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PREDICTOR CONTROL FOR WAVE PDE / NONLINEAR ODE CASCADED SYSTEM WITH BOUNDARY VALUE-DEPENDENT PROPAGATION SPEED

Xiushan Cai, Yuhang Lin, Junfeng Zhang, and Cong Lin

This paper investigates predictor control for wave partial differential equation (PDE) and nonlinear ordinary differential equation (ODE) cascaded system with boundary value-dependent propagation speed. A predictor control is designed first. A two-step backstepping transformation and a new time variable are employed to derive a target system whose stability is established using Lyapunov arguments. The equivalence between stability of the target and the original system is provided using the invertibility of the backstepping transformations. Stability of the closed-loop system is established by Lyapunov arguments.

Keywords: cascaded system, wave dynamics, boundary value-dependent, predictor control, backstepping transformation

Classification: 93Cxx, 93Dxx

1. INTRODUCTION

We consider the cascaded system of wave PDE/nonlinear ODE given by

$$\dot{X}(t) = f(X(t), u(0,t)) \quad (1)$$

$$\partial_{tt}u(x,t) = v(u(0,t))\partial_{xx}u(x,t) \quad (2)$$

$$\partial_x u(0,t) = 0 \quad (3)$$

$$\partial_x u(L,t) = U(t), \quad (4)$$

where $0 \leq x \leq L, t \geq 0$, and $X(\cdot,\cdot) \in \mathbb{R}^n, u(\cdot) \in \mathbb{R}, U(\cdot) \in \mathbb{R}$ are ODE state, PDE state, and control input, respectively, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, with $f(0,0) = 0$ is locally Lipschitz continuous in X and u, and $v : \mathbb{R} \to (0, +\infty)$ is a propagation velocity of wave PDE. Equation (2) is the actuation path for system (1), located at the boundary $x = 0$, with an actuation device acting at the boundary $x = L$. It is clear that $u(0,t)$ cannot be directly controlled, so it is more difficult to control this cascade system compared with [8] and [12].

The initial condition along the actuation path (2) is

$$u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad (5)$$

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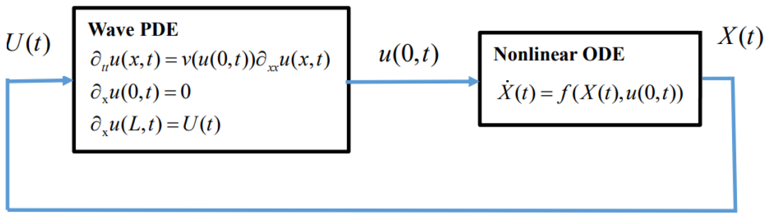

Fig. 1. The Cascaded System of Wave PDE/Nonlinear ODE.

for $x \in [0, L]$. The cascaded system (1) – (4) is depicted as Fig.1.

We design a predictor control that globally stabilizes the cascaded system (1) – (4) by assuming that the propagation velocity is continuously differentiable, strictly positive, and bounded, namely, there are $\varrho_i > 0$, $i = 1, 2$, such that

$$\varrho_2 \geq v(u(0,t)) \geq \varrho_1, \tag{6}$$

for any $u(0,t) \in \mathbb{R}$.

First, the cascaded system is transferred to a coupled 2×2 hyperbolic PDE/ODE cascaded system using some coordinate changes. Secondly, the first-step backstepping transformation is applied such that the coupled 2×2 hyperbolic PDE/ODE cascaded system is transferred to a decoupled 2×2 hyperbolic PDE/ODE cascaded system. Further, using a new time variable, the cascaded system is changed as a pair of transport PDEs and nonlinear ODE cascaded system. Finally, the second-step backstepping transformation is used to design a compensator for the cascaded system. Stability of the closed-loop system is established by Lyapunov arguments.

Wave PDE/nonlinear ODE cascaded system can be used to describe stick-slip oscillation in oil drilling [1] – [5], [10], [13] – [17]. Predictor control of wave PDE/nonlinear ODE is first presented in [1]. In oil drilling model, the ODE is used for modeling friction-dominated drill bit dynamics and wave PDE is used to simulate torsional dynamics of drill string, and the position of the drill bit is a state variable in the ODE, predictor control is studied for the cascaded system with a controlled moving boundary in [6].

In oil drilling model, the domain length of wave PDE varies with time and also depends on the bit speed, a predictor control is presented for wave PDE/nonlinear ODE with a uncontrolled moving boundary in [7]. The propagation velocity of drill string torsional dynamics varies with different pipes, so wave PDE/nonlinear ODE with spatially-varying propagation speed is investigated in [8]. Predictor control for a class of nonlinear ODE/wave PDE cascaded systems with time-varying propagation speed is also studied in [12]. This paper considers predictor control design of system (1) – (4), which is similar but not equivalent to stabilize stick-slip oscillation in drilling systems.

Detailed comparisons with [1, 6, 7, 8, 9, 12] are given in Table 1, including the studied system, propagation velocity constraint, boundary dependent property, whether feasibility conditions are required, and convergence range.

studied system	propagation velocity constraint	boundary dependence	feasibility condition	convergence range
(1) $\partial_{tt}u(x,t)=\partial_{xx}u(x,t)$ (3) (4) in [1]	no	no	no	global
(1) $\partial_{tt}u(x,t)=\partial_{xx}u(x,t)$ $\partial_x u(L(t,X),t)=U(t)$ (4) in [6]	no	a moving controlled boundary $L(t,X)$	$\lvert\frac{\partial L(t,X)}{\partial X}\times f(X,u(0,t))+\frac{\partial L(t,X)}{\partial t}\rvert\leq d<1$	local
$\dot{X}(t)=f(X,u(\delta(X,t),t))$ $\partial_{tt}u(x,t)=\partial_{xx}u(x,t)$ $\partial_x u(\delta(X,t),t)=0$ (4) in [7]	no	a moving uncontrolled boundary $\delta(X,t)$	$0\leq\frac{\partial\delta(X,t)}{\partial X}\times f(X,u(\delta(X,t)))+\frac{\partial\delta(X,t)}{\partial t}\leq c<1$	local
(1) $\partial_{tt}u(x,t)=v(x)\partial_{xx}u(x,t)$ (3),(4) in [8]	$v(x)>0$, $x\in[0,L]$	no	no	global
(1) $\partial_{tt}u(x,t)=v(t)\partial_{xx}u(x,t)$ (3),(4) in [12]	$\exists\varrho_i>0$, $i=1,2,3$ $\varrho_1\leq v(t)\leq\varrho_2$ $\lvert\dot{v}(t)\rvert\leq\varrho_3$	no	no	global
(1) $\partial_t u(x,t)=v(u(0,t))\partial_x u(x,t)$ $u(L,t)=U(t)$ in [9]	$\exists\varrho_1>0$, $v(u(0,t))>\varrho_1$	no	no	global
(1)–(4) Current paper	$\exists\varrho_i>0$, $i=1,2$, $\varrho_2\geq v(u(0,\ t))\geq\varrho_1$	no	no	global

Tab. 1. Comparison of Related Results of PDE/nonlinear ODE Cascaded System.

From Table 1, compared with [9], the same propagation speed $v(u(0,t))$ is studied, we study wave PDE actuator dynamics, but [9] investigates transport PDE actuator dynamics. The constraint on $v(u(0,t))$ is similar to that in [9], a global stability result of the closed-loop system is established in this paper.

Compared with [12], we investigate boundary-value-dependent propagation speed not just time-varying propagation speed $v(t)$ in [11]. Our contribution stands as the first one in which wave actual compensation of a delayed-input-dependent input delay is achieved. A global stability result of the closed-loop system is established. Compared with [12], our main contributions are as follows:

1) The constraint on $v(u(0,t))$ is relaxed, see Table 1.

2) Predictors (77), (78) and (79), (80) are simpler than those in [12].

3) By introducing a new time variable, the target system (84) – (88) is simpler than [12].

This paper is organized as follows: Assumptions and control design are in Section 2. Some preliminary transformations are in Section 3. Stability analysis of the proposed control law is established in Section 4. Simulation is in Section 5. Finally, the conclusion is in Section 6.

Notation. $|\cdot|$ is Euclidean norm, and $u : [0,L] \times \mathbb{R} \to \mathbb{R}$, the norm $\|u(t)\|_{L_\infty[0,L]} = \lim\limits_{n\to\infty} (\int_0^L |u(x,t)|^n \,\mathrm{d}x)^{1/n}$ is the spatial L_∞ norm, written more compactly as $\|u(t)\|_\infty$. For $u : [-L,L] \times \mathbb{R} \to \mathbb{R}$, denote $\|u(t)\|_{\infty,1} = \lim\limits_{n\to\infty} (\int_{-L}^L |u(x,t)|^n \,\mathrm{d}x)^{1/n}$. For $p : [0,L] \times \mathbb{R} \to \mathbb{R}^n$, we use a spatial L_∞-norm $\|p(t)\|_\infty = \sup\limits_{0\le x\le L} (p_1^2(x,t) + \cdots + p_n^2(x,t))^{1/2}$.

2. ASSUMPTIONS AND CONTROL DESIGN

Definitions of class $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{KL}$ functions and input-to-state stable (ISS) are from [11] and [18].

A function $\gamma : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{K}$ if it is continuous strictly increasing and satisfies $\gamma(0) = 0$; it is of class $\mathcal{K}_\infty$, if in addition $\gamma(s) \to \infty$ as $s \to \infty$. Note that if γ is of class $\mathcal{K}_\infty$, then the inverse function γ^{-1} is well defined and is again of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is said to be of class $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s it is decreasing to zero on t as $t \to \infty$.

