On Out of Sample Errors in Portfolio Optimization – A Study of the Portuguese Market

João Bento

Dissertation written under the supervision of Professor Pedro Monteiro e Silva Barroso

Dissertation submitted in partial fulfilment of requirements for the MSc in Finance, at the Universidade Católica Portuguesa, 28/03/2022.

Abstract

Motivated by a long-standing notion of the challenge that is investing successfully in the Portuguese stock exchange, this thesis studies the implementation in Portugal of a new model that tackles one of portfolio optimization's greatest obstacles: estimation errors. The Galton model, originally developed and tested in the United States market, uses a discovered pattern in out of sample errors, the difference between forecasted and realized values in out of sample tests, to correct historical inputs. These errors are mostly disregarded by other optimization efforts. With the corrected inputs, the model delivers a performance upgrade all around - more sensible weighting of the assets, improved return performance, returns concentrated around its mean avoiding extreme occurrences, less turnover and, above all, accurate forecasting of the portfolios' volatility, in the same line as the models' conclusions in the American market.

Keywords: Covariance Matrix; Out of Sample Errors; Portfolio Optimization; Risk Management.

Resumo

Motivado por uma perceção recorrente do desafio que é investir de forma bem-sucedida na bolsa financeira portuguesa, esta tese estuda a implementação em Portugal de um novo modelo que visa resolver um dos maiores obstáculos para a otimização de portfólios: erros de estimação. O modelo Galton, originalmente desenvolvido e testado no mercado dos Estados Unidos da América, utiliza um padrão descoberto nos erros fora da amostra, a diferença entre os valores previstos e os reais, para corrigir os dados históricos inseridos. Estes erros são, maioritariamente, ignorados por outros esforços de otimização. Com os dados corrigidos, o modelo produz uma melhoria no desempenho em todos os aspetos – pesos dos ativos mais contidos, melhoramento a nível de retornos, retornos mais concentrados em volta da sua média evitando ocorrências extremas, menos volume de transações de ativos e, acima de tudo, previsão precisa da volatilidade dos portfólios, em linha com as conclusões do modelo no mercado americano.

Palavras-chave: Erros Fora da Amostra; Gestão de Risco; Matriz de Covariância; Otimização de Portfólios.

Contents List

List of Tables and Figures

Acknowledgements

I deliver this thesis knowing it is a decisive moment of my academic path and of my life.

First of all, I would like to express my gratitude to my family, especially to my parents, Paula and João, to my brother Miguel, to my grandmother, Edite and to my godparents, Fernando and Matilde, for all the support and for providing all the conditions so that I could achieve my full potential. Without them, it would not be possible. To my grandfather Fernando, I made it.

Secondly, I would like to thank my friends - from childhood, from my hometown, from previous schools and from Católica -, who were in it with me throughout the journey. May we be together for future achievements.

Lastly, I would like to thank the Católica - Lisbon School of Business & Economics, where I felt embraced since the first day and where I was provided with the best education and a high demand that led me to leave a better student, professional and person than I was the day I was admitted.

1. Introduction

In this thesis, I study the Galton method from "Lest We Forget: using out-of-sample forecast errors in portfolio optimization" by Pedro Barroso and Konark Saxena in the Portuguese stock market, between January 1988 and September 2021. The intuition behind the method in that paper is that one can learn from past out of sample (OOS) estimation errors to make better predictions in subsequent estimations, since that in simple OOS tests these are discarded. The focus of the method is in improving the estimation of a models' inputs, such as the mean, variance or pairwise correlation of returns.

The correction applied has an initial guess implied that all portfolio optimization inputs are similar across assets. This naturally leads to the 1/N portfolio, as no asset distinguishes itself, and therefore neither do their weights – if the data confirms this assumption. Like DeMiguel et al. (2009) shows, this is not at all farfetched. The results of the original paper, where the model is studied in the American market from the year 1962 until 2016 with data of monthly frequency obtained from the Center for Research in Security Prices (CRSP), show that the Galton produces covariance matrixes implying risk forecasts that are near to the true OOS risk and enhances return predictions, differing greatly from several other models it was compared with, one of them being the simple Markowitz model, that systematically fails to estimate risk and present performances far away from its own expectations.

The method works because the authors observed that inputs tend to regress to the mean in subsequent periods and found that there is a pattern in OOS errors that make them predictable to some degree. The concept of regression to the mean was first proposed by Francis Galton (Galton (1984)), hence the name attributed to the model. One other advantage decurrent from the Galton is that it does not assume a great deal about the data. Its simple approach reveals to be more flexible and robust than a more specialized approach.

To arrive at Barroso and Saxena (2021), a long path has been travelled in the field of portfolio optimization. The concept of a portfolio was not always the same as is today. Around the 1930's, investors had, of course, a group of stocks they would invest in, but the generalized thinking of the time was to find good stocks and buy them at a good price, and nothing else. Later, there were noted notions of diversification with investment firms owning a large number of securities. What was missing was a unifying theory of investment behind it.

The roots of (modern) portfolio theory are in Markowitz (1952). Henry Markowitz brought the concept of risk to a more central role and stated that, for an investor that only cares about the expected return and the risk of its investments, the optimal and most efficient portfolio could be obtained with only the expected returns and the covariance matrix of the assets. Also Roy (1952) proposes making choices on the basis of mean and variance of the portfolio as a whole, but with some differences from Markowitz. Unlike the second, the first directly indicated what would be the only and specific best portfolio.

That was not the end of portfolio theory, and new contributions on this field followed that would evolve the investors' view on their portfolios. William Sharpe presented its Capital Asset Pricing model describing the relationship between systematic risk and expected returns in Sharpe (1964). Ross (1976) proposed its Asset Pricing Theory stating that the expected return could be obtained from quantified specific sensitivity to diverse factors or theoretical market indices. Engle (1982) presented new insights on how return volatility is persistent and predictable and on the diminishing benefits of diversification when in stress events. Jobson and Korkie (1980) and Michaud (1989) show how errors in estimations damage the effort of optimization. Detemple (1986), Dothan and Feldman (1986), Gennote (1986) and Brennan (1998) discussed how investors are unable to obtain with certainty the true expected returns and covariance matrixes the assets. DeMiguel et al. (2009) performed a comprehensive OOS study to find that the estimation error is greater than the benefits of optimization and that no model systematically outperforms the naïve 1/N throughout several industries and countries or even characteristics-sorted portfolios such as size and book-to-market value. It adds, as well, that many optimization models amplify the effect of errors in estimation.

The frequent existence of these forecasting errors and its prejudicial effect on optimization constitutes, still to this day, one of the greatest challenges in portfolio construction.

The Portuguese stock market, Euronext Lisbon, is one of the oldest in Europe, with its origins reaching as far as 1769 with the "Assembly of the Businessmen" ("Assembleia dos Homens de Negócio"). Oporto saw the creation of a stock exchange in 1891 and a derivatives exchange later, but these merged with the Lisbon bourse to create the "Lisbon and Oporto Stock Exchange" ("Bolsa de Valores de Lisboa e Porto"). It was in 2002 that Euronext N.V. acquired it and the stock exchange became part of the Euronext group of exchanges under the name

Euronext Lisbon, reaching its present form. It is, currently, the only regulated market for the activity of trading shares in the country[1].

In the time this thesis is being written and in general terms, the stocks in this bourse have a low book-to-market ratio, have sound balance sheets and low volatility[2]. The utilities industry dominates the market capitalisation of the stock exchange, although the 40% share is split by only three companies. The consumers and the energy industries follow with 20% and 19%, respectively. The basic materials industry reaches 10%. The financials industry only reaches 5%, with the rest being dispersed across many sectors. Relative to other close markets, such as Spain, France, Belgium, Netherlands or Germany, Portugal has a low market capitalisation-to-GDP ratio, that weaves between 20% and mid-30% since the 2008/2009 financial crisis[3].

The PSI 20 is the reference index introduced on the 31st of December 1992 that mirrors the performance of the maximal 20 largest companies listed in Euronext Lisbon and the most commonly used indicator for the Portuguese financial market, with a full capitalization of 72,8 billion euros as of September 2021[4]. Figure 1 shows the index performance since its origin, as well as the American S&P 500's, for comparison, as a proxy of the two countries. While the first presents a Sharpe Ratio of 0.19 for the studied period, the second presents 0.57, yielding a total excess return of 410.82% versus 27.54% from the first.

The Portuguese market has a special set of characteristics and is far less studied than the world's most renowned financial markets, with the United States of America at the front line. As can be observed, overall market performance has been mediocre suggesting that it can be challenging to find (non-market-neutral) profitable and robust investment strategies that can perform consistently and satisfy investors, portfolio managers, traders, bankers, financial advisors and others. The motivation for this thesis was to find a strategy that could perform well and study its behaviour. On top of this, the strategy was originally tested in the United States, so observing how it performs in a market where the underlying stock market has a very different

---

[1] Euronext (2021), *A Euronext Lisbon é uma das bolsas mais antigas da Europa.* Retrieved October 19, 2021, from //www.euronext.com/pt/markets/Lisbon.
[2] MSCI (2021), *MSCI Portugal Index (USD).* Retrieved October 24, 2021, from https://www.msci.com/documents/10199/3037c213-2f17-46b1-9bed-7aac8c359b2b.
[3] OECD (2020), *OECD Capital Market Review of Portugal 2020: Mobilising Portuguese Capital Markets for Investment and Growth, OECD Capital Market Series*. Retrieved October 24, 2021, from http://www.oecd.org/corporate/OECD-Capital-Market-Review-Portugal.htm.
[4] Euronext (2021), *PSI 20.* Retrieved October 26, 2021 from https://live.euronext.com/pt/product/indices/PTING0200002-XLIS/market-information.

long-run performance can offer valuable insights. Chui, Titman and Wei (2010) and Goyal and Wahal (2015) show how strategies are often less effective when tested in different markets. For these reasons, with my thesis I aspire to improve the knowledge on the financial market of my country, Portugal, and to contribute to the academic research endeavour in the field of finance and specifically portfolio optimization.
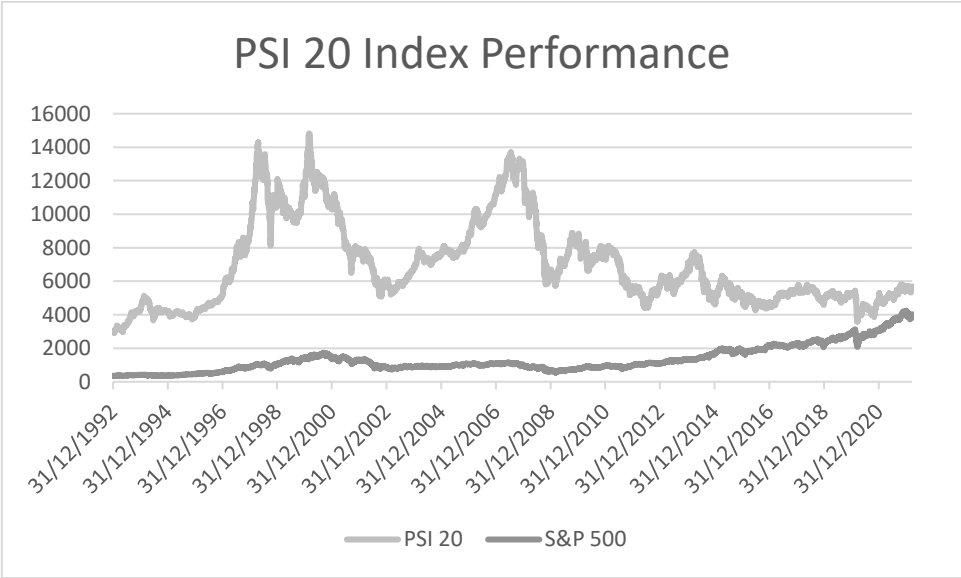
Figure 1. PSI 20 Index Performance. The index quotation of the PSI 20 since its creation, registered on a daily basis, is here presented, as a proxy for the Portuguese financial market. For comparison purposes, the S&P 500 is also exhibited, representing the American market.

My study of the Galton in Portugal confirmed the notion that historical estimates are not perfect forecasters and the method produced reasonable returns and reasonable estimation of them. The Galtons achieved its results with more sensible weights, closer to the market portfolio than to the other optimizing models', and less turnover, an important factor for its real-life applicability. On top of it, their prediction of risk was close to perfect, perhaps constituting its strongest suit. The methods delivered the best cumulative return and Sharpe Ratio among its pairs for the MV group, despite being outperformed in these measures for the GMV group, and did so with lower kurtosis - therefore, with extreme returns occurring much less often and being much less extreme.

This thesis is organized as follows. Section 2 discusses the related literature. Section 3 describes the methodology followed and how the study was conducted. Section 4 presents the results obtained and discusses the performance of the Galton and those of the models it is compared with. Section 5 studies the robustness of the Galton method. Section 6 concludes.

