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UNIFICATION AND FINITE MODEL PROPERTY FOR LINEAR STEP-LIKE TEMPORAL MULTI-AGENT LOGIC WITH THE UNIVERSAL MODALITY[1]

Abstract

This paper proposes a semantic description of the linear step-like temporal multi-agent logic with the universal modality $\mathcal{LTK}.sl_U$ based on the idea of non-reflexive non-transitive nature of time. We proved a finite model property and projective unification for this logic.

Keywords: Multi-agent system, Kripke semantic, unification, modal logic, non-transitive time, step-like, universal modality, finite model property, p-morphism.

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1. Introduction

Temporal logics have been widely used for more than half a century as an effective tool for describing information processes and calculations [15]. Here the most significant role belongs to logical systems $\mathcal{LTL}$ and $\mathcal{CTL}$,

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that constitute today one of the foundation in the theory of program verification [10]. The interweaving of such logics with multi-agent systems makes it possible to model the intellectual reasoning of various nature sites, including social ones. Examples of such logics are $\mathcal{LTK}$ [9], $\mathcal{LKInd}$—as a version with the induction axiom [20], $\mathcal{LTK}_r$—as a logic of a reflexive non-transitive temporal relation [16].

In the field of modern approaches to modeling multi-agent systems, there is a lack of a consistent approach: various methods of interaction between agents, modal operators and valuations are proposed, new versions of combining modal systems are chosen. This situation can be partially explained by the fact that suitable (from the idea of natural modeling of processes) combined systems, after a deeper study, turn out to be complex and even lack some useful properties [7]. Of course, this imposes significant restrictions on the applicability of such systems in real information projects [8].

Most temporal logics are built on the idea of reflexive transitive time, what helps make it possible to effectively apply the developed apparatus of modal logics in their study. However, such systems have a lot of weak points when modeling complex systems, in which we are usually required properties of dynamism, indeterminacy, instability of the information transfer process and taking into account possible errors in the translation process.

In addition, the participants of computational process, described as individual agents, whose knowledge is determined by multiple relations, are able to communicate, make decisions under the influence of public opinion of society or their own independent views, accumulate and expand available information and, at the same time, "forget" or "lose" entire segments over time. In this regard, logical systems based on non-transitive, multiple fragmentary relations look promising.

Among other things, the nature of time itself, as a physical process, in many ways remains a mystery to humanity. The argument in favor of its non-transitivity, at least from the point of view of the technical tools available to us for its modeling, is the step-by-step principle of implementing any computational process—when at any moment we only have today's knowledge and know what actions will be taken to move to the next moment of time. From this point of view, it is of interest to study a non-transitive and non-reflective version of temporal logic $\mathcal{LTL}$, in which, taking into account the specified properties of relations, the temporal process is a step-like

sequential procedure. Thereby, it seems rational to model such logics using methods of relational semantics.

An adequate approach that allows both to enhance the expressive power of a modal language and to bring some clarity to the process of studying the fundamental properties of a deductive system is the universal modality operator. In the case of non-transitive models it allow us to overcome the limitations associated with the finiteness of the modal degree of formulas and expresses statements that are valid "forever" in temporal systems [14].

One of the important properties of any proposition in logic is its unification, i.e. the ability to transform a formula into a theorem by the substitution of variables. In the case of social models, the unification process actually separates an unconditional true fragment from the general information of arbitrary truth values available to the agent. Among the effective approaches to solving the unification problem, the most important are the method of projective formulas and projective approximation [12], the method for describing complete sets of unifiers in terms of n-characteristic models based on reduced form of formulas [19]. From the standpoint of the social interpretation for the unification problem, it becomes clear that it is also useful to define the boundaries of an wittingly non-unifiable fragment: such an approach was proposed in [18] for extensions of modal logics $\mathcal{S}4$ and $(\mathcal{K}4 + [\Box\bot \equiv \bot])$, and later generalized for a cases of linear transitive temporal logics of knowledge [1, 5].

It is clear that the most important task is to find maximal unifiers that allow to build all the others. However, it is also interesting to find minimal—*ground*—unifiers obtained by a substitution of constants. Often, ground unifiers allow us expressing schemes for constructing maximal and even the most general unifiers [11, 6], although this approach is not always possible [13].

S. I. Bashmakov previously described one non-reflexive non-transitive temporal linear logic with universal modality and proved projective unification [2]. Later, in [3] he announced the possibility of generalizing this result for the case of logic enriched with agent's knowledge relations. In this work, we realized the semantic construction of a linear step-like temporal logic of knowledge with a universal modality, proved the finite model property and projective unification. For this logic, we introduce the notation $\mathcal{LTK}.sl_U$. The term "step-like" is an interpretation of the non-reflective non-transitive nature of the temporal relation given to logic.