The system $\dot{X}(t) = f(X,u)$, where f locally Lipschitz in X and u, is input-to-state stable (ISS) if there is a function β of class $\mathcal{KL}$ and a function γ of class $\mathcal{K}$ such that for each measurable essentially bounded control $u(\cdot)$ and each initial state $X(0)$ the solution exists for all $t \ge 0$ and satisfies

$$|X(t)| \le \beta(|X(0)|, t) + \gamma\left(\sup_{0\le\sigma\le t} |u(\sigma)|\right). \tag{7}$$

This paper needs Assumptions 1– 3, which are from [1].

Assumption 2.1. Assume that $\kappa : \mathbb{R}^n \to \mathbb{R}$, $\kappa(0) = 0$ is continuously differentiable with locally Lipschitz derivative $\frac{\partial \kappa(X)}{\partial X}$, and system $\dot{X} = f(X, \kappa(X) + v)$ is ISS with respect to v.

Remark 2.2. Assumption (2.1) means that there exists a control law $\kappa(X)$ such that ODE (1) is ISS to υ.

For the following ODE

$$\dot{Y}(t) = h(Y(t), W(0,t)) \quad (8)$$
$$\partial_t W(0,t) = \varsigma(t), \quad (9)$$

where $h : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $Y(\cdot) \in \mathbb{R}^n$, $W(0,\cdot) \in \mathbb{R}$, $\varsigma(\cdot) \in \mathbb{R}$, if we denote

$$Z(t) = \begin{bmatrix} Y(t) \\ W(0,t) \end{bmatrix}, \quad \varphi(Z(t), \varsigma(t)) = \begin{bmatrix} h(Y(t), W(0,t)) \\ \varsigma(t) \end{bmatrix}, \quad (10)$$

then (8) – (9) can be expressed as

$$\dot{Z}(t) = \varphi(Z(t), \varsigma(t)). \quad (11)$$

Assumption 2.3. For system (11), there are smooth positive definite functions $R_i, i = 1, 2$ and class $\mathcal{K}_\infty$ functions $\alpha_j, j = 1 \cdots, 6$ such that

$$\begin{aligned} \alpha_1(|Z|) &\leq R_1(Z) \leq \alpha_2(|Z|) & (12)\\ \frac{\partial R_1(Z)}{\partial Z}\varphi(Z,\varsigma) &\leq R_1(Z) + \alpha_3(|\varsigma|) & (13)\\ \alpha_4(|Z|) &\leq R_2(Z) \leq \alpha_5(|Z|) & (14)\\ -\frac{\partial R_2(Z)}{\partial Z}\varphi(Z,\varsigma) &\leq R_2(Z) + \alpha_6(|\varsigma|), & (15) \end{aligned}$$

for all $Z \in \mathbb{R}^{n+1}$ and $\varsigma \in \mathbb{R}$.

Remark 2.4. Conditions (12), (13) (or (14), (15)) imply that for every initial condition and every locally bounded input signal the corresponding solution of (11) is defined for all $t \geq 0$ (or $t \leq 0$).

Let

$$\mu_1(Z) = -c_1(Z_2 - \kappa(Z_1)) + \frac{\partial \kappa(Z_1)}{\partial Z_1} f(Z_1, Z_2), \quad (16)$$

with $Z = [Z_1, Z_2] \in \mathbb{R}^n \times \mathbb{R}$, and $c_1 > 0$.

Assumption 2.5. For system $\dot{Z} = \varphi(Z, \mu_1(Z) + \varsigma)$, there exist a smooth positive definite function R_3 and class $\mathcal{K}_\infty$ functions $\alpha_j, j = 7, 8, 9$ such that

$$\alpha_7(|Z|) \leq R_3(Z) \leq \alpha_8(|Z|) \quad (17)$$
$$-\frac{\partial R_3(Z)}{\partial Z}\varphi(Z, \mu_1(Z) + \varsigma) \leq R_3(Z) + \alpha_9(|\varsigma|), \quad (18)$$

for $Z \in \mathbb{R}^{n+1}$ and $\varsigma \in \mathbb{R}$.

Remark 2.6. Assumption (2.5) means that for every initial condition and every locally bounded input signal the corresponding solution of $\dot{Z} = \varphi(Z, \mu_1(Z) + \varsigma)$ is defined for all $t \leq 0$.

For subsystem $\dot{X}(t) = f(X(t), u(0,t))$, if there exists $u(0,t) = \kappa(X(t))$ such that $\dot{X}(t) = f(X(t), \kappa(X(t)))$ is globally asymptotically stable, the predictor control for system (1)−(4) is designed as

$$\begin{aligned} U(t) = &-\frac{\partial_t u(L,t)}{2\sqrt{v(u(0,t))}} + \frac{1}{2}\partial_x u(L,t) \\ &- \frac{\int_0^L K_{11}(L,s,t)s_1(s,t)\,\mathrm{d}s}{2\sqrt{v(u(0,t))}} - \frac{\int_0^L K_{12}(L,s,t)s_2(s,t)\,\mathrm{d}s}{2\sqrt{v(u(0,t))}} \\ &- \frac{c_1}{2}\big(p_2(L,\phi(t)) - \kappa(p_1(L,\phi(t)))\big) + \frac{\partial\kappa(p_1(L,\phi(t)))}{\partial p_1}\frac{f(p_1(L,\phi(t)),\, p_2(L,\phi(t)))}{2\sqrt{v(p_2(y,\phi(t)))}}, \end{aligned} \tag{19}$$

where $p_1 \in \mathbb{R}^n, p_2 \in \mathbb{R}$ are defined as

$$p_1(x,\phi(t)) = X(t) + \int_0^x \frac{f(p_1(y,\phi(t)), p_2(y,\phi(t)))}{\sqrt{v(p_2(y,\phi(t)))}}\mathrm{d}y \tag{20}$$

$$\begin{aligned} p_2(x,\phi(t)) = &\; u(0,t) + \int_0^x \frac{s_1(y,t)}{\sqrt[4]{v(p_2(y,\phi(t)))v(u(0,t))}}\mathrm{d}y \\ &- \int_0^x\int_y^x \frac{s_1(y,t)K_{11}(\sigma,y,t)}{\sqrt[4]{v(p_2(\sigma,\phi(t)))v(u(0,t))}}\mathrm{d}\sigma\mathrm{d}y \\ &- \int_0^x\int_y^x \frac{s_2(y,t)K_{12}(\sigma,y,t)}{\sqrt[4]{v(p_2(\sigma,\phi(t)))v(u(0,t))}}\mathrm{d}\sigma\mathrm{d}y \end{aligned} \tag{21}$$

with $\phi(t) = \int_0^t \sqrt{v(u(0,\sigma))}\mathrm{d}\sigma$ and

$$s_1(y,t) = \partial_t u(y,t) + \sqrt{v(u(0,t))}\,\partial_y u(y,t) \tag{22}$$

and

$$s_2(y,t) = \partial_t u(y,t) - \sqrt{v(u(0,t))}\,\partial_y u(y,t), \tag{23}$$

for all $0 \leq x \leq L, t \geq 0$. The initial conditions of p_1 and p_2 are defined as

$$p_1(x,0) = X(0) + \int_0^x \frac{f(p_1(y,0), p_2(y,0))}{\sqrt{v(p_2(y,0))}}\mathrm{d}y \tag{24}$$

$$\begin{aligned} p_2(x,0) = &\; u_0(0) + \int_0^x \frac{s_1(y,0)}{\sqrt[4]{v(p_2(y,0))v(u_0(0))}}\mathrm{d}y \\ &- \int_0^x\int_y^x \frac{s_1(y,0)K_{11}(\sigma,y,0)}{\sqrt[4]{v(p_2(\sigma,0))v(u_0(0))}}\mathrm{d}\sigma\mathrm{d}y \\ &- \int_0^x\int_y^x \frac{s_2(y,0)K_{12}(\sigma,y,0)}{\sqrt[4]{v(p_2(\sigma,0))v(u_0(0))}}\mathrm{d}\sigma\mathrm{d}y \end{aligned} \tag{25}$$

for all $0 \leq x \leq L$. The gain $c_1 > 0$, and the functions K_{11} and K_{12} are solutions to the kernel PDEs

$$\begin{aligned}
\partial_t K(x,s,t) - \partial_s K(x,s,t)\mathcal{A}(u) - \mathcal{A}(u)\partial_x K(x,s,t) &= -K(x,s,t)\mathcal{B}(u) && (26)\\
\mathcal{A}(u)K(x,x,t) - K(x,x,t)\mathcal{A}(u) &= \mathcal{B}(u) && (27)\\
K_{11}(x,0,t) &= K_{12}(x,0,t) && (28)\\
K_{21}(x,0,t) &= K_{22}(x,0,t), && (29)
\end{aligned}$$

where (26) – (29) is defined on $\{(x,s,t) : 0 \leq s \leq x \leq L, t \geq 0\}$. The boundary value-dependent matrices are given by

$$\mathcal{A}(u) = \begin{bmatrix} \sqrt{v(u(0,t))} & 0 \\ 0 & -\sqrt{v(u(0,t))} \end{bmatrix}, \tag{30}$$

$$\mathcal{B}(u) = \begin{bmatrix} 0 & -\frac{\dot{v}(u(0,t))}{4v(u(0,t))} \\ -\frac{\dot{v}(u(0,t))}{4v(u(0,t))} & 0 \end{bmatrix}, \tag{31}$$

where $\dot{v}(u(0,t)) = \frac{\partial v(u(0,t))}{\partial u}\partial_t u(0,t)$.

