

COMMENSURABILITY CLASSES CONTAINING THREE KNOT COMPLEMENTS

NEIL HOFFMAN

ABSTRACT. This paper exhibits an infinite family of hyperbolic knot complements that have three knot complements in their respective commensurability classes.

1. INTRODUCTION

The study of the commensurability classes of hyperbolic knot complements that contain other knot complements has attracted some recent interest (see [BBW],[CD],[GHH] [HS],[MM], [NR1],[Re], [RW]). A particularly interesting set of examples results from cyclic surgeries on hyperbolic knot complements, since the cyclic surgeries give rise to cyclic covers by other knot complements (see [GW]). Moreover, The Cyclic Surgery Theorem [CGLS] shows that there are at most two non-trivial cyclic surgeries on a hyperbolic knot complement and so a hyperbolic knot complement has at most two non-trivial, finite sheeted covers which are other knot complements. Similarly, if a hyperbolic knot complement, $S^3 - k_1$ is covered by another knot complement, $S^3 - k_2$, then $S^3 - k_1$ admits a cyclic surgery. There are known examples of hyperbolic knot complements with exactly three knot complements in their commensurability classes. For example, the $(-2, 3, 7)$ pretzel knot of [FS] famously admits two non-trivial cyclic surgeries and is therefore covered by two other hyperbolic knot complements.

An infinite family of pairs of commensurable hyperbolic knot complements was constructed by W. Neuman.

For a discussion of this construction, see [GHH].

Finally, two hyperbolic knot complements can be commensurable if they both have hidden symmetries. This property is equivalent to both knot complements

non-normally covering the same orbifold (see § 2.2). The dodecahedral knots of [AR] admit the only known examples of non-arithmetic knot complements with hidden symmetries (see [NR1]) and the figure 8 knot complement is the only arithmetic knot complement (see [Re]).

This discussion motivates the following conjecture of Reid and Walsh (see [RW, Conj 5.2]).

Conjecture. *Let $S^3 - K$ be a hyperbolic knot complement. Then, there are at most two other knot complements in its commensurability class.*

It has been announced by Boileau, Boyer, and Walsh ([BBW, Thm 1.3]) that the conjecture holds for knot complements without hidden symmetries. In their paper, they show that if a hyperbolic knot complement does not admit hidden symmetries, then any commensurable hyperbolic knot complement will cover a common orbifold. Furthermore, this orbifold admits a finite cyclic surgery for each knot complement that covers it. This paper presents a family of such orbifolds that are covered by exactly three hyperbolic knot complements. Specifically, the main theorem of this paper is the following (see § 2 for definitions):

Theorem 1.1. *Let $n \geq 1$ and $(n, 7) = 1$. For all but at most finitely many pairs of integers (n, m) , the result of (n, m) Dehn surgery on the unknotted cusp of the Berge manifold is a hyperbolic orbifold with exactly three knot complements its commensurability classes.*

The infinite family of orbifolds described by Theorem 1.1 which we refer to as $\beta_{n,m}$ (see §2) also has the property that for $n \neq 1$, each knot complement covering $\beta_{n,m}$ admits an n -fold symmetry which does not fix any point on the cusp. In particular, even when $n = 2$, this symmetry is not a strong involution. By [WZ], such a knot complement cannot admit a lens space surgery and so, by the above discussion, is not covered by any other knot complement.

The paper is organized as follows. In addition to some background material and definitions, § 2 we prove a lemma about possible orbifold quotients of the Berge manifold. In § 3, we show that the orbifolds $\beta_{n,m}$ are shown to admit three

cyclic surgeries, and the proof of the main theorem is contained in §4. In § 5, we provide a partial classification of commensurability classes containing three knot complements.

2. PRELIMINARIES

2.1. Two hyperbolic 3-orbifolds, \mathbb{H}^3/Γ_1 and \mathbb{H}^3/Γ_2 , are said to be commensurable if they share a common finite sheeted cover. In terms of groups, $\exists g \in PSL(2, \mathbb{C})$ so that Γ_1 and $g\Gamma_2g^{-1}$ have a common subgroup which is finite index in both groups.

