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# Path-Based Stability Analysis for Monotone Control Systems on Proper Cones

Yu Kawano , Member, IEEE, and Bart Besselink , Member, IEEE

**Abstract**—In this article, we study positive invariance and attractivity properties for nonlinear control systems, which are monotone with respect to proper cones. Monotonicity simplifies such analysis for specific sets defined by the proper cones. Instead of Lyapunov functions, a pair of so-called *paths* in the state space and input space play important roles. As applications, our results are utilized for analysis of asymptotic stability and also input-to-state stability on proper cones. The results are illustrated by means of examples.

**Index Terms**—Monotone systems, nonlinear systems, paths, proper cones, stability.

## I. INTRODUCTION

Dynamical systems that preserve a partial order relationship between initial states over their trajectories are known as *monotone* systems, see [1]–[3]. They appear naturally as models for social dynamics [4] and chemical reaction processes [5], and have become an important tool in stability analysis for large-scale interconnected systems, e.g., [6], [7]. A distinguishing feature of monotonicity is that it has strong implications for the asymptotic behavior of autonomous systems, which has been investigated especially for systems that are monotone with respect to the positive orthant, i.e., for *cooperative* systems [1]. Inspired by such results, stability of cooperative systems has been investigated in various problem settings [8]–[11]. As an example, stability properties of cooperative systems can be characterized by using so-called max-separable Lyapunov functions, i.e., Lyapunov functions that can be written as the maximum of functions with scalar arguments representing the state components [8]–[11]. It is well-known that such max-separable Lyapunov functions are intimately related to so-called *paths* in the positive orthant, which opens a door for path-based stability analysis of cooperative systems, see [6], [7], [10], [12], [13].

Moving beyond cooperativity, nonlinear systems that are monotone with respect to general proper cones have also received considerable attention recently, see, e.g., [3], [14]–[16]. These works typically consider a contraction (or differential) framework for analysis of stability properties. Apart from that, stability of monotone nonlinear systems has not been well-studied. This is in contrast with the detailed development of path-based analysis and separable Lyapunov functions for cooperative systems.

In this note, our objective is to extend path-based stability analysis to monotone systems. As the main stepping stone, we study (robust)

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positive invariance and attractivity of specific sets defined on the basis of the proper cones with respect to which the systems are monotone. First, we show that monotonicity greatly simplifies the analysis of (robust) positive invariance of these sets as the direct verification of (a generalization of) Nagumo’s theorem, e.g., [17], can be avoided. Second, we prove that the existence of a pair of so-called paths in the state space and input space guarantees attractivity. Then, we apply our results to asymptotic stability analysis and input-to-state stability (ISS) analysis of monotone systems.

In fact, we obtain natural extensions of the results based on max-separable Lyapunov functions and path-based analysis for cooperative systems. Especially, our positive invariance condition and attractivity condition can be hypothesized from similar conditions for cooperative systems [12], [13]. However, our analysis of monotone systems requires to handle tangent cones that do not appear in the path-based analysis of cooperative systems, since the Kamke–Müller condition described by a tangent cone is not further simplified for monotone systems, differently from cooperative systems. We regard these generalizations from positive orthants to general proper cones as a first step to enlarge the range of applications of monotonicity-based analysis pursued especially in the systems and control community. Our results are illustrated by examples, including a tunnel-diode circuit, which is not cooperative but monotone with respect to some proper cones. For this circuit, path-based analysis provides less conservative stability conditions than contraction analysis, which suggests that path-based analysis can be one of the central tools for analysis of monotone systems as for cooperative systems.

## II. PROBLEM STATEMENT

### A. Preliminaries

We use notation similar to that in [2] and [17]. Let  $\mathbb{R}$  and  $\mathbb{R}_+$  be the field of real numbers and set of nonnegative real numbers, respectively. We denote a norm on  $\mathbb{R}^n$  as  $|\cdot|$  and denote the closed ball of radius  $r$  with respect to this norm as  $\mathcal{B}_r$ , i.e.,  $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid |x| \leq r\}$ . A continuous function  $\alpha : [0, r) \rightarrow \mathbb{R}_+$  is said to be of class  $\mathcal{K}$  if it is strictly increasing and  $\alpha(0) = 0$ . Moreover, it is said to belong to class  $\mathcal{K}_\infty$  if  $r = \infty$  and  $\alpha(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . A continuous function  $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be of class  $\mathcal{KL}$  if for each fixed  $s$ , the mapping  $\beta(\cdot, s)$  belongs to class  $\mathcal{K}$ , for each fixed  $r$ , the mapping  $\beta(r, \cdot)$  is decreasing and  $\beta(r, s) \rightarrow 0$  as  $s \rightarrow \infty$ .

A closed set  $K \subset \mathbb{R}^n$  is said to be a *proper cone* if it has the following properties: 1) it is a cone, i.e.,  $cK \subset K$  for any  $c \in \mathbb{R}_+$ ; 2) it is convex, i.e.,  $K + K \subset K$ ; 3) it is pointed, i.e.,  $K \cap (-K) = \{0\}$ ; and 4) it is solid, i.e.,  $\text{int } K \neq \emptyset$ .

A proper cone  $K \subset \mathbb{R}^n$  allows for introducing a partial order on  $\mathbb{R}^n$ , where for  $x, x' \in \mathbb{R}^n$ , we write  $x \succeq x'$  if and only if  $x - x' \in K$ . This ordering can be strengthened as  $x \succ x'$  if and only if  $x - x' \in \text{int } K$ . We will also use the dual  $K^*$  of a cone  $K$  defined as  $K^* = \{v \in \mathbb{R}^n \mid \langle v, x \rangle \geq 0 \text{ for all } x \in K\}$ , where  $\langle \cdot, \cdot \rangle$  is the standard inner product.

Finally, for a closed set  $S \subset \mathbb{R}^n$ , the tangent cone to  $S$  at  $x \in S$ , denoted by  $\mathcal{T}_x S$ , is defined as

$$\mathcal{T}_x S := \left\{ z \in \mathbb{R}^n \mid \liminf_{\tau \rightarrow 0^+} \frac{d(x + \tau z, S)}{\tau} = 0 \right\}. \quad (1)$$

Here,  $d(x, S) = \inf_{y \in S} |x - y|$  for the Euclidean norm  $|\cdot|$ . If  $S$  is convex, the  $\liminf$  in (1) can be replaced by  $\lim$ , see [17, p. 102].

## B. Monotone Systems

Consider the nonlinear system

$$\dot{x} = f(x, u), \quad (2)$$

where the vector field  $f : \tilde{\mathcal{X}} \times \mathcal{U} \rightarrow \mathbb{R}^n$  is continuous in  $(x, u)$  and locally Lipschitz continuous in  $x$  locally uniformly in  $u$ . Here,  $\tilde{\mathcal{X}}$  is an open subset of  $\mathbb{R}^n$ , and  $\tilde{\mathcal{X}}$  contains a closed set  $\mathcal{X}$ , which is itself a closure of some open subset. Following [2, Section II] and [1, Remark 3.1.4], we assume that  $\mathcal{X}$  satisfies the following approximability property: for all  $x, x' \in \mathcal{X}$  such that  $x \succeq x'$ , there exists sequences  $\{x_\ell\}, \{x'_\ell\}$  with  $x_\ell, x'_\ell \in \text{int } \mathcal{X}$  such that  $x_\ell \succeq x'_\ell$  for all  $\ell$  and  $x_\ell \rightarrow x$  and  $x'_\ell \rightarrow x'$  as  $\ell \rightarrow \infty$ .

