Towards the null-polygon Wilson loop / spinning three-point function duality from the OPE limit

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Abstract

We calculate loop contributions to correlation functions involving 20' operators in $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory, which are related to Wilson loops via a duality.

We begin by explaining the basics of conformal symmetry. We show why it's useful and interesting to study conformally invariant theories, and $\mathcal{N} = 4$ Supersymmetric Yang-Mills Theory in specific.

We derive the constraints imposed by conformal symmetry on 2- and 3-point correlation functions, and show that these are fully determined in terms of the dimensions Δ_i of the operators and OPE coefficients c_{ijk} of the three-point functions. We also introduce the Operator Product Expansion and show that higher-point functions introduce no new parameters.

We then present the method of asymptotic expansions, which one can use to calculate integrals in the OPE limit. We show how this procedure can be used to express an integral depending on n external points as a sum over asymptotic regions, so that we only need to calculate integrals depending on at most $n - 1$ external points.

We explain how integrals with scalar products involving integration points in the denominator of the integrand can be expressed in terms of integrals without such scalar products, and how to reduce the set of integrals to evaluate to a smaller set of "master integrals" using integration-by-parts identities.

We introduce 20' operators in $\mathcal{N} = 4$ SYM and calculate their four-point function up to two loops, their five-point function up to one loop, and the correlator of four 20' and a Lagrangian operator up to one loop. By comparing with the conformal block expansion, we extract OPE coefficients and anomalous dimensions of twist-two operators.

We finish with a brief discussion of possible continuations of this work.

Resumo

Calculamos funções de correlação envolvendo operadores 20' na teoria $\mathcal{N} = 4$ Yang-Mills supersimétrica, que podem ser relacionadas com Wilson loops por via de uma dualidade.

Começamos por explicar algumas noções básicas sobre simetria conforme. Mostramos as razões pelas quais é útil e interessante estudar teorias com invariância conforme, e a teoria $\mathcal{N} = 4$ Yang-Mills Supersimétrica em específico.

Deduzimos as restrições impostas pela simetria conforme nas funções de correlação de dois e três pontos, e mostramos que estas funções são completamente determinadas em termos das dimensões Δ_i dos operadores e dos coeficientes de OPE c_{ijk} das funções de três pontos. Também introduzimos a Operator Product Expansion e mostramos que funções de correlação de mais pontos não introducem dependências em mais parâmetros.

Apresentamos depois o método de expansões assintóticas, que pode ser utilizado para calcular integrais no limite de OPE. Mostramos como este procedimento pode ser utilizado para exprimir um integral que depende de n pontos externos como uma soma sobre regiões assintóticas, de tal modo que apenas precisamos de calcular integrais que dependem de, no máximo, $n - 1$ pontos externos.

Explicamos como integrais com produtos escalares que envolvam pontos de integração no numerador do integrando podem ser expressos em termos de integrais sem produtos escalares, e como é possível reduzir o conjunto de integrais que temos de avaliar a um conjunto mais pequeno de "integrais mestre" usando identidades de integração por partes.

Introduzimos operadores 20' em $\mathcal{N} = 4$ SYM e calculamos a sua função de 4 pontos até ordem 2 no acoplamento, a sua função de 5 pontos até ordem 1 no acoplamento, e a função de correlação de quatro 20' e um operador Lagrangeano até ordem 1 no acoplamento. Por comparação com a expansão em blocos conformes, conseguimos extrair coeficientes de OPE e dimensões anómalas de operadores com twist 2.

Terminamos com uma breve discussão de possíveis continuações deste trabalho.

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Nomenclature

AdS	Anti-de Sitter
CFT	Conformal Field Theory
IBP	Integration By Parts
OPE	Operator Product Expansion
QFT	Quantum Field Theory
RG	Renormalization Group
SYM	Supersymmetric Yang-Mills

Chapter 1

Introduction

1.1 Conformal field theory

Conformal Field theory (CFT) has been a subject of great interest in the past decades, for many reasons. One of these reasons relates to the Renormalization Group (RG). In this framework, we begin by assuming that a theory has a momentum cutoff Λ, such that the generating functional Z is given by

$$Z = \int [\mathcal{D}\phi]_\Lambda \ e^{\int \mathcal{L} + J\phi} \tag{1.1}$$

where

$$[\mathcal{D}\phi]_\Lambda \equiv \prod_{|k|<\Lambda} d\phi_k \tag{1.2}$$

and ϕ_k is the Fourier component of the field ϕ. We can then integrate over a shell in momentum space, expressing the generating functional as an integral up to a new cutoff $b\Lambda$, where $0 < b < 1$:

$$Z = \int [\mathcal{D}\phi]_{b\Lambda} \ e^{\int \mathcal{L}_{\text{eff}} + J\phi} \tag{1.3}$$

The new Lagrangian $\mathcal{L}_{\text{eff}}$ will have different parameters, and so it will correspond to a different point in parameter space as $\mathcal{L}$. Integrating out this momentum shell is called a step of the RG. If we take successive steps and take infinitely many steps, while taking the step length to be infinitesimally small ($b \approx 1$) we get a continuous "flow" in parameter space, which we call RG flow. This flow continues until the theory reaches a point where taking a step will leave it in the same point. These are called fixed points of RG flow. Conformally invariant theories are interesting because they are fixed points.

Another reason we want to study conformal theories is that they have a large symmetry group, which imposes constraints on correlation functions. This naturally makes the theory simpler to study, and makes it so we can obtain non-perturbative results, while in general Quantum Field Theory (QFT) these are hard to come by.

A third reason is what's known as the AdS-CFT correspondence. This is a conjectured duality between d-dimensional gravity theories in anti-de Sitter (AdS) space and $(d-1)$-dimensional CFTs.

1.2 $\mathcal{N}=4$ supersymmetric Yang-Mills theory

We will focus on the particular case of $\mathcal{N}=4$ supersymmetric Yang-Mills theory (SYM). This is a gauge theory with gauge group $SU(N_c)$. It has 6 real scalar fields

$$\Phi_I, \quad I=1,\ldots,6, \tag{1.4}$$

4 fermionic fields

$$\Psi_A, \quad A=1,\ldots,4, \tag{1.5}$$

and one gauge field A_μ, where μ is a spacetime index. As the name suggests, this theory also exhibits supersymmetry, having 4 sets of supercharges. As is typical for a gauge field theory, all fields are $N_c \times N_c$ matrices:

$$\begin{aligned} \Phi_I &= \Phi_I^a\, t_a \\ \Psi_A &= \Psi_A^a\, t_a \\ A_\mu &= A_\mu^a\, t_a \end{aligned}$$

where t_a are the generators of the gauge group in the adjoint representation and the implicit sum in a runs over the generators. The $\mathcal{N}=4$ SYM Lagrangian is

$$\mathcal{L}=\frac{1}{g_{YM}^2}\mathrm{Tr}\left[\frac{1}{2}\left[D_\mu,D_\nu\right]^2+\left(D_\mu\Phi_i\right)^2-\frac{1}{2}\left[\Phi_i,\Phi_j\right]^2+i\bar{\Psi}\left(\Gamma^\mu D_\mu\Psi+\Gamma^i\left[\Phi_i,\Psi\right]\right)+\right.$$
$$\left.+\,\partial^\mu\bar{c}\,D_\mu c+\zeta\left(\partial^\mu A_\mu\right)^2\right]$$

where D_μ is the covariant derivative, which is defined by

$$D_\mu \Phi = \partial_\mu \Phi - i \left[A_\mu, \Phi\right], \tag{1.6}$$

Γ^μ are matrices, c and $\bar{c}$ are the Faddeev-Popov ghosts, and ζ is the parameter which corresponds to the choice of gauge. This theory is interesting for a number of reasons. First, as mentioned before, it is conformally invariant, and therefore a fixed point of RG flow. Secondly, it is supersymmetric, which means there will be more restrictions to the observables. This makes the theory easier to study, and may even lead to nonperturbative results. Thirdly, it is an integrable theory in the planar limit $N_c \to \infty$, meaning it has an infinite number of conserved charges. This restricts the observables even more, and is another possible way of obtaining nonperturbative results. This theory is also a practical example of the AdS-CFT duality, since it is dual to a type IIB superstring theory in $S_5 \times AdS_5$ space. In $\mathcal{N} = 4$ SYM in particular, there are duality relations between Wilson loops, correlation functions, and scattering amplitudes.

The focus of our work will be related to the duality between correlation functions and null-polygon Wilson loops, which was first proposed in [1]. This duality can be stated as

$$\lim_{x_{i,i+1}^2 \to 0} G_n / G_n^{(0)} \propto W\left[C_n\right] \tag{1.7}$$

where G_n is the n-point function of local gauge-invariant operators

$$G_n = \left\langle \mathcal{O}\left(x_1\right) \cdots \mathcal{O}\left(x_n\right) \right\rangle, \tag{1.8}$$

$W\left[C_n\right]$ is the Wilson loop

$$W\left[C_n\right] = \frac{1}{N_c^2 - 1} \left\langle 0 \middle| \mathrm{tr}_{\mathrm{adj}} P \left\{ \exp\left(ig \oint_{C_n} dx \cdot A\left(x\right) \right) \right\} \middle| 0 \right\rangle, \tag{1.9}$$

and C_n is the piecewise-null polygon defined by the points x_i, $i = 1, \ldots, n$. A specific case of the duality (1.7) is the duality between a six-point function of scalar operators and the null-hexagon Wilson loop, i.e. $n = 6$. By taking OPEs as depicted in figure 1.1 (the so-called snowflake configuration), this duality can be understood as relating the null-hexagon Wilson loop to the three-point function of spinning operators. This particular duality was made exact in [2]. In this paper, the authors verified the duality up to one loop. One of the goals of our work will be to verify this duality at higher orders in the coupling.

In $\mathcal{N} = 4$ SYM, correlation functions and Wilson loops can both be calculated using integrability. Single-trace scalar operators like the ones we will be studying are cyclical, and so can

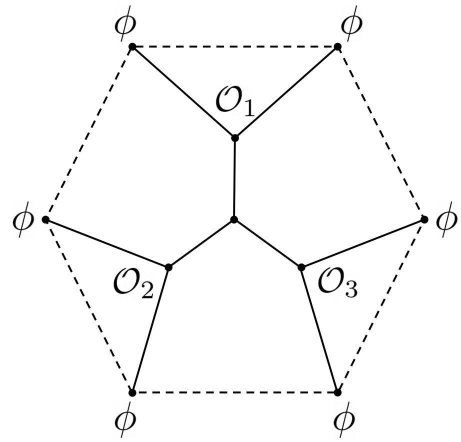

Figure 1.1: OPE decomposition of the six-point function of scalars ϕ in the snowflake configuration. The operators $\mathcal{O}_i$ are those appearing in the $\phi \times \phi$ OPE and have arbitrary spin. The dashed outer lines represent a null separation between points.

be represented as circles. The three-point function of these operators in the planar limit has the topology of a pair of pants (see Figure 1.2). By cutting the pair of pants along the dotted line, it is divided into two hexagons, which make up the front and back of the pants (see Figure 1.2). The hexagons can be computed exactly using integrability [3]. Higher-point functions can also be cut into pair of pants, and therefore the n-point functions of these operators can be calculated with integrability. We can also divide a null-polygon Wilson loop into pentagons (see Figure 1.3). These pentagons can also be calculated using integrability [4].

