

REMARKS ON A BERNSTEIN-TYPE OPERATOR OF ALDAZ,
KOUNCHEV AND RENDER

ANA-MARIA ACU*, HEINER GONSKA† and MARGARETA HEILMANN‡

This brief note is dedicated to our dear friend and long term collaborator Ioan Raşa on the occasion of his 70th birthday. Over the years he did a lot, most fruitful work on quite a number of Bernstein-type and related operators, among others. Here we deal with a then surprising, 12-year old object of this type offering a lot of challenges.

Abstract. The Bernstein-type operator of Aldaz, Kounchev and Render (2009) is discussed. New direct results in terms of the classical second order modulus as well as in a modification following Marsden and Schoenberg are given.

MSC 2010. 41A25, 41A36.

Keywords. Bernstein-type operator; King operator; second order modulus of continuity; Marsden-Schoenberg; modulus of order j .

1. INTRODUCTION

Starting from the classical Bernstein operators B_n defined for $f \in C[0, 1]$ as

$$B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{n,k}(x), \quad \text{where}$$

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},$$

during recent years many, probably much too many, modifications have been considered. One of the present hypes follows a 2003 paper written by J.P. King [16] in order to obtain linear positive operators which preserve two functions different from the classical test functions e_0 and e_1 . Here we used the convention $e_j(x) = x^j$, $j = 0, 1, \dots$

King modified the classical Bernstein operators as follows:

$$(1) \quad f \rightarrow (B_n f) \circ r_n, \quad f \in C[0, 1],$$

*Lucian Blaga University of Sibiu, Department of Mathematics and Informatics, Str. Dr. I. Ratiu, No. 5-7, RO-550012 Sibiu, Romania, e-mail: anamaria.acu@ulbsibiu.ro.

†University of Duisburg-Essen, Faculty of Mathematics, Bismarckstr. 90, D-47057 Duisburg, Germany, e-mail: heiner.gonska@uni-due.de.

‡University of Wuppertal, School of Mathematics and Natural Sciences, Gaußstraße 20, D-42119 Wuppertal, Germany, e-mail: heilmann@math.uni-wuppertal.de.

where

$$r_n(x) = \begin{cases} x^2, & n = 1, \\ \frac{1}{2(n-1)} \left(-1 + \sqrt{1 + 4n(n-1)x^2} \right), & n = 2, 3, \dots \end{cases}$$

These operators preserve the functions $e_i(x) = x^i$ for $i = 0, 2$. A slight extension was considered by Cárdenas *et al.* in [6] where a sequence of operators $B_{n,\alpha}$ that preserve e_0 and $e_2 + \alpha e_1$, $\alpha \in [0, +\infty)$ was introduced. It is clear that the operator of King does not produce polynomials in the general case. Therefore, in their 2005 remarks on the article of King, Gonska and Pițul (see [15]) asked the question if there exist linear and positive **polynomial** operators that reproduce e_2 .

Using a continuous strictly increasing function τ defined on $[0, 1]$ with $\tau(0) = 0$ and $\tau(1) = 1$, $\tau'(x) > 0, x \in [0, 1]$, Cárdenas-Morales *et al.* (see [6]) introduced a modification of the Bernstein operator as follows:

(2)

$$B_n^\tau(f; x) := \sum_{k=0}^n \binom{n}{k} \tau(x)^k (1 - \tau(x))^{n-k} (f \circ \tau^{-1}) \left(\frac{k}{n} \right), \quad f \in C[0, 1], \quad x \in [0, 1].$$

Note that $\tau'(0) > 0$ is essential for their results. A predecessor can be found in Cottin *et al.* [9], but $\tau(x) = \sqrt{x}$ there.

These operators preserve the functions e_0 and τ . So, for $\tau(x) = x^j$, one has

$$B_n^\tau(e_0; x) = e_0 \quad \text{and} \quad B_n^\tau(e_j; x) = e_j.$$

In this case B_n^τ produces polynomials of degree jn . If the function τ is not a polynomial, then the modified operator (2) is not a polynomial operator.