Let $Comm^+(\Gamma) = \{g \in PSL(2, \mathbb{C}) \mid [\Gamma : \Gamma \cap g\Gamma g^{-1}] < \infty \text{ and } [g\Gamma g^{-1} : \Gamma \cap g\Gamma g^{-1}] < \infty\}$ and $N^+(\Gamma)$ be the normalizer of Γ in $PSL(2, \mathbb{C})$. We say that a group Γ has *hidden symmetries* if $[Comm^+(\Gamma) : N^+(\Gamma)] > 1$. A hyperbolic orbifold, M , has *hidden symmetries* if $\pi_1^{orb}(M)$ has hidden symmetries. For this discussion, we consider only orientable manifolds and orbifolds.

2.2. When a hyperbolic knot group has hidden symmetries the associated knot complement non-normally covers some orbifold with a *rigid cusp* i.e. the cusp is $C \times [0, \infty)$ where C is $S^2(2, 3, 6)$, $S^2(3, 3, 3)$ or $S^2(2, 4, 4)$ (see [Re, Lemma 4]).

By [NR1, Prop 2.7], the cusp field of a hyperbolic orbifold is a subfield of the invariant trace field. Thus, if a hyperbolic orbifold has a $S^2(3, 3, 3)$ or $S^2(2, 3, 6)$ cusp, $\mathbb{Q}(\sqrt{-3})$ must be a subfield of the orbifold's invariant trace field and if the cusp is $S^2(2, 4, 4)$, $\mathbb{Q}(i)$ must be a subfield of the orbifold's invariant trace field (see [NR1, Proof of Thm 5.1(iv)]).

Proposition 2.1. *Let $p : O_1 \rightarrow O_2$ be a covering of orbifolds such that O_1 has a rigid cusp C_1 . Then, O_2 has a rigid cusp C_2 such that $p(C_1) = C_2$ and if $x \in C_2$ then $|p^{-1}(x) \cap C_1| = n^2$ for some integer n unless C_1 is $S^2(3, 3, 3)$ and C_2 is $S^2(2, 3, 6)$ then $|p^{-1}(x) \cap C_1| = 2n^2$ for some integer n .*

Proof. First consider the case where C_1 is an $S^2(2, 4, 4)$. In this case, C_2 must also be a $S^2(2, 4, 4)$ cusp. The peripheral subgroup corresponding to C_2 is $P_2 \cong (\mathbb{Z} \times \mathbb{Z}) \rtimes_{\phi} \mathbb{Z}/4\mathbb{Z}$, and so P_2 has an element of order 4 acting on the cusp. Thus, $\phi : \mathbb{Z}/4\mathbb{Z} \rightarrow Aut(\mathbb{Z} \times \mathbb{Z})$ is a faithful representation. Let $P_1 \subset P_2$ be the peripheral subgroup corresponding to C_1 . So $P_1 \cong (n\mathbb{Z} \times m\mathbb{Z}) \rtimes_{\phi} \mathbb{Z}/4\mathbb{Z}$. However, the order

4 automorphism switches the two generators for the $\mathbb{Z} \times \mathbb{Z}$ subgroup of P_2 . Thus, $n = m$ and the degree of the covering is n^2 .

A similar proof carries through if C_1 and C_2 are both either $S^2(3, 3, 3)$ or $S^2(2, 3, 6)$ cusps.

In the case, where C_1 is a $S^2(3, 3, 3)$ and C_2 is a $S^2(2, 3, 6)$ cusp, the $\mathbb{Z}/3\mathbb{Z}$ subgroup of P_1 is index 2 in the $\mathbb{Z}/6\mathbb{Z}$ subgroup of P_2 . Hence, the covering degree is $2n^2$.

□

2.3. For $n \geq 1$ and $(n, 7) = 1$, let $\beta_{n,m}$ be the orbifold obtained by (n, m) Dehn surgery on the unknotted cusp of the Berge manifold (see Figure 1) using a standard framing on the cusps of this link complement as in [Ro].

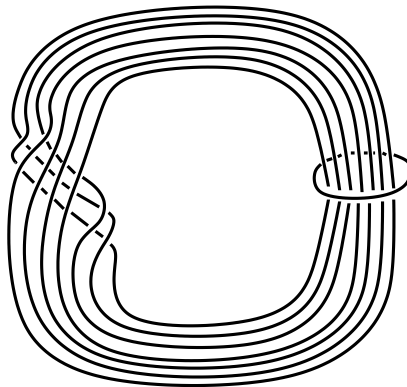


FIGURE 1. The Berge manifold is the complement of this link.