2. Related Literature

In 1952, Markowitz presented a ground-breaking paper on portfolio selection and optimization. What became known as Modern Portfolio Theory turned into the basis of this branch in academia. He stated that investors worry about expected returns and the volatility of them and that for an increase in the volatility of a portfolio, investors were to demand an increase in the expected returns as well. An efficient portfolio would be one that provides the maximum expected returns for a given level of risk. This would translate into an efficient frontier, a combination of all possible values in the risk-return trade-off. The weights of the optimal portfolio, the portfolio that maximizes the risk-return trade-off in a market with access to a risk-free rate, would be given by the expression below. Any other combination of weights, adding up to 1, that not the one that results from this expression ends in a sub-optimal mean-variance result. This was named the Mean-Variance portfolio (MV).

$$w_t^{MV} = \frac{\sum_{H,t}^{-1} \mu_{H,t}}{1_{N\,(H)}\,\sum_{H,t}^{-1} \mu_{H,t}} \tag{1}$$

The notation used is as follows. $\sum_H^{-1}$ indicates the inverted covariance matrix obtained from historical estimates and $\mu_{H,t}$ the matrix of excess expected returns obtained from historical values. $1_{N(H)}$ is an horizontal matrix of ones. The subscript 't' indicates the period.

Also following from Markowitz work, one could obtain the weights for a portfolio that grants the minimum variance for risk-averse investors. This was called Global Minimum Variance portfolio (GMV).

$$w_t^{GMV} = \frac{\Sigma_{H,t}^{-1} \, 1_{N \, (V)}}{1_{N \, (H)} \, \Sigma_{H,t}^{-1} \, 1_{N \, (V)}} \qquad (2)$$

The notation used is the same as above, the only difference being that there is now a vertical vector of ones, $1_{N \, (V)}$.

For the GMV portfolio, the investor needs the covariance matrix but not the expected returns, since that the primary objective of this portfolio is to minimize its risk. This yields an advantage in the sense that it is not affected by errors in its estimation (Jobson and Korkie (1980) and Michaud (1989)). Here onwards, SimpleMV will refer to the Markowitz MV approach and SimpleGMV will refer to the Markowitz GMV strategy, unchanged.

Empirical-Bayes methods are techniques in which the prior distribution is inferred from the data, as opposed to setting it a priori (a Bayesian method). In Barroso and Saxena (2021) this can be seen from the assumption that all assets are similar in its characteristics (mean, volatility and pairwise correlation of returns, for example). Because all are so alike, it is logical to conclude that the same weight would be attributed to each asset, therefore constituting the 1/N. Looking at DeMiguel et al. (2009), this approach appears to be reasonable. If data confirms this assumption, the optimization inputs will be shrunk in the direction of their grand mean. If this does not hold, less shrinkage is performed. The Galton is only anticipated to improve a great extent if this initial guess holds, this is, that the 1/N is a satisfactory initial assumption.

Also seen in Barroso and Saxena (2021) is that the Galton method is applied directly to the Mean-Variance and Global Minimum Variance portfolios from Markowitz. This method's aim is in correcting the estimates and deliver them with less noise and error, with the final objective of improving results. In this study, the Galton combined with Markowitz's Mean-Variance portfolio will simply be called GaltonMV and when combined with Markowitz's Global Minimum Variance portfolio it will simply be called GaltonGMV.

Following DeMiguel et al. (2009) and the difficulty of optimized portfolios to consistently overperform the naïve 1/N (here onwards just '1/N'), I take a closer look at this method. As the name indicates, it consists of an equal-weighted distribution for all assets that constitute a portfolio. Working as a proxy for market behaviour, I also study a value-weighted model (here onwards 'VW'). The weight of each stock in the portfolio is set proportionally to the market capitalization of the respective company in the market. This is the portfolio that

reflects the market development the closest as, for example, in the Portuguese market the four largest companies amount to 85% of the Euronext Lisbon total market capitalization[5], and the portfolios used in this thesis have the 25 largest stocks at a specific moment in time.

There are also models using differently formed covariance matrixes with the objective of optimizing an investor's risk-adjusted return, with different characteristics and for different profiles of investment.

Elton and Gruber (1973)'s covariance matrix originates in a pairwise correlation matrix where the diagonal elements are all ones and the remaining elements are the average of pairwise correlations in the retrieved data for that period. In this thesis, the referred matrix combined with the MV model is referred as MVEG and when combined with the GMV model is referred as GMVEG. Mathematically, it is as follows.

$$\Sigma_{EG,t} = diag(\sigma_{H,t})\rho_{EG,t}diag(\sigma_{H,t}) \tag{3}$$

For the Elton and Gruber covariance matrix, $\Sigma_{EG,t}$, $diag(\sigma_{H,t})$ is a matrix of all zeros where the diagonal values are the standard deviations of each stock retrieved from an historical sample and $\rho_{EG,t}$ is the constant average of pairwise correlations described above.

Other models address the estimation of returns and not the covariance matrix. This is the case with the version of the Stein (1956)'s shrinkage estimator for mean returns from James and Stein (1961), its positive-part improvement, which presents a smaller risk compared to the original one. It uses the historical inputs and, as the name indicates, shrinks them to a more central grand mean of all values, excelling the ordinary least squares approach (OLS), meaning it has a lower or equal mean squared error (MSE) than the ordinary least square estimator. It can be obtained as follows.

$$\mu_{JS} = \mu_0 \, 1_N + g_{1,JS} \, (\mu_H - \mu_0 \, 1_N) \tag{4}$$

[5] MSCI (2021), *MSCI Portugal Index (USD)*. Retrieved October 26, 2021, from
https://www.msci.com/documents/10199/3037c213-2f17-46b1-9bed-7aac8c359b2b.

$$g_{1,JS} = \max(1 - \frac{(N-3)\,\sigma_0^2}{||\mu_H - \mu_0\,1_N||^2}, 0), N \geq 4 \qquad (5)$$

The estimator μ_{JS} is gotten with the μ_H, a matrix of the sample average returns from the historical window and μ_0, the cross-sectional mean. For $g_{1,JS}$ one also needs the grand variance of returns, σ_0^2, and N, the number of periods in the historical sample.

3. Data and Methodology
3.1 Data

In order to conduct this study, I collected the returns of stocks and the market value of all the Portuguese companies listed in the Portuguese stock exchange, Euronext Lisbon, from the Refinitiv Eikon database on a monthly basis spanning from January 1988 until September 2021, in euros. Only the companies that have available information on both simultaneously were kept, this is, if there was information on a company's stock return but not on its market capitalization, the company would not be considered, and vice-versa. Outside this criterion, EDP Renováveis was also added to the database, despite being a Spanish company. The reasons that constituted this decision were that it is part of the EDP group, the Portuguese flagship energy-sector company (EDP is its controlling shareholder), it is only listed in the Portuguese stock exchange, it is part of the PSI 20 index since its IPO in 2008 and is, as of the time this thesis is being written, the largest quoted company by market capitalization in Portugal. Therefore, its exclusion from the dataset would significantly alter the degree of similarity between my database and the real market. The value-weighted portfolio served to validate this, as it presented a very high correlation with the MSCI Portugal and the PSI 20 index, and near equal Sharpe Ratios. Furthermore, the data was cleansed to eliminate the redundant and noisy information caused by dead and delisted stocks, and a total loss of investment added when a stock exits the database. It is important to notice that the returns were obtained from a return index and, for this reason, in January 1988 no returns are presented, meaning that the dataset only truly initiates in February 1988.

To transform returns into excess returns, the risk-free rate for Portugal was also retrieved from Refinitiv Eikon (whose symbol is ECPTE3M). The process followed was to subtract from the returns, for each month, the corresponding risk-free rate.

There are 232 companies that, since the start of the learning period, enter the dataset. Of these, only 117 are capable of achieving the minimum requirements of the model at a given point in time to contribute to the estimation universe, this is, to contribute for the estimation of the model parameters. The maximum number of companies with information available in a certain month is 61, found in March 1999, and the minimum is 21, found in November and December of 2008. From the beginning of the dataset until the year where the maximum is registered, the number of companies gradually increases. But after reaching its peak, it consistently decreases until 2008 and the minimums, slightly recovering and stagnating until the end of the dataset. Regarding the investment universe, there are 67 different companies that at a certain point in time are available to be part of a portfolio, wavering between a monthly minimum of 25 and a monthly maximum of 44. The general trend is of an increase until the peak in mid-2006, followed by a decrease, as time passes, reaching the minimum in the penultimate month.

3.2 Methodology

In-sample (IS) methodologies occur when a model is tested in the same period where it is run. Alternatively, in OOS methodologies one tests the model in a defined period (the in-sample period) and runs it in a subsequent period, that does not overlap the first. This change presents several improvements to the results, as testing a model in the same period it is ran may lead to data mining and a higher sensibility to outliers, as well as only being adapted to the IS period it was tested in. OOS tested models are, then, more flexible and present the advantage of being able to be used in real-time, something IS models cannot achieve.

The OOS methods' process seldom starts by collecting estimates, either with an expanding or a rolling window, this is, a window that grows in size with each iteration or a period-fixed window that rolls over with every iteration. The estimates obtained throughout the process serve to be used in the OOS period, where the model is run. It is important to notice that, if the estimates are not perfect, and they rarely are, errors arise, as the difference between the actual value observed OOS and the corresponding forecast obtained IS. These errors are not

used in any type of way by typical models. One base pillar that distinguishes the Galton model is that it uses these errors in its process.

The Galton requires three measures obtained from the stock returns: mean, variance and pairwise correlation. Therefore, I have repeated all the processes described in this section three times, one for each measure.

This process is composed of five major subsections: the regression of variables and a correction of estimates, portfolio building, computation of the corrected covariance matrix and obtaining portfolio weights and results. Additionally, there is a section that discusses other models computed to better frame the results.

3.2.1 Estimates and Regression

To constitute the estimation universe, firstly, the data is scanned to guarantee that there is sufficient information for the following steps of the model. There needs to be 72 previous periods (months) of data, 60 to compute the historical estimations and 12 to compute the ex-post realization value, both for the mean and variance measures, but also for the pairwise correlation. All companies that do not hold these criteria are removed from the selection. The remaining are used in the process.

Regarding the mean and the variance, I obtain the historical estimate as the mean value of a rolling window of 60 periods. For month t, the value would be the average of the 60 periods comprised from t-12-60 and t-12-1. The ex-post realization is the mean value of a rolling window of 12 periods. So, for month t, it would be the average value of the 12 periods encompassed between t-12 and t-1. On the other hand, for the pairwise correlation, it is the pairwise correlation that I compute for the abovementioned windows. The output, instead of a single row as before, is a correlation matrix of size [25x25], with a diagonal of ones.

To remove outliers, I excluded observations below the 2,5% percentile and above 97,5% percentile for the variance's rolling windows and below the 1% percentile and above the 99% percentile for the mean's rolling windows. This is, for all observations of the 12-month and 60-month windows, separately, I compute the percentiles and remove observations that exceed them. The choice of the thresholds is intended to remove non-representative values while not being too intrusive as to deliberately change results.

On top of the required periods for the estimations, the model also requires a learning period, this is, a set of months that helps the model to present more accurate results. This period is set for 108 months, as in Barroso and Saxena (2021). All things considered, the first period where there are results is the period 181 (108 + 60 + 12 = 180), in February 2003. Since the process is repeated monthly, there can be a maximum of 224 different investment opportunity sets.

Next, the following OLS regression is run as in Fama and Macbeth (1973).

$$X_{E,t,c} = g_{0,t} + g_{1,t}X_{H,t,c} + \varepsilon_{t,c} \qquad (6)$$

The notation I used is as follows. X designates the measure variable (mean, variance or pairwise correlation) and can take various forms. X_H represents the historical estimate and X_E the subsequent ex-post realization. The subscripts t and c indicate the period and the company (or pair of companies, for the pairwise correlation), respectively.

The objective of regressing the ex-post realizations of each monthly portfolio of on its historical estimates is to obtain the g_0 and g_1 coefficients. For each period, I obtain this pair of values, for the 224 periods that exist in the dataset.

3.2.2 Correction

I use the coefficients obtained from the regression to correct the historical estimates of the companies within each portfolio. This is achieved accordingly to the following equation.

$$X_{G,t,c} = G_{0,t} + G_{1,t}X_{HP,t,c} \qquad (7)$$

Following the notation previously used, X_G is the Galton-corrected estimate and G_0 and G_1 are the correcting coefficients. These are obtained according to the ensuing expressions:

$G_{0,t,c} = \frac{\sum_{p=1}^{t} \hat{g}_{0,p}}{t}$ and $G_{1,t,c} = \frac{\sum_{p=1}^{t} \hat{g}_{1,p}}{t}$. $X_{HP,t,c}$ is the historical estimate vector of size [1x25] for the stocks in each portfolio.

