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New Lipschitz–type conditions for uniqueness of solutions of ordinary differential equations

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ABSTRACT

We present some generalized Lipschitz conditions which imply uniqueness of solutions for scalar ODEs. We illustrate the applicability of our results with examples not covered by earlier Lipschitz–type uniqueness tests.

1. Introduction

This paper considers local uniqueness of solutions of initial value problems such as

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{1.1}$$

where $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ is a given function. A good account of this classical subject can be found in [1], see also [7,9].

In order to give a flavor of the kind of results we obtain in this paper our starting point is the one-sided Montel–Tonelli's condition [1,6] that, roughly speaking, implies the local uniqueness for problem (1.1) with $t > t_0$ provided that the nonlinearity f satisfies

$$f(t, y) - f(t, x) \le k(t)\, \varphi(y - x), \quad \text{for } x < y, \tag{1.2}$$

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where k is integrable and $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Our key improvement in Montel–Tonelli's condition is to allow the presence of a nonnegative continuous function g in (1.2), namely

$$f(t, y)g(y) - f(t, x)g(x) \leq k(t)\, \varphi \left(\int_x^y g(s)\, ds \right), \quad \text{for } x < y. \tag{1.3}$$

Remarkably, condition (1.3) is satisfied when g is bounded below by a positive constant and the product $f(t, x)g(x)$ is Lipschitz–continuous with respect to x which does not imply, as we will see, that $f(t, x)$ be Lipschitz continuous with respect to x.

We stress that our condition (1.3) can be employed even when (1.1) has a singularity at t_0 and, of course, can be adapted to deal with solvability on the left of t_0. So in this way we obtain, by means of the two–sided counterpart of condition (1.2), a uniqueness result valid for an interval centered at t_0. We also point out that for simplicity we state and prove our results in the scalar setting but we show how they can be extended to the n-dimensional case.

Another approach, initiated in [14], that we explore in the final part of the paper is the transference of assumptions from the x variable to the t variable in order to ensure the uniqueness of the solution. Inspired by [10] we are able to obtain such transference even if $f(t_0, x_0) = 0$, a situation that is typically avoided by the so–called transversality condition [13].

This paper is organized as follows: in Section 2 we use a generalized Gronwall lemma to prove our main uniqueness result for solutions of (1.1) on the right of t_0, and we discuss some particular cases and variants; in Section 3 we deduce analogous uniqueness results for (1.1) on the left of t_0 by means of a change of variable and derive a two–sided uniqueness result and a multidimensional extension; finally, in Section 4, we transfer assumptions from the x variable to the t variable by means of reciprocal problems, in a sense to be precised there. We also provide several examples through the text to illustrate the applicability of our results.

2. A generalized one–sided Lipschitz condition

Let t_0, x_0, a, $b \in \mathbb{R}$, $a > 0$, $b > 0$, and consider a continuous function

$$f : (t_0, t_0 + a] \times [x_0 - b, x_0 + b] \longrightarrow \mathbb{R}.$$

We are concerned with uniqueness of solution of the initial value problem

$$x' = f(t, x),\ t > t_0, \quad x(t_0) = x_0, \tag{2.4}$$

which might exhibit a singularity at the initial time t_0, thus forcing us to relax the classical notion of a solution as follows.

Definition 2.1. A solution of (2.4) is a continuous function

$$x : [t_0, t_0 + c] \longrightarrow [x_0 - b, x_0 + b], \quad \text{for some } c \in (0, a],$$

such that $x(t_0) = x_0$ and $x'(t) = f(t, x(t))$ for all $t \in (t_0, t_0 + c]$.

As an instance, consider the initial value problem

$$x' = \frac{1}{2\sqrt{t}}, \ t > 0, \quad x(0) = 0,$$

which has a singularity at $t = 0$ and a unique solution $x(t) = \sqrt{t}$, $t \geq 0$.

Our proofs lean on the following generalized Gronwall lemma.

Lemma 2.1. *[4, Lemma 2.1] Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function, with $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$. Let t_1, $t_2 \in \mathbb{R}$, $t_1 < t_2$, $k : [t_1, t_2] \to [0, +\infty)$ measurable, and $\int_{t_1}^{t_2} k(s)ds < +\infty$.*

If $u : [t_1, t_2] \to \mathbb{R}$ is continuous and

$$0 \leq u(t) \leq \int_{t_1}^{t} k(s)\varphi(u(s))ds \quad \text{for all } t \in (t_1, t_2], \tag{2.5}$$

then $u(t) = 0$ for all $t \in [t_1, t_2]$.