The Berge manifold admits several surgery slopes of interest. First if we perform Dehn surgery along the $(1, 0)$ slope of the unknotted cusp of the Berge manifold, we will obtain the $(-2, 3, 7)$ pretzel knot (see [FS]). Also, if we drill out a solid torus along the unknotted cusp of the manifold we would obtain the one of the knots in the solid torus that admits three $D^2 \times S^1$ fillings (see [Be, Cor 2.9]). Furthermore, if we perform Dehn surgery along the $(1, r)$ slope and then drill along the core of the surgered torus, we would also obtain a knot complement in $D^2 \times S^1$ that admits three $D^2 \times S^1$ surgeries. In fact, by the above mentioned corollary, these are the only knots in solid tori with this property.

The above construction shows that Dehn surgery along a $(1, r)$ slope of the unknotted cusp of the Berge manifold produces knot complements that admit three lens space surgeries. In fact, it is well known that the $(1, 0)$, $(18, 1)$ and $(19, 1)$ surgery slopes on the $(-2, 3, 7)$ pretzel knot admit lens space surgeries (see [FS]). By drilling out the unknotted cusp of the Berge manifold, these are also the surgery slopes that produce a solid torus filling. Since the linking number of the knotted cusp and the unknotted cusp is 7, the longitude gets sent to the curve $(49r, 1)$ after $(1, r)$ Dehn surgery on the unknotted cusp while the meridian $(1, 0)$ remains fixed (see [Ro, Sect 9.H]). So the $(1, 0)$, $(18, 1)$, and $(19, 1)$ surgery parameters get sent to $(1, 0)$, $(49r + 18, 1)$, and $(49r + 19, 1)$ respectively after $(1, r)$ Dehn surgery on the unknotted cusp. Furthermore, we can use the surgery parameters to compute the homology of the manifolds resulting from lens space surgeries on the knot complements. In fact, we see that for these knots we obtain S^3 and two lens spaces - one with fundamental group of order $|49r + 18|$ and another of order $|49r + 19|$.

More generally, if we allow Dehn surgery along any (p, q) slope of the unknotted cusp of the Berge manifold where $(p, q) = 1$, and either $(1, 0)$, $(18, 1)$, or $(19, 1)$ Dehn surgery on the knotted cusp, we will also get lens spaces. Again, by [Ro, Sect 9.H], we see that the $(1, 0)$ surgery slope corresponds to a lens space of order $|p|$, $(18, 1)$ surgery slope corresponds to a lens space of order $|49q + 18p|$, and $(19, 1)$ surgery slope corresponds to a lens space of order $|49q + 19p|$.

2.4. Denote $v_0 \approx 1.01494146$ as the volume of the regular ideal tetrahedron. The Berge manifold is comprised of four such tetrahedra and therefore its volume is $4v_0$. Denote by Γ_L as the fundamental group of the Berge manifold. Since the complement of the Berge manifold is comprised of four regular ideal tetrahedra, $\Gamma_L \subset Isom^+(\mathbb{T}) \cong PGL(2, \mathbb{O}_3)$, where \mathbb{T} is a tessellation of \mathbb{H}^3 by regular ideal tetrahedra. Hence, the Berge manifold is arithmetic.

The proof of the following lemma takes advantage of the fact that the Berge manifold has relatively low volume in order to show that it cannot cover an orbifold with a torus cusp and a rigid cusp. Where necessary, we consider all groups as subgroups in $PSL(2, \mathbb{C})$.

Lemma 2.2. *The Berge manifold does not cover an orbifold with a torus cusp and a rigid cusp.*

Proof of 2.2. Assume Q_T is an orbifold with a torus cusp and a rigid cusp covered by the Berge manifold. Since the invariant trace field of the Berge manifold is $\mathbb{Q}(\sqrt{-3})$, the rigid cusp of Q_T must be either $S^2(3, 3, 3)$ or $S^2(2, 3, 6)$. In either case, consideration of the unknotted torus cusp of the Berge manifold covering the rigid cusp shows the degree of such a cover is $3k$ for some integer $k \geq 1$. Also, since the Berge manifold is arithmetic and the class number of $\mathbb{Q}(\sqrt{-3})$ is 1, it follows from [CLR, Thm 1.1], that any maximal group commensurable with the Berge manifold has exactly one cusp. Thus, there exists a one-cusped orbifold Q_M covered by Q_T . By consideration of the cusps of Q_T covering the rigid cusp of Q_M (see Prop 2.1), we see that the covering degree of such a map would be $3l + n^2$ or $3l + 2n^2$ for some integers l, n (In the later case, l must be even).