As a basic tool for describing and study the logic $\mathcal{LTK}.sl_U$, we use the traditional and well-studied relational Kripke semantics of possible worlds, generalized to the case of temporal multi-agent systems. A key object here is a $LTK.sl_U$-*frame*, represented by a tuple of clustered elements and $(n+3)$ binary relations specified on them.

2. Semantics for $\mathcal{LTK}.sl_U$

There are various approaches to describing temporal logic. We will define the logic under study as a multimodal system with the following semantics.

The alphabet of the language $L^{\mathcal{LTK}.sl_U}$ includes a countable set of propositional variables $Prop = \{p_1, \dots, p_n, \dots\}$, constants $\{\top, \bot\}$, brackets $(,)$, basic Boolean operations and the following set of unary modal operators: $\{N, \Box_e, \Box_1, \dots, \Box_n, \Box_U\}$.

The smallest set containing propositional variables from $Prop$ and closed under connectives from the language $L^{\mathcal{LTK}.sl_U}$ will be standardly denoted by $For(L^{\mathcal{LTK}.sl_U})$.

$LTK.sl_U$-*frame* is a tuple $F := \langle W, \mathbf{Next}, R_e, R_1, \dots, R_n, R_u \rangle$, where

- $W = \bigcup_{t \in \mathbb{N}} C_t$ is a disjoing union of clusters C_t indexed by natural numbers: $C_{t_1} \cap C_{t_2} = \emptyset$ if $t_1 \neq t_2$;

- $\mathbf{Next}$ is a (non-reflexive non-transitive) binary relation "next natural number": $\forall a, b \in W : a\mathbf{Next}b \Leftrightarrow \exists t \in \mathbb{N}(a \in C_t \& b \in C_{t+1})$;

- R_e is a binary relation defining equivalence on each cluster:

$$\forall a, b \in W(aR_eb \Longleftrightarrow \exists t \in \mathbb{N}(a, b \in C_t));$$

- $\forall i \in [1, n]\, R_i \subseteq \bigcup_{t \in \mathbb{N}}(C_t)^2$ are an agent's knowledge relations defined on clusters;

- $R_u = W^2$ is a relation of total reachability:

$$\forall a, b \in W : aR_ub.$$

A *model* on a $LTK.sl_U$-frame F is a pair $M := \langle F, V \rangle$, where V is a valuation $V : Prop \mapsto 2^W$, where $Prop$ is a countable set of propositional variables. Then $\forall a \in C_t \subset W, \forall t \in \mathbb{N}$ truth conditions of formulas
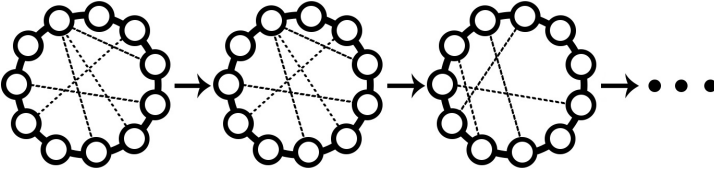
Figure 1. $LTK.sl_U$-frame

containing modal operators are determined in a standard way through the corresponding relations:

- $\langle F, a \rangle \Vdash_V N\varphi \Leftrightarrow \forall b \in C_{t+1} : \langle F, b \rangle \Vdash_V \varphi$;

- $\langle F, a \rangle \Vdash_V \Box_e\varphi \Leftrightarrow \forall b \in C_t : \langle F, b \rangle \Vdash_V \varphi$;

- $\langle F, a \rangle \Vdash_V \Box_i\varphi \Leftrightarrow \forall b \in C_t : a \, R_i \, b \Rightarrow \langle F, b \rangle \Vdash_V \varphi$;

- $\langle F, a \rangle \Vdash_V \Box_U\varphi \Leftrightarrow \forall b \in W : \langle F, b \rangle \Vdash_V \varphi$.

The operator $\Box_U$ is called a *universal modality* and actually sets the truth of a formula on a model; $\Box_e$ is a *Common Knowledge*-operator on each cluster; $\Box_1, \ldots, \Box_n$ are operators of knowledge of agents that they get on each of a frame cluster. We don't impose any special properties on the agent's knowledge, except for the condition that any R_i is a certain limitation of R_e.

We say that a formula φ is *true* in the model $M := \langle F, V \rangle$ (we denote $F \Vdash_V \varphi$) if $V(\varphi) = W$. A formula φ is *valid* on the frame F ($F \Vdash \varphi$) if φ is true in all its models. Finally, φ is *valid* on the class of frames K ($K \Vdash \varphi$), if φ is valid on any frame $F \in K$. Recall that a class of frames is called *characteristic* for a logic $\mathcal{L}$ iff all theorems of a logic are valid on all frames from this class. Let K be the class of all $LTK.sl_U$-frames.