We are already in a position to prove one of the main results in this paper about uniqueness of solutions.

Theorem 2.1. *Assume there exist a continuous function $g : (x_0 - b, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1((t_0, t_0 + a], [0, \infty))$ such that for almost every $t \in (t_0, t_0 + a]$ we have*

$$f(t, y)g(y) - f(t, x)g(x) \leq k(t)\, \varphi\left(\int_x^y g(s)\, ds\right) \quad \text{whenever } x_0 - b < x < y < x_0 + b, \tag{2.6}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (2.4) has at most one solution.

Proof. Let us define the increasing function

$$G(x) = \int_{x_0}^{x} g(s)\, ds, \quad x \in (x_0 - b, x_0 + b),$$

which is continuously differentiable on $(x_0 - b, x_0 + b)$.

Reasoning by contradiction, let us assume that $x(t)$ and $y(t)$ are two different solutions of (2.4). Without loss of generality, we assume that there exist $t_1, t_2 \in [t_0, t_0 + a]$ such that $t_1 < t_2$, $x(t_1) = y(t_1)$ and $x_0 - b < x(t) < y(t) < x_0 + b$ for all $t \in (t_1, t_2]$.

For almost every $t \in (t_1, t_2]$, condition (2.6) yields

$$\begin{aligned}(G \circ y)'(t) - (G \circ x)'(t) &= y'(t)g(y(t)) - x'(t)g(x(t)) \\ &= f(t, y(t))g(y(t)) - f(t, x(t))g(x(t)) \leq k(t)\varphi\left(\int_{x(t)}^{y(t)} g(s)\, ds\right).\end{aligned}$$

Let us denote $u(t) = G \circ y - G \circ x$, and observe that the previous inequality yields

$$0 \leq u(t) \leq \int_{t_1}^{t} k(s)\varphi(u(s))\, ds \quad \text{for all } t \in [t_1, t_2].$$

We deduce from Lemma 2.1 that $u(t) = 0$ on $[t_1, t_2]$, a contradiction with $x < y$ on $(t_1, t_2]$. □

Remark 2.1. Condition (2.6) with $g \equiv 1$, is just the scalar version of Montel–Tonelli's uniqueness criterium, see [6].

Of course, with $g \equiv 1$, k constant and $\varphi(z) = z$ for all $z \geq 0$, we recover the classical one–sided Lipschitz condition

$$f(t, y) - f(t, x) \leq k(y - x), \quad x \leq y.$$

Observe also that if $g \equiv 1$ and $k \equiv 0$ then condition (2.6) reduces to Peano's uniqueness condition, namely, $f(t, x)$ nonincreasing with respect to x.

Remarkably, condition (2.6) is satisfied provided that for some $L \geq 0$ and every t we have

$$f(t, y)g(y) - f(t, x)g(x) \leq L(y - x), \quad x \leq y, \tag{2.7}$$

and there exists $\rho > 0$ such that $g(x) \geq \rho$, $x \in (x_0 - b, x_0 + b)$. Indeed, notice that for $x_0 - b < x < y < x_0 + b$ we have

$$\int_{x}^{y} g(s)\, ds \geq \rho(y - x),$$

so condition (2.6) holds with $k = L\rho^{-1}$ and $\varphi(z) = z$ for all $z \geq 0$.

Notice that (2.7) does not imply that $f(t, x)$ be Lipschitz with respect to x, as we show in the next example.

Example 2.1. We shall prove that problem

$$x' = 1 + \sqrt[3]{x} - x\sqrt{t},\ t \geq 0, \quad x(0) = 0,$$

has a unique solution.

First, Peano's Theorem ensures the existence of at least one solution on some interval $[0, a]$, $a > 0$.

Now, we prove uniqueness on $[0, a]$ by means of Theorem 2.1.

Consider $f(t, x) = 1 + \sqrt[3]{x} - x\sqrt{t}$ for all $(t, x) \in [0, a] \times [-1/2, 1/2]$ and take the positive continuous function

$$g(x) = \frac{1}{1 + \sqrt[3]{x}}, \quad x \in [-1/2, 1/2].$$

Observe that

$$f(t, x)g(x) = 1 - \sqrt{t}\frac{x}{1 + \sqrt[3]{x}},$$

and the function $h(x) = x/(1 + \sqrt[3]{x})$ is increasing on $[-1/2, 1/2]$. Hence, condition (2.7) holds with $L = 0$, which implies condition (2.6).

