto n]; \texttt{skip}$	$\mathcal{S}[a \mapsto n]; \texttt{deref}\,a$	$\xrightarrow{\varepsilon,1}$	$\mathcal{S}; \texttt{n}$

Table 4: Operational Semantics for $\mathrm{IA}^{\mathcal{R}}$ — sequential composition and mutable variables

- "Bad variable" construction — $\texttt{mkvr} : \texttt{nat} \to (\texttt{nat} \to \texttt{com}) \to \texttt{var}$.

The operational semantics for $\mathrm{IA}^{\mathcal{R}}$ extends that given for $\mathrm{PCF}^{\mathcal{R}}$ in Definition 8.1, just as Idealized Algol extends PCF. We define a labelled transition system over the same set of actions, in which *states* are pairs $(\mathcal{S}; P)$ of a store (a sequence $(a_1, n_1), \ldots, (a_n, n_k)$ of pairs of a location name and integer value) and a (ground-type) program. *Transitions* are all instances of $\mathcal{S}; E[M] \xrightarrow{u,a} \mathcal{S}'; E[M']$ where the evaluation context $E[\_]$ is given by the extended grammar:

$$E ::= [\_] \mid E\,M \mid \texttt{succ}(E) \mid \texttt{pred}(E) \mid \texttt{Ifz}(E)$$
$$\mid \texttt{seq}\,E \mid \texttt{new}\,E \mid (\texttt{new}\,\texttt{n})\,E \mid \texttt{set}\,E \mid (\texttt{set}\,\texttt{n})\,E \mid \texttt{deref}\,E$$

and either

- $\mathcal{S} = \mathcal{S}'$ and $M \xrightarrow{u,a} M'$ is an action from Table 2, or
- $\mathcal{S}; M \xrightarrow{u,a} \mathcal{S}; M'$ is an instance of one of the further rules in Table 4.

As for PCF, the relation $\xrightarrow{u,a}$ is deterministic, and so we may define the weight in $\mathcal{R}$ of each configuration with respect to a sequence $u \in \{l, r\}^*$ of branching choices:

$$w_u(\mathcal{S}; P) = \begin{cases} a & \text{if } \mathcal{S}; P \xrightarrow{u_1,a_1} \ldots \xrightarrow{u_n,a_n} \mathcal{S}'; \texttt{skip} \text{ and } (u_1 \cdot \ldots u_n, a_1 \cdot \ldots \cdot a_n) = (u, a) \\ 0 & \text{if there is no such sequence of transitions} \end{cases}$$

As in the case of PCF, the computational meaning of $\mathrm{IA}^{\mathcal{R}}$ depends on the choice of semiring — it may be regarded as a metalanguage for a family of "resource-sensitive" imperative programming languages and their semantics. The weighted games models discussed previously may be viewed as instances of this. Probabilistic games [19] are used to interpret Idealized Algol extended with a constant $\texttt{coin} : \texttt{nat}$ which reduces to either $\texttt{0}$ or $\texttt{1}$, both with probability 0.5. Thus we may interpret probabilistic Algol inside $\mathrm{IA}^{\mathbb{R}_+^\infty}$ by defining $\texttt{coin} \triangleq (0.5).\texttt{0}\,\texttt{or}\,(0.5).\texttt{1}$.

In [33] slot games are used to give an interpretation of Idealized (Concurrent) Algol which is sound with respect to an operational semantics which keeps track of the (time, memory, etc.) costs of evaluation as a natural number — each reduction rule is decorated with such a cost, and the worst-case cost is assigned to each program. Setting $\mathcal{R}$ to be the tropical semiring, we may define a translation of IA into $\mathrm{IA}^{\mathcal{R}}$ which is sound with respect to this notion of evaluation, by applying a weighting to each operation corresponding to the cost of its evaluation.

