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EXTENDING THE SOLVABILITY OF EQUATIONS USING SECANT-TYPE METHODS IN BANACH SPACE

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Abstract. We extend the solvability of equations defined on a Banach space using numerically efficient secant-type methods.The convergence domain of these methods is enlarged using our new idea of restricted convergence region. By using this approach, we obtain a more precise location where the iterates lie than in earlier studies leading to tighter Lipschitz constants. This way the semi-local convergence produces weaker sufficient convergence criteria and tighter error bounds than in earlier works. These improvements are also obtained under the same computational effort, since the new Lipschitz constants are special cases of the old ones.

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1. INTRODUCTION

Let $F : \Omega \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a nonlinear operator, $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and Ω be a convex set. Numerous iterative methods for solving equation $F(x) = 0$ can be written like

$$x_{n+1} = x_n - L_n F(x_n) \text{ for each } n = 0, 1, 2, \dots \tag{1}$$

where $L_n \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ the space of bounded linear operators from $\mathcal{B}_2$ into $\mathcal{B}_1$ for each $n \in \mathbb{N} \cup \{0\}$. The most widely used methods like (1) are Newton's method, where $L_n = F'(x_n)^{-1}$, and the secant method, where $L_n = \delta F(x_n, x_{n-1})^{-1}$ and δF stands for a consistent approximation to the Fréchet-derivative of F [6, 26]. A lot of problems in control theory, optimization, inverse problems theory, Mathematical Physics, Chemistry, Economics, Biology and also in engineering can be brought in the form of equation $F(x) = 0$ using Mathematical modeling [1, 2, 3, 6, 11, 12, 13, 14, 22, 23, 26, 29]. Closed form solutions are preferred but this is rarely possible. Consequently, mostly iterative methods are utilized say like method (1) to generate a sequence approximating a locally unique solution x^* of equation $F(x) = 0$ under some conditions. It is well known from the

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numerical efficiency that it is not advantageous to change the operator L_n at each step of the iterative method. If one keeps the operator piecewise constant more efficient iterative methods can be obtained. Optimal recepts can be obtained based on the dimension of the space [29]. Iterative methods of this type have been studied by Traub [29], Potra and Pták [26], Bosarg and Falb [7, 8], Dennis [9], Potra [26, 27], Amat [1, 2], Ezquerro *et al.* [12], Hernandez *et al.* [14, 15, 16], Argyros [3, 4, 5, 6] (see also the references in the preceding papers).

In this paper we are motivated by the work by Potra [25, 26] who improved the work by [7, 8, 9, 29]. Let us choose

(2) $$L_n \in \{\delta F(x_{p_n}, x_{q_n})^{-1}, \delta F(x_{q_n}, x_{p_n})^{-1}\} \text{ for each } n = 0, 1, 2, \ldots,$$

where $\{p_n\}$ and $\{q_n\}$ are non-decreasing sequences of integers such that

(3) $$q_0 = -1, p_0 = 0, q_n \leq p_n \leq n \text{ for each } n = 1, 2, 3, \ldots.$$

The convergence region of method (1)–(2) is small in general. That is why, we find in this paper a more accurate location containing the iterates $\{x_n\}$ than Ω leading to tighter Lipschitz constants. This way we obtain: weaker sufficient convergence criteria, tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ and at least as precise information on the location of the solution x^*. It is worth noticing that the preceding improvements are obtained under the same computational effort, since in practice the computation of the old Lipschitz constants requires the computation of the new constants as a special cases.

The rest of the paper is structured as follows. Section 2 and Section 3 contain the semi-local convergence of secant-type and Newton-type methods, respectively.

2. SEMI-LOCAL CONVERGENCE $P_N \neq Q_N$ FOR EACH N

We shall study the iterative procedure (1) and (2) in this section for the triplets (F, x_0, x_{-1}) belonging to the class $A(\alpha_0, \alpha, \beta, \gamma)$ defined as follows:

Definition 1. *Let $\alpha_0 > 0$, $\alpha > 0, \beta \geq 0, \gamma \geq 0$ satisfy*

(4) $$\alpha\beta + 2\sqrt{\alpha\beta} \leq 1.$$

We say that the triplet (F, x_0, x_{-1}) belongs to the class $A(\alpha_0, \alpha, \beta, \gamma)$ if :