We will call a frame F adequate to a logic $\mathcal{L}$ if for any formula $\varphi \in \mathcal{L}$ it is true that $F \Vdash \varphi$.

A *linear step-like temporal multi-agent logic with universal modality* $\mathcal{LTK}.sl_U$ is a multimodal logic, defined as follows

$$\mathcal{LTK}.sl_U := \{\varphi \in For(L^{\mathcal{LTK}.sl_U}) \mid \forall F \in K : F \Vdash \varphi\}.$$

3.   Finite model property of $\mathcal{LTK}.sl_U$

A *modal degree* $d(\alpha)$ of a formula α in $\mathcal{LTK}.sl_U$ is a number of nested non-reflexive non-transitive modal operators N in α:

$$d(p) = 0, p \in Prop; d(\circ\alpha) = d(\alpha), \text{ where } \circ \in \{\neg, \Box_e, \Box_U, \Box_i\};$$
$$d(\alpha \odot \beta) = max(d(\alpha); d(\beta)), \text{ where } \odot \in \{\vee, \wedge\}; d(N\alpha) = d(\alpha) + 1.$$

A *length* $d(\alpha)$ of a formula α of the logic $\mathcal{LTK}.sl_U$ is defined as follows: $l(p) = 0$ where $p \in Prop$; $l(\alpha \circ \beta) = l(\alpha) + l(\beta) + 1$, where $\circ \in \{\wedge, \vee\}$; $l(\bullet\alpha) = l(\alpha) + 1$, where $\bullet \in \{N, \neg, \Box_e, \Box_U, \Box_i\}$.

An important property of logical systems is a *finite model property*, which allows us to operate with simpler finite models, instead of their infinite variants. A logic $\mathcal{L}$ is said to have *finite model property*, if $\mathcal{L}$ is complete with respect to the class of finite frames.

In order to prove that the logic $\mathcal{LTK}.sl_U$ has the finite model property, we define a p-morphic mapping of an infinite $\mathcal{LTK}.sl_U$-model M to a finite-by-time one, and then, using the technique of filtering clusters, we construct a model with clusters of finite cardinality on the p-morphic version. This section proves that such a model will preserve the truth of formulas in our logic.

3.1.   p-morphism for $\mathcal{LTK}.\mathbf{sl_U}$

A map f from a frame $F := \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u \rangle$ onto a frame $F' := \langle W', R'_e, R'_1, \ldots, R'_n, R'_u \rangle$ is called a p-morphism, if the following conditions hold $\forall a, b \in W \forall R \subseteq \{\mathbf{Next}, R_e, R_1, \ldots, R_n, R_u\}$:

1. $aRb \Rightarrow f(a)R'f(b)$;

2. $f(a)R'f(b) \Rightarrow \exists c \in W[aRc \wedge f(c) = f(b)]$.

Now we define the finite by the time (by the number of clusters) model N as follows:

$$N := \left\langle \bigcup_{j \in [1, k+1]} C_j, \mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u, V' \right\rangle,$$

where for some $\mathcal{LTK}.sl_U$-model $M = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u, V \rangle$ the following conditions are satisfied:

- $\bigcup_{j \in [1,k]} C_j \subset W$ are finite number of clusters, C_{k+1} is a singleton cluster;

- $R'_e, R'_1, \ldots, R'_n$ are given as limitations of corresponding relations $R_e, R_1, \ldots, R_n$ on clusters $\bigcup_{j \in [1,k]} C_j$ supplemented by the following conditions $\forall R \in \{R_e, R_1, \ldots, R_n\}$:

$$\forall a, b \in W \setminus \{C_1, \ldots, C_k\} \, if \, aRb, then \, a = b \in C_{k+1} \& aR'b;$$

- R'_u coincides on clusters $C_1, \ldots, C_k$ with the relation R_u, and for elements out of these clusters it is given as follows:

$$\forall a \in W \setminus \{C_1, \ldots, C_k\} \forall b \in W \, if \, aR_u b, then \, a \in C_{k+1} \& aR'_u b.$$

- **Next**$'$ is defined as follows: $\forall a \in \{C_1, \ldots, C_k\}$ if $a\textbf{Next}b$, then $b \in \{C_2, \ldots, C_{k+1}\}$, and $\forall a \in W \setminus \{C_1, \ldots, C_k\}$, $\forall b \in W$ if $a\textbf{Next}b$, then $a, b \in C_{k+1} \& a\textbf{Next}'b$;

- $V'(p) = V(p) \cap \bigcup_{j \in [1,k]} C_j$ for $p \in Prop$.

