



Local grand Lebesgue spaces on quasi-metric measure spaces and some applications

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Abstract

We introduce local grand Lebesgue spaces, over a quasi-metric measure space (X, d, μ) , where the Lebesgue space is “aggrandized” not everywhere but only at a given closed set F of measure zero. We show that such spaces coincide for different choices of aggrandizers if their Matuszewska–Orlicz indices are positive. Within the framework of such local grand Lebesgue spaces, we study the maximal operator, singular operators with standard kernel, and potential type operators. Finally, we give an application to Dirichlet problem for the Poisson equation, taking F as the boundary of the domain.

Keywords Grand Lebesgue spaces · Maximal function · Singular integrals · Riesz potential

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1 Introduction

The grand Lebesgue spaces $L^{(p),\theta}(\Omega)$, defined by the norm

$$\|f\|_{L^{(p),\theta}(\Omega)} = \sup_{0 < \varepsilon < p-1} \left(\varepsilon^\theta \int_{\Omega} |f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}},$$

were introduced in [1, 2] when $|\Omega| < \infty$. An approach to aggrandize Lebesgue spaces on sets of infinite measure may be found in [3–6]. Grand spaces have been intensively studied during the last decades; see, for instance, [7–13]. We refer also to [14] and references therein.

We introduce a version of grand Lebesgue spaces which we call *local grand Lebesgue spaces*. The word “local” means that the Lebesgue space is “aggrandized” not everywhere but only at a given set of measure zero. We consider such spaces in the general setting of quasi-metric measure spaces (X, d, μ) . To be more precise, let $F \subset X$ be a closed set of measure zero, in particular in the Euclidean case we can choose the boundary of a domain as the set F . We introduce the space $L_{F,a}^{(p),\theta}(X, \mu)$ by the norm

$$\|f\|_{L_{F,a}^{(p),\theta}(X,\mu)} := \sup_{0 < \varepsilon < \ell} \varepsilon^\theta \left(\int_X |f(x)|^p a(\text{dist}(x, F))^{p\varepsilon} d\mu(x) \right)^{\frac{1}{p}},$$

where the “aggrandizer” $a : [0, \mathfrak{D}] \rightarrow \mathbb{R}_+$, $\mathfrak{D} = \text{diam}(X)$, is a suitable function with $a(0) = 0$. Such local grand Lebesgue spaces in the Euclidean case and F consisting of a unique point where studied in [15]. We also introduce the corresponding *vanishing local grand Lebesgue space* $VL_{F,a}^{(p),\theta}(X, \mu)$.

We prove some properties of such spaces. In particular, we consider relations between such spaces under different choice of the function a and show that two local grand Lebesgue spaces with aggrandizers a and b coincide if the Matuszewska–Orlicz indices of a and b are positive. We also demonstrate that the space $L^p(X)$ is strictly embedded into such local grand Lebesgue space when (X, d, μ) has some regularity properties. If (X, d, μ) is of homogeneous type, within the frameworks of the spaces $L_{F,a}^{(p),\theta}(X)$ and $VL_{F,a}^{(p),\theta}(X, \mu)$, we study the maximal operator, singular operators with standard kernel, and potential type operators. Here we used the results of the recent papers [16, 17], where there were studied Muckenhoupt weights with singularities at a given set which allowed us to prove the boundedness of these operators via interpolation with respect to weights.

Finally, we give an application to Dirichlet problem for the Poisson equation, taking F as the boundary of the domain, using some pointwise estimates obtained in [18].

The paper is organized as follows. In Sect. 2 we supply necessary preliminaries on quasi-metric measure spaces and the notion of Matuszewska–Orlicz indices. In Sect. 3 we introduce the local grand Lebesgue spaces and show that such spaces coincide for different choices of aggrandizers if their Matuszewska–Orlicz indices are positive. In Sect. 4 we recall the notion of lower Assouad codimension and its

connection with power distance functions as Muckenhoupt weights. In Sect. 5 we study the boundedness of the maximal operator, singular operators with standard kernel, and potential operator. In Sect. 6 we give an application to Dirichlet problem for the Laplace equation.

Notation:

- $\delta_F(x) := \text{dist}(x, F)$;
- \mathfrak{N}_F is the lower Assouad codimension of F , see Definition 4.1;
- $\|f\|_{L^p(X,w)} := \|fw\|_{L^p(X)}$;
- $\mathfrak{D} := \text{diam}(X)$;
- $T : X \hookrightarrow Y$ means that T is a continuous mapping from X into Y .

2 Preliminaries

2.1 On quasi-metric measure spaces

By (X, d, μ) we denote a quasi-metric measure space with the quasi-distance d satisfying the standard conditions:

$$d(x, y) \geq 0, \quad d(x, y) = d(y, x), \quad d(x, y) = 0 \Leftrightarrow x = y, \quad d(x, y) \leq \tau[d(x, z) + d(y, z)], \quad (2.1)$$

and by μ a regular Borel measure. The set X may be bounded or unbounded. We denote $B(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$.