9.1. Denotational Semantics of $\mathrm{IA}^{\mathcal{R}}$

We interpret $\mathrm{IA}^{\mathcal{R}}$ in the category of games and $\mathcal{R}$-weighted strategies by extending the semantics of $\mathrm{PCF}^{\mathcal{R}}$ in $\mathbf{G}_!^{\mathcal{R}}$ with the image under $\Phi_0^{\mathcal{R}}$ of the semantics of the types and constants of Idealized Algol defined in [14]. Thus `com` denotes the game Σ over the arena with a single question enabling a single answer, `nat` the arena with a single question enabling answers for each $n \in \mathbb{N}$, and `var` the arena $[\![\mathtt{nat}]\!] \times [\![\mathtt{com}]\!]^{\omega}$. Each of the constants $C : T$ of Idealized Algol denotes a strategy $[\![C]\!] : I \to [\![T]\!]$ in $\mathbf{G}_!$, and thus a $\mathcal{R}$-weighted strategy in $\mathbf{G}_!^{\mathcal{R}}$. In particular, $\mathtt{new\,n}\,\lambda x.M$ denotes the composition of $[\![M]\!]_{x:\mathtt{var}}$ (in $\mathbf{G}$) with a strategy $\mathsf{cell}_n : I \to [\![\mathtt{var}]\!]$ which behaves as a reference cell initialized with the value `n`.

Remark 9.1. *The strategies denoted by sequential composition and the command* `skip` *correspond (up to currying and dereliction) to monoid structure on the games* $[\![\mathtt{com}]\!]$*. The associated monoidal object of numerals* $\mathsf{N}_M = \bigoplus_{n\in\mathbb{N}}$ *is (up to isomorphism) the denotation of* `nat` *in* $\mathbf{G}_0^{\mathcal{R}}$*. The denotation of* `var` *is isomorphic to* $N_M \oplus N_M$ *since* $\times$ *and* $(\_)^{\omega}$ *are biproducts.*

Computational adequacy is now simply the requirement that the weight for a program $P : \mathtt{com}$ computed by the operational semantics is equal to the weight assigned by its denotation to the single well-opened sequence (qa) in $[\![\mathtt{com}]\!]$. Equivalently, letting $\langle\!|P|\!\rangle_u : I \to [\![\mathtt{com}]\!] = \begin{cases} a.[\![\mathtt{skip}]\!] & \text{if } \_; P \overset{u,a}{\Longrightarrow} \mathcal{S}; \mathtt{skip} \text{ for some } u, \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$.

Proposition 9.2 (Adequacy)**.** *For every program* $P : \mathtt{com}$, $[\![P]\!] = \sum_{u\in\{l,r\}^*} \langle\!|P|\!\rangle_u$.

Proof. We extend the proof of adequacy for $\mathrm{PCF}^{\mathcal{R}}$ to include state — defining a bounded abstract machine over the same set of labels in which states are triples $(\mathcal{S};\mathcal{E};P)$ of a store $\mathcal{S} = (a_1,v_1),\ldots,(a_n,v_n)$, a bounded environment $a_1 : \mathtt{var},\ldots,a_n : \mathtt{var} \vdash \mathcal{E}$ and a term P such that $a_1 : \mathtt{var},\ldots,a_n : \mathtt{var}, |\mathcal{E}| \vdash P : \mathtt{com}$, and transitions are given by the rules of Tables 2(a), 3 and 4 and showing that (a) this gives an equivalent notion of weighted reduction path to the unbounded semantics (cf. Proposition 8.14) and (b) its denotational semantics is computationally adequate (cf. Proposition 8.23).

The proof of these results follows the proofs of Propositions 8.14 and 8.23 — the denotation of $(a_1,v_1),\ldots,(a_n,v_n);\mathcal{E};P$ is given by composing $[\![\mathcal{E};P]\!] : [\![\mathtt{var}]\!]^n \to [\![B]\!]$ with $\mathsf{cell}_{v_1} \otimes \ldots \otimes \mathsf{cell}_{v_n} : I \to [\![\mathtt{var}]\!]^{\otimes n}$ We just need to extend Lemma 8.18 with the new reduction rules for the constants of Idealized Algol, for which soundness was established in [14] in proving adequacy for the qualitative game semantics in $\mathbf{G}_0$. □

9.2. Full Abstraction

Following [14], we give a fully abstract model of $\mathrm{IA}^{\mathcal{R}}$ by changing the meaning of types to denote games in which the set of plays contains only justified sequences which are complete (i.e. every question has been answered). It is sufficient to change the meaning of ground types to contain only complete sequences for the denotations of all types to satisfy this condition. Terms now denote maps from sets of complete sequences into $\mathcal{R}$. The proof of adequacy extends to this interpretation by modest adaptation. Full abstraction follows readily from the following *definability property* for the complete play interpretation of Idealized Algol in $\mathbf{G}_0$.