This is a linear correction of the estimates. In the proportion of the strength of the historical values (reflected on how close to 1 is $G_{1,t,c}$), or its incapability to predict the future values (reflected on how close to 0 is $G_{1,t,c}$), directly indicated by the data, I apply the adjustment. It is also possible to understand from these mathematical expressions how the coefficients learn from the data, and the importance of the learning period for the accuracy of the model. Additionally, it should be noted that the coefficients were obtained from all available companies and not only those in every selected portfolio. This means that there may be even companies who no longer exist but that at some point contributed for the estimates.

3.2.3 Portfolio Building

Every month, I build a portfolio of 25 stocks. There needs to be 60 previous periods (months) of returns data and, additionally, the company must have data on its market value for the previous month. The companies that do not fulfil these requirements are removed from the selection. From the batch of the remaining, I choose the 25 largest companies in terms of market capitalization regarding the preceding month.

In this phase, and in the ones before, there are differences to the method described by Barroso and Saxena (2021). The authors select the 50 largest companies in the beginning of the year and then only rebalance the weights monthly in the subsequent 12 months, what demands that every company selected to have market value data uninterrupted for the year they are selected. This is very difficult to achieve in Portugal, due to the much smaller number of companies available and less complete information accessible. For this reason, in this thesis, there can be companies entering and leaving throughout the year, since the selection is repeated every month, as well as the rolling windows that correspond to the historical estimates and ex-post realizations for the estimation universe only require 5/6 of the periods for the mean and the variance and 3/4 for the pairwise correlation to present a result. The number of companies is reduced from 50 to 25 due to the same data constraints. Additionally, the authors do not remove outliers, which I do. These are the only divergences from the original model.

3.2.4 Galton Corrected Covariance Matrix

The Galton Corrected Covariance Matrix resorts to the corrected estimates of the variances and the corrected estimates of the pairwise correlation. Mathematically, is as follows.

$$\Sigma_{G,t} = diag(\sigma_{G,t})\rho_{G,t}diag(\sigma_{G,t}) \tag{8}$$

Following the previous notation, $\sigma_{G,t}$ indicates the Galton-corrected standard deviations. These are obtained as the square root of the Galton-corrected variances. $\rho_{G,t}$ indicates the corrected matrix of the pairwise correlations. The operator 'diag()' symbolizes that the standard deviation values are displayed in a diagonal manner.

In the process it is assured that the $G_{1,t,c}$ coefficient for the variance and pairwise correlation inputs is limited within the interval [0,1]. This is so because it is important to guarantee that this matrix is positive semi-definite, or else portfolios could be constructed with negative variances. For the GaltonMVEB only, the $G_{1,t,c}$ of the mean is additionally restricted, on top of the previously mentioned.

3.2.5 Weights and Results

I compute the weights of each stock in a portfolio like in Markowitz (1952), using the corrected inputs from Galton.

For the Galton mean-variance (MV) portfolio they are result of the following expression.

$$w_{G,t}^{MV} = \frac{\Sigma_{G,t}^{-1}\,\mu_{G,t}}{1_{N\,(H)}\,\Sigma_{G,t}^{-1}\,\mu_{G,t}} \tag{9}$$

$\Sigma_{G,t}^{-1}$ symbolizes the inverted covariance matrix, the $\mu_{G,t}$ is a matrix the forecasted returns and $1_{N\ (H)}$ is a horizontal matrix of ones.

And with respect to the Global Minimum Variance (GMV) variant, they are result of the expression below.

$$w_{G,t}^{GMV} = \frac{\Sigma_{G,t}^{-1}\ 1_{N\ (V)}}{1_{N\ (H)}\ \Sigma_{G,t}^{-1}\ 1_{N\ (V)}} \tag{10}$$

The notation used is the same as above, the only difference being that there is now a vector of ones that is vertical, $1_{N\ (V)}$.

To ensure that the denominator does not switch the sign of the weights of the portfolio, empirically, I compute it as its absolute value, $|1_{N\ (H)}\ \Sigma_{G,t}^{-1}\ 1_{N\ (V)}|$. Having obtained the weights, I achieve the results multiplying each corresponding weight by the return of the company it is computed for.

3.2.6 Other Models

To frame and better understand the results of the Galton, I compute other models, using as input the same dataset and for the same time-period. These were already described theoretically in the section *2. Related Literature*, and are mentioned here from an empirical lens.

First, there is the 1/N simple agnostic model, where an equal weight is given to the stocks in a portfolio, and the VW model, where the weight of each stock is given as the proportion of the market capitalization of the company in the market. The simple Markowitz models are also included. They only use historical estimations, not the corrected ones. Additionally, the James-Stein model is present. This model computes a shrinkage estimator for mean returns to arrive at the weights. Finally, there is also the Elton and Gruber model. This model uses a constant correlation matrix, where all values are the average value of the

covariance matrix, and the diagonal are ones. For the computations it relies on historical estimates.

4. Results and Out of Sample Performance
4.1 Inferences from the Galton Coefficients

From the predictive regressions ran I obtain the Galton coefficients. Table 1 presents the results of these regressions.

Table 1
The Regression Coefficients
By regressing ex-post realizations as in Fama-MacBeth (1973), gotten from a rolling window of 12 months, on the corresponding subsequent historical estimates, gotten from a rolling window of 60 months, I obtain the Galton coefficients. These are presented for the mean, the variance and the pairwise correlation of returns, in each of the columns. The first row exhibits the mean intercept of the regressions (also referred to as g_0) and the third row presents the mean slope of the regressions (also referred to as g_1). Below each of these rows there is the t-stat measure for a statistical test where the null hypothesis is that each coefficient is 0. There is also one additional test for the slope coefficient but with the null hypothesis of it taking the value of 1. The following two rows present the percentage of monthly coefficients greater than 0 and smaller than 1. The last row presents the average R-square of the regressions.

	Mean	Variance	Correlation
Intercept	0.00	0.01	0.11
t-stat (=0)	2.31	20.81	16.96
Slope	0.06	0.45	0.40
t-stat (=0)	1.31	12.32	21.67
t-stat (=1)	-19.33	-263.09	-33.11
Greater than 0 (%)	49.70	92.77	98.19
Smaller than 1 (%)	90.66	88.55	98.19
R-square (%)	8.35	13.22	7.31

The role that the Galton coefficients play in the optimization process is to assess how well each historical input predicts its subsequent ex-post realization. If the historical input has been seen to be a bad forecaster, the slope (g_1) will be smaller, meaning less importance is attributed to it. A smaller slope indicates that all elements are closer to its overall mean. The opposite case is when is when the historical input is a perfect forecaster. The slope, in this setting, would be 1.

xx

The null hypotheses stating that the value of the intercept is 0 is clearly rejected for the variance and the pairwise correlation for every level of confidence. This also happens when I perform the same test on the slope of the regressions.

As said, for the historical estimates to be a perfect predictor of future values, the intercept would have to be 0 and the slope 1 – therefore it is clear that this is not the case. The data seems to indicate precisely that historical estimates alone will not forecast to perfection its future counterparts for these two measures, and therefore less importance is attributed to them.

Regarding the mean variable, the hypotheses that its intercept and slope values are 0 is not rejected for all levels of confidence. The small slope indicates historical means are not good forecasters of its the subsequent values and so, almost if no importance should be given to them. The prediction of returns has always been a conundrum since it started to be studied. The slopes for the variance and the pairwise correlation are considerably higher, indicating that, relative to the mean (but not on its own), their historical inputs forecast its future values better.

The test on the slope that its value is statistically indifferent from 1 is clearly rejected for all variables, given any level of confidence. These results, and the ones presented above, are generally in line with Barroso and Saxena (2021). It can be inferred that the historical values for the inputs should not be good forecasters in the Portuguese market for the studied timeframe.

Based only on the interpretation of the coefficients, models that simply use the historical estimates should be prone to errors in estimation.

4.2 Weights

One other factor of portfolio performance that can be studied is the weighting of the assets it contains. A portfolio may show appealing theoretical results, but if it is based on constant rebalancing, real-life problems appear that can destroy profits, namely transaction costs. Table 2 analyses the portfolios' weights performance.

As might be expected, due to its nature, MV portfolios have higher turnover, more extreme weights and higher mean and standard deviation of monthly volatility of the weights. The EGMV portfolio has an overwhelming minimum weight of -9363.70% and the SimpleMV and the JSMV do not fall far below, with -6735.95% and -2866.50% respectively. The EGMV's

maximum weight reaches 18429.89%, followed by the SimpleMV's 9012.16% and the JSMV's 3604.00%. Exceptions noted are the GaltonMV and the GaltonMVEB, with low turnover and significantly lower extreme weights. The Galton models also present the lowest average value of short positions, with approximately half of the percentage of the remaining of this group.

Table 2
The Weights' Stability

      After the selection of the monthly portfolios and the obtaining of the estimates followed by the respective correction, I compute the weights. For the different models presented, the first column shows the average monthly turnover as a percentage of the market capitalization of the portfolio. The following two columns present the minimum and maximum weight recorded in the entire OOS period. The fourth and fifth columns show the mean and the standard deviation of the monthly volatility within a portfolio, this is, the measure of the variability of the amplitude of its weights. The last column shows the average number of short positions within a portfolio's monthly 25 possible. All values are percentages.

Strategies	Turnover	Min w	Max w	Mean[$\sigma(w)$]	Std [$\sigma(w)$]	w<0
1/N	0.06	4.00	4.00	0.00	0.00	0.00
VW	0.48	0.03	32.83	6.12	0.71	0.00
GaltonMV	0.60	-21.44	31.35	6.71	1.66	25.29
GaltonMVEB	0.53	-21.44	31.35	6.21	2.06	23.16
SimpleMV	62.97	-6735.95	9012.16	118.91	360.90	46.73
EGMV	48.22	-9363.70	18429.89	91.82	420.83	48.04
JSMV	46.06	-2866.50	3604.00	84.44	169.26	45.13
GaltonGMV	0.38	-7.94	22.31	4.25	0.64	19.96
SimpleGMV	0.38	-179.95	174.16	14.07	7.14	40.87
EGGMV	0.67	-7.57	58.10	9.11	1.38	46.05

In contrast to MV portfolios (apart from the two Galtons), GMV portfolios display much lower turnover, smaller extreme weights and mean and standard deviation of monthly variations of the weights. This time around, the weights only go as low as -179.95% and as high as 174.16%, both from the SimpleGMV. The GaltonGMV does not top the 23% threshold. The turnover, for all, does not surpass 0.67%. The short positions' percentages are relatively close to the corresponding MV portfolios although, as seen by the maximum and minimum weights, its expression in terms of exposure is much smaller. The GaltonGMV presents the lowest percentage of short positions of all models, MV and GMV.

It is interesting to notice, as by its design, the Galtons are in every aspect shrunk towards the 1/N, with less extreme weights, less variation of them and a lower turnover. It is also interesting to point out that the Galtons are closer to the 1/N than they are to their counterparts.

4.3 Risk, Return and Predictability

The performance of the Galton and the other models is displayed in Table 3.

Table 3
The Models' Performance
　　Every month, I select the largest 25 companies by market capitalization that fulfil the requirements of having information on its returns for the previous 60 months and of its market value for the previous month. I use rolling windows for the mean, the volatility and the pairwise correlation of the returns to compute the covariance matrix and, after that, the weights, according to Markowitz – the MV and the GMV portfolios – and, lastly, the results. For comparability, I include other models, such as the Markowitz historical approach, the 1/N and the VW models, the constant correlation matrix from Elton and Gruber (1973) and the shrinkage estimator for mean returns from James and Stein (1961). The first two columns show the annualized Sharpe Ratio and the expected one according to each models' prediction. The third column displays the cumulative return of each strategy in the time-period analysed. The three following columns display the annualized average expected monthly return, the annualized average realized monthly return and the ratio of the two. The subsequent three columns show the annualized average expected monthly standard deviation, the annualized average realized monthly standard deviation and the ratio of the two.