3. SOME PRELIMINARY TRANSFORMATIONS

Denote

$$\begin{aligned}
\overline{\zeta}(x,t) &= \partial_t u(x,t) + \sqrt{v(u(0,t))}\,\partial_x u(x,t), && (32)\\
\overline{\eta}(x,t) &= \partial_t u(x,t) - \sqrt{v(u(0,t))}\,\partial_x u(x,t). && (33)
\end{aligned}$$

From (32), (33), we have

$$\begin{aligned}
\partial_t u(x,t) &= \frac{\overline{\zeta}(x,\,t) + \overline{\eta}(x,\,t)}{2}, && (34)\\
\sqrt{v(u(0,t))}\partial_x u(x,t) &= \frac{\overline{\zeta}(x,\,t) - \overline{\eta}(x,\,t)}{2}, && (35)
\end{aligned}$$

system (1) – (4) can be expressed as

$$\begin{aligned}
\dot{X}(t) &= f(X(t), u(0,t)) && (36)\\
\partial_t u(0,t) &= \overline{\zeta}(0,t) && (37)\\
\partial_t \overline{\zeta}(x,t) &= \sqrt{v(u(0,t))}\,\partial_x \overline{\zeta}(x,t)\\
&\quad + \frac{\dot{v}(u(0,t))}{4v(u(0,t))}(\overline{\zeta}(x,t) - \overline{\eta}(x,t)) && (38)\\
\partial_t \overline{\eta}(x,t) &= -\sqrt{v(u(0,t))}\,\partial_x \overline{\eta}(x,t)\\
&\quad - \frac{\dot{v}(u(0,t))}{4v(u(0,t))}(\overline{\zeta}(x,t) - \overline{\eta}(x,t)) && (39)\\
\overline{\eta}(0,t) &= \overline{\zeta}(0,t) && (40)\\
\overline{\zeta}(L,t) &= \overline{\eta}(L,t) + 2\sqrt{v(u(0,t))}\,U(t). && (41)
\end{aligned}$$

Denote $\overline{\xi}(x,t) = [\overline{\zeta}(x,t), \overline{\eta}(x,t)]^T$, using the state transformation

$$\xi(x,t) \quad = \quad \sqrt[4]{\frac{v(u_0(0))}{v(u(0,t))}}\,\overline{\xi}(x,t), \tag{42}$$

with $\xi(x,t) = [\zeta(x,t), \eta(x,t)]^T$, system (36) – (41) is rewritten as

$$\dot{X}(t) = f(X(t), u(0,t)) \tag{43}$$
$$\partial_t u(0,t) = \sqrt[4]{\frac{v(u(0,t))}{v(u_0(0))}}\,\zeta(0,t) \tag{44}$$
$$\partial_t \xi(x,t) = \mathcal{A}(u(0,t))\partial_x \xi(x,t) + \mathcal{B}(u(0,t))\xi(x,t) \tag{45}$$
$$\eta(0,t) = \zeta(0,t) \tag{46}$$
$$\zeta(L,t) = \eta(L,t) + 2\sqrt[4]{v(u_0(0))v(u(0,t))}\,U(t), \tag{47}$$

where $\mathcal{A}(u)$, $\mathcal{B}(u)$ are defined in (30) and (31), respectively. In order to remove the internal coupling terms in (45), the backstepping transformation is employed

$$\omega(x,t) \quad = \quad \xi(x,t) - \int_0^x K(x,s,t)\xi(s,t)\mathrm{d}s, \tag{48}$$

with $\omega(x,t) = [\omega_1(x,t), \omega_2(x,t)]^T$, for all $0 \le x \le L$, $t \ge 0$, where $K(x,s,t) = [K_{ij}(x,s,t)] \in \mathbb{R}^{2\times 2}$ is solution to equations (26) – (29). Differentiating (48) to time t and space x, system (43) – (47) is transformed to the decoupled PDE/ODE cascaded system

$$\dot{X}(t) = f(X(t), u(0,t)) \tag{49}$$
$$\partial_t u(0,t) = \sqrt[4]{\frac{v(u(0,t))}{v(u_0(0))}}\,\omega_1(0,t) \tag{50}$$
$$\partial_t \omega(x,t) = \mathcal{A}(u(0,t))\,\partial_x \omega(x,t) \tag{51}$$
$$\omega_2(0,t) = \omega_1(0,t) \tag{52}$$
$$\begin{aligned}\omega_1(L,t) = {} & \eta(L,t) + 2\sqrt[4]{v(u_0(0))v(u(0,t))}\,U(t) \\ & - \int_0^L K_{11}(L,s,t)\zeta(s,t)\mathrm{d}s \\ & - \int_0^L K_{12}(L,s,t)\eta(s,t)\mathrm{d}s,\end{aligned} \tag{53}$$

for $0 \le x \le L, t \ge 0$, if K_{11}, K_{12} satisfy (26) – (29).

Cascaded system (49) – (53) is transferred to system (43) – (47) by the inverse backstepping transformation

$$\xi(x,t) \quad = \quad \omega(x,t) + \int_0^x L(x,s,t)\omega(s,t)\mathrm{d}s, \tag{54}$$

where $L(x,s,t) = [L_{ij}(x,s,t)] \in \mathbb{R}^{2\times 2}$ is solution to the kernel equations

$$\begin{aligned} \partial_t L(x,s,t) - \partial_s L(x,s,t)\mathcal{A}(u) - \mathcal{A}(u)\partial_x L(x,s,t) &= \mathcal{B}(u)L(x,s,t) && (55)\\ L(x,x,t)\mathcal{A}(u) - \mathcal{A}(u)L(x,x,t) &= \mathcal{B}(u) && (56)\\ L_{11}(x,0,t) &= L_{12}(x,0,t) && (57)\\ L_{21}(x,0,t) &= L_{22}(x,0,t), && (58) \end{aligned}$$

where (55) – (58) is defined on $\{(x,s,t) : 0 \le s \le x \le L, t \ge 0\}$.

Let us redefine time as

$$\tau = \phi(t) = \int_0^t \sqrt{v(u(0,\sigma))}\mathrm{d}\sigma. \qquad (59)$$

Since $v(u(0,\sigma)) > 0$, for any $u(0,\sigma) \in \mathbb{R}$, the inverse function of $\phi(t)$ exists, namely, $t = \phi^{-1}(\tau)$. By (59), system (49) – (53) can be expressed as

$$\frac{dX(\phi^{-1}(\tau))}{d\tau} = \frac{f(X(\phi^{-1}(\tau)), u(0,\phi^{-1}(\tau)))}{\sqrt{v(u(0,\phi^{-1}(\tau)))}} \qquad (60)$$

$$\partial_\tau u(0,\phi^{-1}(\tau)) = \frac{\omega_1(0,\phi^{-1}(\tau))}{\sqrt[4]{v(u(0,\phi^{-1}(\tau)))v(u_0(0))}} \qquad (61)$$

$$\partial_\tau \omega_1(x,\phi^{-1}(\tau)) = \partial_x \omega_1(x,\phi^{-1}(\tau)) \qquad (62)$$

$$\partial_\tau \omega_2(x,\phi^{-1}(\tau)) = -\partial_x \omega_2(x,\phi^{-1}(\tau)) \qquad (63)$$

$$\omega_2(0,\phi^{-1}(\tau)) = \omega_1(0,\phi^{-1}(\tau)) \qquad (64)$$

$$\begin{aligned} \omega_1(L,\phi^{-1}(\tau)) &= \eta(L,\phi^{-1}(\tau)) + 2\sqrt[4]{v(u(0,0))v(u(0,\phi^{-1}(\tau)))}\,U(\phi^{-1}(\tau))\\ &\quad - \int_0^L K_{11}(L,s,\phi^{-1}(\tau))\zeta(s,\phi^{-1}(\tau))\mathrm{d}s\\ &\quad - \int_0^L K_{12}(L,s,\phi^{-1}(\tau))\eta(s,\phi^{-1}(\tau))\mathrm{d}s, \end{aligned} \qquad (65)$$

for $0 \le x \le L, \tau \ge 0$, if K_{11}, K_{12} satisfy (26) – (29).

Remark 3.1. From (59), one has

$$\phi^{'}(\phi^{-1}(\tau)) = \sqrt{v(u(0,\phi^{-1}(\tau)))}. \qquad (66)$$

Noting that (60) is achieved from (49) and (66).

Denote

$$\overline{X}(\tau) = X(\phi^{-1}(\tau)),\ \overline{U}(\tau) = U(\phi^{-1}(\tau)),\ \overline{u}(x,\tau) = u(x,\phi^{-1}(\tau)), \qquad (67)$$

$$\overline{\omega}_1(x,\tau) = \omega_1(x,\phi^{-1}(\tau)),\ \overline{\omega}_2(x,\tau) = \omega_2(x,\phi^{-1}(\tau)), \qquad (68)$$

system (60) – (65) is rewritten as

$$\dot{\overline{X}}(\tau) = \frac{f(\overline{X}(\tau), \overline{u}(0,\tau))}{\sqrt{v(\overline{u}(0,\tau))}} \tag{69}$$

$$\partial_\tau \overline{u}(0,\tau) = \frac{\overline{\omega}_1(0,\tau)}{\sqrt[4]{v(\overline{u}(0,\tau))v(\overline{u}(0,0))}} \tag{70}$$

$$\partial_\tau \overline{\omega}_1(x,\tau) = \partial_x \overline{\omega}_1(x,\tau) \tag{71}$$

$$\partial_\tau \overline{\omega}_2(x,\tau) = -\partial_x \overline{\omega}_2(x,\tau) \tag{72}$$

$$\overline{\omega}_2(0,\tau) = \overline{\omega}_1(0,\tau) \tag{73}$$

$$\begin{aligned} \overline{\omega}_1(L,\tau) = {} & \eta(L,\phi^{-1}(\tau)) + 2\sqrt[4]{v(\overline{u}(0,0))v(\overline{u}(0,\tau))}\,\overline{U}(\tau) \\ & - \int_0^L K_{11}(L,s,\phi^{-1}(\tau))\zeta(s,\phi^{-1}(\tau))\mathrm{d}s \\ & - \int_0^L K_{12}(L,s,\phi^{-1}(\tau))\eta(s,\phi^{-1}(\tau))\mathrm{d}s, \end{aligned} \tag{74}$$

for $0 \le x \le L, \tau \ge 0$, if K_{11}, K_{12} satisfy (26)–(29).