1.3 Thesis outline

This thesis is structured as follows: In chapter 2 we review some properties of conformal field theories. We will define conformal symmetry and determine the generators of the conformal group. We will examine the restrictions imposed by conformal symmetry on correlation functions of local operators, and introduce some tools which will be useful later, like the Operator Product Expansion and decomposition in conformal blocks.

In chapter 3, we will introduce some techniques to calculate the integrals which appear in the correlation functions we will be studying. We will detail the procedure of asymptotic expansions of integrals. We will also explain how to reduce integrals with spin to scalar integrals, and how to find integration-by-parts identities which reduce the number of scalar integrals to be evaluated to a smaller set of master integrals. We will finish the chapter by calculating some example integrals.

In chapter 4 we will calculate some four- and five-point correlation functions of local operators in $\mathcal{N} = 4$ SYM. For the four-point function, we will consider the OPE limit $x_{12}^2 \to 0$, and for the five point functions we will take the double OPE limit $x_{12}^2, x_{34}^2 \to 0$. By comparing the correlator with the conformal block expansion, we will extract the relevant CFT data, and compare the results to the literature.

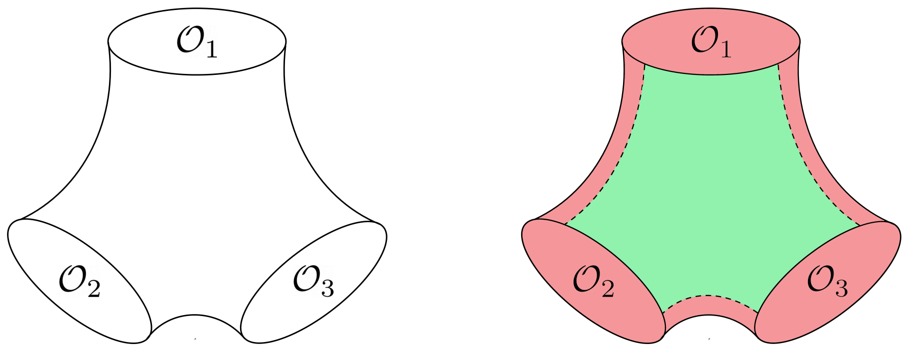

Figure 1.2: The three-point function of scalar single-trace operators in the planar limit. The dotted lines divide the pair of pants into the green and red hexagons.

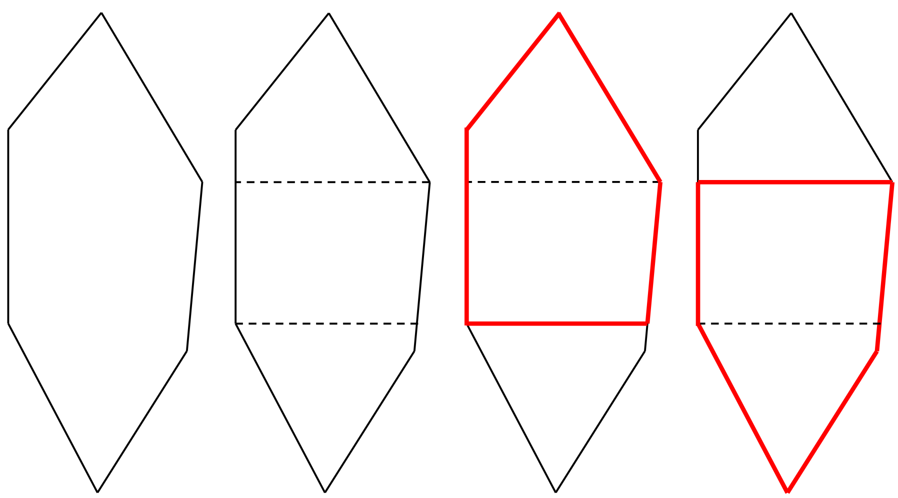

Figure 1.3: The null hexagon Wilson loop can divided into three squares, and any two adjacent squares form a pentagon.

Chapter 2

Conformal Field Theory

Quantum Field Theory is best understood as an effective long-distance limit of some other microscopic theory, which may be arbitrarily complex. This microscopic theory is not expected to have long-distance correlations; these should decay exponentially as:

$$\langle \phi(0)\phi(x) \rangle \sim e^{-\frac{x}{\xi}} \tag{2.1}$$

and so a long-range QFT description is ruled out. However, if the theory is a fixed point of RG flow, then the correlation length ξ diverges, $\xi \to \infty$, and so correlations extend out to ranges much larger than the interatomic spacing. At such a fixed point, the theory must be invariant by a scale transformation:

$$x^\mu \to \lambda x^\mu \tag{2.2}$$

The corresponding transformation of the metric is

$$g_{\mu\nu}(x) \to \lambda^2 g_{\mu\nu}(x) \tag{2.3}$$

This type of transformation - also known as a dilatation - is a special type of what is known as a conformal transformation. A transformation is said to be conformal if the metric transforms like

$$g_{\mu\nu} \to \Omega^2(x) g_{\mu\nu} \tag{2.4}$$

In this way, a conformal transformation is a kind of "local dilatation", with the factor Ω being dependent of x. The change in the metric corresponding to this transformation is

$\delta g_{\mu\nu} = \left(\Omega^2(x) - 1\right) g_{\mu\nu}$. Defining the stress-energy tensor as the response to a metric change, we get:

$$\delta S \propto \int d^D x \, T_{\mu\nu}(x) \delta g^{\mu\nu}(x) = 2 \int d^D x \, \left(\Omega^2(x) - 1\right) T^\mu_\mu(x)$$

Since at a fixed point we must have $\delta S = 0$ for any $\Omega(x)$, we conclude that in a conformally invariant theory we must have $T^\mu_\mu = 0$. Now, if we consider a general infinitesimal change in coordinates

$$x^\mu \to x^\mu + \epsilon^\mu(x), \tag{2.5}$$

we might wonder which choices of ϵ lead to a conformal transformation. For a general coordinate transformation $x \to \tilde{x}$, the metric transforms as

$$g_{\mu\nu} \to \frac{\partial \widetilde{x}^\alpha}{\partial x^\mu} \frac{\partial \widetilde{x}^\beta}{\partial x^\nu} g_{\alpha\beta} \tag{2.6}$$

Inserting equation 2.6 into equation 2.4 and assuming flat spacetime with a Euclidean signature ($g_{\mu\nu} = \delta_{\mu\nu}$), we obtain the condition

$$\frac{\partial \widetilde{x}^\alpha}{\partial x^\mu} \frac{\partial \widetilde{x}^\beta}{\partial x^\nu} \delta_{\alpha\beta} = \Omega^2(x) \delta_{\mu\nu} \tag{2.7}$$

For an infinitesimal transformation $\widetilde{x}^\mu = x^\mu + \epsilon^\mu(x)$, we have

$$\frac{\partial \widetilde{x}^\alpha}{\partial x^\mu} = \delta^\alpha_\mu + \partial_\mu \epsilon^\alpha(x) \tag{2.8}$$

and so equation 2.7 becomes

$$\left(\delta^\alpha_\mu + \partial_\mu \epsilon^\alpha(x)\right) \left(\delta^\beta_\nu + \partial_\nu \epsilon^\beta(x)\right) \delta_{\alpha\beta} = \Omega^2(x) \delta_{\mu\nu} \tag{2.9}$$

$$\iff \delta_{\mu\nu} + \partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) + \mathcal{O}\left(\epsilon^2\right) = \Omega^2(x) \delta_{\mu\nu} \tag{2.10}$$

Neglecting quadratic terms in ϵ, this condition is equivalent to

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \left(\Omega^2(x) - 1\right) \delta_{\mu\nu} \equiv c(x) \delta_{\mu\nu} \tag{2.11}$$

These are the Killing equations for conformal symmetry. Their solutions tell us which vectors ϵ result in conformal transformations for a Euclidean metric. We will now find these solutions. First, we take the trace of both sides:

$$2\partial \cdot \epsilon = dc(x) \tag{2.12}$$

Substituting equation 2.12 into equation 2.11, we obtain

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} \left(\partial \cdot \epsilon\right) \delta_{\mu\nu} \tag{2.13}$$

Next we take the divergence of this equation in both indices:

$$2\left(1 - \frac{1}{d}\right) \partial^2 \left(\partial \cdot \epsilon\right) = 0 \tag{2.14}$$

This tells us that either $d = 1$ or $\partial^2 \left(\partial \cdot \epsilon\right) = 0$. Assume $d \neq 1$. We apply $\partial^\nu \partial_\rho$ to equation 2.13:

$$\partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) + \partial^2 \partial_\rho \epsilon_\mu = \frac{2}{d} \partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) \tag{2.15}$$

Now we symmetrize the equation in ρ and μ:

$$\partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) + \frac{1}{2} \partial^2 \left(\partial_\rho \epsilon_\mu + \partial_\mu \epsilon_\rho\right) = \frac{2}{d} \partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) \tag{2.16}$$

Once again, using equation 2.13, we get

$$\partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) + \frac{1}{d} \partial^2 \left(\partial \cdot \epsilon\right) \delta_{\mu\nu} = \frac{2}{d} \partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) \tag{2.17}$$

Using $\partial^2 \left(\partial \cdot \epsilon\right) = 0$, we obtain:

$$\left(1 - \frac{2}{d}\right) \partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) = 0 \tag{2.18}$$

This tells us that either $d = 2$ or $\partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) = 0$. Assume now that $d \neq 2$. Now we apply $\partial_\gamma \partial_\rho$ to 2.13 and use $\partial_\rho \partial_\mu \left(\partial \cdot \epsilon\right) = 0$, yielding:

$$\partial_\gamma \partial_\rho \partial_\mu \epsilon_\nu + \partial_\gamma \partial_\rho \partial_\nu \epsilon_\mu = 0 \tag{2.19}$$

By taking permutations of ρ, ν, μ, we get three equations, which have the solution

$$\partial_\gamma \partial_\rho \partial_\mu \epsilon_\nu = 0 \tag{2.20}$$

This means that ϵ is at most quadratic in x. The most general quadratic expression is given by

$$\epsilon^\mu (x) = a_1^\mu + c_1 x^\mu + m_\nu^\mu x^\nu + (a_2 \cdot x) x^\mu + n_{\nu\rho}^\mu x^\nu x^\rho + a_3^\mu x^2, \tag{2.21}$$

where c_1 is a constant scalar, a_i^μ are constant vectors, and m_ν^μ and $n_{\nu\rho}^\mu$ are constant higher-order tensors. We can now use previous equations to place restrictions on these constants. We can start by taking a derivative:

$$\partial^\nu \epsilon^\mu = c_1 \delta^{\mu\nu} + m^{\mu\nu} + (a_2 \cdot x) \delta^{\mu\nu} + a_2^\nu x^\mu + 2 n_\rho^{\mu\nu} x^\rho + 2 a_3^\mu x^\nu \tag{2.22}$$