- (c1) *F is a nonlinear operator defined on a convex subset Ω of a Banach space $\mathcal{B}_1$ and with values in a Banach space $\mathcal{B}_2$.*
- (c2) *x_0 and x_{-1} are two points belonging to the interior Ω° of Ω and satisfies the inequality*

(5) $$\|x_0 - x_{-1}\| \leq \beta.$$

- (c3) *F is Fréchet differentiable on Ω° and there exists a mapping δF : $\Omega^\circ \times \Omega^\circ \longrightarrow L(\mathcal{B}_1, \mathcal{B}_2)$ such that:*

$$\text{linear operator } L_0, \text{ where } L_0 \text{ is either } \delta F(x_0, x_{-1})$$

(6) or $\delta F(x_{-1}, x_0)$, is invertible, its inverse $L_0 = P_0^{-1}$ is bounded

(7) $$\|L_0 F(x_0)\| \leq \alpha;$$

(8)
$\|L_0(\delta F(x,y) - F'(x_0))\| \leq \alpha_0(\|x - x_0\| + \|y - x_0\|)$ for each $x, y \in \Omega, a_0 > 0$.

Set $\Omega_0 = \Omega^\circ \cap U(x_0, r_0)$, $r_0 = \frac{1-a_0c}{2a_0}$.

(9) $\|L_0(\delta F(x,y) - F'(z))\| \leq \alpha(\|x - z\| + \|y - z\|)$ for each $x, y, z \in \Omega_0$

and

(10) $$\alpha_0 \leq \alpha.$$

(c4) *The set* $\Omega_\gamma = \{x \in \Omega : F \text{is continuous at } x\}$ *contains the ball* $\bar{U}(x_1, r_1) \subset V$ *with center* $x_1 = x_0 - L_0 F(x_0)$ *and radius* $r_1 = \frac{1}{2\alpha}[1 - \alpha(2\beta + \gamma) - \sqrt{(1 - \alpha\gamma)^2 - 4\alpha\beta}]$.

We associate the class $A(\alpha_0, \alpha, \beta, \gamma)$ with the constant δ and sequence $\{s_n\}_{n \geq -1}$ given by the formulae:

(11) $$s_{-1} = \frac{1 + \alpha\gamma}{2\alpha},\ s_0 = \frac{1 - \alpha\gamma}{2\alpha},\ s_{n+1} = s_n - \frac{s_n^2 - \delta^2}{s_{p_n} - s_{q_n}},\ n = 0, 1, 2, \ldots$$

and $\delta = \frac{1}{2\alpha}\sqrt{(1 - \alpha\gamma)^2 - 4\alpha\beta}$.

Using the above notation we present the following semi-local convergence result.

Theorem 2. *If* $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$ *then the iterative algorithm* (1)–(2) *is well defined, the sequence* $\{x_n\}_{n \geq 1}$ *generated by it converges to a solution* $x^* \in U(x_0, r_1)$ *of the equation* $F(x) = 0$. *Moreover, the following items hold:*

$$\begin{aligned} \|x_n - x^*\| \leq\ & s_0 - \|x_n - x_0\| - [(s_n - \|x_n - x_0\|^2 \\ & -(\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\| \\ & +\|x_{p_{n-1}} - x_{q_{n-1}}\|)\|x_n - x_{n-1}\|]^{1/2} \leq s_n - \delta \end{aligned} \tag{12}$$

and

$$\begin{aligned} \|x_n - x^*\| \geq\ & [(s_0 - \tfrac{1}{2}(\|x_{p_n} - x_{q_n}\| + \|x_{p_n} - x_0\| + \|x_{q_n} - x_0\|) - \|x_n - x_{p_n}\|)^2 \\ & +(2s_0 - \|x_{p_n} - x_0\| - \|x_{q_n} - x_0\|)\|x_n - x_{n+1}\|]^{1/2} \\ & -s_0 + \tfrac{1}{2}(\|x_{p_n} - x_{q_n}\| + \|x_{p_n} - x_0\| + \|x_{q_n} - x_0\|) + \|x_n - x_{p_n}\|. \end{aligned} \tag{13}$$

Proof. The linear operator $M = \delta F(u, v)$ is invertible for each $u, v \in \Omega^\circ$ with