Strategies	SR	E(SR)	r_c (%)	E(r_m) (%)	r_m (%)	r_m/E(r_m)	E(σ) (%)	σ (%)	σ/E(σ)
1/N	0.09		-4.09		1.82			20.07	
VW	0.20		44.12		3.50			17.40	
GaltonMV	0.89	0.24	1551.29	4.58	16.91	3.69	19.26	19.01	0.99
GaltonMVEB	0.82	0.23	1154.67	4.39	15.40	3.51	18.86	18.79	1.00
SimpleMV	0.44	3.67	-100.00	232.21	261.49	1.13	63.27	589.92	9.32
EGMV	0.31	2.98	-100.00	335.94	363.82	1.08	112.58	1190.71	10.58
JSMV	-0.16	0.28	-100.00	27.08	-40.26	-1.49	97.18	244.12	2.51
GaltonGMV	0.27	0.21	77.23	3.62	4.54	1.25	17.30	16.97	0.98
SimpleGMV	0.19	0.95	26.58	8.11	4.57	0.56	8.55	24.49	2.87
EGGMV	0.70	0.76	648.60	9.37	12.36	1.32	12.29	17.58	1.43

All models present a positive Sharpe Ratio but the JSMV. In the MV group, the GaltonMV has the highest, with 0.89 and for the GMV portfolios, the EGGMV presents the highest Sharpe, with 0.70. In this setting, all portfolios but the JSMV and the SimpleGMV outperform the 1/N and the VW. Looking at the expectation for this measure, only the Galtons predict a lower Sharpe than the actual result. The JSMV predicts its Sharpe with the smallest difference in the MV group, despite overestimating it. The Galtons come close and, as mentioned above, are conservative in estimating it. The GaltonGMV and the EGGMV forecast it the closest for the GMVs. The SimpleMV and the EGMV are the two strategies with the greatest difference of all.

The models that predict the best, at first sight, their future returns are the SimpleMV and the EGMV, with an expected to realized returns ratio of 1.13 and 1.08, respectively. This is deceiving. Despite having the best ratios, the SimpleMV only has 19.73% of the months with the difference between the expected and the realized returns in the range [-5%, +5%], and has months with differences as low as -2346.68% and as high as 427.07%, while the EGMV only has 22.42% of the months in the same interval, with a minimum and maximum differences of -4791.64% and 2527.54%. The GaltonGMV, contrastingly, has 74.89% of the differences between expected and realized returns in the abovementioned interval and does not top an absolute difference of 19.04%. Additionally, in absolute terms, the average difference of the SimpleMV is of 43.67% and that of the EGMV is 64.06%, while the GaltonGMV's is 3.56%. In the end, truth is no model predicts returns precisely.

It is a different picture when looking at the volatility of returns. The GaltonMV, the GaltonMVEB and the GaltonGMV predict the risk of their returns almost exactly (the ratios are between 0.98 and 1.00), with the addition of being conservative in doing so, this is, they estimate higher risk than the realized. No other model does this. The SimpleMV and the EGMV stand out but due to their poor forecasting ability, as they fail their estimations by a factor of approximately 9 and 11, respectively. As expected from the Galton coefficients estimation (see section *4.1 Inferences from the Galton Coefficients*), the corrected covariance matrix of the Galton method distinguishes itself in estimating volatility and performs better than the matrixes who rely on historical inputs.

Table 4 presents the hit rates of the models.

One can once again verify the solidness of the Galtons' covariance matrix as opposed to other models. Despite going over expected thresholds, only slightly, they are the only models whose volatility is within the expected. On the opposing side, there are examples such as the SimpleMV model, with more than 17% of volatilities at a level where there should be only 1%. After the Galtons, the JSMV is the model that performs best. But, as seen by the SimpleMV, its matrix is not particularly accurate forecasting risk – so one can deduct that it is the effect of the shrinkage and the selection of stocks that are responsible for these results.

Looking at the cumulative returns, three of the portfolios would lead the investor to lose the entirety of its investment: the SimpleMV, the EGMV and the JSMV. All models except these three and the SimpleGMV outperform the 1N and VW regarding this measure.

Table 4
The Hit Rates
  For every iteration, I compute the Value-at-Risk (VaR) measure, this is, the maximum volatility percentage that could be experienced over the monthly time horizons, so that there is a very low probability that the actual volatility percentage will exceed it, for the 1%, 5%, 95% and the 99% confidence levels. The VaR implicitly assumes that the distribution of returns of the models follow a Normal distribution. Each time the standard deviation surpassed the VaR, I registered it. The values presented are the percentage of months where this happened, this is, the hit rates. All values are percentages.

Strategies	r<Qz(1%)	r<Qz(5%)	r>Qz(95%)	r>Qz(99%)
GaltonMV	1.42	2.36	5.66	2.83
GaltonMVEB	1.89	4.25	5.19	5.19
SimpleMV	17.45	26.42	21.23	15.57
EGMV	10.38	19.34	10.38	7.08
JSMV	3.77	6.60	13.21	6.13
GaltonGMV	2.36	6.13	3.30	0.94
SimpleGMV	12.74	19.34	18.40	14.15
EGGMV	5.19	8.49	9.91	3.77

These three loss-making models are greatly exposed to overfitting the sample, and so greatly exposed to non-normal risk. Therefore, it is also important to observe other factors. Table 5 shows additional characteristics of the distribution of returns of the portfolios.

Table 5
The Returns' Distribution Peculiarities
  The studied models produce a time-series of monthly returns in the 224 months OOS. For each, the first column shows the skewness of the distribution of returns and the second column shows the kurtosis of the distribution of returns.

Strategies	Skewness	Kurtosis
1/N	-0.38	4.22
VW	-0.56	4.46
GaltonMV	-0.07	5.57
GaltonMVEB	-0.31	2.54
SimpleMV	12.23	170.26
EGMV	13.83	202.90
JSMV	-7.79	90.91
GaltonGMV	-0.72	5.12
SimpleGMV	-1.32	14.44
EGGMV	-0.01	5.22

Skewness is a measure of asymmetry in the distribution of returns. Positive skewness indicates that the tail of the distribution is on the right and the bulk of returns are on the left side. This means negative or positive small returns are more frequent with big positive returns happening occasionally. Negative skewness indicates precisely the contrary, with frequent small returns and occasional big losses. Meeting in the middle, a skewness of 0 indicates a symmetric distribution.

The market proxy and the naïve portfolios have small negative skewness, indicating the market predisposition of smaller returns punctuated with larger less frequent losses. The Galtons and EGGMV's returns show similar skewness. Despite negative, the small values (closer to 0) indicate that the difference between extremes, the large losses or gains, is not overly accentuated. The SimpleGMV already presents a more accentuated negative skewness, but it is the JSMV that presents off the chart negative skewness, reflecting the magnitude of the occasional gains, in this case, but with constant negative results – that in this case may be not as close to 0. The EGMV and the SimpleMV present the highest absolute values, but their skewness is positive. This helps to illustrate in a different light the risk associated with the returns of these models. The three with the highest absolute values are the ones that would lead the investor to lose the entirety of its money.

Kurtosis is a measure of the peakiness of the distribution of returns. The higher the kurtosis the larger are the tails and the more elevated is its peak. In these, the probability of extreme returns (returns on the tails) is higher.

The GMV portfolios present lower kurtosis relative to its MV counterparts. The risk-averse design of these portfolios and its avoidance of extreme positions reflects itself on more results around the mean. The GaltonMV also has a low kurtosis, lower than the non-Galton models of the GMV group, but it is the GaltonMVEB that presents a very low value, lower than the 1N and the VW (and lower than of a normal distribution), showing how the Galton correction helps avoiding extreme results. The SimpleMV, the EGMV and the JSMV present very large values for kurtosis, showing once more how they are no strangers to immense volatility and unpredictability.

4.4 Certainty-Equivalent Yield

As discussed before, there is a positive relationship between the risk of a portfolio and the expected return an investor demands for it. The Sharpe Ratio measure may not completely capture the performance for models with a non-normal distribution of returns, so it is important to complement the analysis with other measures, that do so. The Certainty-Equivalent Yield (CE) is the guaranteed yield an investor would not mind receiving rather than taking a chance and invest in a portfolio with a future expectedly higher but uncertain return. In Table 6 I present the Certainty-Equivalent Yields for the different methods studied. Risk preferences vary with each individual, as there are risk-averse investors and also risk-lover investors. To account for this, the CE uses a coefficient of relative risk aversion (γ) that increases as more risk-averse the investor is. For the results presented, it was set to 3, as is used throughout the paper in Barroso and Saxena (2021). This is a power utility function and, so, it has the advantage of incorporating tail risk. Additionally, and for this section only, the raw returns are used, not excess returns.

Table 6
The Certainty-Equivalent Yield
      For all models studied, I compute the rate an investor would be willing to receive with certainty instead of risking investing in the respective model for an investor with a coefficient of relative risk aversion (γ) equal to 3. All values are presented in percentage points.

Strategies	CE
1/N	-4.46
VW	-1.24
GaltonMV	11.44
GaltonMVEB	9.96
SimpleMV	-100.00
EGMV	-100.00
JSMV	-100.00
GaltonGMV	-0.02
SimpleGMV	-7.41
EGGMV	7.39

For the VW, investors would accept to receive a negative return, albeit small, rather than investing in the respective portfolios. Once more, this is indicative of the low prospect of

returns and of the perceived difficulty of investing in this market. The GaltonGMV presents a negative value very close to 0 and the SimpleGMV has a more negative CE of -7.41%. The SimpleMV, the EGMV and the JSMV too present negatives CEs, but in the case of these, the values are significantly negative. This again show the risk that these models carry, such that investors presumably would prefer to lose all their money with certainty rather than investing in these models. The GaltonMV, the GaltonMVEB and the EGGMV all present a positive and significative CEs, with the highest being 11.44%.

One other aspect that can be analysed is the difference between the realized rates and the CE. For the 1/N and the VW model, for the Galtons and for the EGGMV, investors would be willing to take a discount in the interval of 4% to 6.5% for a guaranteed return. These are the models that present the smallest difference. The SimpleGMV follows with a difference of -11.99%. Telling, once again, of the performance of the models are the differences of the EGMV, the SimpleMV and the JSMV, the highest of all.

4.5 Loadings

Regressing monthly excess returns on different factors[6] allows to analyse the models results from a different perspective. The factors data is specific for the Portuguese market and was converted from United States dollars to euros using an exchange rate retrieved from Refinitiv Eikon (whose symbol is EUDOLLR). Table 7 presents this exercise.

From all alphas, only the GaltonMV, the GaltonMVEB, the SimpleMV and the EGGMV's are statistically significant for a 5% level of confidence. The highest of these belongs to the SimpleMV model, far apart from all others.

The GaltonMV, the GaltonMVEB and GaltonGMV's exposure to each factor is very reduced and somewhat balanced. The most significant coefficient of the Galtons is their positive position on the market premium factor.

The other three MV models show much higher exposure to the factors, with coefficients several times higher than the Galtons'. It is worth highlighting the negative exposure of the EGMV to the market premium factor and its positive exposure to the UMD factor, with values

---

[6] AQR (2021), *Betting Against Beta: Equity Factors Data, Monthly.* Retrieved December 27, 2021, from https://www.aqr.com/Insights/Datasets/Betting-Against-Beta-Equity-Factors-Monthly.

of -6.09 and 6.93, respectively. On top of this, with exception from the SimpleMV and the EGMV, the models appear to be steering away from the BAB factor, especially interesting for the GMV models.

Table 7
The Loadings of the Models on Factors' Regression
       For every strategy, I run a regression of its excess monthly returns on different factors. The intercept of that regression is the alpha of the strategy. The following columns are, respectively, the market premium factor (Mkt-Rf), the size premium factor (SMB), the value premium factor (HML) – these three constitute the Fama and French Three-Factor Model from Fama and French (1993)-, the momentum monthly premium – this factor together with the previous three constitutes the Carhart Four-Factor Model from Carhart (1997) – and also the Betting-Against-Beta factor from Frazzini and Pedersen (2014). The last column presents the R-square of the regressions. All alphas are annualized and presented in percentage points. The factors that are statistically significant at the 5% confidence level are highlighted in bold.

Strategies	α (%)	Mkt-Rf	SMB	HML	UMD	BAB	R-square
GaltonMV	**14.05**	**0.37**	-0.01	**-0.20**	0.05	-0.01	0.22
t-stat (=0)	3.3	6.97	-0.20	-3.05	1.00	-0.31	
GaltonMVEB	**12.32**	**0.34**	-0.05	**-0.20**	0.05	0.01	0.19
t-stat (=0)	2.96	6.51	-0.76	-2.96	0.99	0.33	
SimpleMV	**241.32**	2.13	-2.35	0.08	0.43	0.52	0.01
t-stat (=0)	1.66	1.16	-0.99	0.03	0.23	0.45	
EGMV	322.53	**-6.09**	-4.07	5.88	**6.93**	-1.21	0.03
t-stat (=0)	1.11	-1.66	-0.86	1.25	1.88	-0.52	
JSMV	5.24	**0.93**	-0.10	-0.22	-0.60	0.00	0.02
t-stat (=0)	0.14	2.00	-0.17	-0.37	-1.28	0.02	
GaltonGMV	0.12	**0.50**	-0.03	**-0.13**	**-0.07**	0.00	0.45
t-stat (=0)	0.45	12.84	-0.69	-2.54	-1.65	0.19	
SimpleGMV	2.32	**0.31**	**-0.16**	-0.04	-0.01	-0.02	0.09
t-stat (=0)	0.40	4.33	-1.65	-0.47	-0.15	-0.43	
EGGMV	**8.87**	**0.41**	**-0.10**	-0.06	0.03	0.00	0.28
t-stat (=0)	2.42	8.76	-1.72	-1.09	0.55	0.14	

4.6 Results and Outliers

      In section *3.2.1 Estimates and Regression* I mention that I remove the outliers. Table 8 and Figures 2 and 3 show the reason why this happens.

