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GOLDBACH PARTITIONS AND NORMS OF CUSP FORMS

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Abstract. An integral formula for the Goldbach partitions requires uniform convergence of a complex exponential sum. The dependence of the coefficients of the series is found to be bounded by that of cusp forms. Norms may be defined for these forms on a fundamental domain of a modular group. The relation with the integral formula is found to be sufficient to establish the consistency of the interchange of the integral and the sum, which must remain valid as the even integer N tends to infinity.

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1. INTRODUCTION

An equivalence between a formula for the number of Goldbach partitions of an even integer n

$$G(n) = \sum_{p_1,p_2} \int_0^1 e^{2\pi \mathrm{i}(p_1+p_2-n)\alpha} d\alpha = \sum_{p_1,p_2} \delta_{p_1+p_2,n} \tag{1.1}$$

and the contour integral

$$\int_0^1 \sum_{p_1} e^{2\pi \mathrm{i} p_1 \alpha} \sum_{p_2} e^{2\pi \mathrm{i} p_2 \alpha} e^{-2\pi \mathrm{i} n\alpha} d\alpha \tag{1.2}$$

exists if uniform convergence of the double sum can be proven.

The interchange of the double sum and the integral in the estimates of the number of prime partitions of integers through the circle method follows from the choice of a finite upper limit for the sums. However, for the binary partitions of an even integer, the error in the evaluation of the integral over the major and minor arcs is sufficiently large that the formula does not yield a non-zero lower bound for $G(n)$ [1].

Another method, equating the regularized surface integrals in (r,θ) and $(z,\bar{z})$ coordinates in a unit disk, provides a series expansion consisting of singular and nonsingular terms [2]. A finite upper limit again may be set for the sums in the contour integral, and following interchange of the sums

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and the integral, equivalence with the number of Goldbach partitions may be established. With a support function, the equality between the (r,θ) and $(z,\bar{z})$ integrals and cancellation of singular terms will generate nontrivial expressions for the coefficients in its definition and nonsingular relations between $\{G(m)\}$ for $4 \le m \le n$ [3]. Any ambiguities arising from the sums with arbitrarily large upper limits and the interchange of the sum and the integral in the formula for $G(n)$ are resolved in an expansion of the complex series $\sum_{m=4}^{\infty} G(m) e^{2\pi \mathrm{i} mz}$ in the upper-half plane $\mathbb{H}$. The divergence in the sum with an infinite limit is reflected in the singular terms in expansion of the sums in the powers of $y = \operatorname{Im} z$, and, after separating the contribution of the nonsingular terms, the values of contour integrals of related complex series exist. Therefore, although the coefficients in the recursion relation depend on the value of $\frac{\bar{z}}{z}$ in the limit $z \to 0$, a valid non-zero lower bound then can be derived for the number of Goldbach partitions of an even integer.

An evaluation of several sets of terms in the sums reveals that cancellations are required to establish a form of convergence for rational values of α. The double sum may be re-expressed as a series with coefficients related to the partitions of the integers $m \ge n$ into a sum of two primes. The periodicity of the complex exponential function for rational values of α characterizes the sum of the form $\sum_m a_m e^{2\pi \mathrm{i} m \frac{r}{s}}$. When α is irrational, it follows by the equidistribution theorem [4] that the distribution of the values of integer multiples of α modulo 1 is uniform, which would cause similar cancellations in the sum similarly. Bounds derived for finite exponential sums and arbitrary values of α are considered. A refined upper limit for the sums $S(\alpha) = \sum_{n \le N} \Lambda(n) e^{2\pi \mathrm{i} n\alpha}$ is known to have a similar polynomial form with a different exponent [5].

The relation with Fourier series $\sum_{m=4}^{\infty} a_m e^{2\pi i m x}$ then yields upper bounds for the coefficients to achieve absolute and uniform convergence. These inequalities are not satisfied initially by the series $\sum_{m=4}^{\infty} G(m) e^{2\pi i m x}$ in the contour integral (1.2). However, the series does have coefficients that have magnitudes between limits defined for Fourier series with coefficients increasing as $\mathcal{O}(m^c)$ and cusp forms. Analytic continuation of the Fourier series domain in the upper half plane to a neighbourhood of the interval $[0,1]$ therefore will provide a series expansion of bounds for the contour integral in powers of the imaginary coordinate y. Contour integration selects the $\mathcal{O}(1)$ term in the series and the values derived from the limiting series may be viewed as bounds related to the magnitude of $G(n)$.

2. SUMMATION IN CONTOUR INTEGRAL FORMULA

A set of terms in the sum $\sum_{p_1} \sum_{p_2} e^{2\pi \mathrm{i}(p_1+p_2-n)\alpha}$ will be considered for various values of α. First suppose that α is a rational number $\frac{r}{s}$. Then the exponential is equal to 1 if $p_1 + p_2 - n$ is a multiple of s. When r is even and s is odd, there will be an infinite positive contribution to the sum. Similarly, if r is odd and $2 \parallel s$, the exponential is -1 for $p_1+p_2-n = k\cdot\frac{s}{2}$ with k being odd, yielding $-\infty$.

Returning the fraction with even r and odd s, it is not possible to multiply r by a half-integer since p_1+p_2-n must be integer. Consequently, any choice given by a multiple of $\frac{s-1}{2}$ and $\frac{s+1}{2}$, for example, will yield a value that is close to $e^{2\pi \mathrm{i}}=1$. There is no such direct cancellation of sets of terms in the sum for a fixed value of α.

However, $\sum_{m=0}^{s-1} e^{2\pi\mathrm{i}\frac{m}{s}} = 0$, and there is a periodicity with lag s. The double sum can be equated to a series with weighting factors given by the number of Goldbach partitions of each integer. With the asymptotic estimate $G(m) \sim 2C_{twin}\frac{m}{\ln m \ln(m-2)}$, $\sum_{m=2}^{\infty} G(m)e^{2\pi\mathrm{i}(m-n)\frac{r}{s}}$ may be estimated.

Theorem 1. *The sum $\sum_{m=2}^{\infty} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}}$ is bounded below by*

$$\sum_{m=2}^{s-1} G(m)e^{2\pi\mathrm{i}\frac{r}{s}} + \frac{1}{e^{2\pi i\frac{r}{s}}-1}\left[\lim_{N\to\infty} G(N) - G(s)\right]$$
$$-\frac{C_{twin}s}{2(e^{2\pi\mathrm{i}\frac{r}{s}}-1)}\sum_{k=1}^{\infty}\left[\frac{1}{\left(k+\frac{1}{s}\right)(\ln(ks+1))^2\ln(ks-1)} + \frac{1}{\left(k-\frac{1}{s}\right)(\ln(ks+1))(\ln(ks+1))^2}\right]\cdot$$
$$\cdot\left[1-\frac{1}{\ln(ks+1)} - \frac{1}{\left(1-\frac{2}{ks+1}\right)\ln(ks-1)} + \mathcal{O}\left(\frac{1}{(ks+1)^2}\right)\right].$$

for odd r and even s.