(14) $$\|u - x_0\| + \|v - x_0\| < 2s_0.$$

It follows from (7) that

$$\|I - L_0 M\| = \|L_0(M_0 - M)\| \leq \|L_0(M - F'(x_0))\| + \|L_0(F'(x_0) - M_0)\|$$

$$\leq \quad \alpha_0(\|u - x_0\| + \|v - x_0\|) + \|x_0 - x_{-1}\| < 1.$$

Hence, by the Banach Lemma [6, 28, 30, 31, 32] M is invertible and

$$\|(L_0 M)^{-1}\| \leq [1 - \alpha_0(\|u - x_0\| + \|v - x_0\| + \gamma)]^{-1}. \tag{15}$$

Note that the condition (7) implies the following Lipschitz condition for F'

$$\|L_0(F'(u) - F'(v))\| \leq 2\alpha\|u - v\|, \ u, v \in \Omega_0^\circ. \tag{16}$$

Using the integral representation

$$F(x) - F(y) = \int_0^1 F'(y + \theta(x - y))d\theta(x - y) \tag{17}$$

we obtain that

$$\|L_0(F(x) - F(y) - F'(u)(x - y))\| \leq \alpha(\|x - u\| + \|y - u\|)\|x - y\| \tag{18}$$

for all $x, y \in \Omega_0^\circ$.

Finally form (9) and (17) we have

$$\|L_0(F(x) - F(y) - \delta F(u, v)(x - y)\| \leq \alpha(\|x - u\| + \|y - u\| + \|u - v\|)\|x - y\| \tag{19}$$

for all $x, y, u, v \in \Omega_0^\circ$. Estimates (16), (17),(18), by a continuity argument, remain valid if x and/or y belong to Ω_γ. Using the above inequalities we shall prove that

$$\|x_n - x_{n+1}\| \leq s_n - s_{n+1} \tag{20}$$

for $n = -1, 0, 1, \ldots$.

Clearly, the sequence $\{s_n\}_{n\geq 1}$ given by (11) is decreasing and converges to δ. Hence, if (1)-(2) is well defined for $n = 0, 1, 2, \ldots, k$, and if (20) holds for $n \leq k$ than

$$\|x_0 - x_n\| \leq s_0 - s_n < s_0 - \delta$$

for $n \leq k$. That is (14) is satisfied for $u = x_i$ and $v = x_j$ with $i, j \leq k$. Therefore, (1)-(2) will be well defined for $n = k + 1$ as well.

For $n = -1$ and $n = 0$ (20) reduces to $\|x_{-1} - x_0\| \leq \gamma$ and $\|x_0 - x_{-1}\| \leq \beta$ (see (5) and (7)). Suppose (20) holds for $n = -1, 0, 1, \ldots, k$, where $k \geq 0$. Denote $M_n = L_n^{-1}$, where L_n is given by (2). Notice that

$$F(x_{k+1}) = F(x_{k+1}) - F(x_k) - M_k(x_{k+1} - x_k). \tag{21}$$

Then, using (14) and (18) we can write

$$\begin{aligned}
\|x_{k+1} - x_{k+2}\| &= \|L_{k+1}F(x_{k+1})\| = \|(L_0 M_{k+1})^{-1} L_0 F(x_{k+1})\| \\
&\leq \frac{\alpha(\|x_{k+1} - x_{p_k}\| + \|x_k - x_{p_k}\| + \|x_{p_k} - x_{q_k}\|)}{1 - \alpha_0(\|x_{p_{k+1}} - x_0\| + \|x_{q_{k+1}} - x_0\| + \gamma)}\|x_k - x_{k+1}\| \\
&\leq \frac{\alpha((s_{p_k} - s_{k+1} + s_{p_k} - s_k + s_{q_k} - s_{p_k})}{1 - \alpha_0(s_0 - s_{p_{k+1}} - s_0 - s_{q_{k+1}} + s_{-1} - s_0)}(s_k - s_{k+1}) \\
&\leq \frac{s_{p_k} + s_{q_k} - s_{k+1} - s_k}{s_{p_{k+1}} + s_{q_{k+1}}}(s_k - s_{k+1}) = s_{k+1} - s_{k+2}.
\end{aligned}$$

So, (19) holds for each n. $\mathcal{B}_1$ is a complete space. Hence, the sequence $\{x_n\}_{n\geq 0}$ converges to x^* and

$$\|x_n - x^*\| \leq s_n - \delta. \tag{22}$$

Next, by (18) and (20), we obtain

$$\|L_0F(x_{k+1})\| \leq \alpha(\|x_{k+1} - x_{p_k}\| + \|x_k - x_{p_k}\| + \|x_{p_k} - x_{q_k}\|)\|x_k - x_{k+1}\|, \tag{23}$$

so it follows that $F(x^*) = 0$.