      One aspect that stands out in the table is how elevated the slope of the variance is if the outliers are not removed. The value shown is the average of the monthly regressions' coefficients, which in turn result from the ex-post realizations and historical estimates. In Figures 2 and 3 it can be verified how there are estimates with truly extreme values. It is

important to bear in mind that these are monthly estimations. The full unaltered results are presented in Appendix, in the tables 15 through 19.

Table 8

The Unaltered Regression Coefficients

By regressing ex-post realizations as in Fama-MacBeth (1973), gotten from a rolling window of 12 months, on the corresponding historical estimates, gotten from a rolling window of 60 months, I obtain the Galton coefficients. For this table, they are obtained from the unaltered database, this is, without the removal of outliers. These are presented for the mean, the variance and the pairwise correlation of returns, in each of the columns. The first row exhibits the mean intercept of the regressions (also referred to as g_0) and the third row presents the mean slope of the regressions (also referred to as g_1). Below each of these rows there is the t-stat measure for a statistical test where the null hypothesis is that each coefficient is 0. There is also one additional test for the slope coefficient but with the null hypothesis of it taking the value of 1. The following two rows present the percentage of monthly coefficients greater than 0 and smaller than 1. The last row presents the average R-square of the regressions.

	Mean	Variance	Correlation
Intercept	0.00	0.00	0.10
t-stat (=0)	2.59	-0.03	16.30
Slope	0.17	1.81	0.41
t-stat (=0)	2.23	3.23	22.56
t-stat (=1)	-10.75	-14.58	-32.08
Greater than 0 (%)	56.02	93.37	97.89
Smaller than 1 (%)	87.35	74.10	97.59
R-square (%)	9.54	21.59	7.98
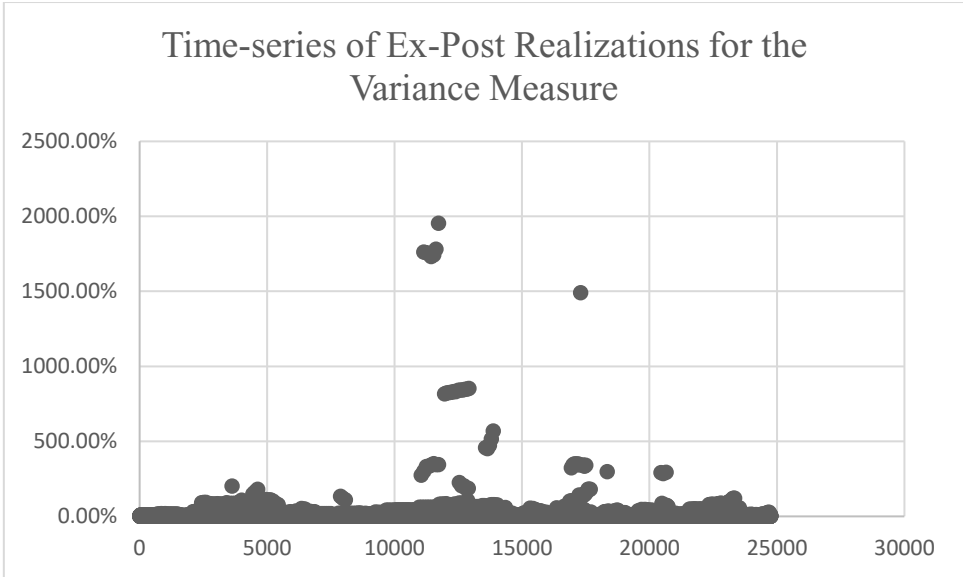
Figure 2. Time-series of Ex-Post Realizations for the Variance Measure. The y-axis represents the average variance of the twelve-periods windows while the x-axis represents the observations, numbered, in a chronological fashion.
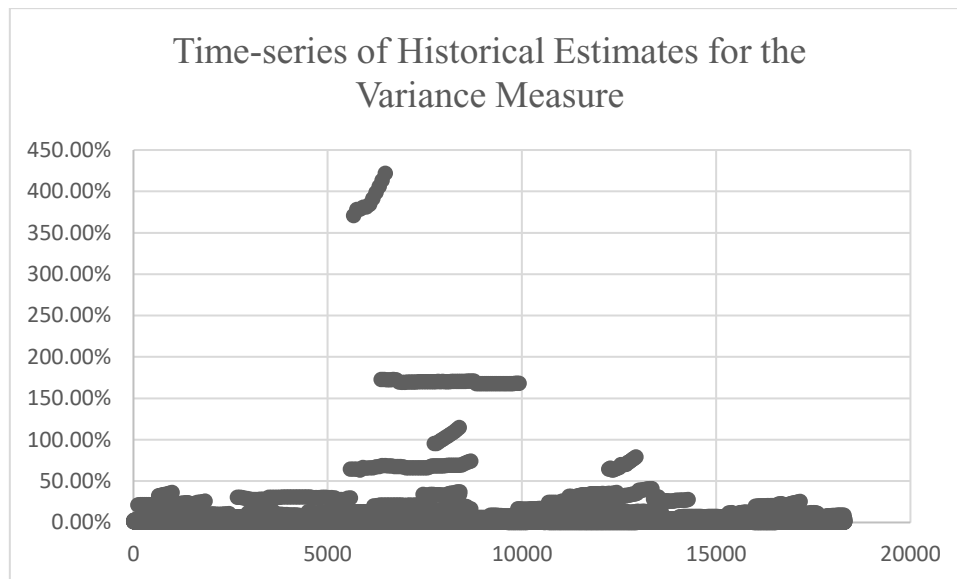
Figure 3. Time-series of Historical Estimates for the Variance Measure. The y-axis represents the average variance of the sixty-periods windows while the x-axis represents the observations, numbered, in a chronological fashion.

Observing the individual monthly slope coefficients of the variance regressions, there are only 89 monthly values that are outside of the interval [-1, 1], this is, 26.81% of the months, but its extremeness is relevant. Throughout the time-series, there seem to be clusters of large values, due to the presence of one or more companies in the selected monthly lots with a large discrepancy between their ex-post realization and historical estimate. After an isolated month with a value of 8.82 in March 1994 comes a set of five months with values higher than 50, registered between July and December of 1999. Between April of 2000 and March of 2001, the coefficients vary between 3 and 7, followed by another period of extreme values from May 2006 until August 2007, where the values are in the interval [-2, 6]. The last two clusters appear between September 2012 and June 2013 and June 2016 and August 2017, with values between 2 and 4. Table 20 in the appendix displays all the coefficients before and after the removal of outliers.

The maximum value is registered in December 1999, where a coefficient of 53.15 is listed. If the removal of the outliers is performed, five companies are removed from the monthly lot: Foncar, Imoleasing, J Soares, Prtg. De Seguros and Salvador Caetano. All five of these have ex-post realizations in outliers' space. Only one, Foncar, repeats the feat for its historical estimate. After this correction, the coefficient for the same month is 0.27. Table 21, in Appendix, shows the individual estimates that contribute for this coefficient.

Over the first occurrence and the following two clusters, the companies excluded are triggered mostly by their ex-post realizations, with values either extremely elevated or extremely low. In the third and fourth clusters, it is the historical estimates mostly that lead to the exclusion of the companies – but always due to particularly low estimations, never elevated ones (this is applicable to all clusters). For the last two clusters, the number of companies breaching the outliers' percentile is very low, and mostly due to their ex-post realizations.

There is one factor all outliers' companies present: the traded volume of their shares is significantly modest. The five removed businesses relative to the maximum coefficient have an average monthly trading volume of their shares in the historical window of 14851 trades and in the ex-post window of 4777 trades. J Soares and Salvador Caetano Com Aut, specifically, had a monthly average in the ex-post window of only 200 transactions – this meant only 4260€ and 566€, respectively. On its hand, some of the largest companies, such as EDP Energias de Portugal, Jeronimo Martins and Banco Comr.Portugues had monthly averages of 38197164 and 39664819 trades for the historical and ex-post windows, respectively. This happens not only in this case but for all outliers' periods. Often low-trading periods result in variances extremely near 0, adding to the occasional enormously high values. And the fact that it is not a select group of companies presenting extreme metrics, but a scattered event, supports a more systemic explanation such as the volume of shares traded.

While the existence of outliers is more severe for the variance measure, this also happens with the mean input, hence the removal of outliers for it as well at a smaller degree. The pairwise correlation measure does not present this problem, so no correction was applied.

5. Robustness

To test the Galton models' robustness, I ran it 9 times each under a different set of parameters. These parameters are the size of the rolling windows for obtaining the historical inputs, denominated H, and the ex-post realizations, designated E, the size of the learning period window, L, and the number of stocks in a portfolio, set as N.

The parameters used in this thesis are the same as in Barroso and Saxena (2021), for comparability purposes. The authors, on its turn, argue that the choice was arbitrary but intended to be plausible.

To fully test the impact of each parameter alone, only one of these is altered from the base-case conditions of the model (H=60, E=12, L=108 and N=25), each time. Additionally, changing time-windows implies different OOS periods for different runs. This would bring other factors to impact the results. For example, there are two major crises in the studied timespan - the Dot Com Bubble in the turning of the century and the 2008/2009 Global Financial Crisis, only felt later in Portugal. Expanding the time-period of the windows would cause some of the runs' OOS periods to not include one of these difficult periods. Naturally, they would present better results, but nevertheless results that are not in any way a responsibility of altering the conditions of the model, therefore giving ways for misinterpretations. To overcome this possibility, all models' OOS period is the same, from October 2009 until September 2021. The base conditions were also replicated in this new period for benchmarking purposes.

Theoretically, longer windows can tap into more information and gain on this knowledge but can also be ineffective if the data generation process changes over time. Table 9 presents the GaltonMV runs' effect on its weights and Table 10 the runs' effect on the performance.

Table 9
The GaltonMV Weights' Robustness
I run the GaltonMV's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main parameters: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns the parameters' window used are indicated. The sixth column presents the average monthly turnover. The following two columns present the minimum and maximum weight recorded in the entire OOS period. The subsequent two columns show the mean and the standard deviation of the monthly standard deviation within a portfolio, this is, the measure of the volatility of its weights. The last column shows the average percentage of short positions within a portfolio. From the fifth to the last column, all values are percentages.

	H	E	L	N	Turn.	Min w	Max w	Mean[σ(w)]	Std [σ(w)]	w<0
	60	12	108	25	3.83	-342.45	532.65	25.38	23.44	37.08
H	80	12	108	25	1.37	-76.11	101.49	16.33	6.94	34.25
	100	12	108	25	1.28	-64.07	106.14	16.32	6.14	34.39
E	60	24	108	25	1.38	-91.94	107.97	15.39	7.46	33.97
	60	36	108	25	0.77	-39.08	54.66	10.11	3.55	30.19
L	60	12	60	25	5.01	-500.17	821.73	25.35	31.47	36.33
	60	12	180	25	0.75	-32.40	37.06	9.72	2.15	30.19
N	60	12	108	20	2.54	-213.38	289.22	26.13	20.29	35.45
	60	12	108	15	3.30	-330.75	498.90	30.93	30.14	30.60

Table 10
The GaltonMV' Performance Robustness

I ran the GaltonMV's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main factors: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns parameters' window used are indicated. The next two columns show the annualized Sharpe Ratio and the expected one according to the model's prediction. The third column displays the cumulative return of each strategy in the time-period analysed. The three following columns display the annualized average expected monthly return, the annualized average realized monthly return and the ratio of the two. The subsequent three columns show the annualized average expected monthly standard deviation, the annualized average realized monthly standard deviation and the ratio of the two.