Theorem 9.3. *[14] For any finite strategy $\sigma : [\![T]\!]$ there exists a term $M_\sigma : T$ of Idealized Algol such that $[\![M_\sigma]\!] = \sigma$.*

A weighted strategy $\phi : A$ is *finitary* if the set $\{s \in P_A \mid \phi(s) \neq 0\}$ of non-zero-weighted complete plays is finite. Note that we may lift the weighting and choice functions from `nat` to any type using the constructors and destructors in the language — e.g. for $M : \texttt{com}$, let $a.M = \texttt{Ifz}(a.(\texttt{seq}\, M\, 0))\, \texttt{skip}\, \texttt{skip}$.

Corollary 9.4 (Definability for $\mathrm{IA}^{\mathcal{R}}$). *For any finitary $\mathcal{R}$-weighted strategy $\phi : [\![T]\!]$ there exists a term $M_\phi : T$ such that $[\![M_\phi]\!] = \phi$.*

Proof. Enumerating the non-zero-weighted sequences of ϕ as $s_1, \ldots, s_n$, we have $\phi = [\![\phi(s_1).M_{\{s_1\}}\, \texttt{or}\, \ldots\, \texttt{or}\, \phi(s_n).M_{\{s_n\}}]\!]$. □

Theorem 9.5 (Full Abstraction)**.** *$M \approx^\Gamma_T N$ if and only if $[\![M : T]\!]_\Gamma = [\![N : T]\!]_\Gamma$.*

Proof. This closely follows the proof in the original model. Soundness holds by Corollary 8.25; for completeness suppose $[\![M]\!] \neq [\![N]\!]$, and thus there exists a complete $s \in P_{[\![T]\!]}$ such that $[\![M]\!](s) \neq [\![N]\!](s)$. By the definability property, the strategy $\phi : [\![T]\!] \to [\![\texttt{com}]\!]$ such that $\phi(t) = 1$ if $t = qsa$ (and 0 otherwise) denotes a term $L : T \to \texttt{com}$ of Idealized Algol and thus $[\![L\, M]\!](qa) = [\![M]\!](s)$ and $[\![L\, N]\!](qa) = [\![N]\!](s)$. Hence by adequacy, $w(L\, M) \neq w(L\, N)$ and so $M \not\approx N$ as required. □

If $\mathcal{R}$ is ordered, then inequational completeness holds via the same arguments.

Corollary 9.6. *Observationally inequivalent terms of* $\mathrm{IA}^{\mathcal{R}}$ *may be separated by a context of Idealized Algol.*

So, for instance, our model is fully abstract for Probabilistic Algol.

10. Conclusions

We have established a series of basic results for quantitative semantics: that Lafont categories with biproducts have uniform fixed points, and that these provide a computationally adequate interpretation of non-deterministic PCF with weights from a complete commutative semiring. We have described a general way of constructing such a category from a qualitative model by change of base, and applied it to a category of games, giving a fully abstract model of weighted Idealized Algol.

These results have been established in rather different ways — using the principles of axiomatic domain theory, enriched category theory and game semantics combined with some basic operational techniques. It is not yet clear how closely these approaches may be combined: e.g. whether computational adequacy may be established by purely axiomatic means.

The representation of fixed point approximants using nested finite multisets suggests that we could extend the resource λ-calculus, and related formalisms such as the differential λ-calculus [11] and differential nets [41] to reason about fixed points. Other avenues include extension of our results to recursive types using principles from [42] or models of linear logic which are not Lafont categories — for example, the notion of "new Lafont category" in [43].

The only really essential properties that we have used from our category of games are that strategies may be viewed as certain cliques in a coherence space and that composition is a stable, linear function. This rules out models of concurrency in which non-determinism arises implicitly, except in the case of idempotent semirings. Adding quantitative weighting to such models is a subject of current research. For the sake of simplicity, we have sidestepped mention of causal order (e.g. prefix order in games), which gives a finer characterization of strategy behaviour. For example, we may enrich categories over event structures [25, 44] (or dI-domains [45]) — thus a monoidal functor adding weights to event structures may be used to change their base but needs to take account of both the ordering and the coherence relation (e.g. the sum of the weights of two events in "immediate conflict" should be less than the weight of their causing event).

References

[1] J. Laird, G. Manzonetto, G. McCusker and M. Pagani, Weighted relational models of typed lambda-calculi, in: Proceedings of LICS '13, 2013.

[2] J.-Y. Girard, Normal functors, power series and λ-calculus, Annals of Pure and Applied Logic 37 (1988) 129–177.