Proof. When the series is partitioned into sets of s elements,

$$\sum_{m=4}^{\infty} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}} = \sum_{m=4}^{s-1} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}} + \sum_{k=1}^{\infty}\sum_{m=ks}^{(k+1)s-1} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}}. \tag{2.1}$$

Multiplication of the finite sum in the second term by $e^{2\pi\mathrm{i}\frac{r}{s}}-1$ gives

$$\begin{aligned}(e^{2\pi\mathrm{i}\frac{r}{s}}-1)\sum_{m=ks}^{(k+1)s-1} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}} &= \\ &= \sum_{m=ks+1}^{(k+1)s} G(m-1)e^{2\pi\mathrm{i}m\frac{r}{s}} - \sum_{m=ks}^{(k+1)s-1} G(m)e^{2\pi\mathrm{i}m\frac{r}{s}} \\ &= G((k+1)s-1) - G(ks) + \sum_{m=ks+1}^{(k+1)s-1}(G(m-1)-G(m))e^{2\pi\mathrm{i}m\frac{r}{s}}.\end{aligned} \tag{2.2}$$

By the asymptotic estimate for $G(m)$,

$$G(m-1)-G(m) \sim 2C_{twin}\left[\frac{m-1}{\ln(m-1)\ln(m-3)} - \frac{m}{\ln m\ \ln(m-2)}\right] = \tag{2.3}$$

$$= -2C_{twin}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}+\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big].$$

Then

$$\sum_{m=ks}^{(k+1)s-1} G(m)e^{2\pi i m\frac{r}{s}} =$$

$$= \frac{G((k+1)s-1)-G(ks)}{e^{2\pi i\frac{r}{s}}-1} \tag{2.4}$$

$$-\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{m=ks+1}^{(k+1)s-1}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big]$$

and

(2.5)

$$\sum_{k=1}^{\infty}\sum_{m=ks}^{(k+1)s-1} G(m)e^{2\pi i m\frac{r}{s}} = \frac{1}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}[G((k+1)s-1)-G(ks)]$$

$$-\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s-1}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big]$$

$$= \frac{1}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}[G((k+1)s)-G(ks)]+\frac{1}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}[G((k+1)s-1)-G((k+1)s)]$$

$$-\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s-1}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big]$$

$$= \frac{1}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}[G((k+1)s)-G(ks)]$$

$$-\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big]$$

$$= \frac{1}{e^{2\pi i\frac{r}{s}}-1}[\lim_{N\to\infty} G(N)-G(s)]$$

$$-\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s}\frac{1}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big].$$

If r is odd and s is even, $e^{2\pi i\left(m+\frac{s}{2}\right)\frac{r}{s}} = -e^{2\pi i m\frac{r}{s}}$, and

$$\sum_{ks+1}^{(k+1)s}\frac{e^{2\pi i m\frac{r}{s}}}{\ln m\ \ln(m-2)}\Big[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\Big] = \tag{2.6}$$

$$= \sum_{ks+1}^{\left(k+\frac{1}{2}\right)s} e^{2\pi i m\frac{r}{s}}\left\{\frac{1}{\ln m\ \ln(m-2)}\left[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m+\frac{s}{2}}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\right]\right.$$
$$\left.-\frac{1}{\ln\left(m+\frac{s}{2}\right)\ln\left(m+\frac{s}{2}-2\right)}\left[1-\frac{1}{\ln\left(m+\frac{s}{2}\right)}-\frac{1}{\left(1-\frac{2}{m+\frac{s}{2}}\right)\ln\left(m+\frac{s}{2}-2\right)}+\mathcal{O}\left(\frac{1}{\left(m+\frac{s}{2}\right)^2}\right)\right]\right\}.$$

Since

(2.7)
$$\frac{1}{\ln m\ \ln(m-2)}-\frac{1}{\ln\left(m+\frac{s}{2}\right)\ln\left(m+\frac{s}{2}-2\right)}=$$
$$=\frac{1}{\ln m\ \ln(m-2)}\left[1-\frac{1}{\left(1+\frac{\ln\left(1+\frac{s}{2m}\right)}{\ln m}\right)\left(1+\frac{\ln\left(1+\frac{s}{2(m-2)}\right)}{\ln(m-2)}\right)}\right]$$
$$=t\frac{1}{\ln m\ \ln(m-2)}\left[t\frac{\ln(1+\frac{s}{2m})}{\ln m}+\frac{\ln\left(1+\frac{s}{2(m-2)}\right)}{\ln(m-2)}+\mathcal{O}\left(\frac{1}{(\ln m)^2}\right)\right]$$
$$\approx\frac{s}{2m(\ln m)^2\ln(m-2)}+\frac{s}{2(m-2)\ln m(\ln(m-2))^2}+\mathcal{O}\left(\frac{1}{m(\ln m)^4}\right).$$

The following upper bound

(2.8)
$$\frac{2C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+\frac{1}{2})s}\left[\frac{s}{2m(\ln m)^2\ln(m-2)}+\frac{s}{2(m-2)\ln m(\ln(m-2))^2}\quad\mathcal{O}\left(\frac{1}{m(\ln m)^4}\right)\right]\cdot$$
$$\cdot\left[1-\frac{1}{\ln m}-\frac{1}{\left(1-\frac{2}{m}\right)\ln(m-2)}+\mathcal{O}\left(\frac{1}{m^2}\right)\right]$$
$$<\frac{C_{twin}}{e^{2\pi i\frac{r}{s}}-1}\sum_{k=1}^{\infty}\frac{s^2}{2}\left[\frac{1}{(ks+1)(\ln(ks+1))^2\ln(ks-1)}+\frac{1}{(ks-1)\ln(ks+1)(\ln(ks-1))^2}+\mathcal{O}\left(\frac{1}{(ks)(\ln(ks))^4}\right)\right]\cdot$$
$$\cdot\left[1-\frac{1}{\ln(ks+1)}-\frac{1}{\left(1-\frac{2}{ks+1}\right)\ln(ks-1)}+\mathcal{O}\left(\frac{1}{(ks+1)^2}\right)\right]$$
$$=\frac{C_{twin}s}{2(e^{2\pi i\frac{r}{s}}-1)}\sum_{k=1}^{\infty}\left[\frac{1}{\left(k+\frac{1}{s}\right)(\ln(ks+1))^2\ln(ks-1)}+\frac{1}{\left(k-\frac{1}{s}\right)\ln(ks+1)(\ln(ks-1))^2}+\mathcal{O}\left(\frac{1}{k\ (\ln(ks))^4}\right)\right]$$
$$\left[1-\frac{1}{\ln(ks+1)}-\frac{1}{\left(1-\frac{2}{ks+1}\right)\ln(ks-1)}+\mathcal{O}\left(\frac{1}{(ks+1)^2}\right)\right].$$

converges since the exponent of the logarithm equals 3. Therefore, the complex exponential series has the given lower bound. □

Then $\sum_{m=4}^{\infty}G(m)e^{2\pi i m\frac{r}{s}}$ diverges and the another method for defining the contour integral would be required when the integrand consists of infinite sums.