Let $x = x_n$ and $y = x^*$ in (17) and denote $A = \int_0^1 F'(x^* + \theta(x_n - x^*))d\theta$. Then, by (19) and (21) we get in turn that

$$\begin{aligned}
\|x_n - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\| &\leq 2\|x_n - x_0\| + \|x_n - x^*\| + \gamma \\
&< 2(\|x_n - x_0\| + \|x_n - x^*\|) \\
&\leq 2(s_0 - s_n + s_n - \delta) + \gamma \\
&\leq 2s_0 + \gamma = \tfrac{1}{\alpha}.
\end{aligned}$$

In view of (8) and the Banach's lemma we show as in (15) that, A is invertible and

$$\|(L_0A)^{-1}\| \leq [1 - \alpha_0(2\|x_n - x_0\| + \|x_n - x^*\| + \gamma)^{-1}. \tag{24}$$

Using (22) and (24), we have

$$\begin{aligned}
\|x_n - x^*\| &= \|A^{-1}F(x_n)\| \leq \|(L_0A)^{-1}\|\|L_0F(x_n)\| \\
&\leq \tfrac{\alpha(\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\| + \|x_{p_{n-1}} - x_{q_{n-1}}\|)}{1 - \alpha_0(2\|x_n - x_0\| + \|x_n - x^*\| + \gamma)}\|x_n - x_{n-1}\|.
\end{aligned}$$

It is easy to see that the above inequality together with (10) and $\|x_n - x^*\| < s_0$ imply the estimate (12).

By the identity

$$x_{n+1} - x_n = x^* - x_n + (L_0M_n)^{-1}L_0(F(x^*) - F(x_n) - M_n(x^* - x_n)),$$

(20) and (19), we obtain

$$\|x_{n+1} - x_n\| \leq \tfrac{\alpha(2\|x_n - x_{p_n}\| + \|x^* - x_n\| + \|x_{p_n} - x_{q_n}\|)}{1 - \alpha_0(\|x_{p_n} - x_0\| + \|x_{q_n} - x_0\| + \gamma}\|x_n - x^*\| + \|x_n - x^*\|,$$

so (12) is shown. □

Corollary 3. *Suppose $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$. Then the equation $F(x) = 0$ has a solution $x^* \in U$ and this is the only solution of the equation in the set $Q_1 = \{x \in \Omega_\gamma : \|x - x_0\| \leq r\}$ if $\delta > 0, r = \delta - s_0 - \gamma + \frac{1}{\alpha_0}$, or in the set $Q_2 = \{x \in \Omega_\gamma : \|x - x_0\| \leq s_0\}$ if $\delta = 0$.*

Proof. The existence has been shown in the Theorem2. Suppose $\delta > 0$ and let $x^* \in U$ and $y^* \in Q_1$ be solutions of the equation $F(x) = 0$. Let us denote $S_* = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$. Using (8), we have

$$\begin{aligned}
\|I - L_0S_*\| &= \|L_0(M_0 - S_*)\| \leq \alpha_0(\|y^* - x_0\| + \|x^* - x_0\| + \|x_0 - x_{-1}\|) \\
&< \alpha_0(r + s_0 - \delta + \gamma) = 1.
\end{aligned}$$

That is S_* is invertible. Then, by (17) we deduce that $x^* = y^*$.