Thus, the covering of Q_M by the Berge manifold is of order $d = 3k(3l + n^2)$ or $d = 3k(3l + 2n^2)$. Now, $d \leq 48$ (see [Me]) and since $k, l, n \geq 1$, we have that $d \geq 12$. Hence, $vol(Q_M) \leq v_0/3$ if Q_M has a $S^2(3, 3, 3)$ cusp and $vol(Q_M) \leq v_0/6$ if Q_M has a $S^2(2, 3, 6)$ cusp.

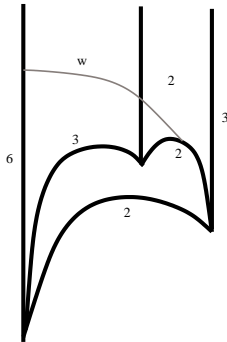


FIGURE 2. The fundamental domain for Γ together with the involution w

It follows that this orbifold must appear on the lists in [A, Thm 3.3, 4.2] and [NR2]. However, none of the orbifolds with $S^2(3, 3, 3)$ cusps appearing on these lists correspond to maximal groups commensurable with the Berge manifold, so we may

assume that Q_M has a $S^2(2, 3, 6)$ cusp. After combining the above restrictions on the degree of a cover and the restrictions from Adams' list, there are two possibilities for Q_M :

either Q_M has volume $v_0/6$ and a $S^2(2, 3, 6)$ cusp (here $k = 1, l = 2, n = 1$) or Q_M has volume $v_0/12$ and a $S^2(2, 3, 6)$ cusp (here $k = 2, l = 2, n = 1$).

First, consider the case where Q_M has volume $v_0/6$. By noting that $\pi_1^{orb}(Q_M)$ has an index 2 subgroup $\Gamma := \langle x, y, z | x^2, y^2, z^3, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3 \rangle$ and $\pi_1^{orb}(Q_M) = \langle \Gamma, w \rangle$ where w is the order 2 rotation on the fundamental domain of Γ , we obtain a presentation for $\pi_1^{orb}(Q_M)$ (see [NR1], [MR] and Figure 2).

Thus, we obtain the following presentation

$$\pi_1^{orb}(Q_M) = \langle w, x, y, z | x^2, y^2, z^3, w^2, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3, (wx)^2, wywyz^{-1} \rangle.$$

However, using GAP, the above group does not have any index 8 subgroups. Thus, there can be no orbifold Q_T .

In second case, $Q_M \cong \mathbb{H}^3/PGL(2, \mathbb{O}_3)$ and the $[PGL(2, \mathbb{O}_3) : \pi_1^{orb}(Q_T)] = 8$. If $\pi_1^{orb}(Q_T) \subset PSL(2, \mathbb{O}_3)$, $[PSL(2, \mathbb{O}_3) : \pi_1^{orb}(Q_T)] = 4$. Using GAP, there is a unique index 4 subgroup G of $PSL(2, \mathbb{O}_3)$. However, G has finite abelianization, and therefore cannot be the orbifold group of Q_T .

Thus, we may assume that $\pi_1^{orb}(Q_T) \not\subset PSL(2, \mathbb{O}_3)$ and deduce that there is a unique subgroup Λ of index 2 in $\pi_1^{orb}(Q_T)$ such that $\Lambda \subset PSL(2, \mathbb{O}_3)$. By covolume considerations Λ has index 8 in $PSL(2, \mathbb{O}_3)$. Also, \mathbb{H}^3/Λ has a torus cusp and an $S^2(3, 3, 3)$ cusp. Since $\mathbb{H}^3/PSL(2, \mathbb{O}_3)$ has an $S^2(3, 3, 3)$ cusp, the degree of the covering $p : \mathbb{H}^3/\Lambda \rightarrow \mathbb{H}^3/PSL(2, \mathbb{O}_3)$ has to be $3l + n^2$ (see Prop 2.1), which is never 8.

This completes the proof. \square

3. CYCLIC SURGERIES ON $\beta_{n,m}$

In this section, we show that for fixed n and m , $\beta_{n,m}$ admits three finite cyclic surgeries. We also show directly it is covered by three knot complements if $n \neq 7$.