Fix $j > 1$, $j \in \mathbb{N}$. Let Π_n be the space of polynomials over $[0, 1]$ of degree less than or equal to n . For every $n \geq j$, Aldaz, Kounchev and Render [3] introduced a **polynomial** Bernstein operator $B_{n,0,j} : C[0, 1] \rightarrow \Pi_n$ that fixes e_0 and e_j , and converges in the strong operator topology to the identity as $n \rightarrow \infty$. The operator is a linear combination of the classical Bernstein basis $\{p_{n,k}\}_{k=0,\dots,n}$, thus produces polynomials of degree n and is explicitly given by

$$(3) \quad B_{n,0,j}(f; x) = \sum_{k=0}^n f \left(t_{n,k}^j \right) p_{n,k}(x),$$

where

$$t_{n,k}^j = \left(\frac{k(k-1)\dots(k-j+1)}{n(n-1)\dots(n-j+1)} \right)^{1/j}.$$

REMARK 1. (i) For $j = 1$ this is the classical Bernstein operator (reproducing e_0 and e_1).

(ii) $B_{n,0,j}$ is linear, positive and of the form

$$(4) \quad \sum_{k=0}^n l_{n,k}(f) p_{n,k}(x),$$

where $l_{n,k}$ are positive linear functionals with $l_{n,k}(e_0) = 1$. Obviously all the $B_{n,0,j}$ s map into $\Pi_n[0, 1]$ (in contrast to the B_n^τ s from above, case $\tau = e_j$).

(iii) For j fixed and $n = j$ we have $(t_{n,k}^j)_{k=0,\dots,n} = (0, \dots, 0, 1)$ with n zeros preceding the 1.

For $n = j + 1$ one obtains $(t_{n,k}^j)_{k=0,\dots,n} = (0, \dots, 0, n^{\frac{1}{j}}, 1)$, and so on. That is, only for n big enough all the nodes will be distinct.

(iv) The notation $B_{n,0,j}$ is motivated by the fact that the operator reproduces e_0 and e_j . As shown by Finta in [10], see Theorem 2.1 there, no sequence (L_n) of type (4) can reproduce e_i and e_j , $i, j \in \{1, 2, \dots\}$, $i < j$. So reproduction of e_0 is a must!

Moreover, there exist infinitely many sequences of operators L_n of type (4) which approximate each continuous function on $[0, 1]$, and have the functions e_0 and e_j as fixed points, where $j \in \{1, 2, \dots\}$ is given (see [10, Th. 2.4]). \square

2. ON DIRECT ESTIMATES FOR $B_{n,0,j}$

We start with some brief history. For the case $j = 2$, $n \geq 2$, an inequality involving ω_1 (following Shisha & Mond) was given by Cardenas-Morales *et al.* in [8, Proposition 3.1] as follows

$$|B_{n,0,2}f(x) - f(x)| \leq \omega_1(f, \delta) \left(1 + \frac{1}{\delta} \sqrt{2x(1-x) \frac{1-(1-x)^{n-1}}{n-1}} \right),$$

$f \in C[0, 1]$, $x \in [0, 1]$ and $\delta > 0$.

Moreover, in 2014 Finta showed (see [11])

$$|B_{n,0,j}(f; x) - f(x)| \leq 2\omega_1 \left(f; \left(\frac{2j(3+4j)}{\sqrt{n}} \right)^{\frac{1}{2j}} \right),$$

$f \in C[0, 1]$, $x \in [0, 1]$ and $n \geq j \geq 2$.