The space (X, d, μ) is said to be *homogeneous*, if

$$\mu B(x, 2r) \leq C \mu B(x, r). \quad (2.2)$$

The measure is said to satisfy *the growth condition*, if there exists $N > 0$ such that

$$\mu B(x, r) \leq Cr^N, \quad 0 < r < d, \quad (2.3)$$

where N is not necessarily an integer.

As is well known, extension of various results from the Euclidean case $X = \Omega \subseteq \mathbb{R}^n$ to the general case of (X, d, μ) often depends on the choice of the assumption (2.2) or (2.3) for the measure μ .

We say that (X, d, μ) is *N -Ahlfors regular*, if there exists $N > 0$ such that

$$C^{-1}r^N \leq \mu B(x, r) \leq Cr^N,$$

where N is not necessarily an integer.

For more details on quasi-metric measure spaces we refer, e.g., to [19, 20].

2.2 The class $G(I)$ of aggrandizers

Let $I = (0, \mathfrak{D})$ with $0 < \mathfrak{D} \leq \infty$. By $L_+^\infty(I)$ we denote the cone of non-negative functions in $L^\infty(I)$. We define the class $G(I)$, following [15], as the set of functions $a \in L_+^\infty(I)$ satisfying the conditions:

- (1) $a(t)$ is continuous in a neighbourhood of the origin and $a(0) = 0$; and
- (2) $\inf_{t \in (\kappa, \mathfrak{D})} a(t) > 0$ for every $\kappa \in (0, \mathfrak{D})$.

We call functions $a \in G(I)$ *aggrandizers*.

2.3 Matuszewska–Orlicz indices

In Sect. 3 we use the notion of Matuszewska–Orlicz indices $m(a)$ and $M(a)$ of a non-negative function a introduced in [21], see also [22] where properties of these indices are given in a form convenient for us. These indices are defined by

$$m(a) := \sup_{0 < x < 1} \frac{\ln \left(\overline{\lim}_{h \rightarrow 0} \frac{a(hx)}{a(h)} \right)}{\ln x} \quad \text{and} \quad M(a) := \sup_{x > 1} \frac{\ln \left(\overline{\lim}_{h \rightarrow 0} \frac{a(hx)}{a(h)} \right)}{\ln x}.$$

Note also that

$$m(t^\alpha) = \alpha, \quad m \left[\left(\ln \frac{D \cdot e}{t} \right)^{\pm 1} \right] = 0 \ (D < \infty), \quad m(t^\alpha a(t)) = \alpha + m(a),$$

$$m(a(t)^\beta) = \beta m(a), \quad m(1/a) = -M(a),$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$.

A non-negative function a on I is called *quasi-monotone* if there exist $\alpha, \beta \in \mathbb{R}$ such that $a(t)t^{-\alpha}$ is almost increasing (a.i.) and $a(t)t^{-\beta}$ is almost decreasing (a.d.). A quasi-monotone function has finite indices and $m(a) = \sup\{\alpha \mid a(t)t^{-\alpha} \text{ is a.i.}\}$ and $M(a) = \inf\{\beta \mid a(t)t^{-\beta} \text{ is a.d.}\}$. Everywhere in the sequel, when considering indices of a function, we suppose that it is quasi-monotone near the origin.

3 Local grand Lebesgue spaces $L_{F,a}^{p,\theta}(\Omega, \mu)$

Let (X, d, μ) be a quasi-metric measure space and $\mathfrak{D} := \text{diam}(X)$ with $0 < \mathfrak{D} \leq \infty$.

Definition 3.1 Let Ω be an open set in X and $F \subset X$ be a closed non-empty set with $\mu F = 0$. For a function $a \in L_+^\infty(I)$, we define the *local grand Lebesgue space* $L_{F,a}^{p,\theta}(\Omega, \mu)$, $0 < p < \infty, \theta > 0$, by the (quasi)-norm

$$\|f\|_{L_{F,a}^{p,\theta}(\Omega, \mu)} := \sup_{0 < \varepsilon < \ell} \varepsilon^\theta \left(\int_\Omega |f(x)|^p a(\delta_F(x))^{p\varepsilon} d\mu(x) \right)^{\frac{1}{p}}, \quad (3.1)$$

where $\delta_F(x) := \inf_{y \in F} d(x, y)$, and

$$\|f\|_{L_{F,a}^{(\infty),\theta}(\Omega,\mu)} = \sup_{0 < \varepsilon < \ell} \varepsilon^\theta \operatorname{ess\,sup}_{x \in \Omega} |f(x)| a(\delta_F(x))^\varepsilon, \quad 0 < \ell < \infty.$$

In particular, we may choose the boundary of Ω as the set F (assuming that $\mu(\partial\Omega) = 0$). In the case the set F consists of a single point $x_0 \in \Omega$, we use the notation $F_0 := \{x_0\}$.