Case $\delta = 0$. Let $M_n = 0$ and $q_n = -1$ for $n = 0, 1, 2, \dots$ then the iterative procedure (1)-(2) becomes

$$x_{n+1} = x_n - L_0 F(x_n), \ n = 0, 1, 2, \dots. \tag{25}$$

By Theorem 2 it follows that the sequence $\{x_n\}_{n \geq 0}$ given by (25) converges to a solution x^* of the equation $F(x) = 0$. It also follows that

$$\|x_n - x_{n+1}\| \leq s_n - s_{n+1}, \tag{26}$$

where

$$s_0 = (\frac{\beta}{\alpha_0})^{1/2}, \ s_{n+1} = s_n - \alpha_0 s_n^2, \ n = 0, 1, 2, \dots. \tag{27}$$

Using induction we get that

$$s_n \geq \frac{(\frac{\beta}{\alpha_0})^{1/2}}{n+1}, \ \ n = 0, 1, 2, \dots. \tag{28}$$

Let $y^* \in Q_2$ be a solution of the equation $F(x) = 0$ and denote $A_n = \int_0^1 F'(y^* + \theta(x_n - y^*))d\theta$. According to (8), (14), (24) and (25) we have

$$\begin{aligned} \|x_{n+1} - y^*\| &= \|L_0(M_0 - A_n)(x_n - y^*)\| \\ &\leq \alpha_0 \|x_n - y^*\|(\|y^* - x_0\| + \|x_n - x_0\| + \|x_0 - x_{-1}\|) \\ &\leq \|x_n - y^*\|(1 - \alpha_0 s_n) \leq \dots \\ &\leq \|x_1 - y^*\| \prod_{j=1}^{n} (1 - \alpha_0 s_j). \end{aligned}$$

By (28), we deduce that $\lim_{n \longrightarrow \infty} \prod_{j=1}^{n} (1 - \alpha_0 s_j) = 0$. Hence, we conclude that $y^* = \lim_{n \longrightarrow \infty} x_n = x^*$. $\square$

Next, we show that the results obtained in this section are sharp within the class $A(\alpha_0, \alpha, \beta, \gamma)$.

Proposition 4. [27] *Suppose $\alpha_0 > 0, \alpha > 0, \beta \geq 0, \gamma \geq 0$ satisfy inequality* (4). *Then, the following items hold:*

- (i) *There exist a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ and two points $x_0, x_{-1} \in \mathbb{R}$ such that the triplet $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$ and for this triplet the estimates* (11) *are attained at each $n = 0, 1, 2, 3, \dots$.*
- (ii) *For each $n = 0, 1, 2, \dots$ there exist a function $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ and two points $x_0, x_{-1} \in \mathbb{R}$ such that the triplet $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$ and for this triplet* (12) *holds with equality.*

Concerning the domain of uniqueness of the solution x^* established in the Corollary 3 we have

Proposition 5. [27] *Suppose $\alpha_0 > 0, \alpha > 0, \beta \geq 0, \gamma \geq 0$ s satisfy(4). Let δ be the constant defined by (11).*

- (i) *If $\delta > 0$ then there exist a function $F : \mathbb{R} \longrightarrow \mathbb{R}$ and four points $x_0, x_{-1}, x^*, y^* \in \mathbb{R}$ such that $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$, $F(x^*) = F(y^*) = 0, |x_0 - x^*| = s_0 - \delta, |x_0 - y^*| = s_0 + \delta$.*

(ii) *If* $\delta = 0$, *then for each* $\epsilon > 0$ *there exist a function* $f_\epsilon : \mathbb{R} \longrightarrow \mathbb{R}$ *and four points* $x_0, x_{-1}, x^*, y^*_\epsilon \in \mathbb{R}$ *such that* $(F, x_0, x_{-1}) \in A(\alpha_0, \alpha, \beta, \gamma)$, $F(x^*) = F(y^*_\epsilon) = 0, |x_0 - x^*| = s_0, |x_0 - y^*_\epsilon| = s_0 + \epsilon$.

Next, we shall consider some particular cases of the iterative procedure (1)-(2) and shall compare the results obtained in the preceding section with some known results.