Symmetrizing both sides:

$$\begin{aligned} &\partial^\nu \epsilon^\mu + \partial^\mu \epsilon^\nu = \\ &= 2 (c_1 + a_2 \cdot x) \delta^{\mu\nu} + (m^{\mu\nu} + m^{\nu\mu}) + (a_2^\nu + 2a_3^\nu) x^\mu + (a_2^\mu + 2a_3^\mu) x^\nu + 2 \left(n_\rho^{\mu\nu} + n_\rho^{\nu\mu} \right) x^\rho = \\ &\qquad = c(x) \delta^{\mu\nu} \end{aligned} \tag{2.23}$$

This tells us that

$$m^{\mu\nu} + m^{\nu\mu} = 0 \tag{2.24}$$
$$a_2^\mu + 2a_3^\mu = 0 \tag{2.25}$$
$$a_2^\nu + 2a_3^\nu = 0 \tag{2.26}$$
$$n_\rho^{\mu\nu} + n_\rho^{\nu\mu} = 0 \tag{2.27}$$

We conclude that $m^{\mu\nu}$ and $n_\rho^{\mu\nu}$ are antisymmetric in μ and ν, and $a_2^\mu = -2a_3^\mu$. However, since $n_\rho^{\mu\nu}$ is also symmetric in ν and ρ by construction, we have:

$$n_{\mu\nu\rho} = -n_{\nu\mu\rho} = -n_{\nu\rho\mu} = n_{\rho\nu\mu} = n_{\rho\mu\nu} = -n_{\mu\rho\nu} = -n_{\mu\nu\rho} \tag{2.28}$$

Therefore we conclude that $n_\rho^{\mu\nu} = 0$. Inserting these conditions into equation 2.21 and relabelling $a_1 \to a$, $a_3 \to b$, $c_1 \to -\lambda$, we get

$$\epsilon^{\mu}(x) = a^{\mu} + m^{\mu}_{\nu}x^{\nu} - \lambda x^{\mu} + b^{\mu}x^{2} - 2(b \cdot x)x^{\mu} \tag{2.29}$$

This is the most general form of ϵ^{μ} for a conformal transformation. We can determine the generators of this symmetry group by considering the effect of this infinitesimal transformation on a function $f(x)$, neglecting quadratic and higher order terms in ϵ:

$$f(x + \epsilon(x)) = f(x) + \left[a^{\mu}\partial_{\mu} + m^{\mu\nu}x_{\nu}\partial_{\mu} - \lambda x^{\mu}\partial_{\mu} + b^{\mu}\left(x^{2}\partial_{\mu} - 2x_{\mu}(x \cdot \partial)\right)\right] f(x) \tag{2.30}$$

Remembering that $m^{\mu\nu}$ is antisymmetric, we can write equation 2.30 as

$$f(x + \epsilon(x)) = f(x) + \left[a^{\mu}P_{\mu} + \frac{1}{2}m^{\mu\nu}M_{\mu\nu} + \lambda D + b^{\mu}K_{\mu}\right] f(x), \tag{2.31}$$

where

$$\begin{aligned} P_{\mu} &= \partial_{\mu} & M_{\mu\nu} &= x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu} \\ D &= -x^{\mu}\partial_{\mu} & K_{\mu} &= x^{2}\partial_{\mu} - 2x_{\mu}(x \cdot \partial) \end{aligned}$$

are the generators of the conformal group, and a^{μ}, $m^{\mu\nu}$, λ and b^{μ} are the corresponding parameters. These generators obey the commutation relations

$$[D, P_{\mu}] = P_{\mu} \qquad [D, K_{\mu}] = -K_{\mu} \qquad [K_{\mu}, P_{\nu}] = 2\delta_{\mu\nu}D - 2M_{\mu\nu} \tag{2.32}$$

$$[M_{\mu\nu}, P_{\rho}] = \delta_{\nu\rho}P_{\mu} - \delta_{\mu\rho}P_{\nu} \qquad [M_{\mu\nu}, K_{\rho}] = \delta_{\nu\rho}K_{\mu} - \delta_{\mu\rho}K_{\nu} \tag{2.33}$$

$$[M_{\mu\nu}, M_{\rho\gamma}] = \delta_{\nu\rho}M_{\mu\gamma} - \delta_{\mu\rho}M_{\nu\gamma} - \delta_{\nu\gamma}M_{\mu\rho} + \delta_{\mu\gamma}M_{\nu\rho} \tag{2.34}$$

Note that the P and K operators obey the same algebra as the ordinary ladder operators a and $a^{\dagger}$ in single-particle quantum mechanics, with D being the equivalent of the number operator $a^{\dagger}a$. Consider an operator $\mathcal{O}$ such that

$$D\mathcal{O} = \Delta\mathcal{O} \tag{2.35}$$

where Δ is a constant. If we act with K_{μ} on this operator:

$$DK_{\mu}\mathcal{O} = ([D, K_{\mu}] + K_{\mu}D)\,\mathcal{O} = (\Delta - 1)K_{\mu}\mathcal{O} \tag{2.36}$$

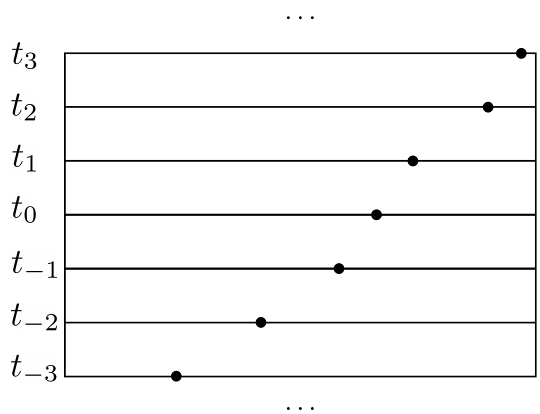

Figure 2.1: Foliation of spacetime through surfaces of constant time.

Because, as will be shown below, we must have $\Delta \geq 0$, there must be operators $\mathcal{O}$ such that $K_\mu \mathcal{O} = 0$. These are called conformal primaries.

2.1 Radial quantization

In general QFTs we can define a spacetime direction as time, say $x^0 = t$, and identify the component of the momentum operator associated with that direction as the Hamiltonian, $P_0 = H$. Then, given an operator $\mathcal{O}(t_0, \mathbf{x})$ defined on the spacetime surface of time t_0, we can determine the form of this operator for arbitrary t, via:

$$\mathcal{O}(t, \mathbf{x}) = e^{(t-t_0)H} \mathcal{O}(t_0, \mathbf{x}) e^{-(t-t_0)H} \tag{2.37}$$

This amounts to foliating spacetime through surfaces of constant time (Figure 2.1).

Because conformal theories also have translation invariance, we can also quantize in this way. However, there is a more convenient method we can use. Instead of a component of the momentum operator, we can use the dilatation operator D.

First, we choose a point of spacetime as the origin and define operators and commutation relations at this point. Then, we define the operator at an arbitrary radius as:

$$\mathcal{O}(r) = e^{rD} \mathcal{O}(0) e^{-rD} \tag{2.38}$$

This amounts to foliating spacetime through surfaces of constant radius (Figure 2.2).

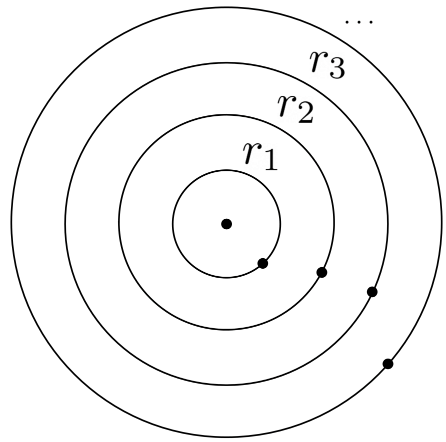

Figure 2.2: Foliation of spacetime through surfaces of constant radius.

In this way an operator is completely determined by its commutation relations with other operators at the origin. We will be mainly interested in so-called primary operators, which are defined by the commutation relations

$$[D, \mathcal{O}(0)] = \Delta\mathcal{O}(0), \qquad [K_\mu, \mathcal{O}(0)] = 0, \qquad [M_{\mu\nu}, \mathcal{O}^a(0)] = (S_{\mu\nu})^a_b\, \mathcal{O}^b(0) \tag{2.39}$$

where $S_{\mu\nu}$ are the generators of the rotation group in the relevant representation and the constant Δ defined by these relations is called the dimension of the operator.

2.2 Restrictions to correlation functions

Correlation functions of local operators are objects of fundamental importance in QFT. They can be interpreted as probabilities of certain interactions occuring, and can also be used to determine scattering amplitudes, which can be measured in particle colliders. It is therefore natural to study these functions. In CFTs, the additional symmetries impose restrictions on correlation functions, which makes them considerably easier to determine. Let us examine some of these restrictions.

2.2.1 Two-point functions

Consider the two-point function of scalar primary operators

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \tag{2.40}$$

If the correlator is invariant under a transformation with generator L and parameter θ to both operators:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \left\langle e^{L\theta}\mathcal{O}_1(x_1)e^{-L\theta}e^{L\theta}\mathcal{O}_2(x_2)e^{-L\theta}\right\rangle \tag{2.41}$$

In the limit $\theta \ll 1$ we can expand the exponentials:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle \quad = \quad \langle (1+L\theta)\,\mathcal{O}_1(x_1)\,(1-L\theta)\,(1+L\theta)\,\mathcal{O}_2(x_2)\,(1-L\theta)\rangle \quad + \quad \mathcal{O}\left(\theta^2\right)$$

Because this equality must hold for arbitrary (but small) θ, we must have

$$\langle [L,\mathcal{O}_1(x_1)]\,\mathcal{O}_2(x_2)\rangle + \langle \mathcal{O}_1(x_1)\,[L,\mathcal{O}_2(x_2)]\rangle = 0 \tag{2.42}$$

In particular, for dilatation symmetry, this means that

$$(-x_1\cdot\partial_1 - x_2\cdot\partial_2 + \Delta_1 + \Delta_2)\,\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = 0 \tag{2.43}$$

Let us look at the restrictions imposed by conformal symmetry on the two-point function (2.40). First of all, it is clear by Poincaré invariance that it must be a function of x_{12}^2:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = f(x_{12}^2) \tag{2.44}$$

Assuming this function f can be expanded in powers of x_{12}^2:

$$f(x_{12}^2) = \sum_\alpha c_\alpha \left(x_{12}^2\right)^\alpha \tag{2.45}$$

and inserting this into equation 2.43 we get

$$\sum_\alpha c_\alpha\,(2\alpha + \Delta_1 + \Delta_2)\left(x_{12}^2\right)^\alpha = 0 \tag{2.46}$$

Therefore we must have

$$c_\alpha = 0 \vee \alpha = -\frac{\Delta_1 + \Delta_2}{2} \tag{2.47}$$

We conclude that the two-point function is of the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2}{2}}} \tag{2.48}$$

For the special conformal generator K_μ we have, if $\mathcal{O}$ is a scalar:

$$[K_\mu, \mathcal{O}(x)] = \left[K_\mu, e^{x\cdot P}\mathcal{O}(0)\, e^{-x\cdot P}\right] = e^{x\cdot P}\left[e^{x\cdot P}K_\mu e^{-x\cdot P}, \mathcal{O}(0)\right] e^{-x\cdot P} = (K_\mu + 2\Delta x_\mu)\,\mathcal{O}(x) \tag{2.49}$$