4. STABILITY ANALYSIS OF THE PROPOSED CONTROL LAW

4.1. Equivalent nominal controller for the target system's ODE

From Assumption 2.1, the control law $u(0,t) = \kappa(X(t))$ globally stabilizes system $\dot{X}(t) = f(X(t), u(0,t))$. Equivalently, the control law $u(0,\phi^{-1}(\tau)) = \kappa(X(\phi^{-1}(\tau)))$ globally stabilizes system $\frac{dX(\phi^{-1}(\tau))}{d\tau} = \frac{f(X(\phi^{-1}(\tau)),u(0,\phi^{-1}(\tau)))}{\sqrt{v(u(0,\phi^{-1}(\tau)))}}$. So the control law $\overline{u}(0,\tau) = \kappa(\overline{X}(\tau))$ globally stabilizes system (69).

It is easy to check that the feedback law

$$\mu(\chi) \quad = \quad \sqrt[4]{v(\overline{u}(0,\tau))v(\overline{u}(0,0))}\mu_1(\chi), \tag{75}$$

where $\chi = [\chi_1, \chi_2] \in \mathbb{R}^n \times \mathbb{R}$ and μ_1 is given by (16), is a nominal stabilizing controller for the ODE

$$\dot{\chi} \quad = \quad \varphi\left(\chi, \frac{1}{\sqrt[4]{v(\overline{u}(0,\tau))v(\overline{u}(0,0))}}\mu\right), \tag{76}$$

where φ is given by (10).

The predictor signal that compensates the PDE actuator dynamics for the target system (69)–(74) is defined by the following vector functions $p(x,\tau)$ and $q(x,\tau)$:

$$p(x,\tau) = \overline{Z}(\tau) + \int_0^x \varphi\left(p(y,\tau), \frac{\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(p_2(y,\tau))v(\overline{u}(0,0))}}\right)\mathrm{d}y, \tag{77}$$

where $p(x,\tau) = [p_1(x,\tau), p_2(x,\tau)]^T$ with the initial condition

$$p(x,0) = \overline{Z}(0) + \int_0^x \varphi\left(p(y,0), \frac{\overline{\omega}_1(y,0)}{\sqrt[4]{v(p_2(y,0))v(\overline{u}(0,0))}}\right)\mathrm{d}y, \tag{78}$$

and

$$\overline{Z}(\tau) = \begin{bmatrix} \overline{X}(\tau) \\ \overline{u}(0,\tau) \end{bmatrix},$$

$$\varphi\left(p(y,\tau), \frac{\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(p_2(y,\tau))v(\overline{u}(0,0))}}\right) = \begin{bmatrix} \frac{f(p_1(y,\tau),p_2(y,\tau))}{\sqrt{v(p_2(y,\tau))}} \\ \frac{\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(p_2(y,\tau))v(\overline{u}(0,0))}} \end{bmatrix},$$

and

$$q(x,\tau) = \overline{Z}(\tau) - \int_0^x \varphi\left(q(y,\tau), \frac{\overline{\omega}_2(y,\tau)}{\sqrt[4]{v(q_2(y,\tau))v(\overline{u}(0,0))}}\right)\mathrm{d}y, \tag{79}$$

where $q(x,\tau) = [q_1(x,\tau), q_2(x,\tau)]^T$, with the initial condition

$$q(x,0) = \overline{Z}(0) - \int_0^x \varphi\left(q(y,0), \frac{\overline{\omega}_2(y,0)}{\sqrt[4]{v(q_2(y,0))v(\overline{u}(0,0))}}\right)\mathrm{d}y, \tag{80}$$

where

$$\varphi\left(q(y,\tau), \frac{\overline{\omega}_2(y,\tau)}{\sqrt[4]{v(q_2(y,\tau))v(\overline{u}(0,0))}}\right) = \begin{bmatrix} \frac{f(q_1(y,\tau),q_2(y,\tau))}{\sqrt{v(q_2(y,\tau))}} \\ \frac{\overline{\omega}_2(y,\tau)}{\sqrt[4]{v(q_2(y,\tau))v(\overline{u}(0,0))}} \end{bmatrix}.$$

4.2. A second-step backstepping transformation towards stability analysis

A second-step backstepping transformation is designed in order to map (69) – (74) into the final target system whose stability will be established. We state the following Lemma.

Lemma 4.1. (Second-Step Backstepping Transform) Let

$$\varpi(x,\tau) = \overline{\omega}_1(x,\tau) - \mu(p(x,\tau)), \tag{81}$$

$$\lambda(x,\tau) = \overline{\omega}_2(x,\tau) - \mu(q(x,\tau)), \tag{82}$$

where μ is defined in (75), and $p(x,\tau), q(x,\tau)$ are given as (77), (79), respectively, and $\overline{U}(\tau)$ is

$$\begin{aligned} \overline{U}(\tau) = & -\frac{1}{2\sqrt[4]{v(\overline{u}(0,\tau))v(\overline{u}(0,0))}}(\eta(L,\phi^{-1}(\tau)) \\ & - \int_0^L (K_{11}(L,s,\phi^{-1}(\tau))\zeta(s,\phi^{-1}(\tau)) + K_{12}(L,s,\phi^{-1}(\tau))\eta(s,\phi^{-1}(\tau)))\mathrm{d}s \\ & - \mu(p(L,\tau))), \end{aligned} \tag{83}$$

map system (69) – (74) into the target system

$$\dot{\overline{Z}}(\tau) = \varphi\left(\overline{Z}(\tau), \frac{\varpi(0,\tau)+\mu(\overline{Z}(\tau))}{\sqrt[4]{v(\overline{u}(0,\tau))v(\overline{u}(0,0))}}\right) \tag{84}$$

$$\partial_\tau \varpi(x,\tau) = \partial_x \varpi(x,\tau) \tag{85}$$

$$\partial_\tau \lambda(x,\tau) = -\partial_x \lambda(x,\tau) \tag{86}$$

$$\lambda(0,\tau) = \varpi(0,\tau) \tag{87}$$

$$\varpi(L,\tau) = 0. \tag{88}$$

The proof of Lemma 4.1 is provided in Appendix A.1.

Remark 4.2. Using (16), (32), (33), (42), (75), it can be deduced that the control law (83) is just (19). In addition, from (10), (32), (33), (48), it is not difficult to find that $p(x,\phi(t))$ defined as (77) is just $[p_1(x,\phi(t)), p_2(x,\phi(t))]^T$ given as (20), (21).

Inverse Backstepping Transforms: Define the vector functions $\pi(x,\tau)$ and $\iota(x,\tau)$ as

$$\pi(x,\tau) = \overline{Z}(\tau) + \int_0^x \varphi\left(\pi(y,\tau), \frac{\varpi(y,\tau)+\mu(\pi(y,\tau))}{\sqrt[4]{v(\pi_2(y,\tau))v(\overline{u}(0,0))}}\right) \mathrm{d}y, \tag{89}$$

where $\pi(x,\tau) = [\pi_1(x,\tau), \pi_2(x,\tau)]^T$, with the initial condition

$$\pi(x,0) = \overline{Z}(0) + \int_0^x \varphi\left(\pi(y,0), \frac{\varpi(y,0)+\mu(\pi(y,0))}{\sqrt[4]{v(\pi_2(y,0))v(\overline{u}(0,0))}}\right) \mathrm{d}y, \tag{90}$$

and

$$\iota(x,\tau) = \overline{Z}(\tau) - \int_0^x \varphi\left(\iota(y,\tau), \frac{\lambda(y,\tau)+\mu(\iota(y,\tau))}{\sqrt[4]{v(\iota_2(y,\tau))v(\overline{u}(0,0))}}\right) \mathrm{d}y, \tag{91}$$

where $\iota(x,\tau) = [\iota_1(x,\tau), \iota_2(x,\tau)]^T$ with the initial condition

$$\iota(x,0) = \overline{Z}(0) - \int_0^x \varphi\left(\iota(y,0), \frac{\lambda(y,0)+\mu(\iota(y,0))}{\sqrt[4]{v(\iota_2(y,0))v(\overline{u}(0,0))}}\right) \mathrm{d}y, \tag{92}$$

where ϖ, λ, μ are defined in (81), (82), (75), respectively.

The inverse backstepping transformations of ϖ, λ are defined as

$$\overline{\omega}_1(x,\tau) = \varpi(x,\tau) + \mu(\pi(x,\tau)), \tag{93}$$

$$\overline{\omega}_2(x,\tau) = \lambda(x,\tau) + \mu(\iota(x,\tau)), \tag{94}$$

where $\pi(x,\tau)$, $\iota(x,\tau)$, $0 \le x \le L$, $\tau \ge 0$, are given as (89), (91), respectively.

The inverse backstepping transformation (93), (94), and the control law (83), transform the target system (84) – (88) into system (69) – (74), it can be deduced from straightforward computations.

4.3. Some Lemmas and A Theorem

Under Assumptions 2.1 – 2.5, and condition (6), we prove stability of the closed-loop system (1) – (4) together with (19) – (21) by using the following Lemmas (Lemmas 4.4 – 4.10).

Theorem 4.3. Consider system (1) – (4) with (19) – (21), if Assumptions 2.1 – 2.5 and condition (6) hold, for any initial condition $u_0(x) \in C_1[0, L]$, $u_1(x) \in C[0, L]$, which is compatible with the feedback law (19) – (21) and such that $u_x(0, 0) = 0$, then the closed-loop system has a unique solution $X(t) \in C_1([0, \infty), \mathbb{R}^n)$, $(u(\cdot, t), u_t(\cdot, t)) \in C([0, \infty), C_1[0, L] \times C[0, L])$, moreover, there is a $\mathcal{KL}$ function $\overline{\beta}$ such that

$$\Omega(t) \quad \leq \quad \overline{\beta}(\Omega(0), t) \tag{95}$$

where

$$\Omega(t) = |X(t)| + \|u(t)\|_\infty + \|u_t(t)\|_\infty + \|u_x(t)\|_\infty,$$

for $t \geq 0$.