Moreover, we consider input functions  $u : \mathbb{R}_+ \rightarrow \mathcal{U}$  for some compact set  $\mathcal{U} \subset \mathbb{R}^m$  and denote the set of all such functions by  $\mathcal{U}_\infty$ . Following [2, Section II], we assume that, for each initial condition  $x(0) = x_0 \in \mathcal{X}$  and each input function  $u \in \mathcal{U}_\infty$ , the corresponding solution  $x(t)$  exists for all  $t \in \mathbb{R}_+$ . In addition, it satisfies  $x(t) \in \mathcal{X}$  for all  $t \in \mathbb{R}_+$ , i.e.,  $\mathcal{X}$  is (robustly) positively invariant with respect to  $\mathcal{U}_\infty$ . In the remainder of this note, we denote the solution (at time  $t \in \mathbb{R}_+$ ) to (2) with initial condition  $x_0$  and for input  $u$  by  $\phi(t, x_0, u)$ .

In this note, we are interested in systems (2) that are *monotone*. To define monotonicity, let  $K \subset \mathbb{R}^n$  and  $K_u \subset \mathbb{R}^m$  be proper cones in the state space and input space, respectively.

**Assumption 2.1 (Monotonicity):** The system (2) is monotone with respect to proper cones  $K$  and  $K_u$ , i.e., the implication

$$x_0 \succeq x'_0, u \succeq u' \Rightarrow \phi(t, x_0, u) \succeq \phi(t, x'_0, u'), \quad \forall t \in \mathbb{R}_+ \quad (3)$$

holds for all  $x_0, x'_0 \in \mathcal{X}$  and  $u, u' \in \mathcal{U}_\infty$ .  $\triangleleft$

In (3), the partial order  $\succeq$  is taken with respect to the respective cone. Specifically,  $u \succeq u'$  in (3) means  $u(t) - u'(t) \in K_u$  for all  $t \in \mathbb{R}_+$ .

The implication (3) reflects that any order in the initial state is preserved over system trajectories for input functions that are ordered as well. A necessary and sufficient condition for monotonicity as in (3) is given by an extension of the so-called Kamke–Müller condition (e.g., [1]) to control systems.

**Lemma 2.2 (see [2]):** The system (2) is monotone as in (3) if and only if the implication

$$x \succeq x', u \succeq u' \Rightarrow f(x, u) - f(x', u') \in \mathcal{T}_{x-x'}K \quad (4)$$

holds for all  $x, x' \in \text{int } \mathcal{X}$  and  $u, u' \in \mathcal{U}$ .  $\triangleleft$

If  $f$  is of class  $C^1$ , an alternative condition for monotonicity of an autonomous system in terms of  $\partial f / \partial x$  is found in [3] and [18], but in Theorem 3.2 below, we show that the condition (4) in terms of tangent cones is helpful to simplify analysis of positive invariance. For autonomous systems, monotonicity with respect to  $\mathbb{R}_+^n$  (also known as cooperativity) simplifies analysis of compact omega limit sets, e.g., [1]. This is also recognized in systems and control theory, where stability analysis of cooperative systems has been developed both for autonomous and control systems. Here, so-called max-separable Lyapunov functions play important roles [3], [8]–[11]. These results however highly depend on the fact that the cones  $K = \mathbb{R}_+^n$  and  $K_u = \mathbb{R}_+^m$  are considered. In this note, our objective is to remove these restrictions and to proceed with stability analysis of monotone systems with respect to general proper cones.

## III. MAIN RESULTS

In this section, we characterize stability properties of monotone control systems through two technical results. The interpretation of these results is postponed to Section IV.

As a first step, we study robust positive invariance of a subset of  $\mathcal{X}$  according to the following definition (see [17, Definition 4.3]).

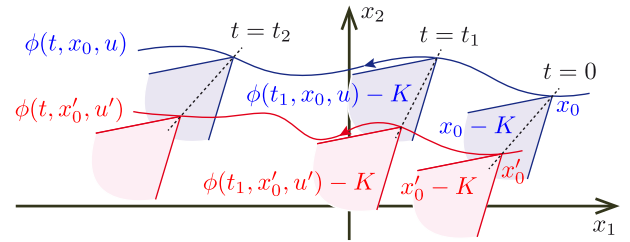


Fig. 1. Trajectories of a monotone system. From monotonicity,  $\phi(\cdot, x'_0, u') \in \phi(\cdot, x_0, u) - K$  for any  $x'_0 \in x_0 - K$  and  $u' \in u - K_u$ . The convexity of  $K$  further implies  $\phi(\cdot, x'_0, u') - K \subset \phi(\cdot, x_0, u) - K$ . That is, the red cone associated with  $\phi(t, x'_0, u')$  is contained in the blue cone associated with  $\phi(t, x_0, u)$  at each time.

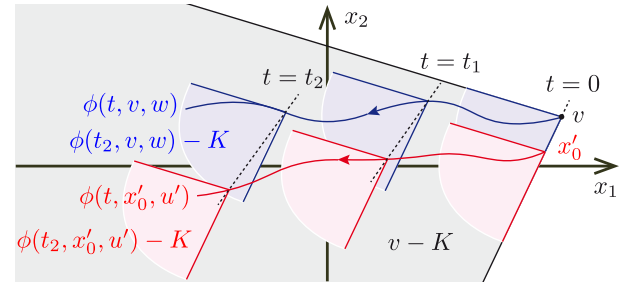


Fig. 2. Interpretation of Theorem 3.2. If (7) holds,  $\phi(t, v, w) \in v - K$  and consequently  $\phi(t, v, w) - K \subset v - K$ ,  $t \in [0, t_1]$  for sufficiently small  $t_1 > 0$ . Moreover, monotonicity implies  $\phi(\cdot, x'_0, u') \in \phi(\cdot, v, w) - K$  for any  $x'_0 \in v - K$  and  $u' \in w - K_u$ . Therefore,  $\phi(t, x'_0, u') \in v - K$  for any  $t \in [0, t_1]$ . This inclusion for any  $x'_0 \in v - K$  and  $u' \in w - K_u$  concludes the positive invariance of  $v - K$ . In other words, from monotonicity, if  $\phi(t, v, w)$  stays in  $v - K$ ,  $t \in [0, t_1]$ , then any trajectory starting from  $v - K$  stays in  $v - K$ , i.e., the blue and red cones associated with trajectories are contained in  $v - K$  for all  $t \in \mathbb{R}_+$ .

**Definition 3.1:** A closed subset  $S \subset \mathcal{X}$  is said to be a *robustly positively invariant set* of the system (2) with respect to  $W_\infty \subset \mathcal{U}_\infty$  if for all  $x_0 \in S$  and  $u \in W_\infty$ , the condition  $\phi(t, x_0, u) \in S$  holds for all  $t \in \mathbb{R}_+$ .  $\triangleleft$

In the remainder, we will be interested in specific sets  $S$  and  $W_\infty$ . Let  $v \in \text{int } \mathcal{X}$ ,  $w \in \mathcal{U}$ , and define

$$S(v) := \{x \in \mathcal{X} \mid v \succeq x\}, \quad (5)$$

$$W(w) := \{u \in \mathcal{U} \mid w \succeq u\}. \quad (6)$$

Similar to before,  $W_\infty(w)$  denotes the class of input functions  $u : \mathbb{R}_+ \rightarrow W(w)$ .

Now, we are ready to state our first result on robust positive invariance; for its interpretation, see Fig. 2.

**Theorem 3.2:** Consider a monotone system (2). Let  $v \in \text{int } \mathcal{X}$ ,  $w \in \mathcal{U}$ , and consider the sets (5) and (6). Then, the set  $S(v)$  is a robustly positively invariant set of the system (2) with respect to  $W_\infty(w)$  if and only if

$$f(v, w) \in \mathcal{T}_v S(v). \quad (7)$$

*Proof:* See Appendix A.  $\blacksquare$

The proof of Theorem 3.2 relies on (a generalization of) Nagumo's theorem. The main contribution of Theorem 3.2 is that the condition (7) only involves the single point  $(v, w)$  (rather than any  $x \in S(v)$  and  $u \in W(w)$  as in Nagumo's theorem in (30)). Positive invariance of a proper cone is also considered in [18, Proposition 1.2], but this features a condition that needs to be verified on the entire boundary of the cone as in Nagumo's theorem. Monotonicity and the use of tangent cones thus simplify the analysis of robust positive invariance.