Finally, observe that $f(t,x)$ is not Lipschitz continuous with respect to x or t on any neighborhood of the initial condition. Moreover, for any $t \in [0,1]$ the mapping $f(t,\cdot)$ is increasing on some interval around the initial condition $x=0$, thus falling outside the scope of Peano's uniqueness theorem.

The following elementary lemma is helpful in order to construct examples involving more general Kamke–Osgood functions φ in condition (2.6).

Lemma 2.2. *If $\psi : [0,\infty) \to [0,\infty)$ is nondecreasing and concave, and $\psi(0)=0$, then for any $a,b \in [0,\infty)$ we have*

$$|\psi(a)-\psi(b)| \le \psi(|a-b|).$$

Proof. The result is trivial if $a=0$ or $b=0$ or $a=b$, so let us assume, without loss of generality, that $0<a<b$. Since ψ is nondecreasing,

$$|\psi(a)-\psi(b)| = \psi(b)-\psi(a).$$

Since ψ is concave, the slope of the secant line through $(a,\psi(a))$ and $(b,\psi(b))$ is less or equal than the slope of the segment $(0,\psi(0))=(0,0)$ and $(b,\psi(b))$, i.e.

$$\frac{\psi(b)-\psi(a)}{b-a} \le \frac{\psi(b)}{b}.$$

Analogously, the slope of the segment joining $(0,\psi(0))=(0,0)$ and $(b,\psi(b))$ is less than or equal to the slope of the segment with endpoints $(0,\psi(0))=(0,0)$ and $(b-a,\psi(b-a))$, i.e.

$$\frac{\psi(b)}{b} \le \frac{\psi(b-a)}{b-a},$$

and the proof is complete. □

Next proposition contains a family of examples for which condition (2.6) is satisfied with a nontrivial Kamke–Osgood function φ.

Proposition 2.1. *Assume there exist a continuous function $g : (x_0-b, x_0+b) \longrightarrow [0,\infty)$, $g(x) \ge \rho > 0$ for all $x \in (x_0-b,x_0+b)$, and a function $k \in L^1((t_0,t_0+a],[0,\infty))$ such that for almost every $t \in (t_0,t_0+a]$ and all $x \in (x_0-b,x_0+b)$ we have*

$$f(t,x)g(x) = k(t)\psi(|h(t,x)|), \tag{2.8}$$

where $h : I \times [x_0-b,x_0+b] \to \mathbb{R}$ is Lipschitz–continuous with respect to x and Lipschitz constant $L>0$ and $\psi : [0,\infty) \to [0,\infty)$ is increasing, concave, $\psi(0)=0$, and there exists

$$\lim_{z\to 0^+} \frac{\psi(L\,\rho^{-1}\,z)}{-z\ln z} \in \mathbb{R}. \tag{2.9}$$

Then, problem (2.4) has at most one solution.

Proof. It suffices to show that condition (2.6) holds for some function φ in the conditions of Theorem 2.1. To do so, observe that for a.a. $t \in (t_0,t_0+a]$ and $x_0-b<x<y<x_0+b$ we have, by Lemma 2.2, that

$$
\begin{aligned}
f(t,y)g(y) - f(t,x)g(x) &= k(t)[\psi(|h(t,y)|) - \psi(|h(t,x)|)] \le k(t)\psi(|h(t,y) - h(t,x)|) \\
&\le k(t)\psi(L|y-x|) \le k(t)\psi\left(L\rho^{-1}\int_x^y g(s)\,ds\right).
\end{aligned}
$$

Therefore, condition (2.6) holds with $\varphi(z) = \psi(L\rho^{-1}z)$, $z \ge 0$, which is nondecreasing, $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and for any $\varepsilon > 0$ we have

$$
\int_0^{\varepsilon} \frac{dz}{\varphi(z)} = +\infty
$$

due to (2.9) and

$$
\int_0^{\mathrm{e}^{-1}} \frac{dz}{-z\ln z} = +\infty. \quad \square
$$

When we have some additional information about all possible solutions we can allow g to have a weak singularity at x_0, that is

$$
\lim_{x\to x_0^+} g(x) = +\infty \quad \text{and} \quad \int_{x_0}^{x_0+b} g(s)\,ds < \infty.
$$

So, we focus on the case of non–negative right–hand sides under the following basic assumption, which avoids constant solutions:

$(H1)$ $f(\cdot, x_0)$ is not identically zero on $(t_0, t_0 + \varepsilon)$ for any $\varepsilon \in (0, a)$.