To simplify notation, we will denote a finite frame defining such a model N as $F_{fin} := \left\langle \bigcup_{j \in [1,k+1]} C_j, \textbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u \right\rangle$. For consistency, we denote an infinite $LTK.sl_U$-frame here as F_{inf}.
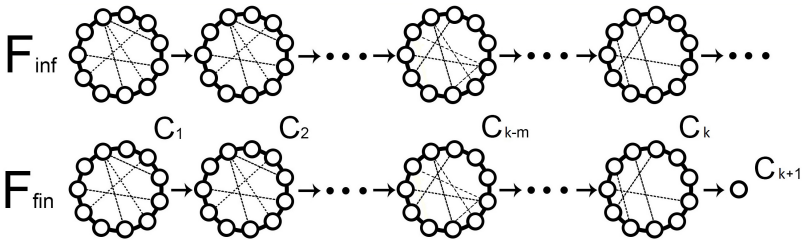

Figure 2. An infinite frame F_{inf} and a finite frame F_{fin}

THEOREM 3.1. *Any F_{fin} is a p-morphic image of F_{inf}.*

PROOF: Let f be a mapping of infinite $LTK.sl_U$-frame

$$F_{inf} = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u \rangle$$

onto a finite

$$F_{fin} = \left\langle \bigcup_{j \in [1,k+1]} C_j, \mathbf{Next}', R_e', R_1', \ldots, R_n', R_u' \right\rangle,$$

given as follows:

1. $\forall x \in \bigcup_{j \in [1,k]} C_j \ f(x) = x$;

2. $\forall x \in W \setminus \bigcup_{j \in [1,k]} C_j \ f(x) = y$, where $y \in C_{k+1}$.

Let us prove that the mapping f is a p-morphism. For this, it is necessary to show the correctness of the given mapping, with respect to the points (1.) and (2.) of the definition.

(1.) $\forall a, b \in W$ if $a\mathbf{Next}b$, hence by the definition of $\mathbf{Next}$, $a \in C_i$ and $b \in C_{i+1}$. If $b \in \{C_2, \ldots, C_k\}$, then $f(a) = a$, $f(b) = b$ and $f(a)\mathbf{Next}'f(b)$. If $b \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a), f(b) \in C_{k+1}$ and $f(a)\mathbf{Next}'f(b)$.

If aR_eb and $a, b \in C_i \subset \{C_1, \ldots, C_k\}$ then $f(a)R_e'f(b)$. If $C_i \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a) = f(b) = y \in C_{k+1}$.

By virtue of $R_i \subseteq R_e \forall i \in [1, \ldots, n]$, for relations $R_1', \ldots, R_n'$ proof is similar to R_e.

By definition, $R_u = W^2$ and then $\forall a, b \in \{C_1, \ldots, C_k\} \ f(a)R_u'f(b)$. If $a \in W \setminus \{C_1, \ldots, C_k\}$ or $b \in W \setminus \{C_1, \ldots, C_k\}$, then $f(a) = yR_u'f(b)$ or $f(a)R_u'y = f(b)$. Respectively, $y \in C_{k+1}$.

(2.) $\forall a, b \in W$ if $f(a)\mathbf{Next}'f(b)$, then the following cases are possible:

- if $f(a)\mathbf{Next}'f(b)$, then, by definition $\mathbf{Next}'$, $f(a) \in C_i, f(b) \in C_{i+1}$, where $i + 1 \in [2, \ldots, k]$. In this case $f(a) = a$, and for $f(b)$ two options are possible:

  - when $f(b) \in \{C_2, \ldots, C_k\}$, $b = c$ and $a\mathbf{Next}c$;
  - when $f(b) \in C_{k+1} \& f(a) \in C_k$, as c we take $\forall c \in C_{k+1}$, then $a\mathbf{Next}c$.

- if $f(a), f(b) \in C_{k+1}$, then $a \in C_j, b \in C_{j+1}$(where $\{C_j, C_{j+1}\} \subset W \setminus \{C_1, \ldots, C_k\}$) and then as c we take $\forall x \in C_{j+1}$. In this case $f(c) = f(b) \in C_{k+1} \subset F_{fin}$.

$\forall a, b \in W$ if $f(a)R'_e f(b)$, then two options are possible:

- if $f(a), f(b) \in C_i$, where $i \in \{1, \ldots, k\}$, then $aR_e b$, where $a, b \in C_i$, $i \in [1, \ldots, k]$. In this case $c \in C_i$;

- if $f(a), f(b) \in C_{k+1}$, then $aR_e b$ and $a, b \in W \setminus \{C_1, \ldots, C_k\}$.

In the case of R_i, the proof trivially repeats the reasoning for R_e.

$\forall a, b \in W f(a)R'_u f(b)$, therefore, as c we can take any element of W.

Therefore, any finite frame F_{fin} is a p-morphic image of F_{inf}. $\qquad\square$

Now let us prove that any formula refutable on $\mathcal{LTK}.sl_U$-model M is refuted also on N.