Remark 6. (a1) In earlier studies on Secant-type methods condition

$$\|L_0(\delta F(x, y) - F'(z))\| \leq \alpha_1(\|x - z\| + \|y - z\|) \tag{29}$$

for each $x, y, z \in \Omega$ together with the condition

$$\alpha_1\gamma + 2\sqrt{\alpha_1\beta} \leq 1 \tag{30}$$

are used [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. Notice that (29) implies (8), (9),

$$\alpha_0 \leq \alpha_1$$
$$\alpha \leq \alpha_1,$$

and $\frac{\alpha_1}{\alpha}$ can be arbitrarily large [3, 4, 5, 6]. Moreover, we have by (4) and (30) that

$$(30) \Longrightarrow (4)$$

but not necessarily vice versa, unless, if $\alpha = \alpha_1$. Furthermore, in [27] $\bar{r} = s_0 + \delta$ and

$$\bar{r} \leq r,$$

so the uniqueness of the solution is extended under our approach. Finally, it follows from the proof of Theorem 2 that sequence $\{s_n\}$ defined by

$$\begin{aligned} s_{-1} &= \tfrac{1+\alpha_0\gamma}{2\alpha_0} > t_{-1}, \\ s_0 &= \tfrac{1-\alpha_0\gamma}{2\alpha_0} > t_0 \\ s_{n+2} &= s_{n+1} + \tfrac{\alpha(s_{p_n} - s_{n+1} + s_{q_n} - s_n)}{1-\alpha_0(s_0 - s_{p_{n+1}} + s_{-1} - s_{q_{n+1}}} \end{aligned} \tag{31}$$

is also a majorizing sequence for $\{x_n\}$ and we can have instead of the corresponding estimates given in Theorem 2, the more precise estimates

$$\|x_n - x^*\| \leq \tfrac{\alpha(\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\| + \|x_{p_{n-1}} - x_{q_{n-1}}\|)}{1-\alpha_0(2\|x_n - x_0\| + \|x_n - x^*\| + \gamma)}\|x_n - x_{n-1}\| \tag{32}$$

and

$$\|x_{n+1} - x_n\| \leq \tfrac{\alpha(2\|x_n - x_{p_n}\| + \|x^* - x_n\| + \|x_{p_n} - x_{q_n}\|)\|x_n - x^*\|}{1-\alpha_0(\|x_{p_n} - x_0\| + \|x_{q_n} - x_0\| + \gamma)} + \|x_n - x^*\|, \tag{33}$$

respectively. Estimates (32) and (33) are clearly more precise than the corresponding ones given in [25, 26] by

$$\|x_n - x^*\| \leq \tfrac{\alpha_1(\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\| + \|x_{p_{n-1}} - x_{q_{n-1}}\|)}{1-\alpha_1(2\|x_n - x_0\| + \|x_n - x^*\| + \gamma)}\|x_n - x_{n-1}\| \tag{34}$$

and

$$\|x_{n+1} - x_n\| \leq \tfrac{\alpha_1(2\|x_n - x_{p_n}\| + \|x^* - x_n\| + \|x_{p_n} - x_{q_n}\|)\|x_n - x^*\|}{1-\alpha_1(\|x_{p_n} - x_0\| + \|x_{q_n} - x_0\| + \gamma)} + \|x_n - x^*\|, \tag{35}$$

respectively. The preceding results are obtained assuming that $\alpha_0 \leq \alpha$. However, if $\alpha \leq \alpha_0$, then the preceding results hold with α_0 replacing α. The advantages in this study were obtained under the same computational cost as in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32], since in practice the computation of α_1 requires the computation if α_0 and α as special cases.

The preceding results can be improved even further, if we replace Ω_0 by Ω_1 defined by

$$\Omega_1 = \Omega_0 \cap U(x_1, r_0 - \|L_0F(x_0)\|).$$

Notice that $\Omega_1 \subseteq \Omega_0$, so the corresponding condition to (9) hold on Ω_1 and the corresponding constant $\bar{\alpha}$ will be such that $\bar{\alpha} \leq \alpha$. Examples where strict inequalities $\alpha_0 < \alpha_1, \alpha_0 < \alpha, \alpha < \alpha_1$ and $\bar{\alpha} < \alpha$ can be found in [3, 4, 5, 6]. In the rest of the remarks our results compare favorable to earlier ones.

(a2) If $p_n = n$ and $q_n = n - 1$ for each $n = 1, 2, \ldots$ then (1)–(2) reduces to the secant method. The error estimates (12) and (13) improves (32) and (33) which in turn improved the ones in [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 29, 30, 31, 32].

(a3) If $p_n = 0$ and $q_n = -1$ for $n = 0, 1, 2, \ldots$ then (1)–(2) reduces to the simplified secant method. The result contained in Theorem 2 improves in this case the result from [27].