Lemma 4.4. (Stability Estimate for Target System) Consider system (84) – (88), if Assumption 2.1 and condition (6) hold, there is a class $\mathcal{KL}$ function β, such that

$$\begin{aligned} &|\overline{Z}(\tau)| + \|\varpi(\tau)\|_\infty + \|\lambda(\tau)\|_\infty \\ &\leq \beta(|\overline{Z}(0)| + \|\varpi(0)\|_\infty + \|\lambda(0)\|_\infty, \tau) \end{aligned} \tag{96}$$

for all $\tau \geq 0$.

The proof of Lemma 4.4 is provided in Appendix A.2.

Lemma 4.5. (Bound on Forward Predictor) Under Assumption 2.3 and condition (6), there exists a class $\mathcal{K}_\infty$ function ρ_1 such that the following holds:

$$\sup_{0 \leq x \leq L} |p(x, \tau)| \leq \rho_1(|\overline{Z}(\tau)| + \|\overline{\omega}_1(\tau)\|_\infty). \tag{97}$$

The proof of Lemma 4.5 is provided in Appendix A.3.

Lemma 4.6. (Bound on Backward Predictor) Under Assumption 2.3 and condition (6), there exists a class $\mathcal{K}_\infty$ function ρ_2 such that the following holds:

$$\sup_{0 \leq x \leq L} |q(x, \tau)| \leq \rho_2(|\overline{Z}(\tau)| + \|\overline{\omega}_2(\tau)\|_\infty). \tag{98}$$

The proof of Lemma 4.6 is provided in Appendix A.4.

Lemma 4.7. (Bound on Extended Forward State Predictor) Under Assumption 2.1 and condition (6), there exists a class $\mathcal{K}_\infty$ function ρ_3 such that the following holds

$$\sup_{0 \leq x \leq L} |\pi(x, \tau)| \leq \rho_3(|\overline{Z}(\tau)| + \|\varpi(\tau)\|_\infty). \tag{99}$$

The proof of Lemma 4.7 is provided in Appendix A.5.

Lemma 4.8. (Bound on Extended Backward State Predictor) Under Assumption 2.5 and condition (6), there exists a class $\mathcal{K}_\infty$ function ρ_4 such that the following holds

$$\sup_{0\le x\le L} |\lambda(x,\tau)| \le \rho_4(|\overline{Z}(\tau)| + \|\lambda(\tau)\|_\infty). \tag{100}$$

The proof of Lemma 4.8 is provided in Appendix A.6.

Lemma 4.9. Consider system (84) – (88), and output maps are (93), (94), if Assumptions 2.1, and 2.5 and condition (6) hold, there is a class $\mathcal{K}_\infty$ function γ_2 such that

$$\begin{aligned} &|\overline{Z}(\tau)| + \|\overline{\omega}_1(\tau)\|_\infty + \|\overline{\omega}_2(\tau)\|_\infty \\ &\le \gamma_2(|\overline{Z}(\tau)| + \|\varpi(\tau)\|_\infty + \|\lambda(\tau)\|_\infty). \end{aligned} \tag{101}$$

The proof is omitted since it is easy to achieve.

Lemma 4.10. Consider system (69) – (74), and output maps are (81), (82), if Assumptions 2.1, and 2.3 and condition (6) hold, then there is a class $\mathcal{K}_\infty$ function γ_3 such that

$$\begin{aligned} &|\overline{Z}(\tau)| + \|\varpi(\tau)\|_\infty + \|\lambda(\tau)\|_\infty \\ &\le \gamma_3(|\overline{Z}(\tau)| + \|\overline{\omega}_1(\tau)\|_\infty + \|\overline{\omega}_2(\tau)\|_\infty). \end{aligned} \tag{102}$$

The proof is omitted since it is easy to achieve.

P r o o f of Theorem 1. Combining Lemma 4.4, Lemma 4.9, Lemma 4.10, one has

$$\begin{aligned} &|\overline{Z}(\tau)| + \|\overline{\omega}_1(\tau)\|_\infty + \|\overline{\omega}_2(\tau)\|_\infty \\ &\le \gamma_2(|\overline{Z}(\tau)| + \|\varpi(\tau)\|_\infty + \|\lambda(\tau)\|_\infty) \\ &\le \gamma_2(\beta(|\overline{Z}(0)| + \|\varpi(0)\|_\infty + \|\lambda(0)\|_\infty, \tau)) \\ &\le \gamma_2(\beta(\gamma_3(|\overline{Z}(0)| + \|\overline{\omega}_1(0)\|_\infty + \|\overline{\omega}_2(0)\|_\infty), \tau)), \end{aligned} \tag{103}$$

for all $\tau \ge 0$. By (67), (68), and (59), from (103), we get

$$\begin{aligned} &|Z(t)| + \|\omega_1(t)\|_\infty + \|\omega_2(t)\|_\infty \\ &\le \gamma_2(\beta(\gamma_3(|Z(0)| + \|\omega_1(0)\|_\infty + \|\omega_2(0)\|_\infty), t)) \end{aligned} \tag{104}$$

for all $t \ge 0$. With the help of (48), (54), from (104) we have

$$\begin{aligned} &|Z(t)| + \|\xi(t)\|_\infty \\ &\le (1+\overline{L})\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|Z(0)| + \|\xi(0)\|_\infty)), t)), \end{aligned} \tag{105}$$

for all $t \ge 0$, and $\overline{L} = \sup_{(x,y,t)\in[0,L]\times[0,L]\times[0,\infty)} |L(x,y,t)|$, $\overline{K} = \sup_{(x,y,t)\in[0,L]\times[0,L]\times[0,\infty)} |K(x,y,t)|$.

Using (32), (33), (34), (35), (42), (105), we derive the estimates below

$$\begin{aligned}
&|X(t)| + |u(0,t)| + \|\partial_t u(t)\|_\infty + \|\partial_x u(t)\|_\infty \\
&\le \sqrt{2}|Z(t)| + \frac{\sqrt{2}}{2}\left(1+\frac{1}{\sqrt{\varrho_1}}\right)\sqrt[4]{\frac{\varrho_2}{\varrho_1}}\|\xi(t)\|_\infty \\
&\le \max\left\{\sqrt{2}, \frac{\sqrt{2}}{2}\left(1+\frac{1}{\sqrt{\varrho_1}}\right)\sqrt[4]{\frac{\varrho_2}{\varrho_1}}\right\}(|Z(t)| + \|\xi(t)\|_\infty) \\
&\le (1+\overline{L})\max\left\{\sqrt{2}, \frac{\sqrt{2}}{2}\left(1+\frac{1}{\sqrt{\varrho_1}}\right)\sqrt[4]{\frac{\varrho_2}{\varrho_1}}\right\} \\
&\times \gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K})(|Z(0)| + \|\xi(0)\|_\infty)), t)) \\
&\le (1+\overline{L})\max\left\{\sqrt{2}, \frac{\sqrt{2}}{2}\left(1+\frac{1}{\sqrt{\varrho_1}}\right)\sqrt[4]{\frac{\varrho_2}{\varrho_1}}\right\} \\
&\times \gamma_2(\beta(\gamma_3(\sqrt{2}\left(1+\overline{K}\right)(|X(0)| + \|u(0)\|_\infty \\
&+ 2\sqrt[4]{\frac{\varrho_2}{\varrho_1}}(1+\sqrt{\varrho_2})(\|\partial_t u(0)\|_\infty + \|\partial_x u(0)\|_\infty)), t)). \qquad (106)
\end{aligned}$$

In addition,

$$u(x,t) = u(0,t) + \int_0^x u_s(s,t)\,\mathrm{d}s, \qquad (107)$$

so, it holds

$$\begin{aligned}
&|X(t)| + \|u(t)\|_\infty + \|\partial_t u(t)\|_\infty + \|\partial_x u(t)\|_\infty \\
&\le |X(t)| + |u(0,t)| + \|\partial_t u(t)\|_\infty + 2\|\partial_x u(t)\|_\infty. \qquad (108)
\end{aligned}$$

Hence, defining the class $\mathcal{K}_\infty$ function

$$\begin{aligned}
&\overline{\beta}(s,t) = 2(1+\overline{L})\max\{\sqrt{2}, \tfrac{\sqrt{2}}{2}\left(1+\tfrac{1}{\sqrt{\varrho_1}}\right)\sqrt[4]{\tfrac{\varrho_2}{\varrho_1}}\} \\
&\times\gamma_2(\beta(\gamma_3(\sqrt{2}(1+\overline{K}) + 2\sqrt[4]{\tfrac{\varrho_2}{\varrho_1}}(1+\sqrt{\varrho_2})s), t)),
\end{aligned}$$

we obtain (95).