Note that the space $L_{F,a}^{(\infty),\theta}(\Omega)$ contains, in general, unbounded functions. Let, for simplicity, $X = \mathbb{R}^n$ and Ω be a bounded domain in \mathbb{R}^n . Then

$$\left(\ln \frac{2\mathfrak{D}}{|x - x_0|} \right)^\gamma \in L_{F,a}^{(\infty),\theta}(\Omega)$$

under the conditions $x_0 \in F$, $a \in G(I)$, $m(a) > 0$, and $\gamma \leq \theta$. It is worth pointing out that a $p = \infty$ -version of the usual well-known grand spaces was studied in [23].

The embedding

$$L^p(\Omega, \mu) \hookrightarrow L_{F,a}^{p,\theta}(\Omega, \mu), \quad 0 < p \leq \infty, \theta > 0,$$

holds, whenever $a \in L_+^\infty(I)$.

To ensure that the local grand space $L_{F,a}^{p,\theta}(\Omega, \mu)$ is larger than $L^p(\Omega, \mu)$, we should restrict the choice of functions $a(t)$, mainly by the assumption that $a(0) = 0$. In most of the statements, we assume that $a \in G(I)$.

The norm (3.1) is equivalent to

$$\begin{aligned} & \sup_{0 < \varepsilon < \ell} \varepsilon^\theta \left(\int_{\delta_F(x) < s} |f(x)|^p a(\delta_F(x))^{p\varepsilon} d\mu(x) \right)^{\frac{1}{p}} \\ & + \left(\int_{\delta_F(x) \geq s} |f(x)|^p d\mu(x) \right)^{\frac{1}{p}}, \end{aligned} \tag{3.2}$$

for every $s \in (0, \mathfrak{D})$.

The norm in the Lemma 3.2 is written as $\|f\|_{L_{F,a,\ell}^{p,\theta}(\Omega,\mu)}$ to underline dependence on the range for ε .

Lemma 3.2 *The space $L_{F,a}^{p,\theta}(\Omega, \mu)$ does not depend on the choice of ℓ , up to equivalence of norms*

$$\|f\|_{L_{F,a,\ell_1}^{p,\theta}(\Omega,\mu)} \leq \|f\|_{L_{F,a,\ell_2}^{p,\theta}(\Omega,\mu)} \leq C \|f\|_{L_{F,a,\ell_1}^{p,\theta}(\Omega,\mu)}, \quad 0 < \ell_1 < \ell_2 < \infty, \tag{3.3}$$

where $C = \max \left\{ 1, \frac{1}{\ell_1^\theta \|a\|_{L^\infty}^{\ell_1}} \sup_{\ell_1 \leq \varepsilon < \ell_2} \varepsilon^\theta \|a\|_{L^\infty}^\varepsilon \right\}$.

Proof The proof is straightforward. □

The corresponding vanishing space $VL_{F,a}^{p,\theta}(\Omega, \mu)$ is defined as the set of functions $f \in L_{F,a}^{p,\theta}(\Omega, \mu)$ such that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon^{\theta p} \int_{\Omega} |f(x)|^p a(\delta_F(x))^{p\varepsilon} d\mu(x) = 0.$$

3.1 Basic properties

The spaces $L_{F,a}^{p,\theta}(\Omega, \mu)$ do not depend much on the “properly chosen” aggrandizer a , as shown in Theorem 3.4. First, we prove the following lemma on embedding between such spaces.

Lemma 3.3 *The following is valid:*

(1) *If there exists a number $\alpha > 0$ such that $a(t) \leq Cb(t)^\alpha$, $t \in I$, then*

$$L_{F,b}^{p,\theta}(\Omega, \mu) \hookrightarrow L_{F,a}^{p,\theta}(\Omega, \mu).$$

(2) *If the function a is almost increasing near the origin, and F_1 and F_2 are closed non-empty sets such that $F_1 \subseteq F_2 \subset \Omega$, then*

$$L_{F_1,a}^{p,\theta}(\Omega, \mu) \hookrightarrow L_{F_2,a}^{p,\theta}(\Omega, \mu).$$

Proof To show (1) it suffices to use Lemma 3.2. The proof of (2) follows from noticing that $\delta_{F_2}(x) \leq \delta_{F_1}(x)$ and then it remains to use the fact that a is almost increasing. □

Theorem 3.4 *Let a and b be quasi-monotone functions on $(0, \kappa)$, for some $\kappa \in (0, \mathfrak{D})$. If $m(a) > 0$ and $m(b) > 0$, then*

$$L_{F,a}^{p,\theta}(\Omega) = L_{F,b}^{p,\theta}(\Omega),$$

up to equivalence of norms.