For sequences equidistributed modulo 1, $\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m}e^{2\pi i k a_i}=0$ [6], which resembles the vanishing of the finite sum for rational values of α. The limit $\lim_{m\to\infty}\sum_{i=1}^{m} f(a_i)$ equals $\int_0^1 f(x)dx$. Since

$$\begin{aligned}(2.9)\quad \sum_{m=2}^{\infty} G(m)e^{2\pi \mathrm{i}(m-n)\alpha} &= \sum_{m=2}^{n} G(m)e^{2\pi \mathrm{i}(m-n)\alpha} + \lim_{\ell'\to\infty}\sum_{\ell=1}^{\ell'} G(\ell+n)e^{2\pi \mathrm{i}\ell\alpha} \\ &= \sum_{m=2}^{n} G(m)e^{-2\pi \mathrm{i}(n-m)\alpha} + \lim_{\ell'\to\infty}\sum_{\ell=1}^{\ell'} G(\ell+n)e^{2\pi \mathrm{i}\eta_\ell}\end{aligned}$$

where $\eta_\ell = \ell\alpha \pmod 1$, the second sum could be re-expressed

$$(2.10)\qquad \int_0^1 G(x)e^{2\pi \mathrm{i}x}$$

with $G(x) = \lim_{q\to\infty} G(\ell_q+n)$, where $\{\ell_q\}$ is a subsequence such that $\lim\limits_{q\to\infty} \ell_q\alpha = x$, although the absence of a continuous limit prevents the integral from being well-defined.

Consider a complex exponential sum $f(\alpha) = \sum_{x_i=1}^{N} \exp(2\pi \mathrm{i}P(x_i)\alpha)$ where $P(x)$ is a polynomial of degree k. Then $\int_0^1 |f(\alpha)|^\lambda d\alpha \ll_{P,\epsilon} N^{\mu(\lambda)}$ where $(\lambda,\mu(\lambda))$ is located on a polygonal line with vertices $(2^\nu, 2^\nu-\nu+\epsilon)$, $\nu = 1,...,k$ [7]. For the set of primes, the polynomial would equal nearly $x\ \ln x$, which may be bounded above by a quadratic function of x. The squared absolute value is not equivalent to $f(\alpha)^2$, although the inequality

$$(2.11)\qquad \int_0^1 f(\alpha)^2 d\alpha \le \int_0^1 |f(\alpha)|^2 d\alpha$$

is valid. Choosing $\lambda = 2$, and supposing that the degree $k = 2$ may be used in the formula, $(2,\lambda(2))$ is located on the line with vertices $(2, 1+\epsilon)$ and $(4, 2+\epsilon)$ for some positive real number ϵ. It would follow that $\int_0^1 f(\alpha)^2 d\alpha \le N^{2+\epsilon}$.

Estimates increasing as a power of N have been found for finite complex exponential sums arising in the formula for the number of ternary Goldbach partitions of an odd integer. Let $S(\alpha) = \sum_{n\le N}\Lambda(n)e^{2\pi \mathrm{i}n\alpha}$. For every $\alpha\in\mathbb{R}$ such that there exists a fraction $\frac{a}{q}$ with $|\alpha-\frac{a}{q}| \le \frac{1}{q^2}$, $q\in\mathbb{Z}^+$, $\gcd(a,q)=1$, the upper bound $|S(\alpha)| \ll (Nq^{-\frac{1}{2}} + N^{\frac{4}{5}} + N^{\frac{1}{2}}q^{\frac{1}{2}})(\log\ N)^4$ [5].

3. UNIFORM CONVERGENCE OF THE FOURIER SERIES

The Fourier series $f(t) \sim \sum_m a_m e^{imt}$ converges uniformly to f in $[a,b]$ if $f(t)$ is continuous and satisfies periodic boundary conditions while $f'(t)$ is piecewise continuous on $[a,b]$. Uniform convergence of a series $\sum_n u_n$ may be determined by the Weierstrass M-test, where $\sup_{x\in I} u_n(x) \le M_n$ and $M_n \le \frac{C}{n^a}$, $C>0$, $a>1$. For functions in the α-Hölder class, the uniform convergence of the series requires $|f(t)-(S_Nf)(t)| \le K\frac{\ln N}{N^\alpha}$, where the partial sum is $(S_Nf)(t) = \sum_{m=-N}^{N} a_m e^{imt}$, and generally, the coefficients must be decreasing with n.

When the domain of the function is changed to the upper half plane, other types of coefficients are allowed. For cusp forms of weight $2k$, with $f(\gamma z) = (cz+d)^{2k}f(z)$, $\gamma z = \frac{az+b}{cz+d}$ and $a_0 = 0$, it can be demonstrated that $a_n = \mathcal{O}(n^k)$.

The L^2 convergence of a Fourier series, or equivalently the finiteness of the L^2 norm, may be extended to the functions $F(z) = \mathrm{Im}(z)^{\frac{k}{2}} f(z)$ [8][9]. These norms are defined in terms of integrals over the fundamental domain of the modular group $PSL(2;\mathbb{Z})$ which does not reach the real line. An analytic modular form in this fundamental domain may be mapped to a nonsingular function on the upper-half plane, although analytic continuation would be necessary for the definition of the function on the interval $[0,1]$ since the image of the boundary is not include a segment on the real axis after a finite number of transformations.

A modular or cusp form with a finite norm related to the series $\sum_{m=4}^{\infty} G(m)e^{2\pi i m\alpha}$ would provide bounds for an absolutely convergent sum in the fundamental domain and the upper-half plane. An example would be Eisenstein series [10]

$$G_{2k}(z) = 2\zeta(2k) + 2(-1)^k \tfrac{(2\pi)^{2k}}{(2k-1)!} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) e^{2\pi i n z},$$

with $\sigma_\ell(n) \sim \zeta(\ell+1)n^\ell + \mathcal{O}(n^{max(0,\ell-1)})$, being the generalized sum-of-divisor function, and coefficients $|a_n| = \mathcal{O}(n^{2k-1})$, where k may be chosen to be $\frac{3}{4}$ for the lower bound and 1 for the upper bound.

The dependence of the coefficients in this series is given by

$$(3.1)\quad 2C_{twin}\tfrac{m}{\ln m \ln(m-2)} = 2C_{twin}\tfrac{m}{e^{\ln\ln m} e^{\ln\ln(m-2)}} = 2C_{twin} m^{1-\frac{\ln(\ln m \ln(m-2))}{\ln m}}.$$

and the exponent would be $1 - \frac{\ln(\ln m \ln(m-2))}{\ln m}$.