[3] T. Ehrhard, A finiteness structure on resource terms, in: Proceedings of LICS '10, 2010.

[4] F. Lamarche, Quantitative domains and infinitary algebras, Theoretical Computer Science 94 (1999) 37–62.

[5] P. Freyd, Algebraically complete categories, in: Proceedings of Category Theory, Como 1990, no. 1488 in LNM, Springer, 1990.

[6] A. Simpson and G. Plotkin, Complete axioms for categorical fixed-point operators, in: Proceedings of LICS '00, IEEE Press, 2000, pp. 30–41.

[7] M. Barr, Algebraically compact functors, Journal of Pure and Applied Algebra 82 (1993) 211–231.

[8] Y. Lafont, Logiques, catégories et machines, Ph.D. thesis, Université Paris 7 (1988).

[9] P. Melliès, N. Tabareau and C. Tasson, An explicit formula for the free exponential modality of linear logic, in: Proc. ICALP '09, no. 5556 in LNCS, 2009, pp. 247–260.

[10] G. Boudol, The lambda-calculus with multiplicities, in: E. Best (Ed.), Proceeedings of CONCUR '93, no. 715 in LNCS, 1993, pp. 1–6.

[11] T. Ehrhard, L. Regnier, The differential lambda-calculus, Theoretical Computer Science 309.

[12] P. Tranquilli, Nets between determinism and non-determinism, Ph.D. thesis, Université Paris 7 (2009).

[13] R. F. Blute, J. R. B. Cockett and R. A. G. Seely, Differential categories, Mathematical. Structures in Comp. Sci. 16.

[14] S. Abramsky and G. McCusker, Linearity, Sharing and State: a fully abstract game semantics for Idealized Algol with active expressions, in: P. O'Hearn, R. Tennent (Eds.), Algol-like languages, Birkhauser, 1997.

[15] D. Verity, Enriched categories, internal categories and change of base, Ph.D. thesis, Cambridge University, republished in *Reprints in Theory and Applications of Categories, No. 20 (2011) pp 1-266* (1992).

[16] G. Cruttwell, Normed spaces and change of base for enriched categories, Ph.D. thesis, Dalhousie University (2008).

[17] J.-Y. Girard, P. Taylor and Y. Lafont, Proofs and Types, Cambridge University Press, 1990.

[18] A. Calderon and G. McCusker, Understanding game semantics through coherence spaces, in: Electronic Notes in Theoretical Computer Science, Vol. 265, 2010, pp. 231– 244.

[19] V. Danos and R. Harmer, Probabilistic game semantics, ACM Transactions on Computational Logic 3 (3) (2002) 359–382.

[20] D. Ghica and G. McCusker, The regular language semantics of second-order Idealised Algol, Theoretical Computer Science 309 (2003) 469 – 502.

[21] J. Laird, G. Manzonetto and G. McCusker, Constructing differential categories and deconstructing categories of games, Information and Computation 222 (2013) 247–264.

[22] A. Bahamonde, Tensor product of partially-additive monoids, Semigroup Forum 32 (1) (1985) 31–53.

[23] R. Guitart, Tenseurs et machines, Cahiers de Topologie et Geometrie Differentielle XXI (1) (1980) 5–62.

[24] T. Ehrhard, Hypercoherence: A strongly stable model of linear logic, in: J.-Y. Girard, Y. Lafont and L. Regnier (Ed.), Advances in Linear Logic, Cambridge University Press, 1995.

[25] G. Winskel, Event structures, in: Advances in Petri Nets, Lecture Notes in Computer Science, Springer, 1987.

[26] F. Lamarche, Sequentiality, games and linear logic, in: Proceedings, CLICS workshop, Aarhus University, DAIMI-397–II, 1992.

[27] J. M. E. Hyland and C.-H. L. Ong, On full abstraction for PCF: I, II and III, Information and Computation 163 (2000) 285–408.

[28] G. McCusker, Games and full abstraction for a functional metalanguage with recursive types, Ph.D. thesis, Imperial College London, cambridge University Press (1996).

[29] R. Harmer, Games and full abstraction for nondeterministic languages, Ph.D. thesis, Imperial College London (1999).

[30] J. Laird, Game semantics and Linear CPS interpretation, Theoretical Computer Science 333 (2005) 199–224.