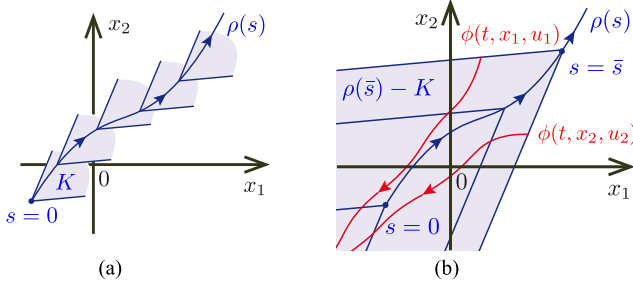


Fig. 3. (a) Example of path  $\rho(s)$ . (b) Interpretation of Theorem 3.8. If (9) holds, any trajectory starting from the boundary of  $S(\rho(s)) = \rho(s) - K$  goes to its interior at each  $s \in (0, \bar{s}]$ , i.e., decreases along  $S(\rho(s))$ . Finally, any trajectory converges to  $S(\rho(0))$ .

*Remark 3.3:* Nagumo's theorem can also directly be exploited for monotonicity analysis. Namely, monotonicity requires that for the system  $\dot{x} = f(x, u)$  and its copy  $\dot{x}' = f(x', u')$ , the set  $\{(x, x') \in \mathcal{X} \times \mathcal{X} \mid x \succeq x'\}$  is robustly positive invariant with respect to the set  $\{(u, u') \in \mathcal{U}_\infty \times \mathcal{U}_\infty \mid u \succeq u'\}$ . This perspective is essentially also taken in [2].  $\triangleleft$

*Remark 3.4:* For linear systems  $\dot{x} = Ax + Bu$ , monotonicity with respect to the cones  $K = \mathbb{R}_+^n$  and  $K_u = \mathbb{R}_+^m$  (often also referred to as positive systems [19]) is well-known to be equivalent to  $A$  being a Metzler matrix (i.e., having nonnegative off-diagonal entries) and  $B$  having nonnegative entries. In this case, (7) reduces to

$$Av + Bw \preceq 0$$

which for  $w = \mathbf{1}$  the all-ones vector, recovers existing results in [20] and [21].  $\triangleleft$

While Theorem 3.2 discusses (robust) positive invariance of sets, we are also interested in attractivity of sets as a stepping stone toward stability properties. This analysis will rely on the introduction of functions  $\gamma$  and  $\rho$  satisfying the following assumption. They are referred to as *paths* by adopting the terminology for analysis in the positive orthants ( $K = \mathbb{R}_+^n$  and  $K_u = \mathbb{R}_+^m$ ), see [6], [9], [22].

*Assumption 3.5 (Strictly Increasing Paths):* Let  $\bar{s} > 0$  be given. The functions  $\rho : [0, \bar{s}] \rightarrow \mathcal{X}$  and  $\gamma : [0, \bar{s}] \rightarrow \mathcal{U}$  are continuous and strictly increasing with respect to  $K$  and  $K_u$ , respectively, i.e., for all  $s, s' \in [0, \bar{s}]$ , we have

$$s > s' \Rightarrow \rho(s) \succ \rho(s'), \quad \gamma(s) \succ \gamma(s'). \quad (8)$$

Moreover,  $\rho(s) \in \text{int } \mathcal{X}$  for all  $s \in (0, \bar{s}]$ .  $\triangleleft$

*Remark 3.6:* Later, we will often consider  $\rho$  satisfying  $\rho(0) = 0$ . In this case, it is clear from Assumption 3.5 that  $\rho : [0, \bar{s}] \rightarrow \{0\} \cup \text{int } K \subset K$ .  $\triangleleft$

The following property of such a strictly increasing function will be used later.

*Lemma 3.7:* Consider a strictly increasing path  $\rho : [0, \bar{s}] \rightarrow \mathcal{X}$ . Then, for any  $x \in \mathcal{X}$  satisfying  $x \prec \rho(\bar{s})$  and  $x \not\prec \rho(0)$ , there exists a unique  $s \in (0, \bar{s})$  such that

$$\rho(s) - x \in \partial K.$$

*Proof:* The proof can be found in Appendix B.  $\blacksquare$

We are now in the position to state the main result on attractivity of sets; Fig. 3 gives an interpretation of this result.

*Theorem 3.8:* Consider a monotone system (2). If there exist strictly increasing paths  $\rho : [0, \bar{s}] \rightarrow \mathcal{X}$  and  $\gamma : [0, \bar{s}] \rightarrow \mathcal{U}$  such that

$$-f(\rho(s), \gamma(s)) \in \text{int } K, \quad \forall s \in (0, \bar{s}], \quad (9)$$

then for any  $\underline{s} \in [0, \bar{s}]$ , the following hold:

- 1) for any  $s \in [\underline{s}, \bar{s}]$ ,  $S(\rho(s))$  is robustly positively invariant with respect to  $W_\infty(\gamma(\underline{s}))$ ;

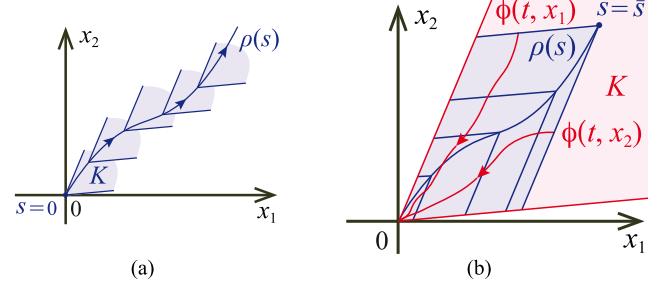


Fig. 4. (a) Example of path  $\rho(s)$  satisfying  $\rho(0) = 0$ . (b) Interpretation of Theorem 4.3. From  $g(0) = 0$ , the cone  $K$  is positively invariant. As explained in Fig. 3, any trajectory starting from the boundary of  $S(\rho(s)) = \rho(s) - K$  goes to its interior at each  $s \in [0, \bar{s}]$  and converges to  $S(\rho(0)) = -K$ . Since  $-K \cap K = \{0\}$ , any trajectory starting from  $S(\rho(s)) \cap K$  converges to the origin.

- 2) for any  $x_0 \in S(\rho(\bar{s}))$  and  $u \in W_\infty(\gamma(\underline{s}))$

$$\lim_{t \rightarrow \infty} d(\phi(t, x_0, u), S(\rho(\underline{s}))) = 0. \quad (10)$$

In the above,  $S(\cdot)$  and  $W_\infty(\cdot)$  are as in (5) and (6).

*Proof:* The proof can be found in Appendix C.  $\blacksquare$

The main result of Theorem 3.8 is that robust positive invariance and attractivity of (specific) sets can be verified by finding paths  $\rho$  and  $\gamma$  satisfying (9). Loosely speaking, the parameter  $\underline{s}$  can be thought of as a bound on the input signals, whereas  $\bar{s}$  characterizes the largest set in state space for which 1) robust positive invariance can be guaranteed, and 2) from which trajectories converge to a smaller set (again characterized by the size of the input  $\underline{s}$ ).

As Theorem 3.8 considers general (proper) cones, it can be regarded as an extension of the results for positive orthants [6], [9], [10], [22]. This will be made more explicit in Section IV, where detailed comparisons will be given.

#### IV. INTERPRETATION

To illustrate both the usefulness of Theorem 3.8 and provide insights in its relevance, this section discusses various applications in the scope of asymptotic stability for autonomous systems as well as ISS-like stability for control systems.

##### A. Asymptotic Stability for Autonomous Systems

Consider the autonomous system

$$\dot{x} = g(x) \quad (11)$$

which is obtained from (2) by defining  $g(x) = f(x, 0)$ . We assume  $g(0) = 0$  and, for simplicity, take  $\mathcal{X} = \mathbb{R}^n$ . Solutions to (11) for initial condition  $x_0$  are denoted by  $\phi(\cdot, x_0)$ .