THEOREM 3.2. *Let $M = \langle F_{inf}, V \rangle$ be an infinite by time $LTK.sl_U$-model, α is an arbitrary formula with the modal degree $d(\alpha) = m, m \in \omega$.*
Then $\forall x \in \bigcup_{j \in [1, k-m]} C_j \subset F_{inf}$ ($m < k$) it is true:

$$\langle M, x \rangle \nVdash \alpha \Leftrightarrow \langle N, x \rangle \nVdash \alpha,$$

where $N = \langle F_{fin}, V' \rangle = \langle \bigcup_{j \in [1, k+1]} C_j, \mathbf{Next}', R'_e, R'_u, R'_1, \ldots, R'_n, V' \rangle$.

PROOF: Let us prove that it is true for all formulas in $\mathcal{LTK}.sl_U$. The proof is by induction on the length of the formula α.

The induction base $l(\alpha) = 0$ corresponds to the case $\alpha = p$. Obviously, in this case the modal degree is also equal to 0 and the statement is true $\forall x \in \bigcup_{j \in [1, k]} C_j$.

Suppose the statement of the theorem is true $\forall \beta$: $l(\beta) < t$, i.e. $\langle M, x \rangle \nVdash \beta \Leftrightarrow \langle N, x \rangle \nVdash \beta$. Let us prove for $l(\alpha) = t$.

The cases $\alpha \in \{\neg\varphi, \Box_U\varphi, \Box_e\varphi, \Box_i\varphi, \varphi \vee \psi, \varphi \wedge \psi\}$ satisfy the conditions of inductive hypothesis due to the fact that the modal degree of the formula α is not increased by adding operators $\{\neg, \Box_U, \Box_e, \Box_i\}$ to the subformula φ of less length, and is potentially increased by adding $\{\vee, \wedge\}$ only up to the value of $max(d(\varphi), d(\psi))$, where φ and ψ are also shorter in length (by the definitions of the truth values of such formulas).

$!\alpha = N\varphi$, $l(\varphi) = l(\alpha) - 1$ and $d(\alpha) = d(\varphi) + 1$. By inductive hypothesis, $\langle M, x \rangle \nVdash \varphi \Leftrightarrow \langle N, x \rangle \nVdash \varphi$, where $x \in \bigcup_{j \in [2, k-(m-1)]} C_j$. By the definition of N, it's true, that $\forall x \in C_i \langle M, x \rangle \Vdash N\varphi \Leftrightarrow \forall y \in C_{i+1}$ (i.e. $x \mathbf{Next} y$) $\langle M, y \rangle \Vdash \varphi$, hence, $\exists x \in C_i \langle M, x \rangle \nVdash N\varphi \Leftrightarrow \exists y \in C_{i+1} \langle M, y \rangle \nVdash \varphi$. Then $\forall \hat{x} \in \bigcup_{j \in [1, k-m]} C_j \langle M, \hat{x} \rangle \nVdash N\varphi \Leftrightarrow \langle N, \hat{x} \rangle \nVdash N\varphi$. $\qquad\square$

4. Filtration for $\mathcal{LTK}.sl_U$

To build a final finite model that is adequate to our logic, we apply the filtering technique to the frame F_{fin}. Let $M = \langle W, \mathbf{Next}, R_e, R_1, \ldots, R_n, R_u, V \rangle$ be a model, built on the infinite $LTK.sl_U$-frame defined above, $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$ is a set of formulas that is closed wrt sub-formulas. We define an equivalence relation $\equiv_\Phi$ on clusters from W as follows: $\forall t \in \mathbb{N}, \forall x, y \in C_t$

$$x \equiv_\Phi y \Longleftrightarrow [\forall \alpha \in \Phi(\langle M, x \rangle \Vdash \alpha \Leftrightarrow \langle M, y \rangle \Vdash \alpha)].$$

In accordance with this definition, below we will use the notation

- $Var(\Phi)$ for a set of all variables of formulas from Φ;

- $[x]_{\equiv_\Phi} := \{y \in W | x \equiv_\Phi y\}$ for equivalence classes;

- $W_\Phi := \{[x]_{\equiv_\Phi} | \forall x \in W\}$ for a set of all such classes;

- $C_{j_\Phi} := \{[x]_{\equiv_\Phi} | \forall x \in C_j \subset F_{fin}\}$, $j \in [1, k+1]$, for each cluster of such classes obtained from each cluster of F_{fin}.