Therefore the function f must satisfy:

$$\left(2x_{1\mu}\left(x_1\cdot\partial_1\right) - x_1^2\partial_{1\mu} + 2x_{2\mu}\left(x_2\cdot\partial_2\right) - x_2^2\partial_{2\mu} + 2\Delta_1 x_{1\mu} + 2\Delta_2 x_{2\mu}\right) f\left(x_{12}^2\right) = 0 \tag{2.50}$$

Using the form (2.48) for f we get the condition

$$\Delta_1 = \Delta_2 \vee C = 0 \tag{2.51}$$

Therefore, we conclude that conformal symmetry implies that the two-point function is given by:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{C\delta_{\Delta_1\Delta_2}}{\left(x_{12}^2\right)^{\Delta_1}} \tag{2.52}$$

The constant C can be absorbed in the definition of the operators, so that the two-point function is completely determined by the symmetry:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{\left(x_{12}^2\right)^{\Delta_1}} \tag{2.53}$$

This result implies that $\Delta_1 \geq 0$. Otherwise, correlation functions would grow infinitely with distance, which is clearly unphysical. For operators with spin, the correlator is a bit more complicated, but still quite simple. The form of the two-point function of symmetric and traceless operators with spin J and dimension Δ is:

$$\left\langle \mathcal{O}_1^{\mu_1\cdots\mu_J}(x_1)\mathcal{O}_2^{\nu_1\cdots\nu_J}(x_2)\right\rangle = \frac{I^{\mu_1\nu_1}\left(x_{12}\right)\cdots I^{\mu_J\nu_J}\left(x_{12}\right)}{\left(x_{12}^2\right)^{\Delta}} + \text{permutations} - \text{traces}, \tag{2.54}$$

where

$$I^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{2x^\mu x^\nu}{x^2} \tag{2.55}$$

The correlator is zero if the operators have different spins or different dimensions. The permutations make the correlator symmetric in the μ_i and in the ν_i, while subtracting the traces makes it traceless. A derivation of this result can be found in [5].

2.2.2 Three-point functions

In the same way as before, we can try to find differential equations for the three-point function of scalar primaries. By Poincaré invariance, the three-point function must only depend on x_{ij}^2, $i, j = 1, 2, 3$:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = f(x_{12}^2, x_{13}^2, x_{23}^2) \tag{2.56}$$

If the three-point function has a symmetry with generator L, then we have

$$\langle [L, \mathcal{O}_1]\, \mathcal{O}_2\mathcal{O}_3\rangle + \langle \mathcal{O}_1\, [L, \mathcal{O}_2]\, \mathcal{O}_3\rangle + \langle \mathcal{O}_1\mathcal{O}_2\, [L, \mathcal{O}_3]\rangle = 0 \tag{2.57}$$

where we omitted the positions of the local operators, $\mathcal{O}_i \equiv \mathcal{O}_i(x_i)$. Using this condition for $L = D$ and $L = K_\mu$, together with the ansatz

$$f(x_{12}^2, x_{13}^2, x_{23}^2) = \sum_{\alpha_1,\alpha_2,\alpha_3} C_{\alpha_1\alpha_2\alpha_3} \left(x_{12}^2\right)^{\alpha_1} \left(x_{13}^2\right)^{\alpha_2} \left(x_{23}^2\right)^{\alpha_3} \tag{2.58}$$

for the three-point function, we conclude that f must be of the form

$$f(x_{12}^2, x_{13}^2, x_{23}^2) = \frac{C}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2-\Delta_3}{2}} \left(x_{13}^2\right)^{\frac{\Delta_1+\Delta_3-\Delta_2}{2}} \left(x_{23}^2\right)^{\frac{\Delta_2+\Delta_3-\Delta_1}{2}}} \tag{2.59}$$

More generally, the correlator of three operators with arbitrary spin is [6]

$$\langle \mathcal{O}_{k_1}(x_1, z_1) \dots \mathcal{O}_{k_3}(x_3, z_3)\rangle = \sum_{l_i} \frac{C^{l_1 l_2 l_3}_{J_1 J_2 J_3} V_{1,23}^{J_1-l_2-l_3} V_{2,31}^{J_2-l_1-l_3} V_{3,12}^{J_3-l_1-l_2} H_{12}^{l_3} H_{13}^{l_2} H_{23}^{l_1}}{\left(x_{12}^2\right)^{\frac{h_1+h_2-h_3}{2}} \left(x_{13}^2\right)^{\frac{h_1+h_3-h_2}{2}} \left(x_{23}^2\right)^{\frac{h_2+h_3-h_1}{2}}}, \tag{2.60}$$

where z_i are null polarization vectors ($z_i^2 = 0$), $h_i = \Delta_k + J_k$, and

$$V_{i,jk} = \frac{(z_i \cdot x_{ij})\, x_{ik}^2 - (z_i \cdot x_{ik})\, x_{ij}^2}{x_{jk}^2} \qquad H_{ij} = (z_i \cdot x_{ij})\,(z_j \cdot x_{ij}) - \frac{x_{ij}^2\,(z_i \cdot z_j)}{2}. \tag{2.61}$$

The sum in l_i is constrained by the fact that negative powers of z_i cannot appear in the correlator. This means that, for example, $l_2 + l_3 \leq J_1$.

2.2.3 Higher points and cross-ratios

We have seen that conformal correlators of two and three points are fixed up to undetermined constants. At four and more points, however, there are conformally-invariant variables on which the correlator can depend, which we call conformal cross-ratios. We might wonder how many independent cross-ratios there are for, say, four points. A simple way to determine this is by using conformal transformations to a special frame of reference, where the number of degrees of freedom of our system will become clearer. The steps to get to this frame are as follows:

- Start with four points in arbitrary positions $x_i,\ i = 1, 2, 3, 4$.
- Use special conformal transformations to send $x_4 \to \infty$.
- Use translations to move x_1 to $(0, \dots, 0)$.
- Use rotations around the origin and dilatations to move x_3 to $(1, 0, \dots, 0)$.
- Use the rotations that leave x_3 invariant to move x_2 to $(x, y, 0, \dots, 0)$.

Now, if a function of x_i is conformally invariant, as conformal correlators must be, then it can only depend on x and y. Therefore, we have determined that the four-point function depends on only two variables. The usual choice for these variables is

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \tag{2.62}$$

Now that we know the four-point case, we may ask how many cross-ratios there are for five or more points. We can follow the same procedure as for four points, and then add another point at an arbitrary position x_5. We have already exploited the translation, dilatation, and special conformal symmetries, as well as part of the rotation symmetry. We can, however, still exploit the part of the rotation group which is orthogonal to the plane we fixed before. We can use these symmetries to fix all but 3 of the coordinates of point x_5. Therefore, we will have 5 independent cross-ratios for 5 points. We will choose them to be

$$u_1 = \frac{x_{12}^2 x_{35}^2}{x_{13}^2 x_{25}^2}, \qquad u_2 = \frac{x_{23}^2 x_{14}^2}{x_{24}^2 x_{13}^2}, \qquad u_3 = \frac{x_{34}^2 x_{25}^2}{x_{35}^2 x_{24}^2}, \qquad u_4 = \frac{x_{45}^2 x_{13}^2}{x_{14}^2 x_{35}^2}, \qquad u_5 = \frac{x_{15}^2 x_{24}^2}{x_{25}^2 x_{14}^2} \tag{2.63}$$

Similarly, if we add another point x_6, we can use the remaining rotation symmetry to fix all but 4 of its components. Adding a point x_7, we can fix all but 5 of its components, and so on. This will continue until we add point x_{d+2}. At this stage, we will already have used the full rotation symmetry, and therefore we will be adding d degrees of freedom for all further points we add. The number of independent cross-ratios for n points will therefore be given by

$$\begin{cases} \frac{n(n-3)}{2} & \text{if } n < d+3 \\ nd - \frac{(d+2)(d+1)}{2} & \text{if } n \geq d+3 \end{cases} \tag{2.64}$$

2.3 State-operator correspondence

In any quantum field theory, given a local operator $\mathcal{O}(x)$, we can define a state on the corresponding Hilbert space by:

$$|\mathcal{O}(x)\rangle = \mathcal{O}(x)|0\rangle \tag{2.65}$$

In general, the opposite is not true. That is, we cannot uniquely define a local operator from a state. However, in a CFT, there is a unique correspondence between states and operators. We will give a proof adapted from [5]. Say we have an eigenstate of the dilatation operator $|\Delta\rangle$:

$$D|\Delta\rangle = \Delta|\Delta\rangle \tag{2.66}$$

This state is defined on a Hilbert space a radius r from a point. We can write it as a path integral

$$|\Delta\rangle = \int D\phi_b |\phi_b\rangle\langle\phi_b|\Delta\rangle, \tag{2.67}$$

where ϕ_b is a scalar field defined only on the surface where the state $|\Delta\rangle$ is defined, and

$$\langle\phi_b|\Delta\rangle = \int_{\phi_{in}(r,\mathbf{n})=\phi_b(\mathbf{n})} D\phi_{in}(r',\mathbf{n})\mathcal{O}_\Delta(x)e^{-S[\phi_{in}]} \tag{2.68}$$

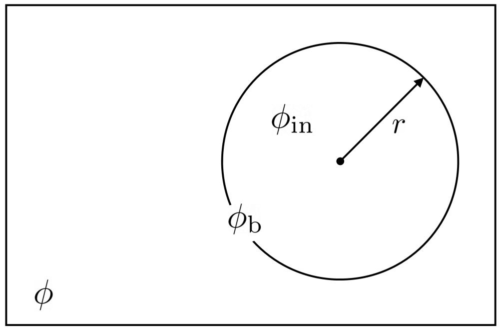

Figure 2.3: Splitting of the path integral by introducing a field ϕ_{in} inside the sphere of radius r. This field is integrated over, with boundary conditions $\phi_{\text{in}}(r, \mathbf{n}) = \phi_{\text{b}}(\mathbf{n})$. The field ϕ outside the sphere is also integrated over, with boundary conditions $\phi(r, \mathbf{n}) = \phi_{\text{b}}(\mathbf{n})$. Finally, the boundary field ϕ_b is also integrated over.

where ϕ_{in} is defined only inside the ball of radius r. This procedure is pictured in figure 2.3.