Stability of the origin is studied according to the following definition, which restricts analysis to trajectories in the cone  $K$  (which is positively invariant under Assumption 2.1).

*Definition 4.1:* Consider the system (11) and a proper cone  $K$ . The origin is said to be stable in  $K$  if, for each  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$x_0 \in \mathcal{B}_\delta \cap K \Rightarrow \phi(t, x_0) \in \mathcal{B}_\varepsilon \cap K.$$

It is said to be asymptotically stable in  $K$  if it is stable in  $K$  and, additionally,  $\delta > 0$  can be chosen such that

$$x_0 \in \mathcal{B}_\delta \cap K \Rightarrow \lim_{t \rightarrow \infty} \phi(t, x_0) = 0.$$

In the study of stability properties, the following lemma on sets of the form  $S(\rho(s)) \cap K$  will turn out to be useful.  $\triangleleft$

*Lemma 4.2:* Let  $\rho: \mathbb{R}_+ \rightarrow K$  be a strictly increasing path with  $\rho(0) = 0$ , and consider (5). Then, there exist functions  $\alpha_1$  and  $\alpha_2$  of class  $\mathcal{K}$  such that

$$\mathcal{B}_{\alpha_1(s)} \cap K \subset S(\rho(s)) \cap K \subset \mathcal{B}_{\alpha_2(s)} \cap K \quad (12)$$

for all  $s \in \mathbb{R}_+$ .

*Proof:* To show  $\mathcal{B}_{\alpha_1(s)} \cap K \subset S(\rho(s)) \cap K$ , define

$$\phi(s) = \sup\{r \in \mathbb{R}_+ \mid x \in \mathcal{B}_r \cap K \Rightarrow x \in S(\rho(s))\}.$$

Note that  $S(\rho(0)) = S(0) = -K$  and recall that  $K$  is pointed, which implies  $\phi(0) = 0$ . For  $s > 0$ , it follows from  $\rho(s) \succ 0$  and  $\text{int } K \neq \emptyset$  ( $K$  is solid) that  $\phi(s) > 0$ . In addition, as  $\rho$  is strictly increasing, we have

$$s > s' \Rightarrow S(\rho(s)) \supseteq S(\rho(s'))$$

which implies that  $\phi(s) \geq \phi(s')$ . Hence,  $\phi$  is continuous (as  $\rho$  is continuous), positive definite, and nondecreasing. As a result (e.g., [23, Lemma 4.3]), there exists a class  $\mathcal{K}$  function  $\alpha_1$  satisfying  $\alpha_1(s) \leq \phi(s)$  for all  $s \in \mathbb{R}_+$ , which satisfies (12).

The bound  $S(\rho(s)) \cap K \subset \mathcal{B}_{\alpha_2(s)} \cap K$  is observed by, first, noting that for  $s = 0$ , we have  $S(\rho(0)) \cap K = \{0\}$  by pointedness of  $K$  and, second, that  $S(\rho(s)) \cap K$  is compact for any  $s \in \mathbb{R}_+$ . Here, compactness can be shown through (a slight extension of) [24, Exercise 2.2.24].

Then, Theorem 3.8 leads to the following result.

*Theorem 4.3:* Consider a monotone system (11). If there exist a strictly increasing path  $\rho: \mathbb{R}_+ \rightarrow K$  satisfying  $\rho(0) = 0$  and

$$-g(\rho(s)) \in \text{int } K, \quad \forall s > 0 \quad (13)$$

then the origin of (11) is asymptotically stable in  $K$ .

*Proof:* To prove stability, let  $\varepsilon > 0$ , consider the functions  $\alpha_1$  and  $\alpha_2$  satisfying (12) in Lemma 4.2, and note that we can always choose  $\alpha_2$  to be of class  $\mathcal{K}_\infty$ . Then, define  $s = \alpha_2^{-1}(\varepsilon)$  and  $\delta = \alpha_1(s)$ . Now, stability in  $K$  follows from (12) and positive invariance of  $K$  and  $S(\rho(s))$ , where the latter is a result of (13) and Theorem 3.8.

Next, to show asymptotic stability, note that (13) implies, through Theorem 3.8, that (10) holds for any  $\underline{s} < \bar{s}$ . After recalling that  $\phi(t, x_0) \in K$  for all  $t \in \mathbb{R}_+$  due to monotonicity and noting that

$$S(\rho(0)) \cap K = \{0\}$$

as  $K$  is pointed, the result follows from taking  $\underline{s} = 0$ . ■

The relevance of Theorem 4.3 is that stability analysis is reduced to finding a path  $\rho$  in  $K$  along which the vector field  $g$  of (11) satisfies (13). As (13) merely involves nonlinear algebraic conditions, this is potentially significantly easier than the construction of a Lyapunov function for stability analysis. The result of Theorem 4.3 essentially states that the attractivity condition (9) also implies stability in  $K$ . In the positive orthant case, this can also be concluded from [12, Lemma 2.3]. Theorem 4.3 can thus be viewed as an extension of this fact, where in addition, we provide an attractivity condition in Theorem 3.8.

*Remark 4.4:* While Theorem 4.3 focuses on asymptotic stability in  $K$ , it follows from the proof of this theorem that the origin of (11) is actually asymptotically stable in any set  $\Omega$  as long as 1)  $\Omega$  is positively invariant; 2)  $\text{int}(\Omega \cap S(\rho(s)))$  is bounded and not empty for each  $s > 0$ ; and 3)  $-K \cap \Omega = \{0\}$ . ◁

The following example illustrates the use of Theorem 4.3.

*Example 4.5:* Consider the autonomous nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_2 x_1 \\ \dot{x}_2 &= -x_2^2/2 \end{aligned} \quad (14)$$

where  $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}$ , which is monotone with respect to

$$K = \left\{ x \in \mathbb{R}^2 \mid x = a_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, a_1, a_2 \geq 0 \right\}.$$

The strictly increasing path  $\rho(s) = [s \ 4s]^\top$  satisfies  $\rho(0) = 0$  and (13). Hence, the origin of (14) is asymptotically stable in  $K$ .

This example also illustrates a subtlety of Definition 4.10. Namely, it is clear from (14) that the set of equilibria is given by  $\mathbb{R} \times \{0\}$ , which implies that the origin is not asymptotically stable in  $\mathbb{R}^2$ . However, when analysis is restricted to  $\Omega := \mathcal{X} \cap K$ , the origin is the unique equilibrium point and Theorem 4.3 guarantees asymptotic stability in  $\Omega$ , i.e., attractivity of the equilibrium point for trajectories starting in  $\Omega$ . ◁

*Remark 4.6:* Although the proof of asymptotic stability in Theorem 4.3 is based purely on the path  $\rho$  and monotonicity, there is a natural way of associating a Lyapunov function to this path. Namely, define  $V: K \rightarrow \mathbb{R}_+$  as

$$V(x) = s,$$

where  $s$  is the unique element of the set

$$\{s \in \mathbb{R}_+ \mid \rho(s) - x \in \partial K\},$$

see Lemma 3.7. Then, under the conditions of Theorem 4.3, it follows from its proof that

$$V(\phi(t, x)) < V(x)$$

for any  $x \in K$  and  $t > 0$ , i.e.,  $V$  is a Lyapunov function for (11). ◁