Theorem 2.2. *Let $f : (t_0, t_0 + a] \times [x_0, x_0 + b] \longrightarrow [0, \infty)$ satisfy $(H1)$.*

Assume that there exist a continuous and integrable function $g : (x_0, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1((t_0, t_0 + a], [0, \infty))$ such that for almost every $t \in (t_0, t_0 + a]$ we have

$$
f(t,y)g(y) - f(t,x)g(x) \le k(t)\,\varphi\left(\int_x^y g(s)\,ds\right) \quad \text{whenever } x_0 < x < y < x_0 + b, \tag{2.10}
$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^{\varepsilon} \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (2.4) has at most one solution.

Proof. Observe that solutions (if any) are nondecreasing, which, along with condition $(H1)$, implies that solutions cannot assume again the value x_0 in the interval $(t_0, t_0 + a]$.

Now, let $x, y : [t_0, t_0 + c] \longrightarrow [x_0, x_0 + b]$ be two different solutions of (2.4). For definiteness, assume that for some $t_2 \in (t_0, t_0 + c)$ we have $x(t_2) < y(t_2) < x_0 + b$. By continuity of x and y, there exists $t_1 \in [t_0, t_2)$ such that

$$
x(t_1) = y(t_1) \quad \text{and} \quad x_0 < x(t) < y(t) < x_0 + b \text{ on } (t_1, t_2].
$$

Now the proof follows exactly as in Theorem 2.1 (integrability of g on $(x_0, x_0 + b)$ is needed to have a well–defined function $G(x) = \int_{x_0}^{x} g(s)\,ds$ for all $x \in (x_0, x_0 + b)$). $\square$

Remark 2.2. Assumption $(H1)$ is fundamental in Theorem 2.2: the function $f(t,x) = \sqrt{x}$ defined on $(0,1] \times [0,1]$ satisfies condition (2.10) (with $g(x) = \frac{1}{\sqrt{x}}$, $k \equiv 0$ and $\varphi(z) = z$) but f does not satisfy $(H1)$ and the associated initial value problem

$$x' = \sqrt{x}, \quad x(0) = 0, \quad t > 0,$$

is a paradigmatic example of nonuniqueness.

There are many non–Lipschitz functions in the conditions of Theorem 2.2. In the next proposition we present a family of functions which satisfy condition (2.10) with $\varphi(z) = z$ and some constant function $k(t) \equiv k \geq 0$.

Proposition 2.2. *Let $f : [t_0, t_0 + a] \times [x_0, x_0 + b] \longrightarrow \mathbb{R}$ be expressible in the form*

$$f(t,x) = F(t,x) + G(t,x)\,(x - x_0)^r, \quad \textit{for some } r \in (0,1).$$

If $F(t,x) \geq 0$ for all $(t,x) \in [t_0, t_0+a] \times [x_0, x_0+b]$ and there exists $L \geq 0$ such that for every $t \in [t_0, t_0+a]$ we have

$$F(t,y) - F(t,x) \leq L\,(y - x) \quad \textit{and} \quad G(t,y) - G(t,x) \leq L\,(y - x), \quad \textit{whenever } x_0 < x < y < x_0 + b, \tag{2.11}$$

then the function $f(t,x)$ satisfies (2.10) with $\varphi(z) = z$, $g(x) = (x - x_0)^{-r}$ and $k = Lb^r + L$.

If moreover $G(t,x) \geq 0$ for all $(t,x) \in [t_0, t_0 + a] \times [x_0, x_0 + b]$ then we also have

$$f(t,x) \geq 0 \quad \textit{for all } (t,x) \in [t_0, t_0 + a] \times [x_0, x_0 + b].$$

On the other hand, it is clear that f satisfies condition $(H1)$ if and only if F does so.

Proof. Let $t \in [t_0, t_0 + a]$ be fixed and $x_0 < x < y < x_0 + b$. We have

$$\frac{F(t,y)}{(y - x_0)^r} - \frac{F(t,x)}{(x - x_0)^r} \leq \frac{F(t,y) - F(t,x)}{(y - x_0)^r} \leq L \frac{y - x}{(y - x_0)^r} \leq L \int_x^y g(s)\,ds,$$

hence

$$f(t,y)g(y) - f(t,x)g(x) = \frac{F(t,y)}{(y - x_0)^r} - \frac{F(t,x)}{(x - x_0)^r} + G(t,y) - G(t,x) \leq L \int_x^y g(s)\,ds + L(y - x).$$

On the other hand,

$$\int_x^y g(s)\,ds = \int_x^y (s - x_0)^{-r}\,ds \geq (y - x_0)^{-r}(y - x) \geq b^{-r}(y - x).$$

Summing up, for each fixed $t \in [t_0, t_0 + a]$ and $x_0 < x < y < x_0 + b$ we have

$$f(t,y)g(y) - f(t,x)g(x) \leq k \int_x^y g(s)\,ds,$$

for $k = Lb^r + L$. $\square$

Proposition 2.2 is very useful in the application of Theorem 2.2, as we show in our next example.