(a4) If $p_{km+j} = km, q_{km+j} = km - 1, (q_{-1} = q_0 = -1), j = 0, 1, \ldots, m - 1, k = 0, 1, \ldots,$ then (1)-(2) reduces to a procedure considered by Traub [29] for scalar equations. A local analysis for this procedure has been done by Potra and Pták, [26], Laasonen [18] made a semi-local analysis for the case $m = 2$.When $p_n = q_n + 1$ the iterative processes (1)–(2) was studied by Dennis [9]. The results obtained in Theorem 2 improves all the above mentioned results. Note that by taking $y_n = x_{nm}$ one contains a sequence $\{y_n\}_{n\geq 0}$ which converges to x^* with $R-$order $(m + \sqrt{m^2+4})/2$. The parameter m can be chosen according to the dimension of the space in order to maximize the numerical efficiency of the procedure (see [25, 26]). □

3. SEMI-LOCAL CONVERGENCE ANALYSIS $p_n = q_n$ FOR EACH n

If $x_{-1} = x_0$ and $p_n = q_n$ for each $n = 0, 1, 2, \ldots,$ then the iterative procedure (1)-(2) becomes

$$x_{n+1} = x_n - F'(x_{p_n})^{-1}F(x_n),\ n = 0, 1, 2, \ldots. \tag{36}$$

In [9, 10], Dennis proved that this iterative procedure converges under the hypotheses of the Kantorovich theorem. This fact follows by taking $\gamma = 0$ in Theorem 2. To be more precise let us consider the class $A'(\alpha_0, \alpha, \beta)$ defined below.

Definition 7. *Let $\alpha_0 > 0, \alpha > 0$ and $\beta \geq 0$ satisfy*

$$4\alpha\beta \leq 1. \tag{37}$$

We say that a pair (F, x_0) *belongs to the class* $A'(\alpha_0, \alpha, \beta)$, *if*

(c1') F *is a nonlinear operator defined on a convex subset* Ω *of a Banach space* $\mathcal{B}_1$ *and with values in a Banach space* $\mathcal{B}_2$.

(c2') x_0 *is a point belonging to the interior* Ω° *of* Ω.

(c3') F *is Fréchet differentiable on* $\Omega^\circ, F'(x_0)$ *is boundedly invertible*

$$\|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq 2\alpha_0 \|x - x_0\| \text{ for each } x, y \in \Omega. \tag{38}$$

Set $\Omega_0 = \Omega \cap U(x_0, r_0)$, $r_0 = \frac{1}{2\alpha_0}$ *and*

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq 2\alpha \|x - y\| \text{ for each } x, y \in \Omega_0. \tag{39}$$

(c4') *The set* $\Omega_c = \{x \in \Omega, F \text{ is continuous at } x\}$ *contains the closed ball* U *with center* $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$ *and radius* $r_1 = \frac{1}{2\alpha}(1 - 2\alpha\beta - \sqrt{1 - 4\alpha\beta})$.

It is easy to see that $(F, x_0) \in A'(\alpha_0, \alpha, \beta)$ if and only if $(F, x_0, x_0) \in A(\alpha_0, \alpha, \beta)$. In this case from Theorem 2 it follows that the iterative procedure (36) converges and the following estimates hold:

$$\begin{aligned} \|x_n - x^*\| \leq\ & s_0 - \|x_n - x_0\| \\ & -[(s_0 - \|x_n - x_0\|)^2 - (\|x_n - x_{p_{n-1}}\| + \|x_{n-1} - x_{p_{n-1}}\|) \\ & \times \|x_n - x_{n-1}\|]^{1/2} \end{aligned} \tag{40}$$

and

$$\begin{aligned} \|x_n - x^*\| \geq\ & [s_0 - \|x_{p_n} - x_0\| - \|x_n - x_{p_n}\|)^2 \\ & +2(s_0 - \|x_{p_n} - x_0\|)\|x_n - x_{n+1}\|]^{1/2} - s_0 + \|x_{p_n} - x_0\|) \\ & \times \|x_n - x_{p_n}\|. \end{aligned} \tag{41}$$

Remark 8. Set $\gamma = 0$ in the cases of Remark 6 to obtain the corresponding improvements for Newton-type methods over the ones in [25, 26, 27] and the works earlier than the preceding. □

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