Lemma 3.1. *The orbifolds $\beta_{n,m}$ are covered by three knot complements. Furthermore, the degrees of the corresponding covering maps are distinct.*

Proof. For a fixed $\beta_{n,m}$, let $r = (n, m)$ and consider $\beta_{n,m}$ as the union of the complement of a knot in a solid torus, T_1 and a solid torus with core a singular locus of order r , T_2 (see Figure 3).

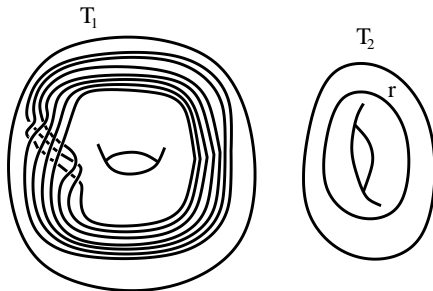


FIGURE 3. The decomposition of a surgered $\beta_{n,m}$ along a torus

By [Be, Cor 2.9], T_1 admits three Dehn surgeries that result in a solid torus. Thus, $\beta_{n,m}$ admits three Dehn surgeries that are homeomorphic to T_2 and a solid torus glued together along their boundaries. Each orbifold O_j ($j \in \{1, 2, 3\}$) resulting from one of these Dehn surgeries has underlying space a lens space with $\pi_1^{orb}(O_j)$ finite cyclic.

In fact, $|\pi_1^{orb}(O_j)|$ is distinct for each choice of j . To see this we observe, as noted above, that O_j is an orbifold with underlying space a lens space. Moreover, this underlying space is a lens space with fundamental group of order either $\frac{n}{r}$, $|49\frac{m}{r} + 18\frac{n}{r}|$, or $|49\frac{m}{r} + 19\frac{n}{r}|$ depending on the choice of surgery on T_1 (see § 2). Splitting O_j into a solid torus coming from the Dehn surgery on T_1 and T_2 the solid torus core a singular curve, we can compute $\pi_1^{orb}(O_j)$ using van Kampen's theorem. Thus, the orders of the each fundamental group increase by a factor of r and $|\pi_1^{orb}(O_j)|$ is either n , $r \cdot |49\frac{m}{r} + 18\frac{n}{r}|$ or $r \cdot |49\frac{m}{r} + 19\frac{n}{r}|$ which take on three distinct values for fixed n , m and r .

In addition, by the Orbifold Theorem (see [BP, Thm 2]) and the above argument that $\pi_1^{orb}(O_j)$ is finite cyclic, each O_j has S^3 as its universal cover. Denote this

covering map $\phi_j : S^3 \rightarrow O_j$. We may view O_j as the union of the solid torus coming from the cusp Dehn filling of $\beta_{n,m}$ and the complement of this solid torus, which we denote by B .

Hence $\phi_j^{-1}(B)$ is a knot or link exterior in S^3 . Since $(n, 7) = 1$ and the singular set of T_2 has linking number 7 with the knotted cusp of $\beta_{n,m}$, the boundary of $\phi_j^{-1}(B)$ is connected. Hence, if $(n, 7) = 1$, $\beta_{n,m}$ will be covered by three knot complements in S^3 . Also, since the orders of $|\pi_1^{orb}(O_j)|$ are distinct, the covering degree of ϕ_j will take on a distinct value for each j .

□

Remark 3.2. *When $n = 1$, the classification of exceptional Dehn surgeries in [MP, Table A.1, Rem A.3] shows that $\beta_{n,m}$ is hyperbolic. Hence, $\beta_{1,m}$ is a hyperbolic knot complement that admits three cyclic surgeries.*

4. PROOF OF THE MAIN THEOREM

In this section, we prove Theorem 1.1. Also for this section, we consider $\Omega_{n,m}$, $\Delta_{n,m}$, and Ω_L as subgroups of $PSL(2, \mathbb{C})$.

Proof of Theorem 1.1. Using Lemma 3.1, each $\beta_{n,m}$ is covered by three knot complements such that the covers are of distinct degrees. Also, the Hyperbolic Dehn Surgery Theorem [Th, Thm 5.8.2] shows that all but at most finitely many of the $\beta_{n,m}$ are hyperbolic. For the rest of the proof we only consider those $\beta_{n,m}$ that are hyperbolic. Given this condition, each $\beta_{n,m}$ we consider is covered by three distinct knot complements. By [BBW, Thm 1.3], to prove Theorem 1.1 it suffices to show that the knot complements covering $\beta_{n,m}$ do not have hidden symmetries.