Now, we want to define an operator $\mathcal{O}_\Delta(x)$ out of the state $|\Delta\rangle$, where x is a point inside the ball. To do so, we need to define correlation functions with this operator. Correlation functions are defined by path integrals. For example, for the two-point function of ϕ:

$$\langle\phi(x_1)\phi(x_2)\rangle = \int D\phi\phi(x_1)\phi(x_2)e^{-S[\phi]} \tag{2.69}$$

where the field ϕ is defined on the entire space. In order to calculate, for example, the correlation funcion

$$\langle\phi(x_1)\phi(x_2)\mathcal{O}_\Delta(x)\rangle \tag{2.70}$$

we can use the auxiliary fields ϕ_{in} and ϕ_{b} to calculate the path integral:

$$\begin{aligned}\langle\phi(x_1)\phi(x_2)\mathcal{O}_\Delta(x)\rangle = \int D\phi\phi(x_1)\phi(x_2)\mathcal{O}_\Delta(x)e^{-S[\phi]} = \\ = \int D\phi_b \int_{\phi|\partial B=\phi_b} D\phi \int_{\phi_{in}|\partial B=\phi_b} D\phi_{in}\phi(x_1)\phi(x_2)\mathcal{O}_\Delta(x)e^{-S[\phi]} = \\ = \int D\phi_b\langle\phi_b|\Delta\rangle \int_{\phi|\partial B=\phi_b} D\phi\phi(x_1)\phi(x_2)e^{-S[\phi]}\end{aligned} \tag{2.71}$$

In the same way, for any correlation function involving $\mathcal{O}_\Delta$:

$$\begin{aligned}\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\mathcal{O}_\Delta(x)\rangle &= \int \left(\prod_{i=1,\dots,n,\Delta} D\mathcal{O}_i\right)\left(\prod_{i=1,\dots,n,\Delta}\mathcal{O}_i\right)e^{-S[\mathcal{O}_i]} = \\ &= \int D\phi_b\langle\phi_b|\Delta\rangle\int_{\phi|_{\partial B}=\phi_b}\left(\prod_{i=1,\dots,n} D\mathcal{O}_i\right)\left(\prod_{i=1,\dots,n}\mathcal{O}_i\right)e^{-S[\mathcal{O}_i]} \end{aligned} \quad (2.72)$$

In this way, we can express any correlator involving $\mathcal{O}_\Delta$ in terms of the state $|\Delta\rangle$. Therefore, we have defined correlation functions involving this operator. This is essentially the same thing as having defined the operator itself, since we don't "measure" an operator by itself, we simply understand local operators through their correlation functions.

2.4 Operator Product Expansion

Another useful property of conformal theories is the operator product expansion (OPE). This property consists of expressing the product of two operators as a sum over conformal primaries:

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{k\text{ primary}}\frac{c_{12k}}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2-\tau_k}{2}}}f_{i_1\cdots i_{J_k}}(x_{12},\partial_2)\mathcal{O}_k^{i_1\cdots i_{J_k}}(x_2) \quad (2.73)$$

where f is a differential operator and J_k is the spin of the primary operator $\mathcal{O}_k$. To prove this, we use the state-operator correspondence. First, we foliate the space around x_2. Then consider a ball centered on x_2 and containing x_1. The operators will generate a state $|\psi_{12}\rangle$ on the surface of the ball. This state can be expressed as a linear combination of dilatation eigenstates:

$$|\psi_{12}\rangle = \sum_k c_{12k}|\Delta_k\rangle \quad (2.74)$$

Because of the state-operator correspondence, each state $|\Delta_k\rangle$ is equivalent to an operator inserted at x_2. Therefore, we have

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_k\frac{c_{12k}}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2-\tau_k}{2}}}\mathcal{O}_k(x_2) \quad (2.75)$$

The operators $\mathcal{O}_k$ are either conformal primaries or descendants, meaning we can turn this sum into a sum over primaries in the following way:

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{k \text{ primary}} \frac{c_{12k}}{\left(x_{12}^2\right)^{\frac{\Delta_1+\Delta_2-\tau_k}{2}}} f_{i_1\cdots i_{J_k}}(x_1, x_2, \partial_2)\mathcal{O}_k^{i_1\cdots i_{J_k}}(x_2) \tag{2.76}$$

where $\tau_k = \Delta_k - J_k$ is the twist of the primary operator $\mathcal{O}_k$. Additionally, because of translation invariance, the only scale allowed in the function f is x_{12}. This proves equation 2.73. So far, we have not determined the differential operators $f(x_{12}, \partial_2)$ acting on the exchanged operator in the OPE, apart from the scaling in x_{12}. However, we know the form of the three-point function in a CFT (equation 2.59), and the OPE must be consistent with this form. Consider the three-point function of identical scalars:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle \tag{2.77}$$

Using the OPE on $\phi(x_1)\phi(x_2)$, this is equal to:

$$\sum_{k \text{ primary}} \frac{c_{\phi\phi k}}{\left(x_{12}^2\right)^{\frac{2\Delta_\phi-\Delta_k+J_k}{2}}} f_{i_1\cdots i_{J_k}}(x_{12}, \partial_2)\langle \mathcal{O}_k^{i_1\cdots i_{J_k}}(x_2)\phi(x_3)\rangle = \frac{c_{\phi\phi\phi}}{\left(x_{12}^2\right)^{\frac{\Delta_\phi}{2}}} f(x_{12}, \partial_2) \frac{1}{\left(x_{23}^2\right)^{\Delta_\phi}} \tag{2.78}$$

Using equation 2.59, this means that

$$f(x_{12}, \partial_2)\frac{1}{\left(x_{23}^2\right)^{\Delta_\phi}} = \frac{1}{\left(x_{13}^2 x_{23}^2\right)^{\frac{\Delta_\phi}{2}}} = \frac{1}{\left(x_{23}^2\right)^{\Delta_\phi}} \sum_{k=0}^{\infty} \binom{-\Delta_\phi/2}{k} \left(\frac{x_{12}^2 + 2x_{12}\cdot x_{23}}{x_{23}^2}\right)^k \tag{2.79}$$

We can use this equation to determine the operator f by expanding the left-hand side in x_{12} and equating the coefficients. Due to rotation invariance, we can write the operator f as:

$$f(x_{12}, \partial_2) = \sum_{n,m=0}^{\infty} a_{nm} x_{12}^{\mu_1} \cdots x_{12}^{\mu_n} \left(x_{12}^2\right)^m \partial_2^{\mu_1} \cdots \partial_2^{\mu_n} \left(\partial_2^2\right)^m \tag{2.80}$$

We can determine the constants a_{nm} explicitly in the case of three identical scalars. We can calculate how the Laplacians and derivatives act on powers:

$$\left(\partial_2^2\right)^n \frac{1}{\left(x_{23}^2\right)^\alpha} = 4^n \frac{(\alpha)_n \left(\alpha - \frac{d}{2} + 1\right)_n}{\left(x_{23}^2\right)^{\alpha+n}} \tag{2.81}$$

$$(x_{12}\cdot\partial_2)^n\,\frac{1}{\left(x_{23}^2\right)^{\alpha}}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-2)^{n-k}(\alpha)_{n-k}\left(x_{12}\cdot x_{23}\right)^{n-2k}\left(x_{12}^2\right)^k}{\left(x_{23}^2\right)^{\alpha+n-k}}\prod_{l=0}^{k-1}\binom{n-2l}{2} \tag{2.82}$$

Expanding both sides of (2.79) in x_{12}^2, we can find the coefficients a_{nm}. There is another kind of OPE we can take, where, instead of taking the coincidence limit $x_{12}\to 0$, we take the lightcone limit $x_{12}^2\to 0$. The lightcone OPE was first written in [7]. It can be written in the form [8]:

$$\phi\left(x_1\right)\phi\left(x_2\right)\approx\sum c_{12k}\int_0^1[dt]\,\frac{\mathcal{O}_k\left(x_1+tx_{12},x_{12}\right)}{\left(x_{12}^2\right)^{\frac{2\Delta_\phi-\tau_k}{2}}}+\cdots \tag{2.83}$$

where the $\cdots$ represent subleading terms in x_{12}^2 and the second argument in $\mathcal{O}_k$ is the vector which is contracted with the open indices of the operator. The integration measure is defined by

$$[dt]=\frac{\Gamma\left(\Delta_k+J_k\right)}{\Gamma^2\left(\frac{\Delta_k+J_k}{2}\right)}\left(t\left(1-t\right)\right)^{\frac{\Delta_k+J_k}{2}-1}dt \tag{2.84}$$

where Δ_k is the dimension of $\mathcal{O}_k$ and J_k is the spin of $\mathcal{O}_k$. We will sometimes refer to the coincidence limit OPE as the Euclidean OPE and the lightcone limit OPE as the Lorentzian OPE. Note that in the Euclidean OPE the primaries being summed over are located at the point around which the theory is quantized (x_2 in equation 2.76), while in the case of the Lorentzian OPE, the leading term receives contributions from operators extending along the light-like segment between both points in the product (x_1 and x_2 in equation 2.83).

2.5 Conformal blocks

The OPE has a very useful application. Consider, for example, a four-point function of scalar operators:

$$\langle\mathcal{O}_1\left(x_1\right)\mathcal{O}_2\left(x_2\right)\mathcal{O}_3\left(x_3\right)\mathcal{O}_4\left(x_4\right)\rangle \tag{2.85}$$

Using the OPE, we can write this as

$$\langle\mathcal{O}_1\left(x_1\right)\mathcal{O}_2\left(x_2\right)\mathcal{O}_3\left(x_3\right)\mathcal{O}_4\left(x_4\right)\rangle=\sum_{\mathcal{O}}c_{12\mathcal{O}}c_{34\mathcal{O}}W_{\mathcal{O}}\left(x_1,x_2,x_3,x_4\right), \tag{2.86}$$

where

$$W_{\mathcal{O}} = f_{i_1 \dots i_{J_k}}\left(x_{12}, \partial_2\right) f_{j_1 \dots j_{J_l}}\left(x_{34}, \partial_4\right) \left\langle \mathcal{O}_k^{i_1 \dots i_{J_k}}\left(x_2\right) \mathcal{O}_l^{j_1 \dots j_{J_l}}\left(x_4\right) \right\rangle . \tag{2.87}$$

are called conformal partial waves. By conformal symmetry, these functions can be written as

$$W_{\mathcal{O}} = \left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_{34}}{2}} \frac{G_{\mathcal{O}}\left(z, \bar{z}\right)}{\left(x_{12}^2\right)^{\frac{1}{2}\left(\Delta_1+\Delta_2\right)} \left(x_{34}^2\right)^{\frac{1}{2}\left(\Delta_3+\Delta_4\right)}} \tag{2.88}$$

These functions $G_{\mathcal{O}}$ are called conformal blocks. For $d = 4$ dimensions, they are given by [6]:

$$\begin{aligned} G_{\mathcal{O}}\left(z, \bar{z}\right) &= \frac{1}{(-2)^{\ell}} \frac{z\bar{z}}{z - \bar{z}} \left[k_{\Delta+\ell}(z) k_{\Delta-\ell-2}(\bar{z}) - k_{\Delta+\ell}(\bar{z}) k_{\Delta-\ell-2}(z)\right], \\ k_{\beta}(x) &\equiv x^{\beta/2} {}_2F_1\left(\frac{\beta - \Delta_{12}}{2}, \frac{\beta + \Delta_{34}}{2}, \beta; x\right), \end{aligned} \tag{2.89}$$

where ${}_2F_1$ is the hypergeometric function

$${}_2F_1\left(a, b, c; z\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{2.90}$$

$(a)_n$ is the Pochhammer symbol

$$(a)_n \equiv \frac{\Gamma\left(a+n\right)}{\Gamma\left(a\right)}, \tag{2.91}$$

and the variables $z, \bar{z}$ are related to the cross-ratios u, v by

$$u = z\bar{z},\ v = (1-z)(1-\bar{z}) \tag{2.92}$$

We will also need to expand five-point functions in conformal blocks:

$$G = \left\langle \mathcal{O}_1\left(x_1\right) \mathcal{O}_2\left(x_2\right) \mathcal{O}_3\left(x_3\right) \mathcal{O}_4\left(x_4\right) \mathcal{O}_5\left(x_5\right) \right\rangle \tag{2.93}$$