Proof It suffices to refer to (3.2) and the fact that, for an arbitrarily small $\eta > 0$, there are constants $c(\eta)$ and $C(\eta)$ such that

$$c(\eta)t^{M(a)+\eta} \leq a(t) \leq C(\eta)t^{m(a)-\eta}, \quad t \in (0, \kappa),$$

see [22, Sect. 6], and apply Lemma 3.3. □

Remark 3.5 Positivity of Matuszewska–Orlicz indices in Theorem 3.4 is, in general, necessary: as shown in the Euclidean case in [15] the spaces defined by $a(t) \sim t$ and $b(t) \sim \frac{1}{\ln(1/t)}$ near the origin are different, see [15, Lemma 2.7].

One can be interested in the strict embedding

$$L^p(\Omega, \mu) \subsetneq L_{F,a}^{p,\theta}(\Omega, \mu). \tag{3.4}$$

By Lemma 3.3 (2) it suffices to consider the case $F = \{x_0\}$. We start with the following lemma, where neither growth nor doubling condition are assumed.

Lemma 3.6 *The function*

$$u(x) = [\mu(B(x_0, d(x, x_0)))]^{-\frac{1}{p}}, \quad x_0 \in \Omega, \tag{3.5}$$

does not belong to $L^p(\Omega, \mu)$, $0 < p < \infty$.

Proof Suppose that $u \in L^p(\Omega, \mu)$. Then

$$\begin{aligned} \int_{B(x_0,r)} u(x)^p d\mu(x) &= \int_{B(x_0,r)} \frac{1}{\mu(B(x_0, d(x, x_0)))} d\mu(x) \\ &\geq \frac{1}{\mu(B(x_0, r))} \int_{B(x_0,r)} d\mu(x) = 1. \end{aligned} \tag{3.6}$$

This contradicts the absolute continuity of the norm in $L^1(\Omega, \mu)$, see [24, Theorem 12.34]. □

Theorem 3.7 *Let μ satisfy $\mu(B(x_0, r)) \sim r^\gamma$ as $r \rightarrow 0$ for some $x_0 \in F$, $\theta \geq 1/p$, and $m(a) > 0$. Then the embedding (3.4) is strict.*

Proof It suffices to consider the case $F = \{x_0\}$ by Lemma 3.3 (2) and we can also take $a(t) \equiv t$ for small values of t by Theorem 3.4. In view of Lemma 3.6, it remains to show that $u \in L_{\{x_0\},a}^{p,\theta}(\Omega, \mu)$. We have

$$S := \varepsilon^{\theta p} \int_{\Omega} u(x)^p d(x, x_0)^{\varepsilon p} d\mu(x) \sim \varepsilon^{\theta p} \int_{\Omega} d(x, x_0)^{\varepsilon p - \gamma} d\mu(x).$$

We can use the norm in equivalence form (3.2) so we can replace Ω by $B(x_0, r)$. Using dyadic decomposition, we obtain

$$\begin{aligned} S &\lesssim \varepsilon^{\theta p} \sum_{k=0}^{\infty} \int_{2^{-k-1}r < d(x_0,x) \leq 2^{-k}r} d(x_0, x)^{\varepsilon p - \gamma} d\mu(x) \\ &\lesssim \varepsilon^{\theta p} \sum_{k=0}^{\infty} (2^{-k}r)^{\varepsilon p - \gamma} \int_{B(x_0, 2^{-k}r)} d\mu(x) \end{aligned}$$

$$\begin{aligned} &\lesssim \varepsilon^{\theta p} \sum_{k=0}^{\infty} (2^{-\varepsilon p})^k \sim \frac{\varepsilon^{\theta p}}{1 - 2^{-\varepsilon p}} \\ &\sim \varepsilon^{\theta p - 1}, \end{aligned}$$

which completes the proof. \square

4 Power of distances as Muckenhoupt weights

In the study of operators of grand local Lebesgue spaces we will use the important Proposition 4.3. We now recall some definitions and introduce the corresponding notations.

Definition 4.1 (*Lower Assouad codimension*) Let $F \subset X$ and

$$F_r := \{x \in X \mid \delta_F(x) < r\}.$$

The *lower Assouad codimension*, denoted by \mathfrak{N}_F , is the supremum of all $\nu \geq 0$ for which there exists a constant $C \geq 1$ such that

$$\frac{\mu(F_r \cap B(x, R))}{\mu B(x, R)} \leq C \left(\frac{r}{R}\right)^\nu,$$

for every $x \in F$ and all $0 < r < R < 2 \operatorname{diam}(X)$.

Note that $\mathfrak{N}_F > 0$ implies that $\mu F = 0$, see [17, p. 6]. For more information about Assouad dimension; see, e.g., [25, 26] and the references given there.

Definition 4.2 (*Muckenhoupt weights*) For $1 < p < \infty$, the class $A_p(X)$ is defined as the set of all weights w on X such that

$$[w]_{A_p} := \sup_{B \subset X} \left(\frac{1}{\mu B} \int_B w d\mu \right) \left(\frac{1}{\mu B} \int_B w^{-\frac{1}{p-1}} d\mu \right)^{p-1} < \infty,$$

where the supremum is taken with respect to all balls B in X . For $p = 1$, the class $A_1(X)$ is defined by the condition $Mw(x) \leq Cw(x)$, where M is the maximal operator, see (5.1); we denote $[w]_{A_1} := \operatorname{ess\,sup}_{x \in X} \frac{Mw(x)}{w(x)}$.