*Remark 4.7:* In the special case of  $K = \mathbb{R}_+^n$ , the path  $\rho: \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$  in Theorem 4.3 has strictly increasing component functions  $\rho_i$  (defined such that  $\rho(s) = [\rho_1(s) \cdots \rho_n(s)]^\top$ ) by Assumption 3.5. In addition, (13) translates to

$$g(\rho(s)) < 0, \quad \forall s > 0,$$

where the inequality is understood elementwise. In this case, the associated Lyapunov function of Remark 4.6 reads

$$V(x) = s = \max_{i \in \mathcal{I}_n} \rho_i^{-1}(x_i), \quad (15)$$

where  $\mathcal{I}_n = \{1, 2, \dots, n\}$ , and  $x_i$  is the  $i$ th component of the vector  $x \in \mathbb{R}_+^n$ . Thus, the result of Theorem 4.3 reduces to [10, Th. 3.1] and  $V$  in (15) is a so-called max-separable Lyapunov function. Such Lyapunov functions are extensively studied in the scope of cooperative systems in, e.g., [8] and [9]. In particular, the relation among asymptotic stability of cooperative systems, paths in the positive orthant, and max-separable Lyapunov functions is well-known and pioneered in the works [6], [7], [25], [26]. By exploiting Theorem 3.8, the result in Theorem 4.3 extends (some of) these results to arbitrary proper cones  $K$ . The function in Remark 4.6 can thus be regarded as an extension of the notion of max-separable Lyapunov function to a proper cone. ◁

*Remark 4.8:* To provide further insight in Theorem 4.3, consider the linear system

$$\dot{x} = Ax, \quad (16)$$

and assume that it is monotone with respect to the cone  $K$ . After choosing a linear function  $\rho(s) = \bar{\rho}s$ , the condition (8) holds if and only if  $\bar{\rho} \in \text{int } K$  and (13) reduces to

$$-A\bar{\rho} \in \text{int } K. \quad (17)$$

It will turn out to be useful to consider the dual version of these conditions. Specifically,  $\bar{\rho} \in \text{int } K$  and (17) are equivalent to

$$\langle v, \bar{\rho} \rangle > 0, \quad \forall v \in K^* \setminus \{0\}, \quad (18)$$

$$\langle v, A\bar{\rho} \rangle = \langle A^\top v, \bar{\rho} \rangle < 0, \quad \forall v \in K^* \setminus \{0\} \quad (19)$$

respectively, where we recall that  $K^*$  is the dual cone to  $K$ . After noting that the system (16) is monotone with respect to  $K$  if and only if the dual system  $\dot{z} = A^\top z$  is monotone with respect to  $K^*$  (see, e.g., [24]), it can be recognized that the conditions (18), (19) are exactly the conditions for stability of  $\dot{z} = A^\top z$  given in [27, Th. 2]. As  $A^\top$  is Hurwitz if and only if  $A$  is Hurwitz, we can view Theorem 4.3 as an extension of (the dual of) [27, Th. 2] toward nonlinear systems.

Finally, we remark that, for  $K = \mathbb{R}_+^n$ ,  $\bar{\rho} \in \text{int } K$  and  $-A\bar{\rho} \in \text{int } K$  translate to the well-known conditions for a Metzler matrix  $A$  to be

Hurwitz given by  $\bar{\rho} > 0$  and  $A\bar{\rho} < 0$  (with the inequalities understood elementwise), e.g., [24] and [20].  $\triangleleft$

### B. ISS for Nonautonomous Systems

In the previous section, we have shown that Theorem 3.8 can be used to guarantee asymptotic stability of autonomous systems on arbitrary proper cones. In this section, we show that Theorem 3.8 leads in fact to stronger results when nonautonomous systems (2) are considered. Specifically, we will show a clear relation with ISS properties.

Similar to the previous section, we assume  $f(0,0) = 0$ , take  $\mathcal{X} = \mathbb{R}^n$ , and restrict analysis to trajectories in the cone  $K$ . This leads to the following definition of (local) ISS.

**Definition 4.9:** Consider the system (2) together with proper cones  $K$  and  $K_u$ . The system is said to be locally input-to-state stable in  $K$  if there exists  $\delta > 0$ , a class  $\mathcal{KL}$  function  $\beta$ , and class  $\mathcal{K}_\infty$  function  $\gamma$  such that

$$\begin{aligned} x_0 \in \mathcal{B}_\delta \cap K, \quad u(t) \in \mathcal{B}_\delta \cap K_u, \quad \forall t \in \mathbb{R}_+ \\ \Rightarrow |\phi(t, x_0, u)| \leq \beta(|x_0|, t) + \gamma(\|u\|_\infty) \end{aligned} \quad (20)$$

for all  $t \in \mathbb{R}_+$ .  $\triangleleft$

In the above,  $\|\cdot\|_\infty$  denotes the  $\mathcal{L}_\infty$  signal norm, i.e.,  $\|u\|_\infty = \sup_{t \in \mathbb{R}_+} |u(t)|$ .

In order to study local ISS properties (in  $K$ ), we will exploit the stability notions of local stability and local asymptotic gain, as formulated next.

**Definition 4.10:** Consider the system (2) together with proper cones  $K$  and  $K_u$ . The system is said to be locally stable in  $K$  if there exists  $\delta > 0$  and nondecreasing functions  $\sigma_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  satisfying  $\sigma_i(0) = 0$  such that

$$\begin{aligned} x_0 \in \mathcal{B}_\delta \cap K, \quad u(t) \in \mathcal{B}_\delta \cap K_u, \quad \forall t \in \mathbb{R}_+ \\ \Rightarrow \sup_{t \geq 0} |\phi(t, x_0, u)| \leq \max\{\sigma_1(|x_0|), \sigma_2(\|u\|_\infty)\}. \end{aligned}$$

The system is said to have the local asymptotic gain property in  $K$  if there exist  $\delta > 0$  and class  $\mathcal{K}$  function  $\gamma$  such that

$$\begin{aligned} x_0 \in \mathcal{B}_\delta \cap K, \quad u(t) \in \mathcal{B}_\delta \cap K_u, \quad \forall t \in \mathbb{R}_+ \\ \Rightarrow \limsup_{t \rightarrow \infty} |\phi(t, x_0, u)| \leq \gamma(\|u\|_\infty). \end{aligned}$$

The relevance of introducing Definition 4.10 is that local stability and the asymptotic gain property imply local ISS, whereas the first two can be shown by using Theorem 3.8. In particular, this leads to the following main result.

**Theorem 4.11:** Consider a monotone system (2). If there exist strictly increasing paths  $\rho : \mathbb{R}_+ \rightarrow K$  and  $\gamma : \mathbb{R}_+ \rightarrow K_u$  satisfying  $\rho(0) = 0, \gamma(0) = 0$ , and

$$-f(\rho(s), \gamma(s)) \in \text{int } K, \quad \forall s > 0, \quad (21)$$

then the system (2) is locally ISS in  $K$ .<sup>1</sup>

*Proof:* It can be concluded from [28] that local stability and the local asymptotic gain property imply local ISS, see also [29]. The fact that analysis is restricted to  $K$  in Definitions 4.9 and 4.10 does not affect this result. In the remainder of the proof, we will therefore show local stability and the local asymptotic gain property.

However, before doing so, recall the result (12) from Lemma 4.2 and note that, similarly

$$\mathcal{B}_{\sigma(s)} \cap K_u \subset W(\gamma(s)) \cap K_u \quad (22)$$

<sup>1</sup>Similar to Theorem 4.3, the conditions actually imply ISS in any robustly positively invariant set  $\Omega \subset \mathcal{X}$  satisfying the conditions of Remark 4.4.

for all  $s \in \mathbb{R}_+$  and some class  $\mathcal{K}$  function  $\sigma$ . Next, define  $\delta > 0$  such that

$$\delta < \lim_{s \rightarrow \infty} \alpha_1(s), \quad \delta < \lim_{s \rightarrow \infty} \sigma(s). \quad (23)$$

(Local Stability): Let  $x_0 \in \mathcal{B}_\delta \cap K$  and  $u$  be such that  $u(t) \in \mathcal{B}_\delta \cap K_u$  for all  $t \in \mathbb{R}_+$ . Then, define

$$\underline{s} = \max\{\alpha_1^{-1}(|x_0|), \sigma^{-1}(\|u\|_\infty)\} \quad (24)$$

and note that both the inverse functions exist due to the definition of  $\delta$  in (23). In fact, we also have  $\underline{s} \leq \delta$ .