[31] P. Baillot, V. Danos, T. Ehrhard and L. Regnier, AJM games are a model of classical linear logic, in: Proceedings of the twelfth International Symposium on Logic In Computer Science, LICS '97, 1997.

[32] R. Harmer and G. McCusker, A fully abstract games semantics for finite non-determinism, in: Proceedings of the Fourteenth Annual Symposium on Logic in Computer Science, LICS '99, IEEE Computer Society Press, 1998.

[33] D. Ghica, Slot games: A quantitative model of computation, in: Proceedings of the 32nd ACM SIGPLAN-SIGACT symposium on Principles of programming languages, 2005, pp. 85 – 97.

[34] J. Adámek, Free algebras and automata realizations in the language of categories, Comment. Math. Univ. Carolinae 16 (1974) 339–351.

[35] J. Beck, Distributive laws, in: Seminar on Triples and Categorical Homology Theory, no. 80 in Lecture Notes in Mathematics, 1969, p. 119 140.

[36] J. Laird, Fixed points in quantitative semantics, in: Proceedings of LICS '16, ACM, 2016, pp. 347 –356.

[37] S. Abramsky, R. Jagadeesan and P. Malacaria, Full abstraction for PCF, Information and Computation 163 (2000) 409–470.

[38] N. Dershowitz and Z. Manna, Proving termination with multiset orderings, Communications of the ACM 22 (1979) 465–476.

[39] T. Ehrhard, M. Pagani and C. Tasson, Full abstraction for probabilistic PCF, Journal of the ACM 65.

[40] J. C. Reynolds, Syntactic Control of Interference, in: Conf. Record 5^{th} ACM Symposium on Principles of Programming Languages, 1978, pp. 39–46.

[41] T. Ehrhard and L. Regnier, Differential interaction nets, Theoretical Computer Science 364 (2) (2006) 166–195.

[42] A. Simpson, Computational adequacy for recursive types in models of intuitionistic set theory, Annals of Pure and Applied Logic 130 (2004) 207–275.

[43] P.-A. Melliès, Categorical Semantics of Linear Logic, no. 27 in Panoramas et Synthèses, Societé Mathématique de France, 2009.

[44] G. Winskel, Concurrent strategies, in: Proceedings of LICS '11, 2011.

[45] G. Berry, Stable models of typed λ-calculi, in: Proceedings of the 5th International Colloquium on Automata, Languages and Programming, no. 62 in LNCS, Springer, 1978, pp. 72–89.

Appendix A: Yoneda Embedding Preserves MTT-Projections

We need to show that the $h_A(p_n) : \mathcal{C}(A, B^{\leq n+1}) \to \mathcal{C}(A, B^{\leq n})$ is the projection part of an e-p pair in **CSpace** for each object A. As we have noted, $h^A(B_\bullet^n)$ is the equalizer in **CSpace** for the group $G = h^A(\mathsf{perm}(B_\bullet^{\otimes n})$ of permutation automorphisms on $\mathcal{C}(A, B_\bullet^{\otimes n}) = h^A(B_\bullet^{\otimes n})$, so it is isomorphic to the explicitly defined equalizer $h^A(B_\bullet^{\otimes n})_{/G}$. Up to this isomorphism, therefore, $h^A(p_n)$ is equivalent to the unique mediating morphism P_n from $h^A(B_\bullet^{\otimes n+1})_{/G}$ to $h^A(B_\bullet^{\otimes n})_{/G}$ for the morphism $\mathsf{eq}; h^A(\breve{p}_i) : h^A(B_\bullet^{\otimes n+1})_{/G} \to h^A(B_\bullet^{\otimes n})$, where $\breve{p}_n : B_\bullet^{\otimes n+1} \to B_\bullet^{\otimes n} \triangleq (B \times I)^{\otimes n} \otimes \pi_r$. So $P_n = \{([c]_G, [d]_G) \mid (c, d) \in h^A(\breve{p}_n)\}$, and it suffices to show that this is a projection. We first establish lemmata about **CSpace**-enriched symmetric monoidal categories.

Lemma A1 *The symmetry isomorphism $\gamma_{B_\bullet, B_\bullet} : B_\bullet \otimes B_\bullet \to B_\bullet \otimes B_\bullet$ satisfies $\gamma_{B_\bullet, B_\bullet}; ((\bot_{B,B} \times I) \otimes (\bot_{B,B} \times I)) = (\bot_{B,B} \times I) \otimes (\bot_{B,B} \times I)$.*

Proof. $\bot_{B,B} \times I = \pi_r; \langle \bot_{I,B}, I\rangle$ and therefore $\gamma; ((\bot_{B,B} \times I) \otimes (\bot_{B,B} \times I)) = (\pi_r \otimes \pi_r); \gamma_{I,I}; (\langle \bot_{I,B}, I\rangle \otimes \langle \bot_{I,B}, I\rangle)$.