Suppose an infinite number of the hyperbolic knot complements that cover $\beta_{n,m}$ admit hidden symmetries. By the discussion in §2.2, every such a knot complement will non-normally cover an orbifold $Q_{n,m}$ with a rigid cusp. Furthermore, on passage to a subset of the $\beta_{n,m}$, we can assume that the orbifolds $Q_{n,m}$ have the same type of rigid cusp, C . Let $\Omega_{n,m} = \pi_1^{orb}(\beta_{n,m})$, $\Delta_{n,m} = \pi_1^{orb}(Q_{n,m})$ and let $P \subset PSL(2, \mathbb{C})$ be the peripheral subgroup of $\Delta_{n,m}$. We may assume that each $\Omega_{n,m}$ is conjugated

so that P has a fixed representation in $PSL(2, \mathbb{C})$. Since $\beta_{n,m}$ has one cusp, notice that $\Delta_{n,m} = P \cdot \Omega_{n,m}$

By Thurston's Hyperbolic Dehn Surgery Theorem [Th, Thm 5.8.2], the volumes of the $\beta_{n,m}$ are bounded from above by the volume of the Berge manifold. In addition, the minimum volume of a non-compact oriented hyperbolic 3-orbifold is $\frac{v_0}{12}$ (see [Me]). Hence, $vol(Q_n) \geq \frac{v_0}{12}$. Thus, we can further subsequence to arrange that $\beta_{n,m}$ covers $Q_{n,m}$, that the $Q_{n,m}$'s have the same type of rigid cusp, and that the covering degree is fixed, say d .

Since $\beta_{n,m}$ is obtained by Dehn surgery on the Berge manifold, the $\Omega_{n,m}$ will converge algebraically and geometrically to Ω_L , the fundamental group of the Berge manifold (see [Th, Thm 5.8.2]). As P was a fixed group in our construction, $\Delta_{n,m}$ also converges algebraically and geometrically to $P \cdot \Omega_L$.

We have the following diagram:

$$\begin{array}{ccc} \Delta_{n,m} & \xrightarrow{(n,m) \rightarrow \infty} & P \cdot \Omega_L \\ \uparrow d & & \uparrow d \\ \Omega_{n,m} & \xrightarrow{(n,m) \rightarrow \infty} & \Omega_L \end{array}$$

Note, $[P \cdot \Omega_L : \Omega_L] = d < \infty$. Let $Q_T = \mathbb{H}^3/P \cdot \Omega_L$. Q_T has two cusps: a torus cusp, corresponding to the cusp created by geometric convergence from Dehn surgery, and a rigid cusp, corresponding to the cusp with peripheral group P .

However by Lemma 2.2, such a limiting Q_L cannot exist. Hence, at most finitely many of the $\beta_{n,m}$ have hidden symmetries.

□

Remark 4.1. *To find explicit examples of hyperbolic knot complements with three knot complements in the commensurability class, we can use the computer program `snap` to show directly that there are no hidden symmetries. Specifically, for $m=0$ and $n=2,3,4,5,6,7$, $\beta_{n,m}$ is hyperbolic and `snap` show us that $\beta_{n,m}$ has an invariant trace field with real embeddings. These fields cannot contain $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$ as subfields. Thus, these knot complements do not have hidden symmetries (recall § 2.2) and there are exactly three knot complements in the commensurability classes.*

5. REMARKS

The following theorem provides a partial classification of hyperbolic orbifolds covered by three knot complements. It can be seen as a direct corollary to a result of [BBW]. However, a proof is provided below for completeness.

Theorem 5.1. *Let O be a closed 3-orbifold and let K be a knot in O that is disjoint from singular locus of O . If $O - K$ is:*

- (1) *hyperbolic,*
- (2) *covered by 3 knot complements,*
- (3) *does not admit hidden symmetries, and*
- (4) *O has non-empty singular locus,*

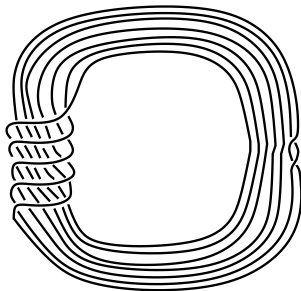
then $O - K \cong \beta_{n,m}$ for some pair (n, m) .