To get only finite clusters, we define a model filtered by a set $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$

$$N_\Phi = \left\langle \bigcup_{j \in [1, k+1]} C_{j_\Phi}, \mathbf{Next}'_\Phi, R'_{e_\Phi}, R'_{1_\Phi}, \ldots, R'_{n_\Phi}, R'_{u_\Phi}, V'_\Phi \right\rangle$$

based on a version of model N with a p-morphic frame F_{fin} and additional following filtration of clusters:

1. $\forall p \in Var(\Phi) \ [V'_\Phi(p) = \{[a]_{\equiv_\Phi} | \langle N, a \rangle \Vdash p\}];$

2. $\forall a, b \in \{\bigcup_{j \in [1, k+1]} C_j\}, \ \forall R' \in [\mathbf{Next}', R'_e, R'_1, \ldots, R'_n, R'_u] \ (aR'b \Rightarrow [a]_{\equiv_\Phi} R'_\Phi [b]_{\equiv_\Phi});$

3. $\forall a, b \in \{\bigcup_{j \in [1, k+1]} C_{j_\Phi}\}$

   (a) $\forall l \in \{e, 1, \ldots, n, u\} \ ([a]_{\equiv_\Phi} R'_{l_\Phi} [b]_{\equiv_\Phi} \implies [\forall \Box_l \alpha \subseteq \Phi \ \langle N, a \rangle \Vdash \Box_l \alpha \Rightarrow \langle N, b \rangle \Vdash \alpha]);$

   (b) $[a]_{\equiv_\Phi} \mathbf{Next}'_\Phi [b]_{\equiv_\Phi} \implies ([\forall N\alpha \subseteq \Phi \ \langle N, a \rangle \Vdash N\alpha \Rightarrow \langle N, b \rangle \Vdash \alpha]).$

Well-known conditions for building the minimal and maximal filtration can also be applied in our case:

— **the minimal filtration**

$$N_\Phi^{min} = \left\langle \bigcup_{j\in[1,k+1]} C_{j\Phi}, \mathbf{Next}_\Phi'^{min}, R_{e\Phi}'^{min}, R_{1\Phi}'^{min}, \ldots, R_{n\Phi}'^{min}, R_{u\Phi}'^{min}, V_\Phi' \right\rangle,$$

where

- $\forall l \in \{e,1,\ldots,n,u\}\ R_{l\Phi}'^{min} = \{([a]_{\equiv_\Phi},[b]_{\equiv_\Phi})|(a,b)\in R_l'\}$,

- $\mathbf{Next}'^{min} = \{([a]_{\equiv_\Phi},[b]_{\equiv_\Phi})|(a,b)\in Next'\}$;

— **the maximal filtration**

$$N_\Phi^{max} = \left\langle \bigcup_{j\in[1,k+1]} C_{j\Phi}, \mathbf{Next}_\Phi'^{max}, R_{e\Phi}'^{max}, R_{1\Phi}'^{max}, \ldots, R_{n\Phi}'^{max}, R_{u\Phi}'^{max}, V_\Phi' \right\rangle,$$

where

- $\forall l \in \{e,1,\ldots,n,u\} : [a]_{\equiv_\Phi} R_{l\Phi}'^{max}[b]_{\equiv_\Phi} \Leftrightarrow [\forall \Box_l\alpha \subseteq \Phi(\langle N,a\rangle \Vdash \Box_l\alpha \Rightarrow \langle N,b\rangle \Vdash \alpha)]$,

- $[a]_{\equiv_\Phi}\mathbf{Next}_\Phi'^{max}[b]_{\equiv_\Phi} \Leftrightarrow ([\forall N\alpha \subseteq \Phi\ \langle N,a\rangle \Vdash N\alpha \Rightarrow \langle N,b\rangle \Vdash \alpha])$.

Due to the choice of the set Φ, the finiteness of the number of relations on a frame and all pairwise variants of their intersections, the clusters C_{t_Φ} obtained as a result of the proposed filtration are also will always have finite cardinality. By virtue of the construction of a filtered model, we assume the true following

LEMMA 4.1. *Let* $N = \langle \bigcup_{j\in[1,k+1]} C_j, \mathbf{Next}', R_e', R_u', R_1', \ldots, R_n', V'\rangle$ *be a p-morphic model N of a $\mathcal{LTK}.sl_U$-model M, $\Phi \subseteq For(L^{\mathcal{LTK}.sl_U})$ is a closed wrt subformulas set of formulas whose modal degree does not exceed m ($m \in \omega, k > m$),*

$$N_\Phi = \left\langle \bigcup_{j\in[1,k+1]} C_{j\Phi}, \mathbf{Next}_\Phi', R_{e\Phi}', R_{1\Phi}', \ldots, R_{n\Phi}', R_{u\Phi}', V_\Phi' \right\rangle$$

be a filtered variant of the model N to the set Φ. Then $\forall x \in \bigcup_{j\in[1,k-m]} C_j$, $\forall \alpha \in \Phi$:

$$\langle N,x\rangle \nVdash \alpha \Leftrightarrow \langle N_\Phi,x\rangle \nVdash \alpha.$$

By virtue of Theorem 2 and Lemma 1, we conclude the finite model property for $\mathcal{LTK}.sl_U$.