The Propositions 4.3 and 4.4 were proved in [17, Theorem 3.4]. It is noteworthy to mention that in [16, Theorem 7] a similar result was obtained, although requiring stronger assumptions on both X and F in terms of Ahlfors regularity.

Proposition 4.3 Let (X, d, μ) be homogeneous, F be a closed non-empty set, and $1 < p < \infty$. Then

$$-\mathfrak{N}_F < \beta < (p - 1)\mathfrak{N}_F \Rightarrow \delta_F^\beta \in A_p(X),$$

where $\delta_F(x) = d(x, F)$ and \mathfrak{N}_F is the lower Assouad codimension of the set F .

Proposition 4.4 *Let (X, d, μ) be homogeneous and F be a closed non-empty set. Then*

$$-\mathfrak{N}_F < \beta \leq 0 \Rightarrow \delta_F^\beta \in A_1(X),$$

where $\delta_F(x) = d(x, F)$ and \mathfrak{N}_F is the lower Assouad codimension of the set F .

In the study of operators in Sect. 5 we will base ourselves on Propositions 4.3 and 4.4. Note that, in applications, when the set F is supplied with a positive measure ν , the Proposition 4.5 is more practical. To this end, we need the following definition.

By $\mathcal{F}_s = \{F\}$ we denote the family of sets $F \subset X$ satisfying the conditions:

- (1) F is closed;
- (2) $\mu F = 0$; and
- (3) the set $F \subset X$ is s -Ahlfors regular with respect to the s -dimensional Hausdorff measure.

Proposition 4.5 ([16, Theorem 7]) *Let (X, d, μ) be a homogeneous quasi-metric measure space N -Ahlfors regular and $F \in \mathcal{F}_s$, with $s < N$. Then $\delta_F^\beta \in A_p(X)$ for $-(N - s) < \beta < (N - s)(p - 1)$, where $A_p(X)$ is the Muckenhoupt class.*

In the case where X is N -Ahlfors regular and $F \in \mathcal{F}_s$, then Proposition 4.3 turns into Proposition 4.5. Non-trivial examples of fractal spaces X and sets F may be found in [16].

Statements of the type of lemma below are known, see for instance [27, Lemma 3.5].

Lemma 4.6 *Let (X, d, μ) be homogeneous, F be a closed non-empty set, and $w(x) := \delta_F(x)$. If $w^\beta \in A_p(X)$ then $\tilde{w}^\beta \in A_p(X)$, where*

$$\tilde{w}(x) = \begin{cases} w(x), & \delta_F(x) < 1, \\ 1, & \delta_F(x) \geq 1. \end{cases}$$

5 On operators in local grand Lebesgue spaces

We start with the simple statement.

Theorem 5.1 *Let $T : L^p(X, \delta_F^\varepsilon) \rightarrow L^q(X, \delta_F^\varepsilon)$ be a bounded sublinear operator, uniformly in $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 > 0$. Then, for a quasi-monotone function a with $m(a) > 0$, we have $T : L_{F,a}^{p,\theta}(X) \hookrightarrow L_{F,a}^{q,\theta}(X)$ and $T : VL_{F,a}^{p,\theta}(X) \hookrightarrow VL_{F,a}^{q,\theta}(X)$.*

Proof By Theorem 3.4, in the definition of the local grand space, we can replace $a(t)$ by $a(t) \equiv t$ when $\mathfrak{D} < \infty$ and by $a(t) \equiv \min\{t, 1\}$ when $\mathfrak{D} = \infty$. After that, the statement becomes obvious. \square

Concerning the boundedness of operators in grand local Lebesgue spaces, we note that one of the principal technical tools, i.e., the Stein–Weiss interpolation theorem, remains valid in the general setting of arbitrary quasi-metric measure spaces, see [28, Cor. 2.2] or [29, 17 and 120 pp.].

Proposition 5.2 ([28, Cor. 2.2]) *Assume that $0 \leq p_0, p_1, q_0, q_1 \leq \infty$ and that*

$$T : \left(L^{p_0}(X, w_0^{1/p_0}), L^{p_1}(X, w_1^{1/p_1}) \right) \rightarrow \left(L^{q_0}(X, \sigma_0^{1/q_0}), L^{q_1}(X, \sigma_1^{1/q_1}) \right)$$

is a continuous linear operator. Then

$$T : L^{p_\theta}(X, w_\theta^{1/p_\theta}) \hookrightarrow L^{q_\theta}(X, \sigma_\theta^{1/q_\theta}),$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad w_\theta = w_0^{(1-\theta)\frac{p}{p_0}} w_1^{\frac{\theta}{p_1}}, \quad \sigma_\theta = \sigma_0^{(1-\theta)\frac{q}{q_0}} \sigma_1^{\frac{\theta}{q_1}},$$