By (24) and the inclusions (12) and (22), we have that

$$x_0 \in S(\rho(\underline{s})), \quad u \in W_\infty(\gamma(\underline{s}))$$

after which item 1) of Theorem 3.8 implies that

$$\phi(t, x_0, u) \in S(\rho(\underline{s}))$$

for all  $t \in \mathbb{R}_+$ . In fact, as (2) is monotone, we also have  $\phi(t, x_0, u) \in K$  for all  $t \in \mathbb{R}_+$ . Turning attention to (12) again, this in fact shows that  $\phi(t, x_0, u) \in \mathcal{B}_{\alpha_2(\underline{s})}$ , i.e.,

$$\begin{aligned} |\phi(t, x_0, u)| &\leq \alpha_2(\underline{s}) \\ &= \max\{\alpha_2(\alpha_1^{-1}(|x_0|)), \alpha_2(\sigma^{-1}(\|u\|_\infty))\} \end{aligned}$$

for all  $t \in \mathbb{R}_+$ . Hence, we have local stability in  $K$  as in Definition 4.10 (with  $\sigma_1 = \alpha_2 \circ \alpha_1^{-1}$  and  $\sigma_2 = \alpha_2 \circ \sigma^{-1}$ ).

(Local Asymptotic Gain): Let  $x_0 \in \mathcal{B}_\delta \cap K$  and  $u$  be such that  $u(t) \in \mathcal{B}_\delta \cap K_u$  for all  $t \in \mathbb{R}_+$ . In a similar manner as in the proof of local stability, define

$$\underline{s} = \sigma^{-1}(\|u\|_\infty), \quad \bar{s} = \sigma^{-1}(\delta),$$

and note that  $\underline{s} \leq \bar{s}$ . Again using (12) and (22), this implies

$$x_0 \in S(\rho(\bar{s})), \quad u \in W_\infty(\gamma(\underline{s}))$$

such that item 2) of Theorem 3.8 gives

$$\limsup_{t \rightarrow \infty} |\phi(t, x_0, u)| \leq \alpha_2(\underline{s}).$$

Here, we have again used (12). This proves the local asymptotic gain property (see Definition 4.10) with  $\gamma = \alpha_2 \circ \sigma^{-1}$ .  $\blacksquare$

**Remark 4.12:** It follows from the proof of Theorem 4.11 that if  $\rho$  and  $\gamma$  are such that the class  $\mathcal{K}$  functions  $\alpha$  in (12) and  $\sigma$  in (22) can in fact be chosen to be of class  $\mathcal{K}_\infty$ , then ISS can be shown globally in  $K$ , i.e., the dependence on the balls  $\mathcal{B}_\delta$  in (20) can be removed.  $\triangleleft$

The paper [13] derives the ISS condition (21) when the state and input cones are positive orthants. A difficulty of generalizing analysis to proper cones is to handle tangent cones for positively invariance analysis as in Theorem 3.2.

We illustrate Theorem 4.11 by an example.

**Example 4.13:** Consider a tunnel-diode circuit [23, Section 1.2.2] modeled as

$$\dot{x} = \begin{bmatrix} -h(x_1)/C + x_2/C \\ -x_1/L - (R/L)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u, \quad (25)$$

where the tunnel diode is characterized by the function  $h$  satisfying  $h(0) = 0$  and  $h(x_1) > 0$  for  $x_1 > 0$ , and thus choose  $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}$ .

The system (25) is monotone with respect to  $K \subset \mathcal{X}$  and  $\mathbb{R}_+$ , where

$$K = \left\{ x \in \mathbb{R}^2 \mid x = \alpha_1 \begin{bmatrix} 1 \\ -k \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \alpha_1, \alpha_2 \geq 0 \right\}, \quad k > 0$$

if and only if

$$-\frac{dh(x_1)}{dx_1} \geq k + \frac{C}{Lk} - \frac{CR}{L}, \quad \forall x_1 \in \mathbb{R}_+. \quad (26)$$

Note that the system (25) is not monotone with respect to  $\mathbb{R}_+^2$  for the state and  $\mathbb{R}_+$  for the input, as the term  $-x_1/L$  prohibits monotonicity with respect to the positive orthant.

Next, we check our condition for local ISS in  $K$ . Choose

$$\rho(s) = \begin{bmatrix} s \\ 0 \end{bmatrix}, \quad \gamma(s) = s/2.$$

Then, for any  $s > s'$

$$\begin{aligned} \rho(s) - \rho(s') &= \begin{bmatrix} s - s' \\ 0 \end{bmatrix} \in \text{int } K, \\ \gamma(s) - \gamma(s') &= s/2 - s'/2 > 0 \end{aligned}$$

such that Assumption 3.5 is satisfied. Moreover

$$-f(\rho(s), \gamma(s)) = \begin{bmatrix} h(s) \\ s - s/2 \end{bmatrix} = \begin{bmatrix} h(s) \\ s/2 \end{bmatrix} \in \text{int } K$$

for any  $s > 0$ . As a result of Theorem 4.11, (25) is locally ISS in  $K$  (for the input cone  $\mathbb{R}_+$ ) if and only if the system is monotone. The right-hand side of (26) is minimized for  $k = \sqrt{C/L}$  that gives the weakest monotonicity condition. It is worth mentioning that even if we do not know  $h$  explicitly, it may be possible to check monotonicity, i.e., (26) for given positive constants  $R$ ,  $L$ , and  $C$ .

We do not impose strict differential positivity, i.e., which would require the strict inequality in (26). Strict differential positivity guarantees multistability of bounded solutions [3]. In this note, we show local ISS in  $K$  without assuming boundedness of a solution. For monotone systems, contraction conditions have been derived [11], but the proposed condition is easier to check and less conservative in this particular example.

Finally, as an illustration of Remark 4.6, we construct the Lyapunov function corresponding to the path  $\rho$ . From Lemma 3.7, we have that, for any given  $x \in K \setminus \{0\}$ , there exists either  $\beta_1 \geq 0$  or  $\beta_2 \geq 0$  such that either

$$\begin{bmatrix} s \\ 0 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (27)$$

or

$$\begin{bmatrix} s \\ 0 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \beta_2 \begin{bmatrix} 1 \\ -k \end{bmatrix} \quad (28)$$

has the unique solution  $s > 0$ . If  $-x_2 \geq 0$ , then by choosing  $\beta_1 = -x_2$ , (27) has the unique solution  $s = x_1$ . Note that  $x \in K \setminus \{0\}$  and  $-x_2 \geq 0$  imply  $x_1 > 0$ . If  $x_2 > 0$ , then by choosing  $\beta_2 = x_2/k$ ,  $k > 0$ , (28) has the unique solution  $s = x_1 + x_2/k$ . Note that  $x \in K$  implies  $x_1 \geq 0$ . Therefore, the corresponding Lyapunov function is constructed as

$$V(x) = \begin{cases} x_1 & \text{if } x_2 \leq 0 \\ x_1 + x_2/k & \text{if } x_2 > 0 \end{cases}, \quad x \in K.$$

The complex structure of this function suggests that path-based analysis provides a simpler way of verifying ISS than searching for a Lyapunov function directly.  $\triangleleft$

## V. CONCLUSION

Stability analysis of monotone systems is considered, resulting in two main differences from existing results. First, our analysis does not require proper cones being positive orthants. Second, our analysis is purely based on paths, i.e., there is no need to construct Lyapunov functions corresponding to paths in order to conclude stability properties. An advantage of this pure path-based analysis is that differentiability arguments of corresponding Lyapunov functions are not needed anymore. Finally, our results are illustrated by the ISS analysis of a tunnel diode circuit, which is not monotone with respect to the positive orthant but is with respect to some proper cones.