We can do a light-cone OPE in (12) and another in (34), as depicted in figure 2.4. We will assume $\mathcal{O}_1 = \mathcal{O}_2$ and $\mathcal{O}_3 = \mathcal{O}_4$, because we will only calculate correlators of this kind. This also allows us to use equation (2.83). Because we will only consider correlators of scalars, we further assume that all $\mathcal{O}_i$ are scalars. Using (2.83), the correlator G will then be given in

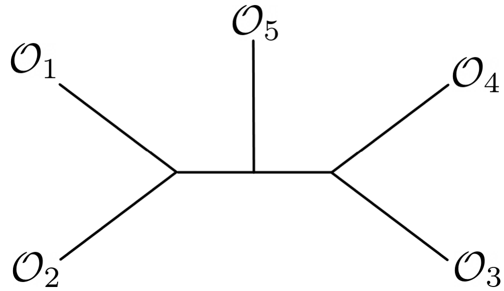

Figure 2.4: OPE decomposition of the five-point function.

terms of three-point functions of spinning operators:

$$G = \sum_{k,j} \frac{c_{12k} c_{34j}}{\left(x_{12}^2\right)^{\frac{2\Delta_1 - \tau_k}{2}} \left(x_{34}^2\right)^{\frac{2\Delta_3 - \tau_k}{2}}} \int_0^1 [dt_1] \int_0^1 [dt_2] \left\langle \mathcal{O}_k\left(x_1 + t_1 x_{12}, x_{12}\right) \mathcal{O}_j\left(x_3 + t_2 x_{34}, x_{34}\right) \mathcal{O}_5\left(x_5\right)\right\rangle \quad (2.94)$$

Using the result for the three-point function of spinning operators, equation (2.60), we can write

$$\left\langle \mathcal{O}_k\left(x_1 + t_1 x_{12}, x_{12}\right) \mathcal{O}_j\left(x_3 + t_2 x_{34}, x_{34}\right) \mathcal{O}_5\left(x_5\right)\right\rangle = = \sum_{l=0}^{\min\{J_k, J_j\}} \frac{c_{kj5}^l V_{k,j5}^{J_k - l} V_{j,5k}^{J_j - l} H_{kj}^l}{\left(x_{kj}^2\right)^{\frac{h_j + h_k - h_5}{2}} \left(x_{k5}^2\right)^{\frac{h_k + h_5 - h_j}{2}} \left(x_{j5}^2\right)^{\frac{h_j + h_5 - h_k}{2}}} \quad (2.95)$$

where

$$x_j = x_1 + t_1 x_{12}, x_{12}, \qquad x_k = x_3 + t_2 x_{34}, x_{34} \quad (2.96)$$

Therefore, we can express the correlator G as

$$G = \frac{1}{\left(x_{12}^2\right)^{\Delta_1} \left(x_{34}^2\right)^{\Delta_3}} \left(\frac{x_{13}^2}{x_{15}^2 x_{35}^2}\right)^{\frac{\Delta_5}{2}} \sum_{k,j} \sum_{l=0}^{\min\{J_k, J_j\}} p_{kjl} G_{kjl}\left(u_i\right) \quad (2.97)$$

where $p_{kjl} = c_{12k} c_{34j} c_{kj5}^l$, and G_{kjl} are the five-point light-cone conformal blocks

$$G_{kjl} = \left(x_{12}^2\right)^{\frac{\tau_k}{2}} \left(x_{34}^2\right)^{\frac{\tau_j}{2}} \left(\frac{x_{15}^2 x_{35}^2}{x_{13}^2}\right)^{\frac{\Delta_5}{2}}$$
$$\int_0^1 [dt_1] \int_0^1 [dt_2] \frac{V_{k,j5}^{J_k-l} V_{j,5k}^{J_j-l} H_{kj}^l}{\left(x_{kj}^2\right)^{\frac{h_j+h_k-h_5}{2}} \left(x_{k5}^2\right)^{\frac{h_k+h_5-h_j}{2}} \left(x_{j5}^2\right)^{\frac{h_j+h_5-h_k}{2}}} \tag{2.98}$$

2.6 Bootstrap equations

Now, if we consider identical operators:

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \tag{2.99}$$

and perform the OPE between $\phi(x_1)$ and $\phi(x_2)$, then this function can be written as:

$$\frac{1}{\left(x_{12}^2 x_{34}^2\right)^{\Delta_\phi}} \sum_k c_{12k} c_{34k} G_{\Delta_k, J_k}(u,v) \tag{2.100}$$

We could, however, have chosen to perform the OPE between $\phi(x_1)$ and $\phi(x_3)$ instead, and we would necessarily obtain the same result. This leads to the equality

$$\sum_k c_{12k} c_{34k} G_{\Delta_k, J_k}(u,v) = u^{\Delta_\phi} \sum_k c_{13k} c_{24k} G_{\Delta_k, J_k}\left(\frac{1}{u}, \frac{v}{u}\right) \tag{2.101}$$

In the same way, performing the OPE between $\phi(x_1)$ and $\phi(x_4)$ must also yield the same result. This leads to the equality

$$\sum_k c_{12k} c_{34k} G_{\Delta_k, J_k}(u,v) = \left(\frac{u}{v}\right)^{\Delta_\phi} \sum_k c_{14k} c_{23k} G_{\Delta_k, J_k}(v,u) \tag{2.102}$$

These are the bootstrap equations. They follow from the symmetry only, and can be used to find restrictions on OPE data of possible conformal theories. We will not be analysing these equations in this thesis, but an introduction to this topic can be found, for example, in [5].

Chapter 3

How to calculate integrals

In order to verify the duality (1.7) we need to calculate the correlation functions in the null-polygon limit, as well as the corresponding Wilson loops. When calculating correlators in perturbation theory, we frequently encounter spacetime integrals of the form:

$$\frac{1}{\pi^{\frac{dl}{2}}}\int\frac{d^dx_{p+1}\cdots d^dx_{p+l}}{D} \tag{3.1}$$

where D is a product of powers x_{ij}^2, where $i = 1, \ldots, p+l$ and $j = p+1, \ldots, p+l$. Therefore, we are interested in calculating these kinds of integrals. In this section we will explain how to do this using the method of asymptotic expansions. For $p = 4$, $l = 2$, the most general integral of the form (3.1) would be:

$$\frac{1}{\pi^d}\int\frac{d^dx_5d^dx_6}{\left(x_{15}^2\right)^{a_{15}}\left(x_{25}^2\right)^{a_{25}}\left(x_{35}^2\right)^{a_{35}}\left(x_{45}^2\right)^{a_{45}}\left(x_{16}^2\right)^{a_{16}}\left(x_{26}^2\right)^{a_{26}}\left(x_{36}^2\right)^{a_{36}}\left(x_{46}^2\right)^{a_{46}}\left(x_{56}^2\right)^{a_{56}}} \tag{3.2}$$

Because $\mathcal{N} = 4$ SYM is a CFT, the integrals we will encounter must be conformally invariant. This means that the integrand must have weight d in each integration point, i.e.

$$\sum_{\substack{i=1\\ i\neq j}}^{p+l} a_{ij} = d, \quad j = p+1, \ldots p+l \tag{3.3}$$

Consider the following integral for $d = 4$ spacetime dimensions:

$$I = \frac{1}{\pi^2}\int\frac{d^4x_5}{x_{15}^2x_{25}^2x_{35}^2x_{45}^2} \tag{3.4}$$

We can use translations to send $x_1 \to 0$. Then we use a special conformal transformation to send $x_4 \to \infty$. We can also use rotations to put x_3 on a coordinate axis, and put x_2 in a coordinate plane. Finally, with a dilatation, we can put $x_3^2 = 1$. This simplifies the integral (3.4):

$$\frac{1}{\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \to \frac{1}{\pi^2}\int \frac{d^4x_5}{x_5^2 x_{25}^2 x_{35}^2} \tag{3.5}$$

These transformations also simplify the cross-ratios u and v:

$$u \to x_2^2, \quad v \to x_{23}^2 \tag{3.6}$$

We will now explain the method of asymptotic expansions and give an example by calculating the integral I.

3.1 Asymptotic Expansions

Each of the integrals in $x_i,\ i \geq 5$ is taken over the whole space. However, there are two special regions of integration when one of the external points is "small" compared to the others (say, $x_2 \ll x_3$, for example). We will make this notion of smallness precise in a moment. In region 1 we have $x_i \sim x_2$, which implies $x_i \ll x_3$, and in region 2 we have $x_i \sim x_3$, which implies $x_i \gg x_2$. We can expand some of the propagators in the integrals in these regions, which simplifies the analysis. In region 1 we can expand:

$$\frac{1}{\left(x_{3i}^2\right)^c} = \sum_{n=0}^{\infty} \binom{-c}{n} \frac{\left(x_i^2 - 2x_3 \cdot x_i\right)^n}{\left(x_3^2\right)^{c+n}} \tag{3.7}$$

Likewise, in region 2 we have the expansion:

$$\frac{1}{\left(x_{2i}^2\right)^c} = \sum_{n=0}^{\infty} \binom{-c}{n} \frac{\left(x_2^2 - 2x_2 \cdot x_i\right)^n}{\left(x_i^2\right)^{c+n}} \tag{3.8}$$

There are two ways that the vector x_2 can be small compared to x_3. The first is $|x_2^\mu| \ll |x_3^\mu|$ for all μ. This happens if we take the Euclidean OPE limit $x_2^\mu \to 0$. If this is true, then powers of x_2^2 and $x_2 \cdot x_3$ will both be subleading. The other way x_2 can be small is $x_2^2 \ll x_3^2$, which corresponds to the Lorentzian OPE limit $x_2^2 \to 0$. In this case, powers of x_2^2 will be subleading, but the same will not happen for powers of $x_2 \cdot x_3$. Therefore, in the Lorentzian limit, higher powers of n in (3.7,3.8) will not necessarily be subleading. In practice, however, one can expand in powers of $x_2 \cdot x_3$ as well. Now, each of the expansions (3.7,3.8) has a finite radius of convergence, so divergences will appear if we integrate the sums over the entire space. However, these divergences will cancel out if we sum over all possible regions. The usefulness in this procedure lies in the fact that, by doing these expansions, we are expresing an n-point integral as a sum of a 2-point integral and an $(n-1)$-point integral, which are easier to compute. Consider, for example, expanding the integral (3.5) in region 1, using (3.7):

$$I^{(1)} = \frac{1}{\pi^{d/2}} \sum_{n=0}^{\infty} \int \frac{d^d x_5 \left(2x_3 \cdot x_5 - x_5^2\right)^n}{x_5^2 x_{25}^2 \left(x_3^2\right)^{n+1}} \tag{3.9}$$

Because this integral diverges in 4 dimensions, we integrate in $d = 4 - 2\epsilon$ dimensions. For simplicity, we can truncate the sum at $n = 0$, keeping in mind that there will be corrections in x_2^2 and $x_2 \cdot x_3$:

$$I^{(1)} = \frac{1}{\pi^{d/2} x_3^2} \int \frac{d^d x_5}{x_5^2 x_{25}^2} + \mathcal{O}\left(x_2 \cdot x_3\right) + \mathcal{O}\left(x_2^2\right) \tag{3.10}$$