In 2018 Aldaz and Render considered a generalization of the classical Bernstein operators on the polynomial spaces, which reproduce $\mathbf{1}$ and a polynomial f_1 , strictly increasing on $(0, 1)$. Denote by $B_n^{f_1}$ this Bernstein type operator. For $f \in C[0, 1]$, $x \in [0, 1]$ (see [4, Remark following Th. 6.5])

$$|B_n^{f_1}(f; x) - f(x)| \leq (c_S + 1)\omega_1 \left(f, n^{-\frac{1}{2}} \right),$$

where $c_S = \frac{4306+837\sqrt{6}}{5832}$, Sikkema's constant. The drawback here is that the latter is shown to be valid only for n sufficiently large.

We will improve the estimates known so far in three different ways.

PROPOSITION 1. *Let $f \in C[0, 1]$, $x \in [0, 1]$, $n \geq j \geq 1$. Then*

$$(5) \quad |f(x) - B_{n,0,j}(f; x)| \leq 1 \cdot \omega_2(f; \frac{1}{\sqrt{n}}) + \omega_1(f; \frac{j-1}{n}).$$

Proof. We have

$$\begin{aligned} |f(x) - B_{n,0,j}(f; x)| &\leq |f(x) - B_n(f; x)| + |B_n(f; x) - B_{n,0,j}(f; x)| \\ &\leq 1 \cdot \omega_2(f; \frac{1}{\sqrt{n}}) + \omega_1(f; \frac{j-1}{n}). \end{aligned}$$

The first summand is taken from Păltănea [18, 19], while the second was given by Acu and Raşa [1]. For $j = 1$ this reduces to a best possible result for classical Bernstein operators. \square

There is a second way to prove a similar inequality. To this end we use Păltănea's general theorem for positive linear operators reproducing constant functions (see [19, Cor. 2.2.1, p. 31]).

$$\begin{aligned} |B_{n,0,j}(f; x) - f(x)| &\leq \frac{1}{h} |B_{n,0,j}(e_1 - x; x)| \cdot \omega_1(f; h) \\ &\quad + \left(1 + \frac{1}{2h^2} B_{n,0,j}((e_1 - x)^2; x)\right) \cdot \omega_2(f; h), \quad 0 < h \leq \frac{1}{2}. \end{aligned}$$

PROPOSITION 2. *Let $f \in C[0, 1]$, $x \in [0, 1]$, $1 \leq j \leq n$. Then*

- a) $|B_{n,0,2}(f; x) - f(x)| \leq \frac{1}{\sqrt{n-1}} d_n(x) \omega_1\left(f; \frac{1}{\sqrt{n-1}}\right) + (1 + x d_n(x)) \omega_2\left(f; \frac{1}{\sqrt{n-1}}\right)$,
 $n \geq 5$, where $d_n(x) := (1-x)[1 - (1-x)^{n-1}]$;
- b) $|B_{n,0,j}(f; x) - f(x)| \leq (j-1) \frac{1}{\sqrt{n}} \omega_1\left(f; \frac{1}{\sqrt{n}}\right) + \left(\frac{1}{8} + j\right) \omega_2\left(f; \frac{1}{\sqrt{n}}\right)$, $n \geq 4$.

Proof. a) In order to use Păltănea's result we estimate $B_{n,0,2}(e_1 - x; x)$ and $B_{n,0,2}((e_1 - x)^2; x)$.

According to [8, (3.7)] there holds

$$\begin{aligned} |B_{n,0,2}(e_1 - x; x)| &= |B_{n,0,2}(e_1; x) - x| \\ &\leq \frac{1-x}{n-1} \left(1 - (1-x)^{n-1}\right) = \frac{d_n(x)}{n-1}. \end{aligned}$$

Furthermore,

$$\begin{aligned} 0 \leq B_{n,0,2}((e_1 - x)^2; x) &= 2x(x - B_{n,0,2}(e_1; x)) \\ &\leq \frac{2x d_n(x)}{n-1}. \end{aligned}$$

Taking $h = \frac{1}{\sqrt{n-1}}$, we get the estimation a).