## APPENDIX

### A. Proof of Theorem 3.2

By Nagumo's theorem, see [17, Th. 4.10] or [2, Th. 4], positive invariance of  $S(v)$  in (5) is equivalent to the condition

$$f(x, u) \in \mathcal{T}_x S(v), \quad \forall x \in S(v), \quad \forall u \in W(w). \quad (29)$$

In the remainder of this proof, we use Lemma 2.1. However, as this result only considers the interior of the invariant set  $\mathcal{X}$ , we cannot directly consider the set  $S(v)$ . In fact, it can be concluded from the results in [2, Section III] that (29) is equivalent to

$$f(x, u) \in \mathcal{T}_x S(v), \quad \forall x \in \bar{S}(v), \quad \forall u \in W(w), \quad (30)$$

where  $\bar{S}(v) := S(v) \cap \text{int } \mathcal{X}$ . In the remainder of the proof, we will therefore show equivalence between (7) and (30).

(Sufficiency) By the definition of the tangent cone, (7) implies

$$\liminf_{\tau \rightarrow 0^+} \frac{\inf_{z \in S(v)} |v + \tau f(v, w) - z|}{\tau} = 0. \quad (31)$$

At the same time, by Assumption 2.1 and Lemma 2.2, we have that, for any  $v \succeq x$  and  $w \succeq u$  (i.e.,  $x \in \bar{S}(v)$  and  $u \in W(w)$ )

$$\lim_{\tau \rightarrow 0^+} \frac{\inf_{y \in K} |v - x + \tau(f(v, w) - f(x, u)) - y|}{\tau} = 0 \quad (32)$$

as is again immediate from (1). Note that  $\liminf$  is replaced by  $\lim$  as  $K$  is convex (recall the remark below (1)).

Next, the triangle inequality gives

$$\begin{aligned} &| -x - \tau f(x, u) - y + z | \\ &\leq |v + \tau f(v, w) - z| \\ &\quad + |v - x + \tau(f(v, w) - f(x, u)) - y| \end{aligned}$$

where the norms in (31) and (32) appear on the right-hand sides. Since taking the infimum does not change the inequality, we have

$$\begin{aligned} &\inf_{(y,z) \in K \times S(v)} | -x - \tau f(x, u) - y + z | \\ &\leq \inf_{(y,z) \in K \times S(v)} (|v + \tau f(v, w) - z| \\ &\quad + |v - x + \tau(f(v, w) - f(x, u)) - y|) \\ &= \inf_{z \in S(v)} |v + \tau f(v, w) - z| \\ &\quad + \inf_{y \in K} |v - x + \tau(f(v, w) - f(x, u)) - y| \quad (33) \end{aligned}$$

for any  $\tau > 0$ , where the equality follows as the two infima are over independent variables ( $z$  and  $y$ , respectively). After dividing the result (33) by  $\tau$  and taking the  $\liminf$ , we obtain

$$\begin{aligned} &\liminf_{\tau \rightarrow 0^+} \frac{\inf_{(y,z) \in K \times S(v)} | -x - \tau f(x, u) - y + z |}{\tau} \\ &\leq \liminf_{\tau \rightarrow 0^+} \left( \frac{\inf_{z \in S(v)} |v + \tau f(v, w) - z|}{\tau} \right. \\ &\quad \left. + \frac{\inf_{y \in K} |v - x + \tau(f(v, w) - f(x, u)) - y|}{\tau} \right) \\ &= \liminf_{\tau \rightarrow 0^+} \frac{\inf_{z \in S(v)} |v + \tau f(v, w) - z|}{\tau} \\ &\quad + \lim_{\tau \rightarrow 0^+} \frac{\inf_{y \in K} |v - x + \tau(f(v, w) - f(x, u)) - y|}{\tau}. \end{aligned}$$

Here, the equality follows from convexity of  $K$ . As a result, we can conclude from (31) and (32) that

$$\liminf_{\tau \rightarrow 0} \frac{\inf_{(y,z) \in K \times S(v)} |-x - \tau f(x,u) - y + z|}{\tau} = 0.$$

To simplify this result, note that the triangular inequality gives

$$|-x - \tau f(x,u) + z| \leq |-x - \tau f(x,u) - y + z| + |y|.$$

By a similar reasoning as earlier, using that  $\inf_{y \in K} |y| = 0$  (from  $0 \in K$ ), we have

$$\liminf_{\tau \rightarrow 0^+} \frac{\inf_{z \in S(v)} |x + \tau f(x,u) - z|}{\tau} = 0. \quad (34)$$

From the definition of the tangent cone, this is exactly (30) at  $(x,u)$ . As  $x \in \bar{S}(v)$  and  $u \in W(w)$  are chosen arbitrarily, this finalizes the proof of sufficiency.

(Necessity) As (30) is a necessary and sufficient condition for robust positive invariance of a closed subset, we let (30) hold and follow a similar idea as in the proof of sufficiency. Indeed, motivated by the norms in (32) and (34), consider the triangle inequality

$$\begin{aligned} & |v + \tau f(v,w) - (y+z)| \\ & \leq |x + \tau f(x,u) - z| \\ & \quad + |v - x + \tau(f(v,w) - f(x,u)) - y|. \end{aligned}$$

In a similar manner as the sufficiency proof, it can be shown that

$$\liminf_{\tau \rightarrow 0} \frac{\inf_{(y,z) \in K \times \bar{S}(v)} |v + \tau f(v,w) - (y+z)|}{\tau} = 0.$$

By using the triangular inequality again, (7) can be obtained.  $\blacksquare$

### B. Proof of Lemma 3.7

Since  $\rho(\bar{s}) - x \in \text{int } K$  (as  $x \prec \rho(\bar{s})$ ) and  $\text{int } K$  is open, there exists  $\varepsilon > 0$  such that, for any  $z \in \mathcal{X}$

$$|\rho(\bar{s}) - x - z| < \varepsilon \Rightarrow z \in \text{int } K.$$

Specifically, we can choose  $z = \rho(s_1) - x$  for some  $s_1 < \bar{s}$ . As  $\rho$  is continuous, for any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $s_1 \in [0, \bar{s})$

$$\bar{s} - s_1 < \delta \Rightarrow |\rho(\bar{s}) - \rho(s_1)| < \varepsilon.$$

In summary, we have constructed  $s_1 < \bar{s}$  such that

$$\begin{aligned} & \bar{s} - s_1 < \delta \\ & \Rightarrow |\rho(\bar{s}) - x - (\rho(s_1) - x)| = |\rho(\bar{s}) - \rho(s_1)| < \varepsilon \\ & \Rightarrow \rho(s_1) - x \in \text{int } K. \end{aligned}$$

By repeating this reasoning, one can construct a strictly decreasing sequence  $\{s_i\}$  satisfying

$$\rho(s_i) - x \in \text{int } K.$$

We note that the assumption  $x \not\prec \rho(0)$  can be written as  $\rho(0) - x \notin K$ . This, together with the fact that  $\rho$  is strictly increasing, implies that  $s_i > 0$ , i.e., the strictly decreasing sequence  $\{s_i\}$  is lower bounded. Thus,  $\{s_i\}$  converges to some  $s \geq 0$ . As  $\rho$  is continuous (recall again Assumption 3.5),  $\{\rho(s_i) - x\}$  is a convergent sequence also, which converges to  $\rho(s) - x$ . Since  $K$  is closed, it follows that  $\rho(s) - x \in K$ . In fact,  $\rho(s) - x \in \partial K$  as  $\rho(s) - x \in \text{int } K$  contradicts the convergence of  $\{s_i\}$ . Continuity of  $\rho$  implies that  $\rho(s) - x$  does not depend on the choice of the sequence  $\{s_i\}$ , such that  $s$  is unique.

We have already shown (by construction of the strictly decreasing sequence) that  $s < \bar{s}$ . The property  $0 < s$  follows from the assumption  $x \not\prec \rho(0)$ .

### C. Proof of Theorem 3.8

Before proving Theorem 3.8, we give two useful lemmas.