If $a_m = \mathcal{O}(m^c)$, for constant c, $f(z) = \sum_{m=0}^{\infty} a_m e^{2\pi i m z} = \mathcal{O}(y^{-c-1})$ because $|f(x+\mathrm{i}y)| < \sum_{m=0}^{\infty} a_m e^{-2\pi m y}$ and

(3.2)
$$\sum_{m=0}^{\infty} m^c e^{-2\pi m y} = \tfrac{(-1)^c}{(2\pi)^c}\tfrac{d^c}{dy^c} \sum_{n=0}^{\infty} e^{-2\pi n y} = \tfrac{(e^{-2\pi m y})^c}{(1-e^{-2\pi y})^{c+1}} + \ldots = (2\pi)^{-c-1} y^{-c-1} + \ldots$$

near the real line. To derive the exponent of the coefficients from the dependence of the function, let $|f(x+iy)| < B(x)y^{-c}$ and consider

(3.3)
$$\frac{1}{\lambda}\int_{\tau_0}^{\tau_0+\lambda}\int_0^{\infty} B(x)y^{-c} e^{\frac{-2\pi i m(x+iy)}{\lambda}}\ dx\ dy = \frac{1}{\lambda}\int_{\tau_0}^{\tau_0+\lambda} B(x)e^{-2\pi m i x} dx \int_0^{\infty} y^{-c} e^{-\frac{2\pi m y}{\lambda}} dy.$$

Let $\tilde{y} = my$. Then

(3.4)
$$\int_0^{\infty} y^{-c} e^{-\frac{2\pi m y}{\lambda}} dy = \int_0^{\infty} m^c \tilde{y}^{-c} e^{-\frac{2\pi m \tilde{y}}{\lambda}} \frac{1}{m} d\tilde{y} \quad = m^{-c-1}\int_0^{\infty} \tilde{y}^{-1} e^{-\frac{2\pi\tilde{y}}{\lambda}} d\tilde{y}.$$

With the introduction of a non-zero lower limit, the integral is $\mathcal{O}(m^{c-1})$.

Given a modular or cusp form of a given weight, there is a corrsponding meromorphic function with isolated poles satisfying a functional relation in s

[11]. This meromorphic function equals

$$\Phi(s) = \left(\tfrac{\lambda}{2\pi}\right)^s \Gamma(s)\varphi(s) \tag{3.5}$$

and there then exists an analytic continuation of the $\varphi(s) = \sum_m \frac{a_m}{m^s}$ and the formula

$$f(\mathrm{i}y) - a_0 = \sum_{m=1}^{\infty} a_m e^{-\frac{2\pi m y}{\lambda}} = \sum_{m=1}^{\infty} a_m \frac{1}{2\pi \mathrm{i}} \int_{\mathrm{Re}\, s=\sigma_0} \Gamma(s) \left(\tfrac{2\pi m y}{\lambda}\right)^{-s} ds \tag{3.6}$$
$$= \int_{\mathrm{Re}\, s=\sigma_0} \Phi(s) y^{-s} ds$$

may be used to determine the analytic continuation of $f(iy)$. For a cusp form with $a_0 = 0$, if y is replaced by $-\mathrm{i}x$,

$$f(x) = \int_{\mathrm{Re}\, s=\sigma_0} \Phi(s)(-ix)^{-s} ds. \tag{3.7}$$

When $a_m = \mathcal{O}(m^c)$, $\mathrm{Re}\, s$ must be selected to be larger than $c+1$ for the series $\varphi(s) = \sum_{m=1}^{\infty} a_m m^{-s}$ to converge. The contour of the integral $\int_{\mathrm{Re}\, s=\sigma_0} \Phi(s) y^{-s} ds$ is located to the right of the singularities of $\Gamma(s)$ or $\varphi(s)$ such that $\Phi(s)$ converges as a series.

The analogue of $f(x)$ when $\varphi(s)$ is $\zeta(2s)$ is $\theta(t) = 12 + \sum_{n=1}^{\infty} e^{\pi \mathrm{i} n^2 t}$. Then $\theta(t+2) = \theta(t)$, $\theta(-\frac{1}{t}) = (\frac{t}{\mathrm{i}})^{\frac{1}{2}} \theta(t)$. Then $\theta(\mathrm{i}t) = \frac{1}{2} + \sum_{m=1}^{\infty} e^{-\pi m^2 t} \to \frac{1}{2}$ as $t \to \infty$ and there is a divergence if $t \to 0$. The theta function is conventionally defined only on the upper half-plane, because the contribution of each of the arcs in the contour integral (3.6) can prevent its evaluation for real t. It may be noted, however, that a direct evaluation of theta function yields $\frac{1}{2}$ as $t \to 1$ if the alternating sum is set equal to zero. For $t \in (0,1)$, the value of $\theta(t)$ would determined by distribution $\{n^2 t (\mathrm{mod}\, 1)\}$. However, if an infinitesimal imaginary part is added to the integration variable, the function would be regularized, since

$$\theta(x + \mathrm{i}y) = \tfrac{1}{2} + \sum_{m=1}^{\infty} e^{-\pi m^2 y} e^{\pi \mathrm{i} m^2 x}. \tag{3.8}$$

There are summation and integral representations of the zeta function that are valid for other regions in the complex plane [12], and the Mellin transform provides another formula for the theta function.

The series $\sum_{m=4}^{\infty} G(m) m^{-s}$ is bounded by $\sum_{m=1}^{\infty} m m^{-s} = \sum_{m=1}^{\infty} m^{-(s-1)} = \zeta(s-1)$ which has a pole at $s = 2$. When $s = 2$,

$$\sum_{m=4}^{\infty} G(m) m^{-s} \simeq \sum_{m=4}^{\infty} \frac{2C_{twin} m}{\ln m \ln(m-2)} m^{-2} = \sum_{m=4}^{\infty} \frac{2C_{twin}}{m \ln m \ln(m-2)} < \infty \tag{3.9}$$

because $\int_4^\infty \frac{dx}{x \ln x \ln(x-2)} < \infty$, and any singularity would be located to the right to the right of the line $\mathrm{Re}\, s = \frac{1}{2}$. Since

$$\sum_{m=4}^{\infty} m^{\frac{1}{2}} e^{-2\pi m y} < \sum_{m=4}^{\infty} \frac{m}{\ln m \ \ln(m-2)} e^{-2\pi m y} < \sum_{m=4}^{\infty} m e^{-2\pi m y}, \tag{3.10}$$

(3.11)
$$(2\pi y)^{-\frac{3}{2}} - (1+\sqrt{2}+\sqrt{3}) + ... < \sum_{m=4}^{\infty} \frac{m}{\ln m \ \ln(m-2)} e^{-2\pi m y} < (2\pi y)^{-2} ... - 6 + ... \ ,$$

and the analytic continuation of a series bounded by two other series in the neighbourhood of an interval is bounded by the analytic continuation of the limiting series.