In order to illustrate the nonsymmetry of the situation, in Fig. 2.1 are the graphs of the function $d_n(x)$ for $n = 5$ and $n = 10$, $x \in [0, 1]$.

b) One has to determine global estimates for $|B_{n,0,j}(e_1 - x; x)|$ and $B_{n,0,j}((e_1 - x)^2; x)$, using again $B_{n,0,j}(e_0; x) = 1$. We have

$$B_{n,0,j}(e_1 - x; x) = \sum_{k=0}^n \left[\left(\frac{k}{n} \cdots \frac{k-j+1}{n-j+1} \right)^{1/j} - \frac{k}{n} \right] p_{n,k}(x).$$

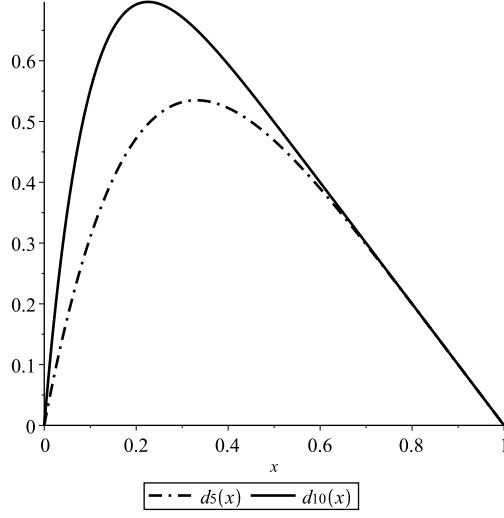


Fig. 2.1. $d_n(x)$ for $n = 5$ and $n = 10$, $x \in [0, 1]$.

The result from [1, Section 4.7], namely

$$-\frac{j-1}{n} \leq \left(\frac{k}{n} \cdots \frac{k-j+1}{n-j+1}\right)^{1/j} - \frac{k}{n} \leq 0,$$

yields

$$(6) \quad |B_{n,0,j}(e_1 - x; x)| \leq \frac{j-1}{n}.$$

The second moment can be written as

$$(7) \quad \begin{aligned} 0 &\leq B_{n,0,j}((e_1 - x)^2; x) \\ &= B_{n,0,j}(e_2; x) - x^2 - 2x [B_{n,0,j}(e_1; x) - x] \\ &\leq B_{n,0,j}(e_2; x) - x^2 + 2x \frac{j-1}{n}, \end{aligned}$$

where we used (6).

We have

$$\begin{aligned} &B_{n,0,j}(e_2; x) - x^2 \\ &= B_{n,0,j}(e_2; x) - B_n(e_2; x) + B_n(e_2; x) - x^2 \\ &= \sum_{k=0}^n \left[\left(\frac{k}{n} \cdots \frac{k-j+1}{n-j+1}\right)^{\frac{2}{j}} - \left(\frac{k}{n}\right)^2 \right] p_{n,k}(x) + \frac{x(1-x)}{n} \\ &= \sum_{k=0}^n \left[\left(\frac{k}{n} \cdots \frac{k-j+1}{n-j+1}\right)^{\frac{1}{j}} - \frac{k}{n} \right] \left[\left(\frac{k}{n} \cdots \frac{k-j+1}{n-j+1}\right)^{\frac{1}{j}} + \frac{k}{n} \right] p_{n,k}(x) + \frac{x(1-x)}{n} \\ &\leq \frac{x(1-x)}{n}. \end{aligned}$$

Together with (7) this yields

$$0 \leq B_{n,0,j} \left((e_1 - x)^2; x \right) \leq \frac{x(1-x)}{n} + 2x \frac{j-1}{n}.$$

So we arrive at

$$(8) \quad \begin{aligned} |B_{n,0,j}(f; x) - f(x)| &\leq \frac{j-1}{n} \frac{1}{h} \omega_1(f; h) + \left\{ 1 + \frac{1}{2h^2} \left[\frac{x(1-x)}{n} + 2x \frac{j-1}{n} \right] \right\} \omega_2(f; h) \\ &\leq \frac{j-1}{n} \frac{1}{h} \omega_1(f; h) + \left\{ 1 + \frac{1}{2nh^2} \left(\frac{1}{4} + 2j \right) \right\} \omega_2(f; h). \end{aligned}$$