	H	E	L	N	SR	E(SR)	r_c (%)	$E(r_m)$ (%)	r_m (%)	$r_m/E(r_m)$	$E(\sigma)$ (%)	σ (%)	$\sigma/E(\sigma)$
	60	12	108	25	0.50	0.15	7725.48	17.60	42.99	2.44	119.47	86.02	0.72
H	80	12	108	25	0.82	0.10	1810.68	9.45	32.41	3.43	97.44	39.47	0.41
	100	12	108	25	1.03	0.11	4291.65	10.54	38.62	3.66	96.67	37.53	0.39
E	60	24	108	25	1.06	0.09	4294.28	8.34	39.65	4.75	96.57	37.50	0.39
	60	36	108	25	1.23	0.06	2000.45	4.70	28.27	6.02	84.71	22.94	0.27
L	60	12	60	25	0.25	0.18	34047.03	20.77	27.79	1.34	115.78	109.01	0.94
	60	12	180	25	1.26	0.07	1824.08	6.15	27.17	4.42	87.44	21.57	0.25
N	60	12	108	20	0.69	0.13	-100.00	15.03	52.73	3.51	115.75	76.96	0.66
	60	12	108	15	0.25	0.11	-100.00	12.85	25.14	1.96	116.31	99.01	0.85

Increasing H reduces turnover, the absolute value of minimum and maximum weights (closer to zero), the volatility of monthly weights and leads to a lower percentage of short positions. Increasing E leads to a similar change with higher expression. For the L scenarios, using a smaller learning period leads to more extreme weights and more volatility, but a larger one leads to the opposite, to less extreme positions. This turns, in fact, to be the most conservative scenario of all tested. Lastly, decreasing the number of stocks has leads to a more cautious position regarding the weights and the volatility, to a greater extent for a decrease of only 5 assets.

It is interesting to notice that the scenarios with the highest number of combined periods (in the H scenarios (80 + 12 + 108 = 200; 100 + 12 + 108 = 220), in the E scenarios (60 + 36 + 108 = 204; 60 + 60 + 108 = 228) and in the second L scenario (60 + 12 + 180 = 252)) lead to the most conservative positions taken by the model.

Regarding the performance of the model, all increases in the windows' number of periods lead to better Sharpe Ratios. With these, nevertheless, the prediction of returns also worsens, as does the volatility forecast's accuracy. Only the scenario with a smaller estimation

window improves these two measures – with its values relevantly accurate. For the N scenarios, while a decrease to 20 sees an improve in the Sharpe Ratio, a decrease to 15 fails to do so. The estimation of returns and of volatility, nevertheless, improves with a smaller number of stocks. Still, an investor would lose the entirety of its money with these smaller portfolios.

In Tables 11 and 12 I repeat the exercise for the GaltonMVEB and in tables 13 and 14 for the GaltonGMV model.

Table 11
The GaltonMVEB Weights' Robustness
        I ran the GaltonMVEB's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main factors: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns the parameters' window used are indicated. The sixth column presents the average monthly turnover. The following two columns present the minimum and maximum weight recorded in the entire OOS period. The subsequent two columns show the mean and the standard deviation of the monthly standard deviation within a portfolio, this is, the measure of the volatility of its weights. The last column shows the average percentage of short positions within a portfolio. From the fifth to the last column, all values are percentages.

	H	E	L	N	Turn.	Min w	Max w	Mean[σ(w)]	Std [σ(w)]	w<0
	60	12	108	25	3.83	-342.45	532.65	25.38	23.44	37.08
H	80	12	108	25	1.37	-76.11	101.49	16.33	6.94	34.25
	100	12	108	25	1.28	-64.07	106.14	16.32	6.14	34.39
E	60	24	108	25	1.38	-91.94	107.97	15.39	7.46	33.97
	60	36	108	25	0.77	-39.08	54.66	10.11	3.55	30.19
L	60	12	60	25	5.01	-500.17	821.73	25.35	31.47	36.33
	60	12	180	25	0.75	-32.40	37.06	9.72	2.15	30.19
N	60	12	108	20	2.54	-213.38	289.22	26.13	20.29	35.45
	60	12	108	15	3.30	-330.75	498.90	30.93	30.14	30.60

Table 12
The GaltonMVEB' Performance Robustness
       I ran the GaltonMVEB's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main factors: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns parameters' window used are indicated. The next two columns show the annualized Sharpe Ratio and the expected one according to the model's prediction. The third column displays the cumulative return of each strategy in the time-period analysed. The three following columns display the annualized average expected monthly return, the annualized average realized monthly return and the ratio of the two. The subsequent three columns show the annualized average expected monthly standard deviation, the annualized average realized monthly standard deviation and the ratio of the two.

	H	E	L	N	SR	E(SR)	r_c (%)	$E(r_m)$ (%)	r_m (%)	$r_m/E(r_m)$	$E(\sigma)$ (%)	σ (%)	$\sigma/E(\sigma)$
	60	12	108	25	0.50	0.15	7725.48	17.60	42.99	2.44	119.47	86.02	0.72
H	80	12	108	25	0.82	0.10	1810.68	9.45	32.41	3.43	97.44	39.47	0.41
	100	12	108	25	1.03	0.11	4291.65	10.54	38.62	3.66	96.67	37.53	0.39
E	60	24	108	25	1.06	0.09	4294.28	8.34	39.65	4.75	96.57	37.50	0.39
	60	36	108	25	1.23	0.06	2000.45	4.70	28.27	6.02	84.71	22.94	0.27
L	60	12	60	25	0.25	0.18	34047.03	20.77	27.79	1.34	115.78	109.01	0.94
	60	12	180	25	1.26	0.07	1824.08	6.15	27.17	4.42	87.44	21.57	0.25
N	60	12	108	20	0.69	0.13	-100.00	15.03	52.73	3.51	115.75	76.96	0.66
	60	12	108	15	0.25	0.11	-100.00	12.85	25.14	1.96	116.31	99.01	0.85

Table 13
The GaltonGMV Weights' Robustness
       I ran the GaltonGMV's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main factors: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns the parameters' window used are indicated. The sixth column presents the average monthly turnover. The following two columns present the minimum and maximum weight recorded in the entire OOS period. The subsequent two columns show the mean and the standard deviation of the monthly standard deviation within a portfolio, this is, the measure of the volatility of its weights. The last column shows the average percentage of short positions within a portfolio. From the fifth to the last column, all values are percentages.

	H	E	L	N	Turn.	Min w	Max w	Mean[σ(w)]	Std [σ(w)]	w<0
	60	12	108	25	0.31	-8.52	22.07	4.11	0.73	20.14
H	80	12	108	25	0.24	-6.58	21.46	4.35	0.61	22.22
	100	12	108	25	0.24	-5.92	20.76	4.30	0.53	22.78
E	60	24	108	25	0.31	-8.46	22.14	4.19	0.70	21.28
	60	36	108	25	0.28	-7.68	20.82	3.83	0.66	16.67
L	60	12	60	25	0.28	-7.01	21.12	3.83	0.64	16.67
	60	12	180	25	0.34	-7.62	22.05	4.32	0.55	21.67
N	60	12	108	20	0.30	-8.51	18.70	4.34	0.75	13.30
	60	12	108	15	0.24	-8.69	19.05	4.47	0.74	4.63

Table 14

The GaltonGMV' Performance Robustness

      I ran the GaltonGMV's model procedures 9 times and registered its outputs. Each time it was ran, I altered one of the four main factors: the historical time-window (H), the ex-post realizations time-window (E), the learning period (L) and the number of stocks in each portfolio (N). The unit for the parameters is months. The first row corresponds to the base case. The following are divided in four groups of two, where in each group a different parameter is tested. From the second to the fifth columns parameters' window used are indicated. The next two columns show the annualized Sharpe Ratio and the expected one according to the model's prediction. The third column displays the cumulative return of each strategy in the time-period analysed. The three following columns display the annualized average expected monthly return, the annualized average realized monthly return and the ratio of the two. The subsequent three columns show the annualized average expected monthly standard deviation, the annualized average realized monthly standard deviation and the ratio of the two.

	H	E	L	N	SR	E(SR)	r_c (%)	$E(r_m)$ (%)	r_m (%)	$r_m/E(r_m)$	E(σ) (%)	σ (%)	σ/E(σ)
	60	12	108	25	0.36	0.04	71.24	2.81	5.76	2.05	71.34	15.89	0.22
H	80	12	108	25	0.25	0.04	39.42	2.85	4.16	1.46	71.03	16.68	0.23
	100	12	108	25	0.20	0.04	26.40	3.09	3.56	1.15	70.04	17.95	0.26
E	60	24	108	25	0.39	0.04	80.50	2.70	6.18	2.29	71.59	15.79	0.22
	60	36	108	25	0.35	0.04	67.25	2.59	5.58	2.15	72.65	16.03	0.22
L	60	12	60	25	0.36	0.05	69.17	3.18	5.66	1.78	66.82	15.91	0.24
	60	12	180	25	0.34	0.05	65.30	3.74	5.46	1.46	77.14	15.85	0.21
N	60	12	108	20	0.23	0.04	34.82	2.88	3.95	1.37	73.13	17.04	0.23
	60	12	108	15	0.28	0.04	47.04	2.73	4.58	1.68	74.46	16.47	0.22

      The GaltonMVEB presents the exact same performance as the GaltonMV. As there are not any negative regression slope mean coefficients, the range limitation is not applied and, therefore, the model behaves in the same way. The GaltonGMV, on the other hand, presents a different behaviour. While the weights, generally, manifest the same patterns – larger windows indicate a more conservative and stable behaviour –, the scenario where the learning period is shortened also leads to less extreme positions and using smaller portfolios, the least, keeps the positions the same, and in cases hurts the conservativeness, plus the performance patterns are also different. For this model, the larger windows indicate worse Sharpe Ratios, as does using less assets. The prediction of volatility is not significantly affected by the changes in the parameters, while that of returns improves with the greater windows.

6. Conclusion

Portfolio optimization is a valiant effort. Countless hours have been spent researching, implementing and evaluating in this search. With time, improvements have arisen, and a long path has been travelled. But there is still room for improvement, with models struggling in out of sample settings.

One of the most recent contributions in this search is the Galton method from Barroso and Saxena (2021). This method uses out of sample errors, mostly disregarded in this kind of tests, to correct the inputs that go into the models. After robust testing in the United States of America, this correction delivers risk-adjusted results that are more solid and predictable that compare favourably to other state-of-the-art models. Even more than higher returns per se, one consistent advantage of the method is the accuracy in predicting the risk of a portfolio almost exactly, allowing for investors to truly manage their expectations.

I tested this method in the very distinct Portuguese market, a much different context economically, culturally and financially. The Galton coefficients indicated that the historical estimates were far from being exact predictors of future values and especially so for mean returns, with a slope coefficient of 0.06. The model (its MV and MVEB variants, particularly) was the top performer in terms of cumulative returns and regarding the Sharpe measure (with the GaltonMV achieving 0.89). Furthermore, the Galton's returns were obtained with much less extreme weights and less turnover, important if one takes into account factors such as trading costs. Whilst the EGMV minimum weight reached -9363.70% and its maximum 18429.89%, the Galtons did not surpass -21.44% on the minimum side or the 31.35% barrier on the maximum side, and its turnover, in the worst-case scenario, was approximately 105 times smaller than the 62.97% average maximum demonstrated by the SimpleMV. These characteristics held both for the MV model and for the GMV model, despite the group of the second, as by design, not presenting as extreme outputs – except performance-wise, where the EGGMV model outperformed its peers. These returns were also substantially less prone to extreme values, especially relative to the other MV models, as seen by the kurtosis measure (the Galtons' values are between 2.5 and 5.5, while the EGMV presents a maximum of 202.90, a minimum of more than 35 times higher). On top of all, the Galtons were able to forecast risk almost perfectly, better than any model and being conservative while doing it. They presented a ratio of realized volatility to expected volatility around 1, the 'worse' predictor of the three presenting 0.98, whilst the EGMV underestimated their risk by a factor of 10.58 times. All in

all, a key take is that the Galton confirmed its credentials and maintained a robust perspective on portfolio optimization.

Running robustness tests on the historical and ex-post realization windows, as well as on the size of the learning period and of the number of stocks on an equal setting allowed to conclude that the model is somewhat flexible to changes in these parameters, especially demonstrating how it improves its results and conservativeness with greater windows' periods. Nevertheless, the models' estimation of returns and volatility does not generally improve with larger windows, indicating that it benefits from the flexibility of shorter periods for this matter.

The methodology developed in this thesis can, nevertheless, be improved upon. Due to limitations in the data available, small adjustments were made in the portfolio selection, namely on admitting gap-months of missing data in the rolling windows, on the necessity of removing outliers and in the monthly selection of stocks as opposed to selecting the lot yearly. It is difficult to exactly quantify the impact of these changes, but to the best of my knowledge, it did not compromise the solutions and its comparability. Nevertheless, the perfect scenario would be one where these questions would not exist. Additionally, the time-period of available data only starts in the year 1988, what leaves a relatively smaller period for the model's execution and its analysis, especially compared to the 54 years period used in Barroso and Saxena (2021). Hopefully, in the future, more favourable conditions allow the deepening of this study.