*Lemma 5.1:* Let the conditions in the statement of Theorem 3.8 hold. Then

$$f(\rho(s), \gamma(s)) \in \mathcal{T}_{\rho(s)} S(\rho(s)), \quad \forall s \in (0, \bar{s}]. \quad (35)$$

*Proof:* Note that  $\rho(s) \in \text{int } \mathcal{X}$  and  $\gamma(s) \in \mathcal{U}$ . Moreover, by the dynamics (2) and robust positive invariance of  $\text{int } \mathcal{X}$  with respect to  $W_\infty(\gamma(s))$  following that of  $\mathcal{X}$  (see [2, Lemma 3.6]), we also have

$$\rho(s) + \int_0^t f(\phi(\tau, \rho(s), \gamma(s)), \gamma(s)) d\tau \in \text{int } \mathcal{X}$$

for all  $t \geq 0$ . Thus, for sufficiently small  $\tau > 0$ , this implies  $\rho(s) + \tau f(\rho(s), \gamma(s)) \in \text{int } \mathcal{X}$ . Moreover, from (9), we obtain that  $\rho(s) \geq \rho(s) + \tau f(\rho(s), \gamma(s))$  for any  $\tau > 0$ . Together, these conditions imply

$$\rho(s) + \tau f(\rho(s), \gamma(s)) \in S(\rho(s)) \quad (36)$$

as follows from the definition of  $S(\cdot)$  in (5). The result (36) in turn implies (35) after recalling the definition of the tangent cone (1).

*Lemma 5.2:* Let the conditions in the statement of Theorem 3.8 hold and define the set

$$\Omega_\delta := \{x \in \mathcal{X} \mid d(x, S(\rho(\underline{s}))) < \delta\}.$$

Then, for each  $\delta > 0$ , there exists  $s > \underline{s}$  such that

$$S(\rho(s)) \subset \Omega_\delta. \quad (37)$$

*Proof:* Let  $\delta > 0$  be given. Then, by continuity of  $\rho$ , there exists  $s > \underline{s}$  such that

$$|\rho(s) - \rho(\underline{s})| < \delta.$$

To show that (37) holds for this  $s$ , take  $x \in S(\rho(s))$  and define

$$y = \rho(\underline{s}) - \rho(s) + x.$$

Then, a direct computation yields the results

$$\begin{aligned} |x - y| &= |\rho(s) - \rho(\underline{s})| < \delta, \\ \rho(\underline{s}) - y &= \rho(s) - x \in K, \end{aligned} \quad (38)$$

where the inclusion in (38) follows as  $x \in S(\rho(s))$ . Thus, we have constructed  $y \in S(\rho(\underline{s}))$  for which  $d(x, y) < \delta$ . Stated differently,  $x \in \Omega_\delta$  as desired.  $\blacksquare$

We are now in the position to prove Theorem 3.8.

(Proof of 1) According to Theorem 3.2, condition (35) is a necessary and sufficient condition for the robust positive invariance of  $S(\rho(s))$  with respect to  $W_\infty(\gamma(s))$ . The increasing property of  $\gamma$  implies  $W_\infty(\gamma(\underline{s})) \subset W_\infty(\gamma(s))$  for any  $s \in [\underline{s}, \bar{s}]$ . Hence, from Definition 3.1,  $S(\rho(s))$  is also robustly positively invariant with respect to  $W_\infty(\gamma(\underline{s}))$ .

(Proof of 2) We now turn to attractivity of  $S(\rho(\underline{s}))$ . Let  $u \in W_\infty(\gamma(\underline{s}))$  and  $x_0 \in S(\rho(\bar{s}))$  and note that  $\phi(t, x_0, u) \in S(\rho(\bar{s}))$  for all  $t \in \mathbb{R}_+$  due to the result of item 1). We now consider three cases.

In the first case, let  $x_0 \in S(\rho(\underline{s}))$ . Then, (10) follows immediately from robust positive invariance (item 1)).

Second, let  $x_0 \notin S(\rho(\underline{s}))$ , but assume that the trajectory  $\phi(\cdot, x_0, u)$  enters  $S(\rho(\underline{s}))$ , i.e., there exists  $T > 0$  such that  $\phi(T, x_0, u) \in S(\rho(\underline{s}))$ . In this case, the result (10) follows again from robust positive invariance of  $S(\rho(\underline{s}))$ .



In the third and final cases, let  $x_0 \notin S(\rho(\underline{s}))$  and  $\phi(t, x_0, u) \notin S(\rho(\underline{s}))$  for all  $t \in \mathbb{R}_+$ . In the remainder of this proof, we will show that (10) still holds in this case, i.e.,  $S(\rho(\underline{s}))$  is attractive.

Consider the solution  $\phi(t_1, x_0, u)$  to the system (2) at some time  $t_1 > 0$ . Note that  $\phi(t_1, x_0, u) \preceq \rho(\bar{s})$  for any  $t_1 \geq 0$  from the robust positive invariance of  $S(\rho(\bar{s}))$ . Then, from a slight modification of Lemma 3.7, there exists  $s_1 \in (\underline{s}, \bar{s}]$  such that

$$\rho(s_1) - \phi(t_1, x_0, u) \in \partial K.$$

In addition, monotonicity (3) implies that

$$\phi(t, \rho(s_1), \gamma(s_1)) \succeq \phi(t + t_1, x_0, u), \quad \forall t \in \mathbb{R}_+. \quad (39)$$

On the other hand, from the continuity of  $f(x, u)$  and  $\phi(t, \rho(s_1), \gamma(s_1))$ , condition (9) implies that there exists a sufficiently small  $t_2 > 0$  such that

$$-\int_0^t f(\phi(r, \rho(s_1), \gamma(s_1)), \gamma(s_1)) dr \in \text{int } K, \quad \forall t \in (0, t_2],$$

and thus

$$\begin{aligned} & \rho(s_1) - \phi(t, \rho(s_1), \gamma(s_1)) \\ &= \rho(s_1) - \left( \rho(s_1) + \int_0^t f(\phi(r, \rho(s_1), \gamma(s_1)), \gamma(s_1)) dr \right) \\ & \in \text{int } K, \quad \forall t \in (0, t_2]. \end{aligned}$$

This inclusion and (39) imply

$$\rho(s_1) \succ \phi(t, \rho(s_1), \gamma(s_1)) \succeq \phi(t + t_1, x_0, u), \quad \forall t \in (0, t_2].$$

By virtue of Lemma 3.7, there exists  $s_2 \in (\underline{s}, s_1)$  such that

$$\rho(s_2) - \phi(t_2 + t_1, x_0, u) \in \partial K.$$

By repeating this procedure, we obtain a strictly decreasing sequence  $\{s_\ell\}$ , which is lower bounded by  $\underline{s}$  and satisfies

$$\rho(s_\ell) - \phi(t_\ell + \dots + t_1, x_0, u) \in \partial K.$$

As  $\{s_\ell\}$  is a lower bounded sequence,  $\{s_\ell\}$  converges to some  $r \geq \underline{s}$ . Now, we show  $r = \underline{s}$  by contradiction. Suppose that  $r > \underline{s}$ . For any  $\phi(\bar{t}_1, x_0, u)$  of the system starting from  $x_0 \in S(\rho(r))$  with  $u \in W_\infty(\gamma(\underline{s}))$  at time  $\bar{t}_1 \geq 0$ , one can find  $\bar{s}_1 \in [s, r]$  such that

$$\rho(\bar{s}_1) - \phi(\bar{t}_1, x_0, u) \in \partial K.$$

Then, we again obtain a strictly decreasing sequence  $\{\bar{s}_\ell\}$ , which contradicts the convergence of  $\{s_\ell\}$ .

Hence, we have constructed a sequence  $\{s_\ell\}$  that converges to  $\underline{s}$ . As a consequence of Lemma 5.2, this is sufficient to show that

$$\lim_{t \rightarrow \infty} d(\phi(t, x_0, u), S(\rho(\underline{s}))) = 0$$

for any  $x_0 \in S(\rho(\bar{s}))$  and  $u \in W_\infty(\gamma(\underline{s}))$ , completing the proof. ■

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