5.   Unification in $\mathcal{LTK}.sl_U$

5.1.   Definitions of unification theory

A formula $\varphi(p_1, \ldots, p_s)$ is called *unifiable* in logic $\mathcal{L}$, if $\exists \sigma : p_i \mapsto \sigma_i$ for every $p_i \in Var(\varphi)$, s.t. $\sigma(\varphi) = \varphi(\sigma_1, \ldots, \sigma_s) \in \mathcal{L}$. A substitution σ is called *unifier* of φ. A *ground* unifier is a constant substitution (i.e. $gu : p_i \mapsto \{\top, \bot\}, \forall p_i \in Var(\varphi))$.

The preorder relation is defined on the set of unifiers: an unifier σ of $\varphi(p_1, \ldots, p_s)$ is called *more general* than σ^1 in $\mathcal{L}$, if there is a substitution γ, s.t. for any p_i: $\sigma^1(p_i) \equiv \gamma(\sigma(p_i)) \in \mathcal{L}$ $(\sigma_1 \preceq \sigma)$.

An unifier σ of $\varphi(p_1, \ldots, p_s)$ is said to be *maximal*, if for any other σ^i, either $\sigma^i \preceq \sigma$, or $(\sigma^i \not\preceq \sigma)\&(\sigma \not\preceq \sigma^i)$. If σ is more general than any other, it is called a *most general* (*mgu*, for short).

A set of unifiers CU for a formula φ is called *complete* in $\mathcal{L}$, if for any unifier σ of φ there is $\sigma_1 \in CU$: $\sigma \preceq \sigma^1$.

In general, the existence of infinite sequences of unifiers with respect to a given preorder is possible. If such chains are admissible, the formula (and hence all logic) has a *nullary* unification type. In other cases, when ascending chains are terminated in a finite number of steps, unification is *infinitary* (case of a countable number of maximal unifiers for some formula), *finitary* (case of a finite number of maximal ones for all formulas) or *unitary* (in case of the existence of mgu for all formulas) type.

A formula $\varphi(p_1, \ldots, p_s)$ is called *projective* in $\mathcal{L}$, if there is an unifier τ of φ, s.t. $\Box\varphi \rightarrow [p_i \equiv \tau(p_i)] \in \mathcal{L}$ for all $p_i \in Var(\varphi)$. An unifier with such specified properties is called *projective*.

As was proved by S. Ghilardi [12], the projective unifier defines mgu of a formula (and, accordingly, CU). Consequently, having established the projectivity of unification in logic, we will obtain a universal scheme for constructing an mgu and a unitary type of unification. The importance of this approach is reinforced by a corollary from a projective unification, which guarantees almost structural completeness of logic [17].

5.2.   Projective unification in $\mathcal{LTK}.sl_U$

To study the unification properties in $\mathcal{LTK}.sl_U$ we need to redefine the notion of a projective formula because of the non-transitive and non-reflective nature of the temporal operator N. Let's do it through the universal modal-

ity $\square_U$: $\varphi(p_1, \ldots, p_s)$ is called projective in $\mathcal{LTK}.sl_U$, if there is an unifier τ for φ, s.t. $\square_U \varphi \to [p_i \equiv \tau(p_i)] \in \mathcal{LTK}.sl_U$ for all $p_i \in Var(\varphi)$.

As the following theorem shows, unifiability of an arbitrary formula $\varphi(p_1, \ldots, p_s)$ in $\mathcal{LTK}.sl_U$ can be effectively establish using constant substitutions: $\forall p_i \in Var(\varphi)$ $\sigma(p_i) \in \{\top, \bot\}$.

THEOREM 5.1. *If a formula φ is unifiable in $\mathcal{LTK}.sl_U$, then φ has a ground unifier.*

PROOF: The proof of this theorem is similar to the proof in [4] for the case of pretabular extensions of $\mathcal{S}4$. Here we describe a sketch of the proof and supplement it with some important comments.

Let's take an arbitrary unifiable in $\mathcal{L}$ formula $\varphi(p_1, \ldots, p_s)$ and $\delta_1(q_1, \ldots, q_r), \ldots, \delta_s(q_1, \ldots, q_r)$ is its unifier. Then it is true that

$$\delta(\varphi) := \varphi(\delta_1(q_1, \ldots, q_r), \ldots, \delta_s(q_1, \ldots, q_r)) \in \mathcal{L}.$$

Any substitution of variables $q_1, \ldots, q_r$ to constants $c_i \in \{\top, \bot\}(i \in [1, r])$ preserves truth values of the formula $\delta(\varphi)$, because of $\delta(\varphi) \in \mathcal{L}$. In particular, $\varphi(gu(p_1), \ldots, gu(p_s)) \in \mathcal{L}$, where $gu(p_i) := \delta_i(c_1, \ldots, c_r) \in \{\top, \bot\}$, is a partial case of $\delta(\varphi)$. Therefore, any substitution of this form is a ground unifier of φ. To check the existence of such an substitution for arbitrary formula $\psi(p_1, \ldots, p_s)$, it suffices to consider no more than 2^s substitutions of $\{\top, \bot\}$ instead of all p_i. If among them there is one s.t. $\psi(gu(p_1), \ldots, gu(p_s)) \equiv_{\mathcal{L}} \top$, then ψ is unifiable in $\mathcal{L}$ and gu is its ground unifier. If for all 2^s options $gu(\psi) \notin \mathcal{L}$, then ψ doesn't have a ground unifier and therefore any other unifier in $\mathcal{L}$. $\square$

We are now ready to prove the main result of this work.