$0 < \theta < 1$ and

$$\|T\|_{L^{p_\theta}(X, w_\theta^{1/p_\theta}) \rightarrow L^{q_\theta}(X, \sigma_\theta^{1/q_\theta})} \leq C \max_{i=0,1} \|T\|_{L^{p_i}(X, w_i^{1/p_i}) \rightarrow L^{q_i}(X, \sigma_i^{1/q_i})}.$$

5.1 Maximal operator

Recall that the maximal operator is defined as

$$Mf(x) := \sup_{r>0} \frac{1}{\mu B(x, r)} \int_{B(x,r)} |f(y)| d\mu(y). \tag{5.1}$$

For $1 < p < \infty$ and $w \in A_p(X)$, we have

$$\|Mf\|_{L^p(X, w^{1/p})} \leq C [w]_{A_p}^{\frac{1}{p-1}} \|f\|_{L^p(X, w^{1/p})}, \tag{5.2}$$

where the constant C does not depend on the weight w or the function f ; see, e.g. [30]. In the case $p = \infty$, for the space $L^\infty(X, w) := \{f \mid \text{ess sup}_{x \in X} |f(x)|w(x) < \infty\}$, we have the following lemma, where

$$[1/w]_{A_1} = \text{ess sup}_{x \in X} w(x) M(1/w)(x).$$

Lemma 5.3 *The maximal operator M is bounded in $L^\infty(X, w)$ if and only if $1/w \in A_1(X)$ and*

$$\|Mf\|_{L^\infty(X, w)} \leq C [1/w]_{A_1} \|f\|_{L^\infty(X, w)}. \tag{5.3}$$

Proof Sufficiency: We have $M(1/w) \leq c/w$, so that

$$wMf(x) = wM\left(\frac{wf}{w}\right) \leq \|wf\|_{L^\infty(X)} \cdot wM\left(\frac{1}{w}\right) \leq C \|f\|_{L^\infty(X,w)}.$$

Necessity: Choose $f_0 = 1/w \in L^\infty(X, w)$. Then $\|Mf_0\|_{L^\infty(X,\mu)} \leq C \|f_0\|_{L^\infty(X,w)}$ yields $\text{ess sup}_{x \in X} w(x)M(1/w)(x) \leq c < \infty$, i.e., $1/w \in A_1(X)$. \square

Theorem 5.4 *Let (X, d, μ) be homogeneous, F be a closed non-empty set, $\mathfrak{N}_F > 0$, $1 < p \leq \infty$, $\theta > 0$, $a \in G(0, \mathfrak{D})$, $\mathfrak{D} = \text{diam}(X)$, and $m(a) > 0$. Then*

$$M : L_{F,a}^{p,\theta}(X, \mu) \hookrightarrow L_{F,a}^{p,\theta}(X, \mu),$$

and

$$M : VL_{F,a}^{p,\theta}(X, \mu) \hookrightarrow VL_{F,a}^{p,\theta}(X, \mu).$$

Proof By Theorem 3.4, in the definition of the local grand space, we can replace $a(t)$ by $a(t) \equiv t$ when $\mathfrak{D} < \infty$ and by $a(t) \equiv \min\{t, 1\}$ when $\mathfrak{D} = \infty$. For simplicity, we take $\mathfrak{D} < \infty$.

By Proposition 4.3, under the choice $\varepsilon_0 < \mathfrak{N}_F/p'$, we have that $\delta_F^{\varepsilon p} \in A_p(X)$, for all ε such that $0 < \varepsilon < \varepsilon_0$. Take ξ such that $\varepsilon_0 p < \xi < (p - 1)\mathfrak{N}_F$. From (5.2), the inequality $[w^\alpha]_{A_p} \leq [w]_{A_p}^\alpha$, $0 < \alpha < 1$, and $[w]_{A_p} \geq 1$, we have

$$\|M\|_{L_{\delta_F^{\varepsilon p}}^p(X) \rightarrow L_{\delta_F^{\xi}}^p(X)} \leq C[\delta_F^{\varepsilon p}]_{A_p}^{\frac{1}{p-1}} \leq C[\delta_F^\xi]_{A_p}^{\frac{\varepsilon p}{\xi(p-1)}} \leq C[\delta_F^\xi]_{A_p}^{\frac{1}{p-1}},$$

where the constants C do not depend on ε . The result now follows from Theorem 5.1. Arguments for $p = \infty$ are similar, being based on Lemma 5.3 and Proposition 4.4. \square