Let $h = \frac{1}{\sqrt{n}}$, $n \geq 4$. This leads to our proposition. \square

REMARK 2. a) The right hand side of inequality (8) is not a pointwise one and so does not express interpolation at the endpoints. There is room for improvement.

b) If one compares the estimate from Proposition 2 b) and the estimate from Proposition 1 it is clear that $\omega_2(f; \frac{1}{\sqrt{n}})$ is always the worse (*i.e.*, dominant) term. In case $f \in C^2[0, 1]$ both inequalities give $\mathcal{O}(\frac{1}{n})$. This cannot be reached by any of the previous inequalities in terms of ω_1 only. \square

The inequalities (5) and those from Proposition 2 do not imply that the function e_j is reproduced by $B_{n,0,j}$ if $j \geq 2$. In order to achieve this we follow an idea of Marsden and Schoenberg [17, p. 82] (see also [14, Corollary 3.4]). Below we will use a variation of an analogue of their $\omega_1^*(f; \delta)$, namely $\omega_j^*(f; \delta)$ given by

$$\omega_j^*(f; \delta) := \inf_l \{ \omega_1(f - l; \delta) \}, \text{ where } l(x) = ae_j(x), \ a \in \mathbb{R}.$$

Note that we won't loose anything in doing so!

PROPOSITION 3. For $f \in C[0, 1]$, $x \in [0, 1]$ and $n \geq j$ we have

$$|f(x) - B_{n,0,j}(f; x)| \leq \inf_l \left\{ \omega_2(f - l; \frac{1}{\sqrt{n}}) + \omega_1(f - l; \frac{j-1}{n}) \right\}.$$

Proof. Since $B_{n,0,j}$ reproduces e_j we have

$$\begin{aligned} |f(x) - B_{n,0,j}(f; x)| &= |(f - l)(x) - B_{n,0,j}(f - l; x)| \\ &\leq 1 \cdot \omega_2 \left(f - l; \frac{1}{\sqrt{n}} \right) + \omega_1 \left(f - l; \frac{j-1}{n} \right). \end{aligned}$$

Since $l = ae_j$ was arbitrary, we pass to the *inf* and get the result.

Moreover, if f is of the form $f(x) = a_0 + a_j x^j$, $a_0, a_j \in \mathbb{R}$, we arrive at

$$|f(x) - B_{n,0,j}(f; x)| = 0.$$

\square

Analogous statements follow from the inequalities in Proposition 2.

REMARK 3. Except for the papers previously cited we did not find any reference mentioning or using ω_l^* or modifications thereof. The similarity to K-functionals is obvious.

In [14] it is mentioned that

$$\omega_1^*(f; \delta) \leq c \cdot \omega_2(f; \sqrt{\delta}). \quad \square$$

3. ITERATES OF $B_{n,0,j}$

Using the approach of Agratini and Rus (see [2]) the following result for the sequence of iterates $(B_{n,0,j}^m)_{m \geq 1}$ is obtained.

THEOREM 1. For $n \geq j$ fixed and $m \geq 1$, the iterates sequence $(B_{n,0,j}^m)_{m \geq 1}$ satisfies

$$\lim_{m \rightarrow \infty} B_{n,0,j}^m(f; x) = f(0) + [f(1) - f(0)] e_j(x).$$

Proof. Denote

$$X_{\alpha,\beta} := \{f \in C[0,1] \mid f(0) = \alpha, f(1) = \beta\}, \quad (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}.$$

The operators $B_{n,0,j}$ interpolate 0 and 1:

$$\begin{aligned} B_{n,0,j}(f; 0) &= f(t_{n,0}^j) p_{n,0}(x) = f(0), \\ B_{n,0,j}(f; 1) &= f(t_{n,n}^j) p_{n,n}(x) = f(1). \end{aligned}$$