Example 2.2. The initial value problem

$$x' = \sqrt{t}x^4 + \sqrt[3]{t} + (t + x^2)\sqrt[5]{x^4}, \ t \geq 0, \quad x(0) = 0,$$

has a unique solution.

Once again, existence follows from Peano's theorem. Now if $x : [0, c] \longrightarrow [0, \infty)$ is a solution, we take $b > x(c)$ and we observe that the right–hand side of the ODE can be written as

$$f(t, x) = F(t, x) + G(t, x)x^{4/5},$$

for $F(t, x) = \sqrt{t}x^4 + \sqrt[3]{t}$ and $G(t, x) = t + x^2$ for all $(t, x) \in [0, c] \times [0, b]$. By virtue of Proposition 2.2, $f(t, x)$ satisfies (2.10) with $\varphi(z) = z$ for $z \geq 0$, $g(x) = x^{-4/5}$ for $x \in (0, b)$, and a sufficiently large constant $k > 0$. Moreover $G(t, x) \geq 0$ and F satisfies $(H1)$ so all the assumptions of Theorem 2.2 are satisfied.

Observe that $f(t, x)$ is not Lipschitz continuous with respect to x, nor with respect to t, on any neighborhood of the initial condition $(0, 0)$, thus falling outside the scope of the results in [3,5,10,12,14].

There is an analog to Theorem 2.2 for negative right–hand sides.

Theorem 2.3. *Let $f : (t_0, t_0 + a] \times [x_0 - b, x_0] \longrightarrow (-\infty, 0]$ satisfy $(H1)$.*

Assume there exist a continuous and integrable function $g : (x_0 - b, x_0) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1((t_0, t_0 + a], [0, \infty))$ such that for almost every $t \in (t_0, t_0 + a]$ we have

$$f(t, y)g(y) - f(t, x)g(x) \leq k(t)\,\varphi\left(\int_x^y g(s)\,ds\right) \quad \text{whenever } x_0 - b < x < y < x_0, \tag{2.12}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (2.4) has at most one solution.

We close this section with an alternative version of Theorem 2.2 which guarantees uniqueness of the constant solution $x(t) = x_0$ when $f(\cdot, x_0) \equiv 0$ and replaces the Lipschitz–type condition by just a bound on $f(t, y)g(y)$.

Theorem 2.4. *Let $f : (t_0, t_0 + a] \times [x_0, x_0 + b] \longrightarrow [0, \infty)$ be such that $f(t, x_0) = 0$ for all $t \in (t_0, t_0 + a]$.*

Assume that there exist a continuous and integrable function $g : (x_0, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1((t_0, t_0 + a], [0, \infty))$ such that for almost every $t \in (t_0, t_0 + a]$ we have

$$f(t, y)g(y) \leq k(t)\,\varphi\left(\int_{x_0}^y g(s)\,ds\right) \quad \text{whenever } x_0 < y < x_0 + b, \tag{2.13}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (2.4) only has the trivial solution $x(t) = x_0$ for all $t \in (t_0, t_0 + a]$.

Proof. If follows as in Theorem 2.1 with $x(t) = x_0$ and taking into account that now solutions are nondecreasing since f is assumed nonnegative. □

3. Backwards uniqueness and a generalized two–sided Lipschitz condition

We recall that backwards uniqueness just needs a change of variable: a function $x(t)$, $t \in [t_0 - c, t_0]$ ($c \in (0, a]$), is a solution of

$$x'(t) = f(t, x(t)),\ t < t_0,\ x(t_0) = x_0, \tag{3.14}$$

if and only if $y(t) = x(2t_0 - t)$ is a solution of (2.4) with $f(t, z)$ replaced by $-f(2t_0 - t, z)$.

The previous observation yields the following corollary of Theorem 2.1.