Proof. Let γ be the singular locus of O . Denote $|O|$ the underlying space of O . By [BBW, Thm 1.2] and the assumptions, we know that $|O|$ is a lens space, γ is a non-empty subset of the cores of a genus 1 Heegaard splitting of $|O|$, and if $S^3 - K$ covers $O - K$ then it does so cyclically and corresponds to a finite cyclic filling of $O - K$. Finally, denote $M = O - \gamma - K$

First assume γ has one component. Each of the three knot complements covering $O - K$ will correspond to a $S^1 \times D^2$ filling on knotted cusp of M . Again, we appeal to the fact that there is a unique family of knots in solid tori that admits 3 non-trivial $S^1 \times D^2$ fillings (see [Be, Cor 9.1]). Hence, M is obtained by performing $(1, m)$ surgery on the unknotted cusp of the Berge manifold then drilling out the core of the surgered torus. Gluing back in the neighborhood of the fixed point set of $\langle \gamma \rangle$ gives us $\beta_{n,m}$ for some n, m .

Now, assume that γ has two components γ_1 and γ_2 . $M = T^2 \times I - K'$, where K' is a knot. Each of the three finite cyclic on $O - K$ corresponds M admitting a $T^2 \times I$ filling. Hence, Dehn filling along the cusp corresponding to γ_1 will produce a knot complement in $D^2 \times S^1$ with three $D^2 \times S^1$ fillings.

Denote l_1 to be the linking number of γ_1 and K' and l_2 to be the linking number of γ_2 and K' . If l_1 is zero, K' would be a knot in a solid torus that is not a 1-braid

FIGURE 4. The $K(7,5,2,-1)$

after $(1, 0)$ on γ_2 but has two non-trivial $S^1 \times D^2$ fillings. This contradicts [Be, Cor 9.1]. Hence, we may assume $l_1 \neq 0$ and $l_2 \neq 0$.

Also, $(1, n)$ surgery on γ_2 will produce a knot K'' in a solid torus that has linking number $l_2 + n \cdot l_1$ with γ_2 . In particular for large enough n $l_2 + n \cdot l_1 \neq 7$. Hence, it cannot be in the family of knots that admit two non-trivial $S^1 \times D^2$ fillings. \square

One might hope to relax condition (4) above. However, Brandy Guntel pointed out that the $K(7, 5, 2, -1)$ knot complement (see Figure 4) is hyperbolic and admits two non-trivial cyclic surgeries. The fundamental group of one of these lens spaces is of order 32. By our original discussion in §2.3, knot complements obtained by Dehn surgery on the unknotted cusp of the Berge manifold have lens spaces of order $|49r - 18|$ and $|49r - 19|$ neither of which can be 32. Hence, the $K(7, 5, 2, -1)$ complement is not one of the $\beta_{n,m}$. However, since the invariant trace field of the $K(7, 5, 2, -1)$ is an odd degree extension of \mathbb{Q} , we see that this knot complement does not admit hidden symmetries and the $K(7, 5, 2, -1)$ has exactly three knot complements in its comensurability class (see [RW, Cor 5.4]).

As mentioned above $(1, m)$ surgery on the unknotted cusp of the Berge manifold produces Berge knots. It seems natural to ask if any hyperbolic Berge knots can have hidden symmetries. More generally, we might ask if any hyperbolic knot complements can have hidden symmetries and admit non-trivial lens space surgeries. As discussed in § 1, there are three hyperbolic knot complements known to have hidden symmetries: the complements of the two dodecahedral knots of Aitchison and Rubinstein, and the figure eight knot complement (see [AR],[NR1]).

Using SnapPea one can see that both dodecahedral knots are amphichiral. Thus, by [CGLS, Cor 4] they cannot admit a lens space surgery. Additionally, it is well known that the figure eight knot complement does not admit a lens space surgery (see [Ta] for example).

6. ACKNOWLEDGMENTS

First, I would like to thank Alan Reid for raising the questions that lead to this paper and thoughtfully guiding this work from its formative stages to completion. I also would like to thank Cameron Gordon for showing me the family of knot complements with three lens space surgeries $\beta_{1,m}$. Third, I would like to thank Genevieve Walsh and Steve Boyer for a number of enlightening conversations and their suggestions on early versions of this paper. Finally, I would like to thank my fellow graduate students for a number of helpful conversations.