Let $a_m = m$, $\varphi(s) = \zeta(s-1)$, $\Phi(s) = \left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)\zeta(s-1)$ and $\Phi(3-s) = \left(\frac{\lambda}{2\pi}\right)^{3-s} \Gamma(3-s)\zeta(2-s)$. Given $\xi(s) = \pi^{-\frac{\pi}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$ and the functional relation $\xi(s) = \xi(1-s)$, $\pi^{-\frac{s-1}{2}} \Gamma\left(\frac{s-1}{2}\right)\zeta(s-1) = \pi^{-\frac{2-s}{2}} \Gamma\left(\frac{2-s}{2}\right)\zeta(2-s)$. There are no constant values of λ and k yielding an equality between $\Phi(s)$ and $\Phi(3-s)$, and the Fourier series is not a modular form. Similarly, if $a_m = m^{\frac{1}{2}}$, $\varphi(s) = \zeta\left(s - \frac{1}{2}\right)$, $\Phi(s) = \left(\frac{\lambda}{2\pi}\right)^s \Gamma(s)\zeta\left(s - \frac{1}{2}\right)$ and $\Phi\left(\frac{5}{2} - s\right) = \left(\frac{\lambda}{2\pi}\right)^{\frac{5}{2}-s} \Gamma\left(\frac{5}{2} - s\right)\zeta\left(\frac{3}{2} - s\right)$, and $\Phi(s) \neq \Phi\left(\frac{5}{2} - s\right)$ for constant λ and k, and this lower bound is not a modular form.

Nevertheless, these Fourier series will be sufficient to establish bounds for $\sum_{m=4}^{\infty} G(m) m^{-s}$ by removal of the singularities in the upper and lower limits as $y \to 0$. This result has been established generally for Dirichlet series with almost period coefficients through a bound by the integral representation for Lerch transcendents [13]. The integral of the series with coefficients $G(n)$ would be well-defined over the interval $[0,1]$ because the pole is not located at $s = 1$.

Since a Fourier series $f(z) = \sum_{m=1}^{\infty} a_m e^{2\pi i m z}$ diverges as $\mathcal{O}(y^{-c-1})$ when the coefficients a_m increase as $\mathcal{O}(m^c)$, it may be expanded in powers of y, such that the terms with negative exponents reflect the singularity on the real axis. A full expansion of the limiting series in Eq. (3.6) yields

$$\begin{aligned} &2C_{twin}\left[(2\pi y)^{-\frac{3}{2}} + ... - (1+\sqrt{2}+\sqrt{3}) + ...\right] < \\ &< \sum_{m=4}^{\infty} G(m) e^{-2\pi m y} < 2C_{twin}\left[(2\pi y)^{-2} y^{-2} - \tfrac{73}{12} + \tfrac{113}{4} y + ... \ \right]. \end{aligned} \tag{3.12}$$

Inclusion of the complex argument when $x = \frac{r}{s}$ in the upper bound for the Fourier series gives

(3.13)
$$\sum_{m=4}^{\infty} m e^{-2\pi m y} e^{2\pi i m \frac{r}{s}} = \sum_{m=4}^{s-1} m e^{-2\pi m y} e^{2\pi i m \frac{r}{s}} + \sum_{k=1}^{\infty} \sum_{m=ks}^{(k+1)s-1} m e^{-2\pi m y} e^{2\pi i m \frac{r}{s}}.$$

Theorem 2. *The expansion of the upper limit for the complex series* $\sum_{m=4}^{\infty} G(m)e^{2\pi \mathrm{i} mx}e^{-2\pi my}$ *in powers of* y *begins at* $\mathcal{O}\left(\frac{1}{y^2}\right)$.

A calculation similar to that for $G(m)$ yields

$$\sum_{m=4}^{s-1} me^{-2\pi my}e^{2\pi \mathrm{i} m\frac{r}{s}} + \frac{1}{e^{2\pi i\frac{r}{s}}-1}[\lim_{N\to\infty} Ne^{-2\pi Ny} - se^{-2\pi sy}]- \tag{3.14}$$

$$- \frac{1}{e^{2\pi \mathrm{i}\frac{r}{s}}-1}\sum_{r=1}^{\infty}\sum_{m=ks+1}^{(k+1)s} e^{2\pi \mathrm{i} m\frac{r}{s}}(me^{-2\pi my} - (m-1)e^{-2\pi(m-1)y})$$

Near $y = 0$,

(3.15)

$$me^{-2\pi my} - (m-1)e^{-2\pi(m-1)y} = 1 - 2\pi(2m-1)y + 2\pi^2(3m^2 - 3m + 1)y^2 - ...$$

and

(3.16)

$$- \lim_{y\to 0}\frac{1}{e^{2\pi \mathrm{i}\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s} e^{2\pi \mathrm{i} m\frac{r}{s}}[1-2\pi(2m-1)y+2\pi^2(3m^2-3m+1)y^2+...]$$

$$= \lim_{y\to 0}\frac{1}{e^{2\pi \mathrm{i}\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s} [4\pi mye^{2\pi \mathrm{i} m\frac{r}{s}} - 6\pi^2 m(m-1)y^2e^{2\pi \mathrm{i} m\frac{r}{s}} + ...] = 0$$

and $\lim_{N\to\infty} Ne^{-2\pi Ny} = 0$, such that the remainder of Eq.(3.14) is

$$\lim_{y\to 0}\left\{\frac{1}{e^{2\pi \mathrm{i}\frac{r}{s}}-1}(-se^{-2\pi sy}) + \sum_{m=4}^{s-1} me^{-2\pi my}e^{2\pi \mathrm{i} m\frac{r}{s}}\right\} \tag{3.17}$$

Since $se^{-2\pi sy}$ is maximized at $s = \frac{1}{2\pi y}$, let the denominator of the fractions have this value such that this expression is

$$\begin{aligned}
& - \frac{1}{2\pi ey}\frac{1}{e^{4\pi^2 \mathrm{i} ry}-1} + \sum_{m=4}^{\frac{1}{2\pi y}-1} me^{-2\pi my+4\pi^2 \mathrm{i} mry} \\
&= \frac{\mathrm{i}}{8\pi^3 ery^2}\left(1 + 2\pi^2 \mathrm{i} ry - 4\pi^4 r^2 y^2 + ...\right) \\
&\quad - (e^{-2\pi y+4\pi^2 \mathrm{i} ry} + 2e^{-4\pi y+8\pi^2 \mathrm{i} ry} + 3e^{-6\pi y+12\pi^2 \mathrm{i} ry}) \\
&\quad - \frac{1}{2\pi y(1-2\pi \mathrm{i} r)}\left(1 - \frac{(2\pi-4\pi^2 \mathrm{i} r)}{2}y + ...\right)\left(1-\frac{1}{e}\right) - \frac{1}{2\pi ey} \\
&= \frac{\mathrm{i}}{8\pi^3 ery^2}\left(1 + 2\pi^2 \mathrm{i} y - 4\pi^2 r^2 y^2 + ...\right) \\
&\quad - (e^{-2\pi y+4\pi^2 \mathrm{i} ry} + 2e^{-4\pi y+8\pi^2 \mathrm{i} ry} + 3e^{-6\pi y+12\pi^2 \mathrm{i} ry}) \\
&\quad - \frac{\left(1-\frac{1}{e}\right)}{2\pi y(1-2\pi \mathrm{i} r)} - \frac{1}{2\pi ey} + \frac{1}{2}\left(1-\frac{1}{e}\right) + \dots\ .
\end{aligned} \tag{3.18}$$