But $\gamma_{I,I}$ is the identity on $I \otimes I$ in any symmetric monoidal category, and so this is equal to $(\pi_r \otimes \pi_r); (\langle \bot_{I,B}, I\rangle \otimes \langle \bot_{I,B}, I\rangle) = (\bot_{B,B} \times I) \otimes (\bot_{B,B} \times I)$ as required. □

Lemma A2 *If $(c, d), (c', d') \in h^A(\breve{p}_n)$ such that $c \sim_G c'$ then there exists a permutation π on $n+1$ such that $c' = \pi(c)$ and $\pi(n+1) = n+1$.*

Proof. If $c' \sim_G c$ then there is a permutation π on $n+1$ such that $\pi(c) = c'$. If $\pi(n+1) = n+1$ then we are done. So suppose without loss of generality that $\pi(n+1) = n$. We claim that $c' = c'; (B_\bullet^{\otimes n-1} \otimes \gamma_{B_\bullet, B_\bullet}) = \pi(c); (B_\bullet^{\otimes n-1} \otimes \gamma_{B_\bullet, B_\bullet})$ — so that if π' is the permutation with $\pi(n+1) = n+1$, $\pi'(n) = \pi(n+1)$ and $\pi'(i) = \pi(i)$ for $i < n$ then $\pi'(c) = c'$ as required.

Observe that if $(c', d') \in h^A(\breve{p}_n)$ then $(c', c') \in h^A(\breve{p}_n; \breve{e}_n) = h^A(B_\bullet^{\otimes n} \otimes (\perp_{B,B} \times I))$ — i.e. $c'; (B_\bullet^{\otimes n} \otimes (\perp_{B,B} \times I)) = c'$.

We also have $(B_\bullet^{\otimes n} \otimes (\perp \times I)); \theta_\pi = \theta_\pi; (B_\bullet^{\otimes n-1} \otimes (\perp_{B,B} \times I) \otimes B)$, and therefore $c'; (B_\bullet^{\otimes n-1} \otimes (\perp_{B,B} \times I) \otimes B) = (c; \theta_\pi); (B_\bullet^{\otimes n-1} \otimes (\perp_{B,B} \times I) \otimes B_\bullet) = c; (B_\bullet^{\otimes n} \otimes (\perp_{B,B} \times I)); \theta_\pi = c; \theta_\pi = c'$.

Hence $c'; (B_\bullet \otimes n - 1 \otimes (\perp_{B,B} \times I) \otimes (\perp_{B,B} \times I)) = c'$. But we have already shown (Lemma A1) that $\gamma_{B_\bullet, B_\bullet}; (\perp_{B,B} \times I) \otimes (\perp_{B,B} \times I) = (\perp_{B,B} \times I) \otimes (\perp_{B,B} \times I)$ and so $c'; (B_\bullet^{\otimes n-1} \otimes \gamma_{B_\bullet, B_\bullet}) = c'$ as required. □

We can now prove our main result:

Lemma 5.9 *For each object A in a* **CSpace***-enriched SMC with coherent symmetric tensor powers of $B_\bullet$, $h^A(p_n) : \mathcal{C}(A, B_\bullet^{n+1}) \to \mathcal{C}(A, B_\bullet^n)$ is a projection.*

Proof. In **CSpace**, a morphism $f : D \to C$ is a projection (and its converse relation is the corresponding embedding) if and only if it satisfies

1. Coherence — If $(c, d), (c', d') \in f$ then $c \frown c'$ if and only if $d \frown d'$.
2. Surjectivity — For every $d \in D$ there exists $c \in C$ with $(c, d) \in f$.
3. Injectivity — If $(c, d), (c', d') \in f$ then $c = c'$ if and only if $d = d'$.