It can be deduced that under Assumptions 2.1 − 2.5, and condition (6), for $u_0(x) \in C_1[0,L]$, $u_1(x) \in C[0,L]$, which is compatible with the feedback law (19) − (21), the closed-loop system has a unique solution $X(t) \in C_1([0,\infty), R^n)$, $(u(\cdot,t), u_t(\cdot,t)) \in C([0,\infty), C_1[0,L] \times C[0,L])$. □

5. SIMULATION

For a third-order system

$$\begin{aligned}
\dot{X}_1(t) &= X_2(t) + X_3^2(t) \qquad (109)\\
\dot{X}_2(t) &= X_3(t) + X_3(t)u(0,t) \qquad (110)\\
\dot{X}_3(t) &= u(0,t), \qquad (111)
\end{aligned}$$

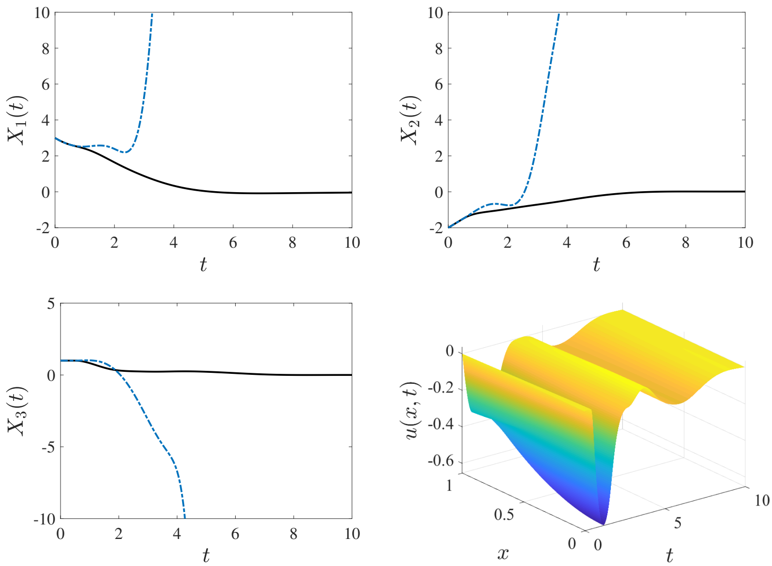

Fig. 2. Responses of the ODE state X and PDE state $u(x,t)$ under the proposed control (solid line) and the nominal control (dotted line).

the nominal feedback law for (109) – (111) is

$$\begin{aligned} u(0,t) = &-X_1(t) - 3X_2(t) - 3X_3(t) - \frac{3}{8}X_2(t)^2 \\ &+ \frac{3}{4}X_2(t)\left(-X_1(t) - 2X_2(t) + \frac{1}{2}X_3(t) + \frac{X_2(t)X_3(t)}{2}\right. \\ &\left.+\frac{5}{8}X_3(t)^2 - \frac{1}{4}X_3(t)^3 - \frac{3}{8}\left(X_2(t) - \frac{X_3(t)^2}{2}\right)^2\right). \end{aligned} \tag{112}$$

Now system (109) – (111) cascading with (2) – (4) with

$$v(u(0,t)) = \left(1 + \frac{1}{1+u(0,t)^2}\right)^2, \tag{113}$$

is controlled by (19) – (21).

In simulation, $L = 1$, $c_1 = 5$, $X_1(0) = 3$, $X_2(0) = -2$, $X_3(0) = 1$ and $u_0(x) = 0$, $u_1(x) = 0$ for $x \in [0,1]$, responses of the ODE states X_1, X_2, X_3 and wave PDE state $u(x,t)$ under the proposed control are given in Figure 2. Response of the predictor control (19) – (21), and the uncompensated control (112) are given in Figure 3. The proposed control stabilizes the cascaded system while the uncompensated control (112) leads to instability.

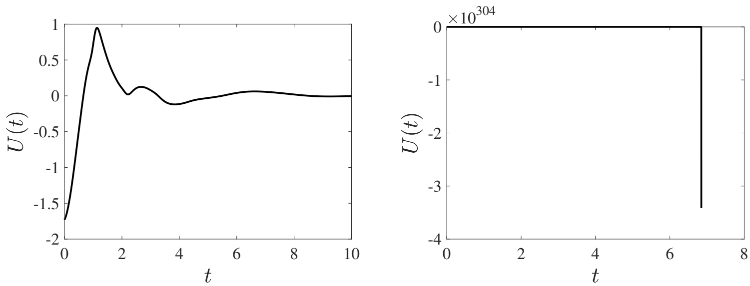

Fig. 3. Responses of the proposed control (left) and the nominal control (right).

6. CONCLUSION

Predictor control is investigated for wave PDE/nonlinear ODE cascaded system with boundary value-dependent propagation speed. The controller design and stability analysis are based on a two-step backstepping transformation and introducing a new time variable. A novel two-step backstepping transformation is employed to derive a target system whose stability is established using Lyapunov arguments. The resulting boundary controller is a predictor-feedback control law, which compensates the wave actuator dynamics and guarantees globally asymptotic stability of the closed-loop system.

A. PROOF OF THE LEMMAS

A.1. Proof of Lemma 4.1

First, from (77), $p(0,\tau)=\overline{Z}(\tau)$ and using (81), (69), we derive (84). Second, from (81), (82), (73), it is easy to deduce (87). In addition, with the help (74), (81), (75), it can be deduced (88). Finally, we will prove (85), and (86). The transformation (77) can be expressed as follows:

$$p_1(x,\tau)=\overline{X}(\tau)+\int_0^x \frac{f(p_1(y,\tau),p_2(y,\tau))}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y, \tag{A.1}$$

$$p_2(x,\tau)=\overline{u}(0,\tau)+\int_0^x \frac{\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(\overline{u}(0,0))v(p_2(y,\tau))}}\mathrm{d}y. \tag{A.2}$$

Differentiating (A.1) with respect to τ and x, we have

$$\begin{aligned}
&\partial_\tau p_1(x,\tau)\\
&=\frac{f(\overline{X}(\tau),\overline{u}(0,\tau))}{\sqrt{v(\overline{u}(0,\tau))}}+\int_0^x \frac{\partial_{p_1} f(p_1(y,\tau),p_2(y,\tau))\partial_\tau p_1(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\
&+\int_0^x \frac{\partial_{p_2} f(p_1(y,\tau),p_2(y,\tau))\partial_\tau p_2(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\
&-\int_0^x \frac{f(p_1(y,\tau),p_2(y,\tau))\partial_{p_2} v(p_2(y,\tau))\partial_\tau p_2(y,\tau)}{2(v(p_2(y,\tau)))^{3/2}}\mathrm{d}y,
\end{aligned} \tag{A.3}$$

and

$$\begin{aligned}&\partial_x p_1(x,\tau)\\&=\frac{f(X(\tau),\overline{u}(0,\tau))}{\sqrt{v(\overline{u}(0,\tau))}}+\int_0^x\frac{\partial_{p_1}f(p_1(y,\tau),p_2(y,\tau))\partial_y p_1(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\&+\int_0^x\frac{\partial_{p_2}f(p_1(y,\tau),p_2(y,\tau))\partial_y p_2(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\&-\int_0^x\frac{f(p_1(y,\tau),p_2(y,\tau))\partial_{p_2}v(p_2(y,\tau))\partial_y p_2(y,\tau)}{2(v(p_2(y,\tau)))^{3/2}}\mathrm{d}y.\end{aligned}\tag{A.4}$$

Defining

$$H_1(x,\tau)=\partial_\tau p_1(x,\tau)-\partial_x p_1(x,\tau),\tag{A.5}$$

$$H_2(x,\tau)=\partial_\tau p_2(x,\tau)-\partial_x p_2(x,\tau),\tag{A.6}$$

and combining (A.3) and (A.4), we arrive at

$$\begin{aligned}&H_1(x,\tau)\\&=\int_0^x\frac{\partial_{p_1}f(p_1(y,\tau),p_2(y,\tau))H_1(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\&+\int_0^x\frac{\partial_{p_2}f(p_1(y,\tau),p_2(y,\tau))H_2(y,\tau)}{\sqrt{v(p_2(y,\tau))}}\mathrm{d}y\\&-\int_0^x\frac{f(p_1(y,\tau),p_2(y,\tau))\partial_{p_2}v(p_2(y,\tau))H_2(y,\tau)}{2(v(p_2(y,\tau)))^{3/2}}\mathrm{d}y.\end{aligned}\tag{A.7}$$

Differentiating (A.2) with respect to τ, one has

$$\begin{aligned}&\partial_\tau p_2(x,\tau)\\&=\partial_\tau\overline{u}(0,\tau)+\int_0^x\frac{\partial_\tau\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(\overline{u}(0,0))v(p_2(y,\tau))}}\mathrm{d}y\\&-\int_0^x\frac{\overline{\omega}_1(y,\tau)\partial_{p_2}v(p_2(y,\tau))\partial_\tau p_2(y,\tau)}{4\sqrt[4]{v(\overline{u}(0,0))(v(p_2(y,\tau)))^5}}\mathrm{d}y.\end{aligned}\tag{A.8}$$

Differentiating (A.2) with respect to x, it holds

$$\begin{aligned}\partial_x p_2(x,\tau)&=\frac{\overline{\omega}_1(x,\tau)}{\sqrt[4]{v(\overline{u}(0,0))v(p_2(x,\tau))}}\\&=\frac{\overline{\omega}_1(0,\tau)}{\sqrt[4]{v(\overline{u}(0,0))v(\overline{u}(0,\tau))}}+\int_0^x\frac{\partial_y\overline{\omega}_1(y,\tau)}{\sqrt[4]{v(\overline{u}(0,0))v(p_2(y,\tau))}}\mathrm{d}y\\&-\int_0^x\frac{\overline{\omega}_1(y,\tau)\partial_{p_2}v(p_2(y,\tau))\partial_y p_2(y,\tau)}{\sqrt[4]{v(\overline{u}(0,0))(v(p_2(y,\tau)))^5}}\mathrm{d}y.\end{aligned}\tag{A.9}$$

Combining (A.8) and (A.9), and using (70), we get

$$H_2(x,\tau)=-\int_0^x\frac{\omega_1(y,\tau)\partial_{p_2}v(p_2(y,\tau))H_2(y,\tau)}{\sqrt[4]{v(\overline{u}(0,0))(v(p_2(y,\tau)))^5}}\mathrm{d}y.\tag{A.10}$$