References

Barroso, P., & Saxena, K. (2021). Lest we forget: using out-of-sample forecast errors in portfolio optimization. *The Review of Financial Studies* (forthcoming).

Brennan, M. J. (1998). The role of learning in dynamic portfolio decisions. *European Finance Review*, 1(3):295-306.

Carhart, M. M. (1997). "On Persistence in Mutual Fund Performance". *The Journal of Finance*. 52 (1): 57–82.

Chui, A. C., Titman, S., & Wei, K. J. (2010). Individualism and momentum around the world. *The Journal of Finance*, *65*(1), 361-392.

DeMiguel, V., Garlappi, L., and Uppal, R. (2009). Optimal versus naive diversification: How ineffcient is the 1/n portfolio strategy? *The Review of Financial Studies*, 22(5):1915-1953.

Detemple, J. B. (1986). Asset pricing in a production economy with incomplete information. *The Journal of Finance*, 41(2):383-391.

Dothan, M. U. and Feldman, D. (1986). Equilibrium interest rates and multiperiod bonds in a partially observable economy. *The Journal of Finance*, 41(2):369-382.

Elton, E. J. and Gruber, M. J. (1973). Estimating the dependence structure of share prices implications for portfolio selection. *The Journal of Finance*, 28(5):1203-1232.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica: Journal of the econometric society*, 987-1007.

Fama, E. F.; French, K. R. (1993). "Common risk factors in the returns on stocks and bonds". *Journal of Financial Economics*. 33: 3–56.

Fama, Eugene F.; MacBeth, James D. (1973). "Risk, Return, and Equilibrium: Empirical Tests". *Journal of Political Economy*. 81 (3): 607–636

Frazzini, A., & Pedersen, L. H. (2014). Betting against beta. *Journal of Financial Economics*, 111(1), 1-25.

Galton, F. (1894). *Natural inheritance*. Macmillan.

Gennotte, G. (1986). Optimal portfolio choice under incomplete information. *The Journal of Finance*, *41*(3): 733-746.

Goyal, A., & Wahal, S. (2015). Is momentum an echo? *Journal of Financial and Quantitative Analysis*, *50*(6), 1237-1267.

James, W., Stein, C. M. (1961). Estimation with quadratic loss. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*. 1:361–380.

Jobson, J. D. and Korkie, B. (1980). Estimation for markowitz efficient portfolios. *Journal*

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7(1):77-91.

Michaud, R. O. (1989). The markowitz optimization enigma: is `optimized' optimal? *Financial Analysts Journal*, 45(1):31-42.
of the American Statistical Association, 75(371):544-554.

Ross, S. A. (2013). The arbitrage theory of capital asset pricing. In *Handbook of the fundamentals of financial decision making: Part I* (pp. 11-30).

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The Journal of Finance*, *19*(3), 425-442.

Stein, C. (1956), "Inadmissibility of the usual estimator for the mean of a multivariate distribution", *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 1, pp. 197–206.

Appendix

Table 15
The Weights' Stability on the Unaltered Database

After the selection of the monthly portfolios and the obtaining of the estimates from the database without the outlier's adjustment followed by the respective correction, I compute the weights. For the different models presented, the first column shows the average monthly turnover as a percentage of the market capitalization of the portfolio. The following two columns present the minimum and maximum weight recorded in the entire OOS period. The fourth and fifth columns show the mean and the standard deviation of the monthly volatility within a portfolio, this is, the measure of the variability of the amplitude of its weights. The last column shows the average number of short positions within a portfolio's monthly 25 possible. All values presented are percentages.

Strategies	Turnover	Min w	Max w	Mean[σ(w)]	Std [σ(w)]	w<0
1/N	0.05	4.17	4.17	0.00	0.00	0.00
VW	0.38	0.03	30.34	5.90	0.73	0.00
GaltonMV	0.48	-18.27	98.79	10.84	2.50	28.53
GaltonMVEB	0.48	-18.27	98.79	10.84	2.50	28.53
SimpleMV	81.69	-11485.96	13135.39	163.45	544.97	46.61
EGMV	25.13	-3920.24	6067.03	71.05	183.67	47.17
JSMV	126.84	-32718.03	23295.52	193.10	1159.33	45.61
GaltonGMV	0.35	-6.51	97.10	9.90	2.26	28.94
SimpleGMV	0.35	-34.94	100.25	14.69	2.63	42.93
EGGMV	0.47	-6.18	101.31	9.63	1.56	44.12

Table 16
The Models' Performance on the Unaltered Database

For the database without the removal of outliers, every month, I select the largest 25 companies by market capitalization that fulfil the requirements of having information on its returns for the previous 60 months. I use rolling windows for the mean, the volatility and the pairwise correlation of the returns to compute the covariance matrix and, after that, the weights, according to Markowitz – the MV and the GMV portfolios – and, lastly, the results. For comparability I include other models, such as the Markowitz historical approach, the 1/N and the VW models, the constant correlation matrix from Elton and Gruber (1973) and the shrinkage estimator for mean returns from James and Stein (1961). The first two columns show the annualized Sharpe Ratio and the expected one according to each models' prediction. The third column displays the cumulative return of each strategy in the time-period analysed. The three following columns display the annualized average expected monthly return, the annualized average realized monthly return and the ratio of the two. The subsequent three columns show the annualized average expected monthly standard deviation, the annualized average realized monthly standard deviation and the ratio of the two.

Strategies	SR	E(SR)	r_c (%)	E(r_m) (%)	r_m (%)	r_m/E(r_m)	E(σ) %()	σ (%)	σ/E(σ)
1/N	0.10		0.38		2.04			19.94	
VW	0.22		54.89		3.81			16.99	
GaltonMV	1.09	0.76	803.66	9.18	18.72	2.04	12.00	17.25	1.44
GaltonMVEB	1.09	0.76	803.66	9.18	18.72	2.04	12.00	17.25	1.44
SimpleMV	0.41	4.08	-100.00	319.42	231.36	0.72	78.35	558.87	7.13
EGMV	0.31	2.82	-100.00	260.82	135.30	0.52	92.55	429.72	4.64
JSMV	-0.36	0.40	-100.00	75.60	-447.58	-5.92	187.98	1255.78	6.68
GaltonGMV	0.55	0.64	150.33	6.35	8.30	1.31	9.92	15.18	1.53
SimpleGMV	0.07	0.93	-21.26	7.42	1.52	0.20	7.94	22.77	2.87
EGGMV	0.67	0.86	491.13	9.36	10.87	1.16	10.94	16.14	1.48

Table 17
The Hit Rates of the Unaltered Database
 Using as base the database where the outliers were not removed, for every iteration, I compute the Value-at-Risk (VaR) measure, this is, the maximum volatility percentage that could be experienced over the monthly time horizons, so that there is a very low probability that the actual volatility percentage will exceed it, for the 1%, 5%, 95% and the 99% confidence levels. The VaR implicitly assumes that the distribution of returns of the models follow a Normal distribution. Each time the standard deviation surpassed the VaR, I registered it. The values presented are the percentage of months where this happened, this is, the hit rates. All values are percentages.

Strategies	r<Qz(1%)	r<Qz(5%)	r>Qz(95%)	r>Qz(99%)
GaltonMV	3.52	6.34	12.68	5.63
GaltonMVEB	3.52	6.34	12.68	12.68
SimpleMV	16.98	24.53	21.23	17.45
EGMV	15.49	26.06	16.90	11.27
JSMV	6.60	9.43	10.38	5.66
GaltonGMV	3.52	14.79	12.68	7.75
SimpleGMV	14.62	24.53	23.58	17.92
EGGMV	4.25	10.38	12.74	5.66

Table 18
The Returns' Distribution Peculiarities with the Unaltered Database
 The studied models, using the database before the removal of outliers, produce a time-series of monthly returns in the 224 months OOS. For each, the first column shows the skewness of the distribution of returns and the second column shows the kurtosis of the distribution of returns.

Strategies	Skewness	Kurtosis
1/N	-0.39	4.16
VW	-0.50	4.21
GaltonMV	0.29	4.85
GaltonMVEB	0.29	1.85
SimpleMV	5.44	76.12
EGMV	1.11	40.61
JSMV	-10.00	108.98
GaltonGMV	0.24	4.00
SimpleGMV	-1.27	11.31
EGGMV	-0.37	6.65

Table 19
The Certainty-Equivalent Yield for the Unaltered Database
     For all models, based on the database before the removal of the outliers, I compute the rate an investor with be willing to receive immediately instead of risking investing in the respective model for an investor with a coefficient of relative risk aversion (γ) equal to 3. All values are presented in percentage points.

Strategies	CE
1/N	-4.17
VW	-0.68
GaltonMV	14.38
GaltonMVEB	14.38
SimpleMV	-100.00
EGMV	-100.00
JSMV	-92.45
GaltonGMV	4.91
SimpleGMV	-7.95
EGGMV	6.84

Table 20
The Monthly Slope Coefficients on the Databases
     By regressing ex-post realizations as in Fama-MacBeth (1973), gotten from a rolling window of 12 months (the period corresponding to [*"Presented Month"* – 12, *"Presented Month"* - 1]) on the corresponding historical estimates, gotten from a rolling window of 60 months (the period corresponding to [*"Presented Month"* – 12 - 60, *"Presented Month"* – 12 - 1]), I obtain the Galton coefficients. In this table, the slope coefficients of the variance measure are presented. The first and fifth columns' values are obtained from the database when there is not any removal of outliers, whilst the second and sixths' columns are a result of the same process after the removal of the outliers.

Date	g_1 with outliers	g_1 without outliers	Date	g_1 with outliers	g_1 without outliers
28/02/1994	0.99	0.84	31/12/2007	0.33	0.34
31/03/1994	8.82	1.19	31/01/2008	0.33	1.17
29/04/1994	0.74	1.06	29/02/2008	0.28	0.12
31/05/1994	0.91	1.31	31/03/2008	0.25	-0.02
30/06/1994	1.89	1.18	30/04/2008	0.29	0.01
29/07/1994	2.05	1.19	30/05/2008	0.32	-0.04
31/08/1994	2.08	1.26	30/06/2008	0.36	-0.10
30/09/1994	1.23	0.83	31/07/2008	0.35	-0.39
31/10/1994	0.41	0.64	29/08/2008	0.27	-0.35
30/11/1994	0.37	0.38	30/09/2008	0.25	0.29
30/12/1994	0.31	0.31	31/10/2008	0.23	-0.26
31/01/1995	0.29	0.29	28/11/2008	0.29	-0.67
28/02/1995	0.18	0.18	31/12/2008	0.80	-0.44
31/03/1995	0.19	0.19	30/01/2009	1.26	-1.07
28/04/1995	0.15	0.15	27/02/2009	1.38	-0.60
31/05/1995	0.13	0.13	31/03/2009	1.37	-0.81
30/06/1995	0.71	0.12	30/04/2009	1.33	0.16
31/07/1995	0.72	0.11	29/05/2009	1.17	1.22
31/08/1995	0.77	0.19	30/06/2009	1.15	2.25
29/09/1995	0.59	-0.02	31/07/2009	1.33	1.92
31/10/1995	0.46	0.12	31/08/2009	1.39	2.15