Corollary 3.1. *Let $f : [t_0 - a, t_0) \times [x_0 - b, x_0 + b] \longrightarrow \mathbb{R}$.*

Assume there exist a continuous function $g : (x_0 - b, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1([t_0 - a, t_0), [0, \infty))$ such that for almost every $t \in [t_0 - a, t_0]$ we have

$$f(t, x)g(x) - f(t, y)g(y) \le k(t)\, \varphi \left(\int_x^y g(s)\, ds \right) \quad \text{whenever } x_0 - b < x < y < x_0 + b, \tag{3.15}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (3.14) has at most one solution.

Proof. The reversed problem

$$y' = -f(2t_0 - t, y),\ t > 0,\ y(t_0) = x_0, \tag{3.16}$$

has at most one solution by virtue of Theorem 2.1. Indeed, condition (3.15) implies that the right–hand side in the ODE in (3.16) satisfies condition (2.6). □

Obviously, we have similar corollaries of Theorem 2.2 and Theorem 2.3. Pay attention to the domains and signs specified for the nonlinear part in the corresponding statements (for instance, unlike Theorem 2.2, its corollary applies for negative nonlinearities only).

Corollary 3.2. *Let $f : [t_0 - a, t_0) \times [x_0, x_0 + b] \longrightarrow (-\infty, 0]$ satisfy $(H1)$.*

Assume that there exist a continuous and integrable function $g : (x_0, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1([t_0 - a, t_0), [0, \infty))$ such that for almost every $t \in [t_0 - a, t_0]$

$$f(t, x)g(x) - f(t, y)g(y) \le k(t)\, \varphi \left(\int_{x_0}^x g(s)\, ds \right) \quad \text{whenever } x_0 < x < y < x_0 + b, \tag{3.17}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (3.14) has at most one solution.

The corresponding corollary of Theorem 2.3 reads as follows (being analogous the corresponding corollary of Theorem 2.4).

Corollary 3.3. *Let $f : [t_0 - a, t_0) \times [x_0 - b, x_0] \longrightarrow [0, \infty)$ satisfy $(H1)$.*

Assume there exist a continuous and integrable function $g : (x_0 - b, x_0) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1([t_0 - a, t_0), [0, \infty))$ such that for almost every $t \in [t_0 - a, t_0]$

$$f(t,x)g(x) - f(t,y)g(y) \le k(t)\,\varphi\left(\int_{x_0}^{x} g(s)\,ds\right) \quad \text{whenever } x_0 - b < x < y < x_0, \tag{3.18}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^{\varepsilon} \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then problem (3.14) has at most one solution.

We can ensure conditions (2.6) and (3.15) at one stroke by means of a generalized two–sided Lipschitz condition. In this case we also include information about existence of solutions, which follows immediately from Peano's theorem.

Corollary 3.4. *For $(t_0, x_0) \in \mathbb{R}^2$ and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b],$$

and let $f : U \longrightarrow \mathbb{R}$ be a continuous function.

Assume there exist a continuous function $g : (x_0 - b, x_0 + b) \longrightarrow [0, \infty)$, $g(x) > 0$ almost everywhere, and a function $k \in L^1((t_0 - a, t_0 + a), [0, \infty))$ such that for almost every $t \in [t_0 - a, t_0 + a]$ we have

$$|f(t,y)g(y) - f(t,x)g(x)| \le k(t)\,\varphi\left(\int_{x}^{y} g(s)\,ds\right), \quad \text{whenever } x_0 - b < x < y < x_0 + b, \tag{3.19}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^{\varepsilon} \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then the initial value problem (1.1) has a unique solution defined on some interval $(t_0 - \nu, t_0 + \nu)$, with $\nu > 0$.

Finally, we establish a multidimensional version of the previous result. Its proof follows similar ideas to that of Theorem 2.1, but we include it for the sake of completeness.

Theorem 3.1. *For $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n})$, and positive numbers a and b, define*

$$U = [t_0 - a, t_0 + a] \times [x_{0,1} - b, x_{0,1} + b] \times \cdots \times [x_{0,n} - b, x_{0,n} + b],$$

and let $f : U \longrightarrow \mathbb{R}^n$ be a continuous function.