Differentiating (A.7) with respect to x, the following ODE in x is deduced

$$\begin{aligned}\partial_x H_1(x,\tau) = {} & \frac{\partial_{p_1} f(p_1(x,\tau),p_2(x,\tau))H_1(x,\tau)}{\sqrt{v(p_2(x,\tau))}} \\ & + \frac{\partial_{p_2} f(p_1(x,\tau),p_2(x,\tau))H_2(x,\tau)}{\sqrt{v(p_2(x,\tau))}} \\ & - \frac{f(p_1(x,\tau),p_2(x,\tau))\partial_{p_2} v(p_2(x,\tau))H_2(x,\tau)}{2(v(p_2(x,\tau)))^{3/2}}, \end{aligned} \tag{A.11}$$

and

$$H_1(0,\tau) = 0. \tag{A.12}$$

Differentiating (A.10) with respect to x, the ODE in x is acquired

$$\partial_x H_2(x,\tau) = -\frac{\overline{\omega}_1(x,\tau)\partial_{p_2} v(p_2(x,\tau))H_2(x,\tau)}{\sqrt[4]{v(u(0,0))(v(p_2(x,\tau)))^5}}, \tag{A.13}$$

and

$$H_2(0,\tau) = 0. \tag{A.14}$$

From (A.11), (A.12), and (A.13), (A.14), it is easy to deduce

$$H_1(x,\tau) = 0, \;\; H_2(x,\tau) = 0, \tag{A.15}$$

for all $x \in [0,L], t \geq 0$. Knowing (A.15), it is clear that

$$\partial_\tau p(x,\tau) = \partial_x p(x,\tau). \tag{A.16}$$

Taking the time and the spatial derivative of the backstepping transformation (81), and using (A.16), relation (85) is deduced. Relation (86) can be derived similarly. □

A.2. Proof of Lemma 4.3

Let us introduce a new variable $z(x,\tau), x \in [-L,L]$ such that

$$z(x,\tau) = \begin{cases} \varpi(x,\tau), & \text{for all } x \in [0,L], \\ \lambda(-x,\tau), & \text{for all } x \in [-L,0]. \end{cases} \tag{B.1}$$

From (85), (86) and (88), we get

$$\partial_\tau z(x,\tau) = \partial_x z(x,\tau), \tag{B.2}$$

for all $x \in [-L,L]$, and $z(L,\tau) = 0$. Now, defining the functional $\Gamma_{g,n}(\tau)$ as follows

$$\Gamma_{g,n}(\tau) = \int_{-L}^{L} e^{2ng(L+x)} z(x,\tau)^{2n} \mathrm{d}x, \tag{B.3}$$

where $g > 0$ and n is a positive integer, and using integration by parts, the derivative of $\Gamma_{g,\,n}(\tau)$ is given by

$$\begin{aligned}&\dot{\Gamma}_{g,n}(\tau)\\&= \int_{-L}^{L} 2ne^{2ng(L+x)} z(x,\tau)^{2n-1} \partial_\tau z(x,\tau)\,\mathrm{d}x\\&\le -2ng\int_{-L}^{L} e^{2ng(L+x)} z(x,\tau)^{2n}\mathrm{d}x.\end{aligned} \tag{B.4}$$

From (B.3), we derive the following estimate

$$\int_{-L}^{L} z(x,\tau)^{2n} dx \le \Gamma_{g,\,n}(\tau) \le e^{4ngL}\int_{-L}^{L} z(x,\tau)^{2n}\mathrm{d}x. \tag{B.5}$$

Integrating (B.4) and using (B.5), we obtain

$$\int_{-L}^{L} z(x,\tau)^{2n} dx \le e^{-2ng(\tau-s)} e^{4ngL}\int_{-L}^{L} z(x,s)^{2n}\mathrm{d}x. \tag{B.6}$$

Further, it can be established that

$$\begin{aligned}&\left(\int_{-L}^{L} z(x,\tau)^{2n} dx\right)^{\frac{1}{2n}}\\&\le e^{-g(\tau-s)} e^{2gL}\left(\int_{-L}^{L} z(x,s)^{2n}\mathrm{d}x\right)^{\frac{1}{2n}}.\end{aligned} \tag{B.7}$$

Taking the limit of (B.7) as n goes to infinity, the following inequality holds

$$\|z(\tau)\|_{\infty,1} \le e^{-g(\tau-s)} e^{2gL}\|z(s)\|_{\infty,1}, \tag{B.8}$$

for all $\tau \ge s \ge 0$. Using (B.1), from (B.8), it can be deduced that

$$\begin{aligned}&\frac{1}{2}(\|\varpi(\tau)\|_\infty + \|\lambda(\tau)\|_\infty)\\&\le \|z(\tau)\|_{\infty,1}\\&\le e^{-g(\tau-s)} e^{2gL}(\|\varpi(s)\|_\infty + \|\lambda(s)\|_\infty),\end{aligned} \tag{B.9}$$

for all $\tau \ge s \ge 0$. Noting that $\varpi(0,s) = \lambda(0,s)$, from (B.9), we get

$$\sup_{s\in[0,\tau]} |\varpi(0,s)| \le e^{2gL}(\|\varpi(0)\|_\infty + \|\lambda(0)\|_\infty). \tag{B.10}$$

Under Assumption 2.1, there exist a class $\mathcal{KL}$ function β_1 and a class $\mathcal{K}_\infty$ function γ_1, such that the solutions to (84) satisfy

$$|\overline{Z}(\tau)| \le \beta_1(|\overline{Z}(s)|, \tau - s) \quad +\gamma_1\left(\sup_{\sigma\in[s,\tau]}\left(\frac{1}{\sqrt[4]{v(\overline{u}(0,\sigma))v(\overline{u}(0,0))}}\,|\varpi(0,\sigma)|\right)\right), \tag{B.11}$$

for all $\tau \geq s \geq 0$, using (6), we get

$$|\overline{Z}(\tau)| \leq \beta_1(|\overline{Z}(s)|, \tau - s) + \gamma_1 \left(\frac{1}{\sqrt{\varrho_1}} \sup_{\sigma \in [s,\tau]} |\varpi(0,\sigma)| \right), \tag{B.12}$$

for all $\tau \geq s \geq 0$. Finally, combining (B.12), (B.10), one has

$$|\overline{Z}(\tau)| \leq \beta_1(|\overline{Z}(0)|, \tau) + \gamma_1 \left(\frac{e^{2gL}}{\sqrt{\varrho_1}} (\|\varpi(0)\|_\infty + \|\lambda(0)\|_\infty) \right) \tag{B.13}$$

for all $\tau \geq 0$. Now defining

$$\beta(s,\tau) = \beta_1(s,\tau) + \gamma_1 \left(\frac{e^{2gL}}{\sqrt{\varrho_1}} s \right) + 2e^{2gL} s,$$

with $g > 0$, from (B.9), (B.13), we deduce (96). □

A.3. Proof of Lemma 4.4

Differentiating (77) with respect to x, we get

$$\partial_x p(x,\tau) = \varphi \left(p(x,\tau), \frac{\overline{\omega}_1(x,\tau)}{\sqrt[4]{v(p_2(x,\tau))v(\overline{u}(0,0))}} \right), \tag{C.1}$$

$$p(0,\tau) = \overline{Z}(\tau), \tag{C.2}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. With the help of (13), we obtain the following relation:

$$\frac{\partial R_1(p(x,\tau))}{\partial p} \varphi \left(p(x,\tau), \frac{\overline{\omega}_1(x,\tau)}{\sqrt[4]{v(p_2(x,\tau))v(\overline{u}(0,0))}} \right)$$
$$\leq R_1(p(x,\tau)) + \alpha_3 \left(\frac{|\overline{\omega}_1(x,\tau)|}{\sqrt[4]{v(p_2(x,\tau))v(\overline{u}(0,0))}} \right). \tag{C.3}$$

Using (C.1), from (C.3), with the help of (6), we have

$$\frac{\partial R_1(p(x,\tau))}{\partial x} \leq R_1(p(x,\tau)) + \alpha_3 \left(\frac{|\overline{\omega}_1(x,\tau)|}{\sqrt{\varrho_1}} \right). \tag{C.4}$$

Using (14), from (C.4), we deduced that

$$\alpha_1(|p(x,\tau)|) \leq R_1(p(x,\tau))$$
$$\leq e^x \alpha_2(|\overline{Z}(\tau)|) + (e^x - 1)\alpha_3 \left(\frac{1}{\sqrt{\varrho_1}} \|\overline{\omega}_1(\tau)\|_\infty \right), \tag{C.5}$$

for all $0 \leq x \leq L, \tau \geq 0$. Defining the function

$$\rho_1(s) = \alpha_1^{-1} \left(e^L \alpha_2(s) + (e^L - 1)\alpha_3 \left(\frac{1}{\sqrt{\varrho_1}} s \right) \right),$$

we obtain (97), which completes the proof. □

A.4. Proof of Lemma 4.5

Differentiating (79) with respect to x, we get

$$\partial_x q(x,\tau) = -\varphi\left(q(x,\tau), \frac{\overline{\omega}_2(x,\tau)}{\sqrt[4]{v(q_2(x,\tau))v(\overline{u}(0,0))}}\right), \tag{D.1}$$

$$q(0,\tau) = \overline{Z}(\tau), \tag{D.2}$$

for all $0 \le x \le L$, $\tau \ge 0$. With the help of (15), we obtain the following relation:

$$\begin{aligned} &-\frac{\partial R_2(q(x,\tau))}{\partial q}\varphi\left(q(x,\tau), \frac{\overline{\omega}_2(x,\tau)}{\sqrt[4]{v(q_2(x,\tau))v(\overline{u}(0,0))}}\right) \\ &\le R_2(q(x,\tau)) + \alpha_6\left(\left|\frac{\overline{\omega}_2(x,\tau)}{\sqrt[4]{v(q_2(x,\tau))v(\overline{u}(0,0))}}\right|\right). \end{aligned} \tag{D.3}$$