To show that P_n satisfies these properties we use the fact that $h^A(\breve{p}_n)$ is itself a projection — we can define $\breve{e}_n : B_\bullet^{\otimes n} \to B_\bullet^{\otimes n+1} \triangleq B_\bullet^{\otimes n} \otimes \langle \perp_{I,B}, I \rangle$ making $(\breve{e}_n, \breve{p}_n)$ — and thus $(h^A(\breve{e}_n), h^A(\breve{p}_n))$ — e-p pairs. So P_n satisfies property 1: if $([c]_G, [d]_G), ([c']_G, [d']_G) \in P_n$ then $[c]_G \frown [c']_G$ if and only if $c \frown c'$ (by the coherence of the group of permutations) if and only if $d \frown d'$ (because $h^A(\breve{p}_n)$ satisfies 1) if and only if $[d]_G \frown [d']_G$.
2 follows for P_n from the surjectivity of $h^A(\breve{p}_n)$: for any $[d]_G \in h^A(B_\bullet^{\otimes n})_{/G}$ there exists $c \in h^A(B_\bullet^{\otimes n+1})$ such that $(c, d) \in h^A(\breve{p}_n)$ and so $([c]_G, [d]_G) \in P_n$.

So it remains to prove property 3 (injectivity) for P_n. This is equivalent to showing that if $(c, d), (c', d') \in h^A(\breve{p}_n)$ then $c \sim_G c'$ iff $d \sim_G d'$. Observe that if π is a permutation on n, and $\widehat{\pi}$ is the permutation on $n+1$ which extends π by setting $\pi(n+1) = n+1$ then $\breve{p}_n; \theta_\pi = \theta_{\widehat{\pi}}; \breve{p}_n$, and so $(c, d) \in h^A(\breve{p}_n)$ if and only if $(\widehat{\pi}(c), \pi(d)) \in h^A(\breve{p}_n)$. Hence for $(c, d), (c', d') \in h^A(\breve{p}_n)$, $d \sim_G d'$ implies $c \sim_G c'$. The converse is a consequence of Lemma A2. □

Appendix B: Soundness for Silent Reductions

We need to show that any configuration which is neither fully evaluated nor an explicit choice denotes the weighted sum of denotations of all of its reducts. The key cases are the rules in Table 3 — in particular for unwinding the fixedpoint by calling a variable in the environment, for which we require the following lemma based on the identities in Lemma 6.16. Writing $s(X)$ for the support of the multiset X:

Lemma B1 For $f : !A \to A$, $f^X; \delta_{!A}; (\mathsf{der}_A \otimes !A) = \sum_{Y \in s(X)} (f^Y; f \otimes f^{X-[Y]})$.

Proof. $f^X; \delta_{!A}; (\mathsf{der}_A \otimes !A) = \sum_{Z \subseteq X} (f^Z \otimes f^{X-Z}); (\mathsf{der}_A \otimes !A)$
$= \sum_{Z \subseteq X} (f^Z; \mathsf{der} \otimes f^{X-Z}) = \sum_{y \in s(X)} (f^Y; f \otimes f^{X-[Y]})$, since for $Z \subseteq X$,
$f^Z; \mathsf{der}_A = \begin{cases} f^Y; f & \text{if } Z = [Y] \text{ for some } Y \in s(X) \\ 0 & \text{otherwise} \end{cases}$ □

Lemma B2 $[\![(x, M, X), \mathcal{E}; E[x]]\!] = \sum_{Y \in s(X)} [\![(y, M[y/x], Y), (x, M, X-[Y]), \mathcal{E}; E[M[y/x]]]\!]$.

Proof. Suppose $x : T \vdash M : T$.
Then $[\![(x, M, X), \mathcal{E}''; E[x]]\!] = [\![M]\!]^X; \delta_{[\![T]\!]}; (![\![T]\!] \otimes [\![\mathcal{E}]\!]); [\![E[x]]\!]$
$= [\![M]\!]^X; \delta_{[\![T]\!]}; ([\![T]\!] \otimes [\![\mathcal{E}]\!]); (\delta_{[\![T]\!]}; (\mathsf{der}_{[\![T]\!]} \otimes ![\![T]\!]) \otimes [\![|\mathcal{E}''|]\!]); [\![E[\bullet]]\!]$ by Prop. 8.17
$= [\![M]\!]^X; \delta_{[\![T]\!]}; (\mathsf{der}_{[\![T]\!]} \otimes [\![T]\!]); ([\![T]\!] \otimes (\delta_{[\![T]\!]}; (![\![T]\!] \otimes [\![\mathcal{E}]\!]))); [\![E[\bullet]]\!]$
$= \sum_{Y \in s(X)} ([\![M]\!]^Y; [\![M]\!]) \otimes [\![M]\!]^{X-[Y]}; ([\![T]\!] \otimes (\delta_{[\![T]\!]}; (![\![T]\!] \otimes [\![\mathcal{E}]\!]))); [\![E[\bullet]]\!]$ by Lemma B1
$= \sum_{Y \in s(X)} (([\![M]\!]^Y \otimes ([\![M]\!]^{X-Y}; ([\![M]\!] \otimes [\![\mathcal{E}]\!]))); [\![E[[\![\bullet]\!]]]\!]$
$= \sum_{Y \in s(X)} [\![(y, M[y/x], Y), (x, M, X-[Y]), \mathcal{E}; E[M[y/x]]]\!]$ by Lemma 8.17.
(Since x does not occur in $M[y/x]$.) □