5.2 Singular operators

We consider the Calderón–Zygmund singular operator K with the standard kernel, as defined in [19] and [20, p. 502]. Let $k : X \times X \setminus \{(x, x) \mid x \in X\} \rightarrow \mathbb{R}$ be a measurable function satisfying the conditions:

$$|k(x, y)| \lesssim \frac{1}{\mu B(x, d(x, y))}, \quad x, y \in X, \quad x \neq y;$$

and

$$|k(x_1, y) - k(x_2, y)| + |k(y, x_1) - k(y, x_2)| \lesssim \omega\left(\frac{d(x_2, x_1)}{d(x_2, y)}\right) \frac{1}{\mu B(x_2, d(x_2, y))},$$

for every $x_1, x_2 \in X$ such that $d(x_2, y) \gtrsim d(x_1, x_2)$. Here ω is a positive, non-decreasing function on $(0, \infty)$ satisfying the Δ_2 -condition (i.e., $\omega(2t) \leq c\omega(t)$ for all $t > 0$ and some $c > 0$ independent of t) and the Dini condition $\int_0^1 \frac{\omega(t)}{t} dt < \infty$. We also assume as well that for some $p_0, 1 < p_0 < \infty$, and all $f \in L^{p_0}(X)$ the limit

$$Kf(x) := \lim_{\varepsilon \rightarrow 0^+} \int_{X \setminus B(x, \varepsilon)} k(x, y)f(y)dy$$

exists a.e. and that the operator K is bounded in $L^{p_0}(X)$.

The following proposition was proved in [31], see also [20, Theorem A, p. 503].

Proposition 5.5 *Let (X, d, μ) be a space of homogeneous type and $1 < p < \infty$. If $w \in A_p(X)$, then the operator K is bounded in $L^p(X, w^{1/p})$.*

Note that the boundedness of the Calderón–Zygmund operators in $L^p(X, w)$ follows from the boundedness of the maximal operator M in $L^p(X, w)$ and $[L^p(X, w)]'$ by the known arguments, see e.g. [32, Lemma 3.2].

Theorem 5.6 *Let (X, d, μ) be homogeneous, F be a closed non-empty set, $\mathfrak{R}_F > 0$, $1 < p < \infty$, $\theta > 0$, $a \in G(0, \mathfrak{D})$, $\mathfrak{D} = \text{diam}(X)$, and $m(a) > 0$. Then, under the assumptions of Proposition 5.5,*

$$K : L_{F,a}^{(p),\theta}(X, \mu) \leftrightarrow L_{F,a}^{(p),\theta}(X, \mu)$$

and

$$K : VL_{F,a}^{(p),\theta}(X, \mu) \leftrightarrow VL_{F,a}^{(p),\theta}(X, \mu).$$

Proof We apply the interpolation Proposition 5.2 with $p_0 = p_1 = p$, $w_0(x) \equiv 1$, and $w_1^p(x) = \delta_F(x)^{p\varepsilon_0}$, corresponding to the cases $\varepsilon = 0$ and $\varepsilon = \varepsilon_0$ in the definition of the norm of the space. We also keep in mind that one can replace ℓ by ε_0 , according to Lemma 3.2. The boundedness of the Calderón–Zygmund operator, at the endpoints, follows from Proposition 4.3 (taking $\varepsilon_0 < \mathfrak{R}_F/p'$) and the boundedness of the operator K in the space $L^p(X, w^{1/p})$, $w \in A_p(X)$, see Proposition 5.5. \square

Corollary 5.7 *Let $X = \Omega \subset \mathbb{R}^n$ be an open set satisfying the doubling condition and μ the Lebesgue measure. Let $\partial\Omega$ denote the boundary of Ω and assume that $\mathfrak{R}_{\partial\Omega} > 0$. Then the maximal operator M and the Calderón–Zygmund operators K from Proposition 5.5, related to Ω , are bounded in the space $L_{\partial\Omega,a}^{(p),\theta}$ if $a \in G(I)$ and $m(a) > 0$.*

Note that, in the case of “nice” boundaries, the codimension is always positive and equals to $n - 1$ in case of Lipschitz boundary.

5.3 Potential operators

For $\alpha > 0$, the Riesz potential I^α is defined by

$$I^\alpha f(x) = \int_X \frac{f(y)d(x, y)^\alpha}{\mu B(x, d(x, y))} d\mu(y).$$

For a given $\eta > 0$, we say that μ satisfies a *reverse doubling condition* if

$$\frac{\mu B(y, r)}{\mu B(x, R)} \leq C \left(\frac{r}{R}\right)^\eta \tag{5.4}$$

holds, whenever $B(y, r) \subset B(x, R) \subset X$.

The following proposition was proved in [17, Theorem 4.1], see also [33] for the optimal exponent for the A_p constant in the Euclidean case.