We can see that this integral no longer depends on x_{35}, which is a considerable simplification. In general, going to higher orders in the expansion will not add dependences in the denominator. It will, at most, introduce powers of $x_2 \cdot x_3$ in the numerator, which we will handle next. However, let's first consider the effect of this expansion on an integral depending on more points. Consider a general five-point, one-loop integral

$$I = \frac{1}{\pi^{\frac{d}{2}}} \int \frac{d^d x_6}{\left(x_{16}^2\right)^{a_{16}} \left(x_{26}^2\right)^{a_{26}} \left(x_{36}^2\right)^{a_{36}} \left(x_{46}^2\right)^{a_{46}} \left(x_{56}^2\right)^{a_{56}}} \tag{3.11}$$

If the integral is conformally invariant, we can send $x_5 \to \infty$ and set $x_1 = (0, \ldots, 0)$:

$$I = \frac{1}{\pi^{\frac{d}{2}}} \int \frac{d^d x_6}{\left(x_6^2\right)^{a_{16}} \left(x_{26}^2\right)^{a_{26}} \left(x_{36}^2\right)^{a_{36}} \left(x_{46}^2\right)^{a_{46}}} \tag{3.12}$$

In the same way as before, we can expand in region 1:

$$I^{(1)} = \frac{1}{\left(x_3^2\right)^{a_{36}} \left(x_4^2\right)^{a_{46}}} \int \frac{d^d x_6}{\left(x_6^2\right)^{a_{16}} \left(x_{26}^2\right)^{a_{26}}} + \mathcal{O}\left(x_2 \cdot x_3\right) + \mathcal{O}\left(x_2 \cdot x_4\right) + \mathcal{O}\left(x_2^2\right) \tag{3.13}$$

We have actually removed two external point dependences in the integral. However, if we expand in region 2 (using (3.8)) we get

$$I^{(2)} = \int \frac{d^d x_6}{\left(x_6^2\right)^{a_{16}+a_{26}} \left(x_{36}^2\right)^{a_{36}} \left(x_{46}^2\right)^{a_{46}}} + \mathcal{O}\left(x_2 \cdot x_3\right) + \mathcal{O}\left(x_2 \cdot x_4\right) + \mathcal{O}\left(x_2^2\right) \tag{3.14}$$

In this region, the integral still depends on three external points. In general, in the region where all the integration points are large, an integral depending on n external points will originate integrals with $n-1$ external points. Likewise, when all integration points are small, there will only be integrals with two external points. In the mixed regions, where some integration points are large and some are small, we will get products of lower-loop integrals. The most complicated case we'll have to consider is a product of integrals depending on 2 and $(n-2)$ points, respectively.

This procedure can yield integrals which depend on fewer than two external points. These integrals evaluate to zero, as we can see by taking the general result for one-loop integrals, equation (3.38), and analytically continuing it to $a_1 = 0$ or $a_2 = 0$.

3.2 Tensor reduction

Because we can perform successive asymptotic expansions until we have a product of two-point integrals, we will only need to evaluate integrals of the form:

$$\int d^d x_5 \frac{x_5^{\mu_1} \cdots x_5^{\mu_J}}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2}}, \tag{3.15}$$

where x_α is an external vector. By rotation covariance, we can write this integral as:

$$\int d^d x_5 \frac{x_5^{\mu_1} \cdots x_5^{\mu_J}}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2}} = \sum_{k=0}^{J/2} I_k H_k^{\mu_1 \cdots \mu_J} \tag{3.16}$$

where

$$H_k^{\mu_1 \cdots \mu_J} = \delta^{\mu_1 \mu_2} \cdots \delta^{\mu_{2k-1} \mu_{2k}} x_\alpha^{\mu_{2k+1}} \cdots x_\alpha^{\mu_J} \tag{3.17}$$

and I_k is a scalar quantity, which we can determine by contracting both sides of equation (3.16) with the $H_k^{\mu_1 \cdots \mu_J}$. This gives a set of linear equations for the I_k which we can solve straightforwardly, for example using matrix methods. The answer will be given in terms of integrals with $x_\alpha \cdot x_5$ in the numerator, but we can rewrite this as $\frac{1}{2}\left(x_\alpha^2 + x_5^2 - x_{\alpha 5}^2\right)$ and pull x_α^2 out of the integrals, so that, in the end, we only need to evaluate integrals of the form

$$I_{a_1, a_2}(x_\alpha) = \frac{1}{\pi^{d/2}} \int \frac{d^d x_5}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2}}. \tag{3.18}$$

This procedure is known as "tensor reduction" of integrals, and it works similarly at higher loops. For the case where $a_1 = a_2 = 1$ in (3.15) there is a simple expression for the integral:

$$\int \frac{d^d x_5 \left(x_5 \cdot x_i\right)^J}{x_5^2 x_{25}^2} = \frac{(1)_J}{2^J \left(d-2\right)_J} C_J^{d/2-1} \left(\frac{x_2 \cdot x_i}{\left(x_2^2 x_i^2\right)^{1/2}} \right) \left(x_2^2 x_i^2\right)^{J/2} \int \frac{d^d x_5}{x_5^2 x_{25}^2} \tag{3.19}$$

where x_i is an external vector and $C_n^\nu(x)$ are Gegenbauer polynomials. If we are interested only in the leading order term as $x_2^2 \to 0$ or $x_i^2 \to 0$, we can take the limit in the Gegenbauer polynomial to obtain

$$\int \frac{d^d x_5 \left(x_5 \cdot x_i\right)^J}{x_5^2 x_{25}^2} \sim \frac{\left(\frac{d}{2}-1\right)_J}{\left(d-2\right)_J} \left(x_2 \cdot x_i\right)^J \int \frac{d^d x_5}{x_5^2 x_{25}^2} \tag{3.20}$$

3.3 IBP identities

3.3.1 1 loop

We can evaluate integrals of the form (3.18) for general a_1, a_2. However, for higher loops - i.e. $l \geq 2$ in (3.1) - this is still complicated. We can, however, simplify the problem even further by using integration-by-parts (IBP) identities. We can derive these identities by acting on the integrand of (3.18) with $\partial_5 \cdot x_\alpha$ and $\partial_5 \cdot x_5$. Acting with $\partial_5 \cdot x_\alpha$:

$$\begin{aligned}
&\frac{1}{\pi^{\frac{d}{2}}} \int d^d x_5 \, x_\alpha \cdot \partial_5 \left(\frac{1}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2}} \right) = \\
&\qquad = \frac{1}{\pi^{\frac{d}{2}}} \int d^d x_5 \left[\frac{-2a_1 x_\alpha \cdot x_5}{\left(x_5^2\right)^{a_1+1} \left(x_{\alpha 5}^2\right)^{a_2}} + \frac{2a_2 x_\alpha \cdot x_{\alpha 5}}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2+1}} \right] = \\
&\qquad = \frac{1}{\pi^{\frac{d}{2}}} \int d^d x_5 \left[\frac{a_1 (x_{\alpha 5}^2 - x_\alpha^2 - x_5^2)}{\left(x_5^2\right)^{a_1+1} \left(x_{\alpha 5}^2\right)^{a_2}} + \frac{a_2 (x_{\alpha 5}^2 - x_5^2 + x_\alpha^2)}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2+1}} \right] = \\
&\qquad = a_1 \left(I_{a_1+1, a_2-1}(x_\alpha) - x_\alpha^2 I_{a_1+1, a_2}(x_\alpha) - I_{a_1, a_2}(x_\alpha) \right) + \\
&\qquad\qquad + a_2 \left(I_{a_1, a_2}(x_\alpha) - I_{a_1-1, a_2+1}(x_\alpha) - x_\alpha^2 I_{a_1, a_2+1}(x_\alpha) \right)
\end{aligned}$$

Because the original integral is the integral of a total derivative and the boundary terms vanish at infinity, it equals zero. Using this fact, we are left with the identity:

$$\begin{aligned}
&a_1 \left(I_{a_1+1, a_2-1}(x_\alpha) - x_\alpha^2 I_{a_1+1, a_2}(x_\alpha) - I_{a_1, a_2}(x_\alpha) \right) + \\
&\qquad + a_2 \left(I_{a_1, a_2}(x_\alpha) - I_{a_1-1, a_2+1}(x_\alpha) - x_\alpha^2 I_{a_1, a_2+1}(x_\alpha) \right) = 0
\end{aligned}$$

Likewise, acting with $\partial_5 \cdot x_5 = \delta_{\mu\nu}\partial_5^\nu x_5^\mu$:

$$\frac{\delta_{\mu\nu}}{\pi^{\frac{d}{2}}} \int d^d x_5\, \partial_5^\nu \left(\frac{x_5^\mu}{\left(x_5^2\right)^{a_1} \left(x_{\alpha 5}^2\right)^{a_2}} \right) = \\ = \left(d - 2a_1 - a_2\right) I_{a_1,a_2}\left(x_\alpha\right) - a_2 I_{a_1-1,a_2+1}\left(x_\alpha\right) + a_2 x_\alpha^2 I_{a_1,a_2+1}\left(x_\alpha\right)$$

Again, the original integral equals zero, so we get the identity

$$\left(d - 2a_1 - a_2\right) I_{a_1,a_2}\left(x_\alpha\right) - a_2 I_{a_1-1,a_2+1}\left(x_\alpha\right) + a_2 x_\alpha^2 I_{a_1,a_2+1}\left(x_\alpha\right) = 0 \tag{3.21}$$

As discussed before, we also know that scaleless integrals vanish, which yield the boundary conditions:

$$I_{0,a_2} = I_{a_1,0} = 0 \tag{3.22}$$

Using these identities, we can greatly reduce the number of integrals we need to evaluate. Because these equations are linear in the integrals, we can use matrix methods to solve them and determine their values in terms of a smaller set of so-called master integrals. In practice, however, the number of equations can get quite large and is very tedious to do by hand, especially for higher loops. There are, however, some publicly available codes which we can use to find and solve the relations between integrals. Two such codes are LiteRed [9] and FIRE [10]. In this work we used LiteRed. At one loop, it tells us that there is only one master integral, which we choose to be $I_{1,1}$. This is only one possible choice for the master integral, because we could simply use the relations to rewrite everything in terms of another integral.

3.3.2 2 loops

At two loops, we can still perform asymptotic expansions and tensor reductions, so that we only need to evaluate scalar integrals depending on a single external vector. The most general form of these integrals is:

$$I_{a_1,a_2,a_3,a_4,a_5}\left(x_\alpha^2\right) = \frac{1}{\pi^d} \int \frac{d^d x_5 d^d x_6}{\left(x_5^2\right)^{a_1} \left(x_6^2\right)^{a_2} \left(x_{\alpha 5}^2\right)^{a_3} \left(x_{\alpha 6}^2\right)^{a_4} \left(x_{56}^2\right)^{a_5}}, \tag{3.23}$$

where x_α is an external vector. We can again generate the IBP identities by acting with $\partial_i \cdot x_j$ on the integrand, where $i = 5, 6$ and $j = \alpha, 5, 6$. This gives us 6 identities. The boundary conditions this time are determined by the fact that the integral is zero if two out of a_1, a_3, a_5

are zero, or two out of a_2, a_4, a_5 are zero. At two loops, LiteRed tells us that there are two master integrals. We will choose the integrals $I_{0,1,1,0,1}$ and $I_{1,1,1,1,0}$ as our masters.