THEOREM 5.2. *Any formula unifiable in $\mathcal{LTK}.sl_U$ is projective.*

PROOF: Let $\varphi(p_1, \ldots, p_s)$ be unifiable in $\mathcal{LTK}.sl_U$. Then $\forall p_i \in Var(\varphi)$ we define the following substitution $\sigma(p_i)$:

$$\sigma(p_i) := (\square_U \varphi \wedge p_i) \vee (\neg \square_U \varphi \wedge gu(p_i)),$$

where $gu(p_1), \ldots, gu(p_s)$ is an arbitrary ground unifier for $\varphi(p_1, \ldots, p_s)$.

Let's take an arbitrary $LTK.sl_U$-model $M := \langle F, V \rangle$. If σ is an unifier for φ, then $\sigma(\varphi) \in \mathcal{LTK}.sl_U$ and $\forall x \in F$ $\langle M, x \rangle \Vdash \sigma(\varphi)$. Let us prove that σ is indeed an unifier for φ in $\mathcal{LTK}.sl_U$.

1. If $\forall x \in F : \langle M, x \rangle \Vdash \varphi$, then $\langle M, x \rangle \Vdash_V \square_U \varphi$ and, therefore, the second disjunctive member will be refuted on x. If $\langle M, x \rangle \Vdash p_i$, then $\langle M, x \rangle \Vdash \square_U \varphi \wedge p_i$, hence, $\langle M, x \rangle \Vdash \sigma(p_i)$. If $\langle M, x \rangle \Vdash \neg p_i$, then $\langle M, x \rangle \nVdash \square_U \varphi \wedge p_i$ and, therefore, $\langle M, x \rangle \Vdash \neg \sigma(p_i)$. As a consequence, the truth value $\varphi(p_1, \ldots, p_s)$ on an arbitrary element x wrt V coincides with the truth value $\varphi(\sigma(p_1), \ldots, \sigma(p_s))$ on the same element with the same valuation V and, in this case, $\langle M, x \rangle \Vdash \sigma(\varphi)$.

2. If $\exists x \in F : \langle M, x \rangle \Vdash \neg \varphi$, then $\langle M, x \rangle \nVdash \square_U \varphi$. In this case, the second disjunctive member can be valid, but the first is refuted on x. Then truth values of all $\sigma(p_i)$ on x coincide with $gu(p_i)$ (i.e. $\sigma(\varphi) \equiv gu(\varphi)$), and since $\langle M, x \rangle \Vdash gu(\varphi)$ (due to the choice of the ground unifier $gu(\varphi) \in \mathcal{LTK}.sl_U$), again $\langle M, x \rangle \Vdash \sigma(\varphi)$. Hence, $\sigma(\varphi) \in \mathcal{LTK}.sl_U$ for φ unifiable in $\mathcal{LTK}.sl_U$.

Let us prove that $\sigma(\varphi)$ is a projective unifier. By the definition, if $\sigma(p_i)$ is a projective unifier for φ, we obtain the following: $\forall p_i \in Var(\varphi)$

$$\square_U \varphi \rightarrow (p_i \leftrightarrow [(\square_U \varphi \wedge p_i) \vee (\neg \square_U \varphi \wedge gu(p_i))]) \in \mathcal{LTK}.sl_U. \qquad (5.1)$$

Suppose the opposite: let σ not be a projective unifier and hence 5.1 is refuted at some model. Then it is not difficult to verify that if the premise of the implication is true, it is impossible to refute the conclusion, and therefore we get a contradiction.

Consequently, σ is a projective unifier for φ in $\mathcal{LTK}.sl_U$, so φ is a projective formula. $\qquad \square$

From the proved theorems and mentioned results by S. Ghilardi, hold

COROLLARY 5.3. Let φ be an arbitrary unifiable formula in $\mathcal{LTK}.sl_U$. Then

1. $\sigma(p_i) := (\square_U \varphi \wedge p_i) \vee (\neg \square_U \varphi \wedge gu(p_i))$ is a projective unifier and, hence, mgu for φ;

2. The logic $\mathcal{LTK}.sl_U$ has a unitary unification type.

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