Proposition 5.8 *Let $\alpha > 0$. Assume that the reverse doubling condition (5.4) holds with the exponent $\eta = \alpha$ and that there exists $Q > \alpha$ such that $\mu B \geq \text{crad}(B)^Q$ for all balls $B \subset X$. Let $\emptyset \neq F \subset X$ be a closed set, $1 < p < Q/\alpha$, and $\beta \in \mathbb{R}$ be such that*

$$-\mathfrak{N}_F \frac{p}{p^\sharp} < \beta < \mathfrak{N}_F(p - 1), \tag{5.5}$$

where $p^\sharp := \frac{Qp}{Q-\alpha p}$ and \mathfrak{N}_F is the lower Assouad codimension of F . Then

$$I^\alpha : L^p(X, \delta_F(\cdot)^{\frac{\beta}{p}}) \hookrightarrow L^{p^\sharp}(X, \delta_F(\cdot)^{\frac{\beta}{p}}),$$

Theorem 5.9 *Let (X, d, μ) be homogeneous, F be a closed non-empty set, $\mathfrak{N}_F > 0$, $1 < p < \infty$, $\theta > 0$, $a \in G(0, \mathfrak{D})$, $\mathfrak{D} = \text{diam}(X)$, and $m(a) > 0$. Then, under the assumptions of Proposition 5.8,*

$$I^\alpha : L_{F,a}^{(p),\theta}(X, \mu) \hookrightarrow L_{F,a}^{p^\sharp,\theta}(X, \mu)$$

and

$$I^\alpha : VL_{F,a}^{(p),\theta}(X, \mu) \hookrightarrow VL_{F,a}^{p^\sharp,\theta}(X, \mu).$$

Proof The proof follows closely the one given in Theorem 5.6. We apply the interpolation Proposition 5.2 with $p_0 = p$, $p_1 = p^\sharp$, $w_0(x) = \sigma_0(x) \equiv 1$, $w_0^p(x) = \delta_F(x)^{p\varepsilon_0}$, and $w_1^{p^\sharp} = \delta_F(x)^{p^\sharp\varepsilon_0}$. The boundedness of the Riesz potential operator at the endpoints is given in Proposition 5.8, under the choice $\varepsilon_0 < \mathfrak{N}_F/p'$ for the $L^{p_1}(X, w_1^{1/p_1}) \rightarrow L^{q_1}(X, \sigma_1^{1/q_1})$ -boundedness. This completes the proof. \square

6 An application to Dirichlet problem

In this section we give an application to Dirichlet problem basing ourselves on some results from [18] for classical weighted Lebesgue spaces. Recall that the Dirichlet problem is given by

$$\begin{cases} -\Delta u = f, & \Omega, \\ u = 0, & \partial\Omega. \end{cases} \tag{6.1}$$

The operator K_{ji} is defined as

$$K_{ji} f(x) = \lim_{\varepsilon \rightarrow 0} \int_{|x-y|>\varepsilon} \partial_{ji} \Phi(x-y) f(y) dy, \tag{6.2}$$

where

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \ln |x|, & n = 2, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}, & n \geq 3, \end{cases}$$

and the corresponding maximal singular operator K_{ji}^* by

$$K_{ji}^* f(x) = \sup_{\varepsilon > 0} \left| \int_{|x-y|>\varepsilon} \partial_{ji} \Phi(x-y) f(y) dy \right|. \tag{6.3}$$

The following pointwise estimate was obtained in [18, Lemma 2.3].

Proposition 6.1 *Let Ω be a bounded C^2 domain. There exists a constant C depending only on n and Ω such that, for any $x \in \Omega$,*

$$\begin{aligned} |u(x)| + |\partial_j u(x)| &\leq CMf(x), \\ |\partial_{ji} u(x)| &\leq C(K_{ji}^* f(x) + Mf(x) + |f(x)|), \end{aligned}$$

where M is the maximal operator and K_{ji}^* is defined in (6.3).

We now introduce the Sobolev grand local space $W_{F,a}^{k,p,\theta}(\Omega)$ as

$$\|f\|_{W_{F,a}^{k,p,\theta}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|\partial_\alpha f\|_{L_{F,a}^{p,\theta}(\Omega)}^p \right)^{\frac{1}{p}} < \infty.$$

In the following theorem, the set F may be any closed non-empty set with $\mathfrak{N}_F > 0$ in $\overline{\Omega}$, although the most interesting case seems to be when $F = \partial\Omega$.

Theorem 6.2 Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain, $1 < p < \infty$, $F \subset \overline{\Omega}$ be a closed non-empty set, $\mathfrak{N}_F > 0$, $\theta > 0$, $a \in G(I)$, and $m(a) > 0$. If u is the solution of problem (6.1), then there exists a constant C depending only on n and Ω such that

$$\|u\|_{W_{F,a}^{2,p,\theta}(\Omega)} \leq C \|f\|_{L_{F,a}^{p,\theta}(\Omega)}. \quad (6.4)$$

Proof The result follows immediately from the pointwise inequalities from Proposition 6.1 and the boundedness of the maximal and the maximal singular operators. The boundedness of the maximal operator is given in Theorem 5.4 and the boundedness of the maximal singular operator follows from the known estimate

$$\int_{\mathbb{R}^n} K^* f(x)^p w(x) dx \leq C_{p,w} \int_{\mathbb{R}^n} Mf(x)^p w(x) dx, \quad w \in A_p,$$

where K^* is the maximal singular operator related with the singular operator K , see [34, Theorem 2, p. 205]. \square

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