Let $f, g \in X_{\alpha,\beta}$. Then, for $x \in [0, 1]$,

$$\begin{aligned} |B_{n,0,j}(f; x) - B_{n,0,j}(g; x)| &= \left| \sum_{k=1}^{n-1} [f(t_{n,k}) - g(t_{n,k})] p_{n,k}(x) \right| \\ &\leq (1 - p_{n,0}(x) - p_{n,n}(x)) \cdot \|f - g\|_\infty \\ &\leq \left(1 - \frac{1}{2^{n-1}}\right) \|f - g\|_\infty. \end{aligned}$$

Therefore, $B_{n,0,j}|_{X_{\alpha,\beta}} : X_{\alpha,\beta} \rightarrow X_{\alpha,\beta}$ is a contraction for every $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}$ and $n \geq j$ fixed.

The function $p_{\alpha,\beta}^* := f(0) + (f(1) - f(0)) e_j$ belongs to $X_{f(0),f(1)}$ and is a fixed point of $B_{n,0,j}$.

By contradiction we obtain the claim. \square

The above result was also obtained by Gavrea and Ivan as an example for a general theorem concerning the limit of the iterates of positive linear operators (see [13]). No corresponding quantitative version is known to us.

4. VORONOVSKAYA RESULT FOR THE BERNSTEIN-TYPE OPERATORS $B_{n,0,j}$

The asymptotic formula of the operator $B_{n,0,j}$ was stated as a conjecture by Cárdenas-Morales *et al.* [8] and proved by Birou in [5].

THEOREM 2 (see [8], [5]). *For all $f \in C[0, 1]$, $x \in (0, 1)$ and $j \geq 1$, whenever $f''(x)$ exists,*

$$(9) \quad \lim_{n \rightarrow \infty} n(B_{n,0,j}(f; x) - f(x)) = \frac{x(1-x)}{2} f''(x) - \frac{(j-1)(1-x)}{2} f'(x).$$

A quantitative **pre-Voronovskaya theorem** for the Bernstein type operator $B_{n,0,j}$ was proved by Finta [12] using the first order Ditzian-Totik modulus of smoothness defined by

$$\omega_{\varphi}^1(f; \delta) = \sup_{0 < h \leq \delta} \sup_{x \pm \frac{1}{2}h\varphi(x)} \left| f(x + \frac{1}{2}h\varphi(x)) - f(x - \frac{1}{2}h\varphi(x)) \right|,$$

where $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0, 1]$. We call this "pre-Voronovskaya" since it appears to point into a good direction but the limit is not explicitly given.








THEOREM 3. [12] *There exists $C > 0$ depending only on j such that*







$$\left| n(B_{n,0,j}(f; x) - f(x)) + f'(x)nB_{n,0,j}(xe_0 - e_1; x) - \frac{1}{2}f''(x)nB_{n,0,j}((e_1 - xe_0)^2; x) \right| \leq C\omega_{\varphi}^1(f''; \frac{1}{\sqrt{n}}),$$

for all $x \in [0, 1]$, $f \in C^2[0, 1]$ and $n \geq j \geq 1$.

No quantitative version of a Voronovskaya-type theorem for the operators of Aldaz *et al.* is known to us.