30/11/1995	0.43	0.16	30/09/2009	1.31	1.93
29/12/1995	0.69	0.39	30/10/2009	1.64	1.72
31/01/1996	0.69	0.38	30/11/2009	1.52	1.67
29/02/1996	0.57	0.27	31/12/2009	0.72	1.50
29/03/1996	0.58	0.27	29/01/2010	0.72	1.44
30/04/1996	0.38	0.06	26/02/2010	0.65	1.56
31/05/1996	0.48	0.13	31/03/2010	0.75	1.59
28/06/1996	0.25	0.24	30/04/2010	1.35	1.45
31/07/1996	0.14	0.13	31/05/2010	0.62	0.69
30/08/1996	-0.02	0.08	30/06/2010	0.37	0.43
30/09/1996	0.20	0.32	30/07/2010	0.24	0.23
31/10/1996	0.04	0.16	31/08/2010	0.45	0.44
29/11/1996	0.04	0.22	30/09/2010	0.69	0.71
31/12/1996	-0.02	0.17	29/10/2010	0.79	0.67
31/01/1997	0.07	0.27	30/11/2010	1.50	1.57
28/02/1997	-0.01	0.22	31/12/2010	1.70	0.94
31/03/1997	0.02	0.24	31/01/2011	0.73	0.81
30/04/1997	0.02	0.30	28/02/2011	0.40	0.47
30/05/1997	0.05	0.25	31/03/2011	0.43	0.51
30/06/1997	0.11	0.21	29/04/2011	0.47	0.53
31/07/1997	0.40	0.56	31/05/2011	0.45	0.54
29/08/1997	0.50	0.72	30/06/2011	0.49	0.56
30/09/1997	0.52	0.54	29/07/2011	0.48	0.49
31/10/1997	0.53	0.64	31/08/2011	0.49	0.51
28/11/1997	0.53	0.60	30/09/2011	0.33	0.35
31/12/1997	0.51	0.56	31/10/2011	0.92	0.95
30/01/1998	0.51	0.54	30/11/2011	1.03	1.11
27/02/1998	0.53	0.72	30/12/2011	1.08	0.95
31/03/1998	0.10	0.73	31/01/2012	0.98	1.91
30/04/1998	0.05	0.43	29/02/2012	1.50	1.45
29/05/1998	0.04	0.43	30/03/2012	1.46	1.45
30/06/1998	0.04	0.46	30/04/2012	1.30	1.25
31/07/1998	0.03	0.28	31/05/2012	1.30	2.00
31/08/1998	0.00	0.18	29/06/2012	1.40	2.14
30/09/1998	0.01	0.09	31/07/2012	1.76	1.69
30/10/1998	0.00	0.04	31/08/2012	1.91	1.87
30/11/1998	0.01	0.13	28/09/2012	3.09	0.58
31/12/1998	0.02	0.21	31/10/2012	3.09	1.68
29/01/1999	0.01	0.13	30/11/2012	3.63	2.43
26/02/1999	0.02	-0.03	31/12/2012	3.87	2.78
31/03/1999	0.01	0.00	31/01/2013	2.26	0.74
30/04/1999	0.02	0.17	28/02/2013	2.13	0.61
31/05/1999	0.20	0.24	29/03/2013	2.19	0.51
30/06/1999	0.23	0.21	30/04/2013	2.11	0.55
30/07/1999	51.98	0.34	31/05/2013	2.18	0.68
31/08/1999	52.11	0.36	28/06/2013	2.11	0.51
30/09/1999	51.77	0.30	31/07/2013	0.57	0.48
29/10/1999	51.64	0.42	30/08/2013	1.88	2.11
30/11/1999	51.95	0.37	30/09/2013	0.29	0.38
31/12/1999	53.15	0.27	31/10/2013	0.30	0.27
31/01/2000	0.44	0.62	29/11/2013	0.43	0.39
29/02/2000	0.75	0.64	31/12/2013	0.47	0.31
31/03/2000	0.76	0.53	31/01/2014	0.45	0.46
28/04/2000	6.09	0.77	28/02/2014	0.44	0.40
31/05/2000	4.77	0.03	31/03/2014	0.44	0.40
30/06/2000	4.11	0.09	30/04/2014	0.61	0.58
31/07/2000	3.94	-0.03	30/05/2014	0.54	0.51
31/08/2000	4.49	0.01	30/06/2014	0.67	0.65
29/09/2000	3.66	0.02	31/07/2014	0.72	0.78
31/10/2000	3.57	0.01	29/08/2014	0.73	0.69

30/11/2000	3.64	0.16	30/09/2014	0.75	0.72
29/12/2000	3.91	0.21	31/10/2014	0.81	0.68
31/01/2001	4.33	0.10	28/11/2014	0.82	0.68
28/02/2001	-0.06	0.41	31/12/2014	0.83	0.79
30/03/2001	5.23	0.55	30/01/2015	0.83	0.80
30/04/2001	0.01	0.37	27/02/2015	0.74	0.42
31/05/2001	0.01	0.35	31/03/2015	0.75	0.35
29/06/2001	0.00	0.38	30/04/2015	0.53	0.51
31/07/2001	0.00	0.43	29/05/2015	0.40	0.36
31/08/2001	0.01	0.38	30/06/2015	0.31	0.22
28/09/2001	0.01	0.53	31/07/2015	0.29	0.25
31/10/2001	0.04	0.54	31/08/2015	0.62	0.58
30/11/2001	-0.01	0.85	30/09/2015	0.43	0.37
31/12/2001	0.00	0.85	30/10/2015	0.37	0.32
31/01/2002	-0.02	0.78	30/11/2015	0.59	0.51
28/02/2002	-0.05	0.57	31/12/2015	0.68	0.52
29/03/2002	0.00	-0.02	29/01/2016	0.67	0.63
30/04/2002	0.01	0.46	29/02/2016	0.35	0.17
31/05/2002	0.01	0.58	31/03/2016	0.10	0.00
28/06/2002	0.01	0.58	29/04/2016	0.14	0.04
31/07/2002	0.01	0.58	31/05/2016	0.24	0.14
30/08/2002	0.00	0.07	30/06/2016	0.21	0.11
30/09/2002	0.00	0.10	29/07/2016	0.17	0.06
31/10/2002	0.00	-0.01	31/08/2016	2.81	0.05
29/11/2002	0.00	-0.02	30/09/2016	3.00	0.32
31/12/2002	0.00	-0.02	31/10/2016	3.03	0.38
31/01/2003	0.00	0.04	30/11/2016	3.16	0.38
28/02/2003	0.00	0.05	30/12/2016	3.35	0.34
31/03/2003	0.00	0.05	31/01/2017	3.02	0.22
30/04/2003	0.00	0.09	28/02/2017	3.07	0.02
30/05/2003	0.00	0.10	31/03/2017	2.85	0.28
30/06/2003	0.00	0.05	28/04/2017	3.16	0.33
31/07/2003	0.00	0.09	31/05/2017	3.23	0.33
29/08/2003	0.00	0.07	30/06/2017	3.61	0.50
30/09/2003	0.00	0.04	31/07/2017	0.58	0.56
31/10/2003	0.00	0.04	31/08/2017	0.52	0.52
28/11/2003	0.00	0.01	29/09/2017	0.41	0.40
31/12/2003	0.00	0.04	31/10/2017	0.55	0.55
30/01/2004	0.00	-0.08	30/11/2017	0.40	0.49
27/02/2004	0.00	-0.08	29/12/2017	0.59	0.42
31/03/2004	0.01	0.00	31/01/2018	0.81	0.46
30/04/2004	0.01	0.04	28/02/2018	0.40	0.40
31/05/2004	0.01	0.05	30/03/2018	0.41	0.40
30/06/2004	0.01	0.06	30/04/2018	0.44	0.43
30/07/2004	0.01	0.07	31/05/2018	0.42	0.41
31/08/2004	0.01	0.06	29/06/2018	0.41	0.40
30/09/2004	0.01	0.09	31/07/2018	0.22	0.21
29/10/2004	0.01	0.10	31/08/2018	0.26	0.25
30/11/2004	0.01	0.08	28/09/2018	0.37	0.37
31/12/2004	0.01	0.08	31/10/2018	0.34	0.34
31/01/2005	0.02	0.21	30/11/2018	0.32	0.32
28/02/2005	0.02	0.24	31/12/2018	0.32	0.32
31/03/2005	0.02	0.13	31/01/2019	0.32	0.32
29/04/2005	0.02	0.12	28/02/2019	0.08	0.07
31/05/2005	0.02	0.13	29/03/2019	0.45	0.43
30/06/2005	0.02	0.16	30/04/2019	0.48	0.47
29/07/2005	0.02	0.13	31/05/2019	0.16	0.15
31/08/2005	0.02	0.10	28/06/2019	0.18	0.18
30/09/2005	0.02	0.08	31/07/2019	0.22	0.22
31/10/2005	0.02	0.08	30/08/2019	0.22	0.22

30/11/2005	0.02	0.07	30/09/2019	0.62	0.47
30/12/2005	0.02	0.07	31/10/2019	0.63	0.45
31/01/2006	0.00	0.13	29/11/2019	0.31	0.31
28/02/2006	0.00	0.06	31/12/2019	0.51	0.51
31/03/2006	0.00	0.20	31/01/2020	0.58	0.58
28/04/2006	0.13	0.09	28/02/2020	0.61	0.61
31/05/2006	5.27	0.10	31/03/2020	0.62	0.62
30/06/2006	5.43	0.09	30/04/2020	0.52	0.52
31/07/2006	5.48	0.07	29/05/2020	0.59	0.58
31/08/2006	5.90	0.03	30/06/2020	0.58	0.56
29/09/2006	4.94	0.06	31/07/2020	0.63	0.62
31/10/2006	-2.31	0.02	31/08/2020	0.68	0.67
30/11/2006	-2.16	0.07	30/09/2020	0.53	0.52
29/12/2006	-2.19	-0.19	30/10/2020	0.54	0.53
31/01/2007	1.40	-0.22	30/11/2020	0.61	0.60
28/02/2007	1.91	0.02	31/12/2020	0.48	0.46
30/03/2007	2.00	-0.55	29/01/2021	0.45	0.42
30/04/2007	4.38	0.25	26/02/2021	0.74	0.77
31/05/2007	3.83	0.24	31/03/2021	1.19	1.13
29/06/2007	3.65	0.23	30/04/2021	1.52	1.49
31/07/2007	3.84	0.24	31/05/2021	1.33	1.31
31/08/2007	3.38	0.67	30/06/2021	1.37	1.36
28/09/2007	0.59	0.36	30/07/2021	1.12	0.51
31/10/2007	0.48	0.52	31/08/2021	1.80	1.46
30/11/2007	0.46	0.38	30/09/2021	0.56	0.50

Table 21

The Study of the Maximum Slope Coefficient

      To obtain the Galton variance coefficient in December 1999, I regress ex-post realizations as in Fama-MacBeth (1973), gotten from a rolling window of 12 months, this is, from 12/1998 until 11/1999, on the corresponding historical estimates, gotten from a rolling window of 60 months, this is, from 12/1993 until 11/1998. For this table, they are obtained from the unaltered database, this is, without the removal of outliers. Each row corresponds to one of the companies present in the estimation universe in December 1999. The names are presented as they are in the Refinitiv Eikon data. The first column presents the ex-post realization values and the second the historical estimates. The third corresponds to a ratio of the two.

	X_E (%)	X_H (%)	X_H/X_E
A Silva & Silva	1.12	1.47	1.31
Atlantis Cristais De Alcobaua	0.69	1.11	1.60
B A Barbosa & Almeida	0.30	0.75	2.48
Banco BPI	0.72	1.20	1.67
Banco Comr.Portugues	0.13	0.82	6.56
Banco Espirito Santo	0.14	0.68	4.98
Banco Essi	0.09	1.29	14.90
Banco Mello	0.51	1.03	2.01
Banco Totta Acores	1.24	1.00	0.80
Banif-SGPS	0.08	0.77	9.99
Bicc Cel-Cat	1.18	0.86	0.73
Bnc.Prtg.Atlantico	0.47	1.42	3.02
Cellulose Do Caima	0.43	1.70	3.92
Centralcer	2.70	1.15	0.43
Cin	0.83	0.79	0.95
Cipan	1.59	5.62	3.53
Cires	0.35	1.58	4.48
Construcciones Do Tamega	2.06	2.18	1.06
Corticeira Amorim	0.13	1.10	8.31
Cr.Lyonn.Portl.	0.40	1.26	3.12
Credito Predial	0.77	0.75	0.97
Efacec	0.61	0.94	1.53
Espart	0.34	0.79	2.31
Estoril Sol	0.67	1.72	2.56
F Ramada	1.05	1.08	1.03
Fenalu Gst.De Invms.E Partp.	4.30	10.50	2.44
Fisipe	0.46	1.21	2.66
Foncar	1780.23	29.94	0.02
Hidroprojecto	0.11	5.20	47.81
Imobiliaria Construtora Grao Para	0.46	1.62	3.52
Imoleasing	0.05	0.46	9.77
Inapa Investimentos Participacoes E Gestao	0.26	0.52	2.00
J Soares	0.00	5.33	13313.49
Jeronimo Martins	0.60	1.32	2.20
Litho Formas Portuguesa	7.25	1.49	0.21
Locapor	0.32	1.07	3.33

Modelo Continente	0.66	1.39	2.10
Mundial Confianca	1.83	1.36	0.75
Mundicenter	1.76	0.64	0.36
Papelaria Fernandes	0.54	1.03	1.93
Prtg. De Seguros	23.66	2.48	0.10
Reditus Sdt.Dptp. Scs.	0.62	3.00	4.86
Salvador Caetano Com Aut	60.56	1.38	0.02
Sdc Investimentos	1.38	1.14	0.82
Seguros Imperio	0.26	0.89	3.39
Sociedade Comercial Orey Antunes	0.87	1.07	1.23
Somague SGPS	1.47	1.45	0.98
Sonae Industria SGPS	0.53	2.38	4.49
Sonae SGPS	0.93	1.37	1.48
Soporcel	1.09	1.20	1.10
Sumol Compal	0.55	1.21	2.20
Tertir	1.02	1.17	1.14
Toyota Caetano	0.47	0.86	1.85
Tranquilidade	0.22	0.65	2.94
Unicer	0.31	0.56	1.79
Vaa Vista Alegre Atlanti	0.54	0.88	1.62