Assume there exist continuous functions $g_i : (x_{0,i} - b, x_{0,i} + b) \longrightarrow [0, \infty)$, $i = 1, \ldots, n$, with $g_i(x) > 0$ almost everywhere, and a function $k \in L^1((t_0 - a, t_0 + a), [0, \infty))$ such that for almost every t and all x, y with $(t,x), (t,y) \in U$ we have

$$\sum_{i=1}^{n} |f_i(t,y)g_i(y_i) - f_i(t,x)g_i(x_i)| \le k(t)\,\varphi\left(\sum_{i=1}^{n}\left|\int_{x_i}^{y_i} g_i(s)\,ds\right|\right), \tag{3.20}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^{\varepsilon} \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then the initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0, \tag{3.21}$$

has a unique solution defined on some interval $(t_0 - \nu, t_0 + \nu)$*, with* $\nu > 0$*.*

Proof. Let us define the continuously differentiable function

$$G(x) = G(x_1, \ldots, x_n) = \left(\int_{x_{0,1}}^{x_1} g_1(s)\, ds, \ldots, \int_{x_{0,n}}^{x_n} g_n(s)\, ds \right), \quad x_i \in (x_{0,i} - b, x_{0,i} + b), \quad i = 1, \ldots, n.$$

Let us assume that x and y are two solutions of (3.21) such that $(t, x(t)), (t, y(t)) \in U$ for all $t \in [t_0, t_1]$. Denote by $\|\cdot\|$ the 1-norm on $\mathbb{R}^n$, i.e., $\|x\| = \sum_{i=1}^n |x_i|$.

For almost every $t \in [t_0, t_1]$, condition (3.20) yields

$$\begin{aligned} \|(G \circ y)'(t) - (G \circ x)'(t)\| &= \sum_{i=1}^n |f_i(t, y(t)) g_i(y_i(t)) - f_i(t, x(t)) g_i(x_i(t))| \\ &\leq k(t) \varphi \left(\sum_{i=1}^n \left| \int_{x_i(t)}^{y_i(t)} g_i(s)\, ds \right| \right). \end{aligned}$$

Finally, let us denote $u = \|G \circ y - G \circ x\|$, and observe that

$$0 \leq u(t) \leq \int_{t_0}^{t} k(s) \varphi\left(u(s)\right)\, ds \quad \text{for all } t \in [t_0, t_1].$$

We deduce from Lemma 2.1 that $u(t) = 0$ on $[t_0, t_1]$, which implies that $x = y$ on $[t_0, t_1]$. $\square$

Remark 3.1. Observe that condition (3.20) with $g \equiv (1, \ldots, 1)$ reduces to Montel-Tonelli's condition, namely,

$$\|f(t, y) - f(t, x)\| \leq k(t) \varphi\left(\|y - x\|\right),$$

where $\|\cdot\|$ stands for the 1-norm on $\mathbb{R}^n$.

4. Uniqueness through reciprocal problems

Roughly speaking, when $f(t, x)$ is positive (or negative) then assumptions can be transferred to the time variable just by studying a reciprocal problem, see [2,3,5,8,10,12,14]. Our next result is a somewhat sharper form of [5, Theorem 2.1].

Theorem 4.1. *For* $(t_0, x_0) \in \mathbb{R}^2$ *and positive numbers* a *and* b*, define*

$$U = (t_0, t_0 + a] \times [x_0 - b, x_0 + b].$$

Let $f : U \longrightarrow \mathbb{R}$ *be a continuous function satisfying the following conditions:*

(1) $f(t, x) \neq 0$ *whenever* $x \neq x_0$*;*

(2) $f(t, x_0)$ is not identically zero on $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$.

Then, either $f(t, x) \geq 0$ for all $(t, x) \in U$, or $f(t, x) \leq 0$ for all $(t, x) \in U$, and if the reciprocal problem

$$t'(x) = \frac{1}{f(t(x), x)}, \; x \neq x_0, \quad t(x_0) = t_0, \tag{4.22}$$

has at most one solution defined on the right of x_0 (if $f \geq 0$) or on the left of x_0 (if $f \leq 0$), then the initial value problem (2.4) has at most one solution.

Proof. We shall prove that all possible solutions of (1.1) are strictly monotone and their inverses solve (4.22) on the same side of x_0, thus proving that (1.1) cannot have more than one solution.

First, observe that conditions (1) and (2) imply that f has constant sign on U (that is, either $f(t, x) \geq 0$ for all $(t, x) \in U$ or $f(t, x) \leq 0$ for all $(t, x) \in U$). To prove it, note that condition (2) ensures that $f(t_1, x_0) \neq 0$ for some $t_1 \in (t_0, t_0 + a)$. Now, fix an arbitrary point $(t, x) \in U$, $x \neq x_0$, which implies $f(t, x) \neq 0$ by condition (1). If $f(t_1, x_0) \cdot f(t, x) < 0$, then, by continuity of f, the segment with endpoints (t_1, x_0) and (t, x) contains a point (t_2, y) such that $f(t_2, y) = 0$, a contradiction with condition (1).