Integration of the variable x over $[0,1]$ can be split into a sum over the numerator r and a summation over irrational values. As

$$\sum_{\ell=1}^{\infty}\sum_{r=0}^{\frac{1}{2\pi y}-1}\tfrac{1}{r} < 1+\int_1^{\frac{1}{2\pi y}-1}\tfrac{dx}{x} = 1+\ln\left(\tfrac{1}{2\pi y}-1\right) \tag{3.19}$$
$$\sum_{r=0}^{\frac{1}{2\pi y}-1}\tfrac{1}{r}ry = \tfrac{1}{2\pi y}\cdot y = \tfrac{1}{2\pi}$$
$$\sum_{r=0}^{\frac{1}{2\pi y}-1}\tfrac{1}{r}r^2y^2 = \tfrac{1}{8\pi^2}-\tfrac{y}{4\pi},$$

the summation over r in the first term in Eq.(3.18) gives

(3.20)

$$\sum_r \tfrac{\mathrm{i}}{8\pi^3 ery^2}(1-2\pi^2\mathrm{i}ry-...) =$$
$$= \tfrac{\mathrm{i}}{8\pi^3 ey^2}\Big[\left(1+\ln\left(\tfrac{1}{2\pi y}-1\right)\right) \qquad -2\pi^2\mathrm{i}\left(\tfrac{1}{2\pi}\right)-\tfrac{4}{3}\pi^4\left(\tfrac{1}{8\pi^2}-\tfrac{y}{4\pi}\right)+...\Big].$$

The expansion of $\frac{1}{e^{4\pi^2\mathrm{i}ry}-1}$ follows from $e^{4\pi^2\mathrm{i}ry}-1 = 4\pi^2\mathrm{i}ry\sum_{k=0}^{\infty}\frac{(4\pi^2\mathrm{i}ry)^k}{(k+1)!}$ and

(3.21)

$$\frac{1}{\sum_{k=0}^{\infty}\frac{(4\pi^2\mathrm{i}ry)^k}{(k+1)!}} = \frac{1}{1+\sum_{k=1}^{\infty}\frac{(4\pi^2ry)^k}{(k+1)!}} =$$
$$= 1-\sum_{k=1}^{\infty}\frac{(4\pi^2\mathrm{i}ry)^k}{(k+1)!}+\left(\sum_{k=1}^{\infty}\frac{(4\pi^2\mathrm{i}ry)^k}{(k+1)!}\right)^2-\left(\sum_{k=1}^{\infty}\frac{(4\pi^2\mathrm{i}ry)^k}{(k+1)!}\right)^3+...$$
$$= 1+\sum_{k=1}^{\infty}(4\pi^2\mathrm{i}ry)^k\sum_{\ell=1}^{\infty}(-1)^\ell\sum_{k_1+...+k_\ell=k}\frac{1}{(k_1+1)!...(k_\ell+1)!}.$$

Similarly,

$$\sum_{r=0}^{\frac{1}{2\pi y}-1}\tfrac{1}{r}(ry)^k = y^k\sum_{r=0}^{\frac{1}{2\pi y}-1}r^{k-1} = \tfrac{1}{k}\left(\tfrac{1}{2\pi}\right)^k+\mathcal{O}(y) \tag{3.22}$$

Then

(3.23)

$$\sum_r \tfrac{\mathrm{i}}{8\pi^3 ery^2}(1-2\pi^2\mathrm{i}ry+...) =$$
$$= \tfrac{\mathrm{i}}{8\pi^3 ey^2}\left[1+\ln\left(\tfrac{1}{2\pi y}-1\right)+\sum_{k=1}^{\infty}\tfrac{(2\pi\mathrm{i})^k}{k}\sum_{\ell=1}^{\infty}(-1)^\ell\sum_{k_1+...+k_\ell=k}\tfrac{1}{(k_1+1)!...(k_\ell+1)!}\right]+\mathcal{O}\left(\tfrac{1}{y}\right).$$

and the summation over r of the entire expression (3.18) is

(3.24)

$$\frac{\mathrm{i}}{8\pi^3 ey^2}\left[1+\ln\left(\frac{1}{2\pi y}-1\right)+\sum_{k=1}^{\infty}\frac{(2\pi\mathrm{i})^k}{k}\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{k_1+\ldots+k_\ell=k}\frac{1}{(k_1+1)!\ldots(k_\ell+1)!}\right]+\ldots$$
$$-\sum_{r=0}^{\frac{1}{2\pi y}-1}\left(e^{-2\pi y+4\pi^2\mathrm{i}ry}+2e^{-4\pi y+8\pi^2\mathrm{i}ry}+3e^{-6\pi y+12\pi^2\mathrm{i}ry}\right)$$
$$-\frac{\left(1-\frac{1}{e}\right)}{2\pi y}\left[1+\frac{1}{2\pi}\sum_{k=2}^{\infty}\left(\frac{\mathrm{i}^{k-1}}{ky^k}\right)\right]-\frac{1}{4\pi^2 ey^2}+\frac{1}{4\pi y}\left(1-\frac{1}{e}\right)+\ldots\,.$$

The sums $\sum_{r=0}^{\frac{1}{2\pi y}-1} e^{4\ell\pi^2\mathrm{i}ry}$ vanish, leaving a series beginning at $\mathcal{O}\left(\frac{1}{y^2}\right)$,

(3.25)

$$\frac{\mathrm{i}}{8\pi^3 ey^2}\left[1+\ln\left(\frac{1}{2\pi y}-1\right)+\sum_{k=1}^{\infty}\frac{(2\pi\mathrm{i})^k}{k}\sum_{\ell=1}^{\infty}(-1)^{\ell}\sum_{k_1+\ldots+k_\ell=k}\frac{1}{(k_1+1)!\ldots(k_\ell+1)!}\right]+\ldots$$
$$-\frac{\left(1-\frac{1}{e}\right)}{2\pi y}\left[1+\frac{1}{2\pi}\sum_{k=2}\left(\frac{i^{k-1}}{ky^k}\right)\right]-\frac{1}{4\pi^2 ey^2}+\frac{1}{4\pi y}\left(1-\frac{1}{e}\right)+\ldots\,.$$

because $\sum_{k\geq 2}\frac{\mathrm{i}^{k-1}}{ky^k}=-\mathrm{i}\sum_m\frac{(-1)^m}{2my^{2m}}+\sum_m\frac{(-1)^m}{(2m+1)y^{2m+1}}\approx\left(\frac{\mathrm{i}}{2}-\frac{1}{2y}\right)\ln\left(1-\frac{1}{y^2}\right)$.