We now prove soundness for all silent reductions.

Lemma 8.18 If $\mathcal{E}; P$ is silent then $[\![\mathcal{E}; P]\!] = \sum \{a.[\![\mathcal{E}'; P']\!] \mid (\mathcal{E}; P) \xrightarrow{(\varepsilon, a)} (\mathcal{E}'; P')\}$.

Proof. Suppose $P = E[x]$, and $\mathcal{E} = \mathcal{E}', (x, M, X), \mathcal{E}''$, so that $[\![\mathcal{E}; P]\!] = [\![E]\!]; [\![(x, M, X), \mathcal{E}'']\!]_{|\mathcal{E}'|}$, and so by Lemma B2, we have
$[\![\mathcal{E}; P]\!] = \sum_{Y \in S(X)} [\![(y, M[y/x], Y), (x, M, X-[Y]), \mathcal{E}; E[M[y/x]]]\!]$ as required.

Suppose $P = E[\mu x.N]$. Then by Proposition 8.17

$$[\![\mathcal{E}; E[\mu x.M]]\!] = [\![\mathcal{E}]\!]; \delta_{[\![|\mathcal{E}|]\!]}; ([\![|\mathcal{E}|]\!] \otimes [\![\mu x.M]\!]_{|\mathcal{E}|}); [\![E[\bullet]]\!]_{|\mathcal{E}|}$$

$$= [\![\mathcal{E}]\!]; \delta_{[\![|\mathcal{E}|]\!]}; ([\![|\mathcal{E}|]\!] \otimes (\delta_{|\mathcal{E}}; ([\![|\mathcal{E}|]\!] \otimes [\![\mu x.M]\!]^{\dagger}_{|\mathcal{E}|}); [\![M]\!]_{|\mathcal{E}|,x})); [\![E[\bullet]]\!]_{|\mathcal{E}|}.$$

by definition of the fixed point, and so by Definition 6.15, this is

$$\sum_{X \in \mathbb{M}} [\![\mathcal{E}]\!]; \delta_{[\![|\mathcal{E}|]\!]}; ([\![|\mathcal{E}|]\!] \otimes (\delta_{|\mathcal{E}}; ([\![|\mathcal{E}|]\!] \otimes [\![M]\!]^X_{|\mathcal{E}|,x}); [\![M]\!]_{|\mathcal{E}|,x})); [\![E[\bullet]]\!]_{|\mathcal{E}|}$$

$= \sum_{X \in \mathbb{M}} [\![\mathcal{E}, (x, \mu x.M, X); E[M]]\!]$ by Proposition 8.17.

Suppose $P = E[(\lambda x.M)\, N]$ — then by an essentially similar argument to the fixed point case above, $[\![\mathcal{E}; E[(\lambda x.M)\, N]]\!] = \sum_{X \in \mathbb{M}} [\![\mathcal{E}, (x, N, X); E[M]]\!]$.

Suppose $P = E[a.N]$ — then $[\![\mathcal{E}; P]\!] = a.[\![\mathcal{E}; E[N]]\!]$ by Proposition 8.17 and bilinearity of composition in $\mathcal{C}$.

Suppose $P = E[\texttt{Ifz}(\texttt{n})]$ or $P = E[\texttt{pred}(\texttt{n}+\texttt{1})]$ for some $n \in \mathbb{N}$, — then $\mathcal{E}; P \xrightarrow{(\varepsilon,1)} \mathcal{E}; P'$ for a unique P' such that $[\![\mathcal{E}; P]\!] = [\![\mathcal{E}; P']\!]$ by definition of an object of numerals. □