Now, if $x(t)$ is a solution of (1.1) on some interval $I = [t_0, t_0 + c]$, we either have $x'(t) \geq 0$ for all $t \in I$ or $x'(t) \leq 0$ for all $t \in I$, hence x is monotone on I. Let us prove that $x'(t) \neq 0$ for all $t \in I$, $t \neq t_0$. Reasoning by contradiction, assume that for some $t^* \in I$, $t^* \neq t_0$, we have $0 = x'(t^*) = f(t^*, x(t^*))$. Then we deduce from condition (1) that $x(t^*) = x_0$. Since x is monotone and $x(t_0) = x_0$, we deduce that x is constant between t_0 and t^*, hence $0 = x'(t) = f(t, x(t))$ for all $t \in (t_0, t^*)$, but this is impossible due to condition (2).

Summing up, x is strictly monotone on $I = [t_0, t_0 + c]$, with nonzero derivative everywhere on $(t_0, t_0 + c]$, and therefore its inverse function $t = x^{-1} : J = x(I) \longrightarrow I$ solves the reciprocal IVP (4.22), either on the right of x_0 (if f is nonnegative) or on the left of x_0 (if f is nonpositive). $\square$

Plainly, imposing on (4.22) the assumptions of the results in the previous section, we obtain new uniqueness results for (2.4) *via* Theorem 4.1. As a sample, we include the following.

Corollary 4.1. *For $(t_0, x_0) \in \mathbb{R}^2$ and positive numbers a and b, define*

$$U = (t_0, t_0 + a] \times [x_0, x_0 + b].$$

Let $f : U \longrightarrow \mathbb{R}$ be a continuous function satisfying the following three conditions:

(1) $f(t, x) > 0$ whenever $x > x_0$;
(2) $f(t, x_0)$ is not identically zero on any interval $(t_0, t_0 + \varepsilon)$ for $0 < \varepsilon < a$;
(3) there exist a continuous and integrable function $g : (t_0, t_0 + a) \longrightarrow [0, \infty)$, $g(t) > 0$ for a.e. t, and a function $k \in L^1((x_0, x_0 + b), [0, \infty))$ such that for almost every $x \in (x_0, x_0 + b)$ we have

$$\frac{g(t)}{f(t, x)} - \frac{g(s)}{f(s, x)} \leq k(x)\, \varphi\left(\int_s^t g(r)\, dr\right) \quad \text{whenever } t_0 < s < t < t_0 + a, \tag{4.23}$$

where $\varphi : [0, +\infty) \to [0, +\infty)$ is a nondecreasing function such that $\varphi(0) = 0$, $\varphi(z) > 0$ for $z > 0$, and $\int_0^\varepsilon \frac{dz}{\varphi(z)} = +\infty$ for every $\varepsilon > 0$.

Then the initial value problem (1.1) has at most one solution.

Proof. Since f is nonnegative, we need uniqueness of solution of (4.22) on the right of x_0, which follows from condition (3) and Theorem 2.2. $\square$

Finally, we illustrate the applicability of Corollary 4.1.

Example 4.1. The singular initial value problem

$$x' = f(t,x) = \frac{1}{\sqrt{t}} + \sqrt[3]{x^2},\ t > 0,\ x(0) = 0, \tag{4.24}$$

has at most one solution. Indeed, take $g(t) = 1/\sqrt{t}$, $t > 0$, and observe that for $0 < s < t$ and any $x \geq 0$, we have

$$\frac{g(t)}{f(t,x)} - \frac{g(s)}{f(s,x)} = \frac{1}{1+\sqrt{t}\sqrt[3]{x^2}} - \frac{1}{1+\sqrt{s}\sqrt[3]{x^2}} \leq 0 < \int_s^t g(r)\,dr,$$

so condition (4.23) in Corollary 4.1 is satisfied with $k \equiv 1$ and $\varphi(z) = z$ for all $z > 0$.

In this case, we can also obtain information about the existence of solution to (4.24) through its reciprocal problem, namely,

$$t' = \frac{\sqrt{t}}{1+\sqrt{t}\sqrt[3]{x^2}},\ x > 0,\ t(0) = 0. \tag{4.25}$$

Observe that (4.25) has a classical everywhere differentiable solution because the right–hand side in the ODE can be extended to a continuous function for all $(t,x) \in \mathbb{R}^2$, hence Peano's existence theorem applies. Now, the inverse of the classical solution of (4.25) is a solution of (4.24) in the sense of Definition 2.1.

The approach in this section is applicable with many other uniqueness conditions imposed on (4.22), such as those in [1,6,11].

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