One estimate of the contribution of the irrational values of x follows from the

$$\sum_{m=4}^{\infty} me^{-2\pi my}e^{2\pi\mathrm{i}mx}=\sum_{m=4}^{\infty} me^{-2\pi my}e^{2\pi\mathrm{i}\eta_m} \tag{3.26}$$

where $\{\eta_m = mx (\mathrm{mod}\,1)\}$ is an equidistributed sequence over the unit interval. It is evident that the series $\sum_{m=4}^{\infty} e^{-2\pi my}e^{2\pi\mathrm{i}\eta_m}$ is absolutely convergent for $y>0$ since

$$\sum_{m=4}^{\infty} e^{-2\pi my}=\frac{1}{e^{2\pi y}-1}-\left(e^{-2\pi y}e^{2\pi\mathrm{i}\eta_1}+e^{-4\pi y}e^{2\pi\mathrm{i}\eta_2}+e^{-6\pi y}e^{2\pi\mathrm{i}\eta_3}\right). \tag{3.27}$$

The derivative of the complex series is

$$\frac{d}{dx}\sum_{m=4}^{\infty} e^{-2\pi my}e^{2\pi\mathrm{i}mx}=2\pi\mathrm{i}\sum_{m=4}^{\infty} me^{-2\pi my}e^{2\pi\mathrm{i}\eta_m}. \tag{3.28}$$

In the limit as $y\to 0$, absolute convergence is not preserved. Furthermore, there does not exist N_0 such that the partial sum $S_N=\sum_{m=4}^{N} e^{2\pi imx}$ differs from S_M for $M,\ N>N_0$ by less than ϵ for a given N_0, and the sum is not uniformly convergent. Consequently, the derivative and the limit cannot be interchanged. Nevertheless,

$$(3.29)\quad \frac{1}{2\pi\mathrm{i}}\frac{d}{dx}\sum_{m=4}^{\infty} e^{-2\pi my}e^{2\pi\mathrm{i}mx}$$
$$= \frac{1}{2\pi\mathrm{i}}\frac{d}{dx}\left(\frac{1}{e^{2\pi y}e^{-2\pi\mathrm{i}x}-1}\right) - (e^{-2\pi y}e^{2\pi\mathrm{i}\eta_1} + 2e^{-4\pi y}e^{2\pi\mathrm{i}\eta_2} + 3e^{-6\pi y}e^{2\pi\mathrm{i}\eta_3})$$
$$= \frac{e^{2\pi y}e^{-2\pi\mathrm{i}x}}{(e^{2\pi y}e^{-2\pi\mathrm{i}x}-1)^2} - (e^{-2\pi y}e^{2\pi\mathrm{i}\eta_1} + 2e^{-4\pi y}e^{2\pi\mathrm{i}\eta_2} + 3e^{-6\pi y}e^{2\pi\mathrm{i}\eta_3}),$$

and, since the limit can be interchanged with finite sums, estimates of partial sums $\sum_{m=4}^{N} me^{2\pi\mathrm{i}\eta_m}$ exist.

Although the limit

$$\lim_{y\to 0}\frac{1}{e^{2\pi\mathrm{i}\frac{r}{s}}-1}\sum_{k=1}^{\infty}\sum_{m=ks+1}^{(k+1)s} e^{2\pi\mathrm{i}\frac{r}{s}}(me^{-2\pi my} - (m-1)e^{-2\pi(m-1)y})$$

had been shown to be zero, if s is set equal to $\frac{1}{2\pi y}$, it equals

$$(3.30)\quad \lim_{y\to 0}\frac{1}{e^{4\pi^2\mathrm{i}ry}-1}\sum_{k=1}^{\infty}\sum_{m=\frac{k}{2\pi y}+1}^{\frac{k+1}{2\pi y}} [4\pi ye^{4\pi^2\mathrm{i}mry} - 6\pi^2 m(m-1)y^2e^{4\pi^2\mathrm{i}mry} + ...]$$

which will include $\mathcal{O}(1)$ terms in the expansion in powers of y.

The beginning of the series at $\mathcal{O}\left(\frac{1}{y^2}\right)$ is evident in the formula

$$\frac{1}{e^{2\pi\mathrm{i}\frac{r}{s}}-1}\sum_{m=s}^{\infty} 4\pi mye^{2\pi\mathrm{i}m\frac{r}{s}}\Big|_{s=\frac{1}{2\pi y}} =$$
$$(3.31)$$
$$= \frac{4\pi y}{e^{4\pi^2\mathrm{i}ry}-1}\lim_{N\to\infty}\frac{1}{4\pi^2\mathrm{i}r}\frac{d}{dy}\sum_{m=s}^{N} e^{4\pi^2\mathrm{i}mry}$$
$$= \frac{4\pi y}{e^{4\pi^2\mathrm{i}ry}-1}\lim_{N\to\infty}\frac{\left[(e^{4\pi^2\mathrm{i}ry}-1)(N+1)\cdot 4\pi^2\mathrm{i}r\cdot e^{4\pi^2\mathrm{i}(N+1)ry} - (e^{4\pi^2\mathrm{i}(N+1)ry}-1)\cdot 4\pi^2\mathrm{i}re^{4\pi^2\mathrm{i}ry}\right]}{(e^{4\pi^2\mathrm{i}ry}-1)^2}.$$

With $\lim_{N\to\infty}\overline{e^{4\pi^2\mathrm{i}(N+1)ry}} = 0$, the remainder is

$$(3.32)\quad \frac{4\pi ye^{4\pi^2\mathrm{i}ry}}{(e^{4\pi^2\mathrm{i}ry}-1)^3} = \frac{4\pi ye^{4\pi^2\mathrm{i}ry}}{(4\pi^2\mathrm{i}ry - 8\pi^4r^2y^2 + ...)^3}$$

which has a leading-order dependence on y of $\mathcal{O}\left(\frac{1}{y^2}\right)$. Summation over r with the upper limit $\frac{1}{2\pi y} - 1$ does not alter the initial exponent of the series. □