REFERENCES

- [A] C. Adams, *Non-compact 3-orbifolds of small volume* in *Topology '90*, Ohio State Univ. Math. Res. Inst. Publ. Vol. 1, de Gruyter (1992) pp. 273-310.
- [AR] I. R. Aitchison and J. H. Rubinstein, *Combinatorial cubings, cusp and the dodecahedral knots*, in *Topology '90*, Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter (1992), pp. 17-26.
- [Be] J. Berge, *The knots in $D^2 \times S^1$ with nontrivial Dehn surgeries yielding $D^2 \times S^1$* , *Topology Appl.* Vol. 38, (1991) pp. 1-19.
- [BBW] M. Boileau, S. Boyer, and G. Walsh, *On commensurability of knot complements*, preprint.
- [BP] M. Boileau, J. Porti, *Geometrization of 3-orbifolds of cyclic type*, *Astérisque* 272 (2001).
- [CD] D. Calegari and N. M. Dunfield, *Commensurability of 1-cusped hyperbolic 3-manifolds*, *Trans. Amer. Math. Soc.* Vol. 354 (2002), pp. 2955-2969.
- [CGLS] M. Culler, C.McA Gordon, J. Leucke and P. Shalen, *Dehn surgery on knots*, *Ann. of Math.* Vol. 125 (1989) pp. 237-300.
- [CLR] T. Chinburg, D. Long, and A. W. Reid, *Cusps of minimal non-compact arithmetic hyperbolic 3-orbifolds*, *Pure and Applied Math Quarterly*, Vol. 4 (2008) pp. 1013-1031.
- [FS] R. Fintushel and R. Stern, *Constructing lens spaces by surgery on knots*, *Math. Z.* Vol. 175 (1980) pp. 33-51.
- [GHH] O. Goodman, D. Heard, and C. Hodgson, *Commensurators of Cusped Hyperbolic Manifolds*, *Experimental Mathematics*, Vol. 17 no 3 (2008), pp 283-306.

- [GW] F. González-Acuña and W.C. Whitten, *Imbeddings of three-manifold groups*, Mem. Amer. Math. Soc. Vol. 474 (1992).
- [HS] J. Hoste and P. Shanahan, *Commensurability classes of twist knots*, J. Knot Theory Ramifications Vol. 14 no. 1,(2005) pp. 91-100.
- [MR] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3-Manifolds*, Springer New York (2003).
- [MP] B. Martelli and C. Petronio, *Dehn filling of the "magic" 3-manifold*, Comm. Anal. Geom. Vol. 14, No. 5 (2006), p. 969-1026.
- [Me] R. Meyerhoff, *The cusped hyperbolic 3-orbifold of minimum volume*, Bull. Amer. Math. Soc. Vol. 13 (1985) pp. 154-156.
- [MB] J. W. Morgan and H. Bass (editors), *The Smith Conjecture*, Academic Press, New York (1984).
- [MM] M. Macasieb and T. Mattman, *Commensurability classes of $(-2,3,n)$ pretzel knot complements*, Alg. & Geom. Top. 8(3) (2008), pp. 1833-1853.
- [NR1] W. D. Neuman and A. W. Reid, *Arithmetic of hyperbolic manifolds*, in Topology '90, Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter (1992), pp. 273-310.
- [NR2] W. D. Neuman and A. W. Reid, *Notes on Adams' small volume orbifolds*, in Topology '90, Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter (1992), pp. 311-314.
- [Re] A. W. Reid, *Arithmeticity of knot complements*, J. London Math. Soc. (2) (1991), pp. 171-184.
- [RW] A. W. Reid, G. S. Walsh, *Commensurability classes of 2-bridge knot complements*, Alg. & Geom. Top. Vol. 8 No. 2 (2008), pp. 1031-1057.
- [Ro] D. Rolfsen, *Knots and Links*. Publish or Perish Press, Berkeley, (1976).
- [Ta] M-o Takahashi, *Two-bridge knots have property P*, Mem. Amer. Math. Soc. 29 (1981)
- [Th] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University, 1977, Mimeographed lecture notes.
- [WZ] S. Wang and Q. Zhou, *Symmetry of knots and cyclic surgery*, Trans. Amer. Math. Soc. Vol. 330 No. 2 (1992), pp. 665-676.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, AUSTIN, TX 78712, U.S.A.

E-mail address: `nhoffman@math.utexas.edu`