Using (D.1), from (D.3), we have

$$\frac{\partial R_2(q(x,\tau))}{\partial x} \le R_2(q(x,\tau)) + \alpha_6\left(\left|\frac{\overline{\omega}_2(x,\tau)}{\sqrt[4]{v(q_2(x,\tau))v(\overline{u}(0,0))}}\right|\right). \tag{D.4}$$

With the help of (6), inequality (D.4) holds

$$\frac{\partial R_2(q(x,\tau))}{\partial x} \le R_2(q(x,\tau)) + \alpha_6\left(\frac{1}{\sqrt{\varrho_1}}\left|\overline{\omega}_2(x,\tau)\right|\right), \tag{D.5}$$

so we have

$$\begin{aligned} &R_2(q(x,\tau)) \\ &\le e^x R_2(\overline{Z}(\tau)) + (e^x - 1)\alpha_6\left(\frac{1}{\sqrt{\varrho_1}}\left|\overline{\omega}_2(x,\tau)\right|\right). \end{aligned} \tag{D.6}$$

Using (14), from (D.6), we arrive at

$$\begin{aligned} \alpha_4(|q(x,\tau)|) &\le R_2(q(x,\tau)) \\ &\le e^x\alpha_5(|\overline{Z}(\tau)|) + (e^x - 1)\alpha_6\left(\frac{1}{\sqrt{\varrho_1}}\|\overline{\omega}_2(\tau)\|_\infty\right), \end{aligned} \tag{D.7}$$

for all $0 \le x \le L, \tau \ge 0$. Defining

$$\rho_2(s) = \alpha_4^{-1}\left(e^L\alpha_5(s) + (e^L - 1)\alpha_6\left(\frac{1}{\sqrt{\varrho_1}}s\right)\right),$$

we derive (98), which completes the proof. □

A.5. Proof of Lemma 4.6

Under Assumption 1, it can be deduced that the control law μ_1 given in (16) is such that the following system

$$\dot{Z} = \varphi(Z, \mu_1(Z) + \varsigma) = \begin{bmatrix} f(X, \xi) \\ \mu_1(Z) + \varsigma \end{bmatrix} \tag{E.1}$$

with $Z = [X^T, \xi]^T$ is input-to-state stable with respect to ς. Thus, there exist a smooth positive definite function R_4 and class $\mathcal{K}_\infty$ functions α_{10}, α_{11}, α_{12} such that

$$\alpha_{10}(|Z|) \leq R_4(Z) \leq \alpha_{11}(|Z|), \tag{E.2}$$

$$\frac{\partial R_4(Z)}{\partial Z}\varphi(Z, \mu_1(Z) + \varsigma) \leq R_4(Z) + \alpha_{12}(|\varsigma|), \tag{E.3}$$

for $Z \in R^{n+1}$ and $\varsigma \in R$.

Differentiating (89) with respect to x, we get

$$\partial_x \pi(x, \tau) = \varphi\left(\pi(x, \tau), \frac{\mu(\pi(x, \tau)) + \varpi(x, \tau)}{\sqrt[4]{v(\pi_2(x, \tau))v(\overline{u}(0, 0))}}\right), \tag{E.4}$$

$$\pi(0, \tau) = \overline{Z}(\tau), \tag{E.5}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. From (E.3), we can deduce that

$$\begin{aligned} &\frac{\partial R_4(\pi(x, \tau))}{\partial \pi}\varphi\left(\pi(x, \tau), \frac{\mu(\pi(x, \tau)) + \varpi(x, \tau)}{\sqrt[4]{v(\pi_2(x, \tau))v(\overline{u}(0, 0))}}\right) \\ &\leq R_4(\pi(x, \tau)) + \alpha_{12}\left(\left|\frac{\varpi(x, \tau)}{\sqrt[4]{v(\pi_2(x, \tau))v(\overline{u}(0, 0))}}\right|\right), \end{aligned} \tag{E.6}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. With the help of (6), (E.4), we have

$$\frac{\partial R_4(\pi(x, \tau))}{\partial x} \leq R_4(\pi(x, \tau)) + \alpha_{12}\left(\left|\frac{\varpi(x, \tau)}{\sqrt{\varrho_1}}\right|\right), \tag{E.7}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. Hence, the following relation holds:

$$R_4(\pi(x, \tau)) \leq e^L R_4(\pi(0, \tau)) + (e^L - 1) \sup_{0 \leq x \leq L} \alpha_{12}\left(\frac{1}{\sqrt{\varrho_1}}|\varpi(x, \tau)|\right), \tag{E.8}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. Using (E.2), from (E.8), we get

$$|\pi(x, \tau)| \leq \alpha_{10}^{-1}\left(e^L \alpha_{11}(|\overline{Z}(\tau)|) + (e^L - 1)\alpha_{12}\left(\frac{1}{\sqrt{\varrho_1}}\|\varpi(\tau)\|_\infty\right)\right), \tag{E.9}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. Defining

$$\rho_3(s) = \alpha_{10}^{-1}\left(e^L \alpha_{11}(s) + (e^L - 1)\alpha_{12}\left(\frac{s}{\sqrt{\varrho_1}}\right)\right),$$

(99) is obtained, which completes the proof. □

A.6. Proof of Lemma 4.7

Differentiating (91) with respect to x, we get

$$\partial_x \iota(x,\tau) = -\varphi\left(\iota(x,\tau), \frac{\lambda(x,\tau)+\mu(\iota(x,\tau))}{\sqrt{v(\iota_2(x,\tau))v(\overline{u}(0,0))}}\right), \tag{F.1}$$

$$\iota(0,\tau) = \overline{Z}(\tau), \tag{F.2}$$

for all $0 \le x \le L$, $\tau \ge 0$. Introducing the following change of variable:

$$y = \int_0^x \frac{\mathrm{d}s}{\sqrt{v(\iota_2(s,\tau))}}, \tag{F.3}$$

for $0 \le x \le L$. Since the transport velocity v is assumed to be strictly positive, the function y is monotonically increasing with respect to x. Thus, it admits an inverse function as $x = \chi(y)$ and system (F.1), (F.2) can be rewritten as

$$\partial_y \iota(\chi(y),\tau) = -\varphi\left(\iota(\chi(y),\tau), \frac{\lambda(\chi(y),\tau)+\mu(\iota(\chi(y),\tau))}{\sqrt{v(\iota_2(\chi(y),\tau))v(\overline{u}(0,0))}}\right), \tag{F.4}$$

$$\iota(0,\tau) = \overline{Z}(\tau), \tag{F.5}$$

for all $0 \le y \le \int_0^L \frac{\mathrm{d}s}{\sqrt{v(\iota_2(s,\tau))}}$, $\tau \ge 0$. Noting that

$$\mu_1(\iota(\chi(y),\tau)) = \frac{\mu(\iota(\chi(y),\tau))}{\sqrt{v(\iota_2(\chi(y),\tau))v(\overline{u}(0,0))}}, \tag{F.6}$$

and using (18), we have

$$\begin{aligned}
&-\frac{\partial R_3(\iota(\chi(y),\tau))}{\partial \iota}\varphi\left(\iota(\chi(y),\tau), \frac{\lambda(\chi(y),\tau)+\mu(\iota(\chi(y),\tau))}{\sqrt{v(\iota_2(\chi(y),\tau))v(\overline{u}(0,0))}}\right) \\
&\le R_3(\iota(\chi(y),\tau)) + \alpha_9\left(\left|\frac{\lambda(\chi(y),t)}{v(\iota_2(\chi(y),\tau))v(\overline{u}(0,0))}\right|\right),
\end{aligned} \tag{F.7}$$

for $\iota(\chi(y),\tau) \in \mathbb{R}^n$ and $\lambda(\chi(y),\tau) \in \mathbb{R}$. With the help of (F.4), we deduce that

$$\begin{aligned}
&\frac{\partial R_3(\iota(\chi(y),\tau))}{\partial y} \\
&\le R_3(\iota(\chi(y),\tau)) + \alpha_9\left(\left|\frac{\lambda(\chi(y),\tau)}{v(\iota_2(\chi(y),\tau))v(\overline{u}(0,0))}\right|\right),
\end{aligned} \tag{F.7}$$

for $0 \le y \le \int_0^L \frac{\mathrm{d}s}{\sqrt{v(\iota_2(s,\tau))}}$, $\tau \ge 0$. Hence, the following relation holds:

$$\begin{aligned}
&R_3(\iota(\chi(y),\tau)) \\
&\le e^y R_3(\iota(\chi(0),\tau)) + (e^y-1)\sup_{0\le s\le y}\alpha_9\left(\left|\frac{\lambda(\chi(s),\tau)}{v(\iota_2(\chi(s),\tau))v(\overline{u}(0,0))}\right|\right),
\end{aligned} \tag{F.8}$$

for all $0 \leq y \leq \int_0^L \frac{\mathrm{d}s}{\sqrt{v(\iota_1(s,\tau))}}$, $\tau \geq 0$. With the help of (6), (F.3), the following holds

$$R_3(\iota(x,\tau)) \leq e^L R_3(\iota(0,\tau)) + (e^L - 1) \sup_{0\leq x\leq L} \alpha_9 \left(\left| \frac{\lambda(x,\tau)}{\sqrt{\varrho_1}} \right| \right), \tag{F.9}$$

for all $0 \leq x \leq L$, $\tau \geq 0$. Finally, using (17), from (F.9), defining

$$\rho_4(s) = \alpha_7^{-1} \left(e^L \alpha_8(s) + (e^L - 1)\alpha_9 \left(\frac{s}{\sqrt{\varrho_1}} \right) \right),$$

(100) is obtained, which completes the proof. □

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