The $\mathcal{O}\left(\frac{1}{y^\ell}\right)$ terms with $\ell > 0$ generally have arguments which do not cancel with $e^{-2\pi\mathrm{i}n\alpha}$, and the integrals over α would vanish. This leaves an $\mathcal{O}(1)$ term that must generate the asymptotic formula for the number of Goldbach partitions. Therefore, separating the series in the form

$$\sum_{m=4}^{N} m e^{-2\pi m y} e^{2\pi i m x} + \sum_{m=N+1}^{\infty} e^{-2\pi m y} e^{2\pi i m x}, \tag{3.33}$$

the first sum in the limit $y \to 0$ equals

$$\sum_{m=4}^{N} m e^{2\pi i x} = (N+1)\tfrac{e^{2\pi i(N+1)x}}{e^{2\pi i x}-1} - (e^{2\pi i x} + 2e^{4\pi i x} + 3e^{6\pi i x}). \tag{3.34}$$

Since $\sum_{\substack{p_1,\ p_2 \\ 4\le p_1+p_2\le N}} e^{2\pi i(p_1+p_2-n)\alpha} = e^{-2\pi i n\alpha}\sum_{m=4}^{N} G(m)e^{2\pi i m\alpha}$, the integral over the majorized sum

$$\begin{aligned}\int_0^1 e^{-2\pi i n x}\Big[\tfrac{(N+1)e^{2\pi i(N+1)x}}{e^{2\pi i x}-1} - (e^{2\pi i x} + 2e^{4\pi i x} + 3e^{6\pi i x})\Big] = \\ = \int_0^1 \Big[(N+1)e^{2\pi i(N-n)x}(1 + e^{-2\pi i x} + e^{-4\pi i x} + \ldots) \\ - (e^{-2(n-1)\pi i x} + 2e^{-2(n-2)\pi i x} + 3e^{-2(n-3)\pi i x})\Big]dx\end{aligned} \tag{3.35}$$

equals $N+1$ for $N \ge n$ and $n \ge 4$, which is an upper bound for $G(n)$.

A bound for the contour integral (1.2) over the interval $[0,1]$ can be defined by removing the singular terms proportional to negative powers of y in Eq.(3.25). The other limit would follow from the estimate of the series $\sum_{m=4}^{\infty} m^{\frac{1}{2}} e^{-2\pi m y} e^{2\pi i m x}$. Finiteness of the integral could be considered with respect to the formula (1.1) with the sum and integral interchanged. However, it serves only as a verification of the asymptotic estimate of the number of Goldbach partitions which must be used for all of the integers in the sum.

The contour integral (1.2) may be converted to surface integrals over the unit disk [2]. Cancellation of the singular terms in the (r,θ) and $(z,\bar{z})$ integrals yields a set of nonsingular equations that determine a relation between the numbers of binary Goldbach partitions of even integers. The method of the removal of divergences is similar to that required for a finite remainder that may be related to $G(n)$. It may be recalled that a support function had been necessary to derive nontrivial relations between $\{G(m)\},\ m \ge n$. Although there is no direct correlation because a conversion to surface integrals is necessary, these integrals in the $z, \bar{z}$ coordinates may be represented through a Cauchy formula that includes derivatives of the function at an interior point and contour integrals.

Evaluation of the contour integrals before the summation generically selects specific values of the index as a consequence of the vanishing the integral of an exponential with non-zero argument over the circle. It is this identity that causes the series representing the number of Goldbach partitions to be truncated in Eq. (1.1). The representations of the number of binary prime partitions of even integers in Eq. (1.2) and the number of ternary prime partitions of odd integers are regularized conventionally through finite upper limits for the sums, such that the bounded sums and integrals may be interchanged. This technique is valid because the sums do not continue indefinitely, with a given value being selected in the contour integral. It follows, therefore, that the use of the formula with finite upper limits in the sums would allow the method of conversion of the integral to a surface integral and the subsequent derivation of recursion relations for number of Goldbach partitions.

4. CONCLUSION

There exists a formula for the number of Goldbach partitions $G(n)$ which is given by a summation over prime pairs of the contour integrals of the exponential $e^{2\pi \mathrm{i}(p_1+p_2-n)\alpha}$, $\alpha \in [0,1]$. Interchange of the sum and the integral yields an equivalent method for evaluating this number if the series is uniformly convergent. Conventional techniques consist of replacing the infinite upper limit by a finite value, such that the interchange is valid, and estimating the contributions to the contour integral. However, in the limit $n \to \infty$, the condition of the interchange of the integral and sum must be satisfied for a valid representation of the number of binary prime partitions of all even integers greater than or equal to four.

The double sum may be equated with $e^{-2\pi \mathrm{i} nx} \sum_{m=4}^{\infty} G(m) e^{2\pi \mathrm{i} mx}$ which is divergent for real x. A Fourier series $\sum_m a_m e^{2\pi \mathrm{i} mx}$ is convergent only for decreasing coefficients a_m. Complex series $\sum_m a_m e^{2\pi \mathrm{i} mz}$ with coefficients increasing as $a_m = \mathcal{O}(m^c)$ are characteristic of cusp forms, which converge in the upper half-plane $\mathrm{Im}\, z > 0$ and have norms defined on fundamental regions of a modular group. Similarly, the series $\sum_{m=4}^{\infty} G(m) e^{2\pi i m z}$ will be uniformly convergent in $\mathbb{H}$, even though the coefficients are increasing.

The divergence on the real line prevents a direct analytic continuation to the interval $[0,1]$. The meromorphic function $\Phi(z) = (2\pi)^{-z}\Gamma(z)\varphi(z)$, where $\varphi(z) = \sum_{m=4}^{\infty} \frac{G(m)}{m^z}$ is well-defined for $\mathrm{Re}\, z \geq 2$ and can be analytically continued to the rest of the complex plane. The function represented by the series $\sum_{m=4}^{\infty} G(m) e^{2\pi \mathrm{i} mz}$ then might be determined through a Mellin transform of $\Phi(z)$, although the contribution of each arc in the contour integral could prevent its evaluation. The theta function, for example, which is derived for $\varphi(s) = \zeta(2s)$, does not have a standard definition on the real line.

An alternative technique for defining the complex series begins with an expansion of the series in powers of $y = \mathrm{Im}\, z$. It is demonstrated that the series in y begins at $\mathcal{O}\left(y^{-2}\right)$ and will diverge as $y \to 0$. However, this formula allows the interpretation of $G(n)$ in terms of the contour integral of the nonsingular part after removal of the singular terms. The integral of the $\mathcal{O}(1)$ terms would give the number of Goldbach partitions and this is verified by its evaluation for the limiting series $\sum_{m=4}^{\infty} m e^{-2\pi m y} e^{2\pi i m x}$ for the series representation of the integrand in the formula for the asymptotic estimate of $G(n)$.

There is no strict ordering of the complex series. There is an ordering of the real sums of the magnitudes with the asymptotic estimates of $G(m)$. This method therefore receives confirmation from the validity of the estimate of $G(n)$ for each value of n. It cannot be used to prove relations for the exact value of $G(n)$ without valid bounds. However, the existence of singular terms in the expansion in powers of y and their removal in the evaluation of the contour integral provide support for this technique more generally.

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