REFERENCES

- [1] A.M. ACU, I. RAŞA, *New estimates for the differences of positive linear operators*, Numer. Algor., **73** (2016) no. 3, pp. 775–789, <https://doi.org/10.1007/s11075-016-0117-8>. 
- [2] O. AGRATINI, I.A. RUS, *Iterates of a class of discrete linear operator via contraction principle*, Comment. Math. Univ. Carol., **44** (2003) 3, 555–563.
- [3] J.M. ALDAZ, O. KOUNCHEV, H. RENDER, *Shape preserving properties of generalized Bernstein operators on extended Chebyshev spaces*, Numer. Math., **114** (2009) no. 1, pp. 1–25, <https://doi.org/10.1007/s00211-009-0248-0>. 
- [4] J.M. ALDAZ, H. RENDER, *Generalized Bernstein operators on the classical polynomial spaces*, Mediterr. J. Math., **15** (2018) art. id. 222, <https://doi.org/10.1007/s00009-018-1266-x>. 
- [5] M. BIROU, *A proof of a conjecture about the asymptotic formula of a Bernstein type operator*, Results Math., **72** (2017), pp. 1129–1138, <https://doi.org/10.1007/s00025-016-0608-x>. 
- [6] D. CÁRDENAS-MORALES, P. GARRANCHO, F.J. MUÑOZ-DELGADO, *Shape preserving approximation by Bernstein-type operators which fix polynomials*, Appl. Math. Comput., **182** (2006), 1615–1622, <https://doi.org/10.1016/j.amc.2006.05.046>. 
- [7] D. CÁRDENAS-MORALES, P. GARRANCHO, I. RAŞA, *Bernstein-type operators which preserve polynomials*, Comput. Math. Appl., **62** (2011), 158–163, <https://doi.org/10.1016/j.camwa.2011.04.063>. 
- [8] D. CÁRDENAS-MORALES, P. GARRANCHO, I. RAŞA, *Asymptotic formulae via a Korovkin-type result*, Abstract Appl. Anal., **2012** (2021), art. ID 217464, 12 pp., <https://doi.org/10.1155/2012/217464>. 

- [9] C. COTTIN, I. GAVREA, H. GONSKA, D. KACSÓ, DING-XUAN ZHOU, *Global smoothness preservation and the variation-diminishing property*, J. Inequal. Appl., **4** (1999) no. 2, 91–114, <https://doi.org/10.1155/s1025583499000314>. 
- [10] Z. FINTA, *Bernstein type operators having 1 and x^j as fixed points*, Cent. Eur. J. Math., **11** (2013), 2257–2261.
- [11] Z. FINTA, *Note on a Korovkin-type theorem*, J. Math. Anal. Appl., 415 (2014), 750–759, <https://doi.org/10.1016/j.jmaa.2014.02.010>. 
- [12] Z. FINTA, *A quantitative variant of Voronovskaja's theorem for King-type operators*, Constructive Mathematical Analysis, **2** (2019) no. 3, 124–129, <https://doi.org/10.33205/cma.553427>. 
- [13] I. GAVREA, M. IVAN, *Asymptotic behaviour of the iterates of positive linear operators*, Abstract Appl. Anal., **2011** (2011), art. ID 670509, 11 pp., <https://doi.org/10.1155/2011/670509>. 
- [14] H. GONSKA, *On approximation by linear operators: improved estimates*, Rev. Anal. Numér. Théor. Approx., **14** (1985), 7–32, <https://ictp.acad.ro/jnaat/journal/article/view/1985-vol14-no1-art2>. 
- [15] H. GONSKA, P. PIȚUL, *Remarks on an article of J.P. King*, Comment. Math. Univ. Carol., **46** (2005), 645–652.
- [16] J.P. KING, *Positive linear operators which preserve x^2* , Acta Math. Hungar., **99** (2003) no. 3, 203–208, <https://doi.org/10.1023/a:1024571126455>. 
- [17] M. MARSDEN, I.J. SCHOENBERG, *On variation diminishing spline approximation methods*, Mathematica, **8** (31) (1966) no. 1, 61–82.
- [18] R. PĂLTĂNEA, *Optimal constant in approximation by Bernstein operators*, J. Comput. Anal. Appl., **5** (2003), 195–235.
- [19] R. PĂLTĂNEA, *Approximation Theory using Positive Linear Operators*, Birkhäuser, Boston, 2004.

Received by the editors: received: August 13, 2021; accepted: September 16, 2021; published online: November 8, 2021.