3.4 Calculating master integrals

3.4.1 1 loop

As was previously said, there is only one master integral at 1 loop. However, it is just as easy to calculate the general one-loop scalar integral (3.18), so we will go through the general case here. We start by using the Schwinger parametrization. Using the fact that

$$\frac{1}{y^n} = \frac{1}{\Gamma(n)} \int_0^\infty du\ u^{n-1} e^{-uy} \tag{3.24}$$

we can rewrite the integral in expression (3.18) as:

$$\frac{1}{\pi^{\frac{d}{2}} \Gamma(a_1)\Gamma(a_2)} \int d^d x_5 \int_0^\infty du \int_0^\infty dv\ u^{a_1-1} v^{a_2-1} e^{-ux_5^2 - vx_{i5}^2} \tag{3.25}$$

We can now easily complete the square in the exponent and perform the Gaussian integral in position space, yielding:

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty du \int_0^\infty dv\ \frac{u^{a_1-1} v^{a_2-1}}{(u+v)^{\frac{d}{2}}} e^{-\frac{uv}{u+v} x_i^2} \tag{3.26}$$

We can simplify this expression further using the identity

$$1 = \int_0^\infty d\lambda\ \delta(\lambda - u - v) \tag{3.27}$$

Inserting this in expression (3.26), we get

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)} \int_0^\infty du \int_0^\infty dv \int_0^\infty d\lambda\ \delta(\lambda - u - v) \frac{u^{a_1-1} v^{a_2-1}}{(u+v)^{\frac{d}{2}}} e^{-\frac{uv}{u+v} x_i^2} \tag{3.28}$$

Rescaling the Schwinger parameters:

$$u \to \lambda u,\ \ v \to \lambda v \tag{3.29}$$

and using the delta function property

$$\delta(cx) = \frac{1}{|c|}\delta(x) \tag{3.30}$$

we get:

$$\frac{1}{\Gamma(a_1)\Gamma(a_2)}\int_0^\infty du \int_0^\infty dv \int_0^\infty d\lambda\ \lambda^{a_1+a_2-\frac{d}{2}-1}\delta\left(1-u-v\right)\frac{u^{a_1-1}v^{a_2-1}}{(u+v)^{\frac{d}{2}}}e^{-\lambda\frac{uv}{u+v}x_i^2} \tag{3.31}$$

We can now rescale $\lambda \to \left(\frac{uv}{u+v}x_i^2\right)^{-1}\lambda$ and perform the integral in λ:

$$\frac{\Gamma\left(a_1+a_2-\frac{d}{2}\right)}{\Gamma(a_1)\Gamma(a_2)}\left(x_i^2\right)^{a_1+a_2-\frac{d}{2}}\int_0^\infty du \int_0^\infty dv\ \left(\frac{uv}{u+v}\right)^{-a_1-a_2+\frac{d}{2}}\delta\left(1-u-v\right)\frac{u^{a_1-1}v^{a_2-1}}{(u+v)^{\frac{d}{2}}} \tag{3.32}$$

Finally, taking advantage of the delta to perform the integral in v, we get:

$$\frac{\Gamma\left(a_1+a_2-\frac{d}{2}\right)}{\Gamma(a_1)\Gamma(a_2)}\left(x_i^2\right)^{a_1+a_2-\frac{d}{2}}\int_0^1 du\ \left(u(1-u)\right)^{-a_1-a_2+\frac{d}{2}}u^{a_1-1}(1-u)^{a_2-1} \tag{3.33}$$

The integral in u now runs only form 0 to 1, because for $u > 1$we have $1-u < 0$ and therefore the integral in v vanishes in that region. Simplifying, we obtain:

$$\frac{\Gamma\left(a_1+a_2-\frac{d}{2}\right)}{\Gamma(a_1)\Gamma(a_2)}\left(x_i^2\right)^{-a_1-a_2+\frac{d}{2}}\int_0^1 du\ u^{-a_2+\frac{d}{2}-1}(1-u)^{-a_1+\frac{d}{2}-1} \tag{3.34}$$

We can use integration by parts to calculate integrals of the form

$$\int_0^1 du\ u^\alpha(1-u)^\beta \tag{3.35}$$

Integrating by parts we get:

$$\int_0^1 du\ u^\alpha(1-u)^\beta = \frac{\beta}{\alpha+1}\int du\ u^{\alpha+1}(1-u)^{\beta-1} = \frac{\beta}{\alpha+1}\int du\ u^{\alpha+1}(1-u)^{\beta-1} \tag{3.36}$$

We can use successive integrations by parts, and we get:

$$\int_0^1 du\ u^\alpha(1-u)^\beta = \frac{\beta(\beta-1)\cdots 1}{\alpha(\alpha+1)\cdots(\alpha+\beta+1)} = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \tag{3.37}$$

Using this result in expression (3.34), we get the final result

$$\frac{1}{\pi^{d/2}} \int d^d x_5 \frac{1}{\left(x_5^2\right)^{a_1} \left(x_{i5}^2\right)^{a_2}} = \left(x_i^2\right)^{-a_1-a_2+\frac{d}{2}} G\left(a_1, a_2\right) \tag{3.38}$$

where

$$G\left(a_1, a_2\right) = \frac{\Gamma\left(a_1 + a_2 - \frac{d}{2}\right) \Gamma\left(\frac{d}{2} - a_1\right) \Gamma\left(\frac{d}{2} - a_2\right)}{\Gamma(a_1)\Gamma(a_2)\Gamma\left(d - a_1 - a_2\right)} \tag{3.39}$$

Note that the RHS diverges for some values of a_1, a_2. To deal with this, we set $d = 4 - 2\epsilon$ and expand in powers of ϵ. The result will, naturally, still be divergent as $\epsilon \to 0$ - i.e, there will be poles in ϵ - but, when we sum over all the asymptotic regions, these poles will remarkably cancel out.

3.4.2 2 loops

At two loops, our master integrals are:

$$I_{0,1,1,0,1}\left(x_\alpha^2\right) = \frac{1}{\pi^d} \int \frac{d^d x_5 d^d x_6}{x_6^2 x_{\alpha 5}^2 x_{56}^2} \tag{3.40}$$

$$I_{1,1,1,1,0}\left(x_\alpha^2\right) = \frac{1}{\pi^d} \int \frac{d^d x_5 d^d x_6}{x_5^2 x_6^2 x_{\alpha 5}^2 x_{\alpha 6}^2} \tag{3.41}$$

Both of these can be calculated using the one-loop result (3.38). We start with the first one:

$$\frac{1}{\pi^d} \int \frac{d^d x_5 d^d x_6}{x_6^2 x_{\alpha 5}^2 x_{56}^2} = \frac{G\left(1,1\right)}{\pi^{\frac{d}{2}}} \int \frac{d^d x_5}{\left(x_5^2\right)^{2-\frac{d}{2}} x_{\alpha 5}^2} = \left(x_\alpha^2\right)^{d-3} G\left(1,1\right) G\left(2 - \frac{d}{2}, 1\right) \tag{3.42}$$

As for the second one:

$$\frac{1}{\pi^d} \int \frac{d^d x_5 d^d x_6}{x_5^2 x_6^2 x_{\alpha 5}^2 x_{\alpha 6}^2} = \left(x_\alpha^2\right)^{\frac{d}{2}-2} G\left(1,1\right) \frac{1}{\pi^{\frac{d}{2}}} \int \frac{d^d x_5}{x_5^2 x_{\alpha 5}^2} = \left(x_\alpha^2\right)^{d-4} G\left(1,1\right)^2 \tag{3.43}$$

So we have determined the master integrals at two loops. Together with the IBP identities, we can determine all two-loop integrals.

3.5 An example at one loop

We are finally ready to calculate the original integral (3.5). We include a prefactor x_3^2 for convenience:

$$I = \frac{x_3^2}{\pi^{\frac{d}{2}}} \int \frac{d^d x_5}{x_5^2 x_{25}^2 x_{35}^2} \tag{3.44}$$

Because our goal will be to test the duality (1.7) in the limit where separations between points are null, we only need to calculate the integral to leading order in x_2^2. The first step is to do asymptotic expansions in this limit. Since this is a one-loop integral, there are only two regions:

$$R_1 : \; x_5^2 \ll x_3^2, \qquad\qquad R_2 : \; x_5^2 \gg x_2^2 \tag{3.45}$$

We also need to truncate the sums (3.7) and (3.8) at some finite order in x_2. For simplicity, we will keep only the first order terms. This lets us rewrite I as:

$$I = I^{(1)} + I^{(2)} \tag{3.46}$$

where

$$\begin{aligned} I^{(1)} &= \frac{1}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{1}{\left(x_3^2\right)^k} \int \frac{d^d x_5 \; \left(2x_3 \cdot x_5 - x_5^2\right)^k}{x_5^2 x_{25}^2} \\ I^{(2)} &= \frac{x_3^2}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \int \frac{d^d x_5 \; \left(2x_2 \cdot x_5 - x_2^2\right)^k}{\left(x_5^2\right)^{k+2} x_{35}^2} \end{aligned}$$

To leading order in x_2^2, we can neglect the squares in the numerator:

$$\begin{aligned} I^{(1)} &\sim \frac{1}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{1}{\left(x_3^2\right)^k} \int \frac{d^d x_5 \; \left(2x_3 \cdot x_5\right)^k}{x_5^2 x_{25}^2} \\ I^{(2)} &\sim \frac{x_3^2}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \int \frac{d^d x_5 \; \left(2x_2 \cdot x_5\right)^k}{\left(x_5^2\right)^{k+2} x_{35}^2} \end{aligned}$$

We can use equation (3.19) to do the tensor reduction in region 1:

$$I^{(1)} \sim \frac{1}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{\left(\frac{d}{2}-1\right)_k}{\left(x_3^2\right)^k (d-2)_k} \left(2x_2 \cdot x_3\right)^k \int \frac{d^d x_5}{x_5^2 x_{25}^2} + \mathcal{O}\left(x_2^2\right) \tag{3.47}$$

Substituting $2x_2 \cdot x_3 \to x_3^2 - x_{23}^2$ in the numerator (we are neglecting powers of x_2^2), expanding, and doing the integral in x_5, we get:

$$I^{(1)} \sim \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} \frac{\left(\frac{d}{2}-1\right)_k}{(d-2)_k} \left(\frac{-x_{23}^2}{x_3^2}\right)^l \left(x_2^2\right)^{\frac{d}{2}-2} G(1,1) \tag{3.48}$$

For region 2, we can restore point x_1 and rewrite the expression as:

$$I^{(2)} \sim \frac{x_{13}^2}{\pi^{\frac{d}{2}}} \sum_{k=0}^{\infty} \frac{(-y \cdot \partial_1)^k}{(k+1)!} \int \frac{d^d x_5}{\left(x_{15}^2\right)^2 x_{35}^2} \Bigg|_{y=x_{12}} \tag{3.49}$$

Evaluating the integral, we get

$$I^{(2)} \sim x_{13}^2 \sum_{k=0}^{\infty} \frac{(-y \cdot \partial_1)^k}{(k+1)!} \left(x_{13}^2\right)^{\frac{d}{2}-3} G(1,2) \Bigg|_{y=x_{12}} \tag{3.50}$$

Acting with the derivatives on the power and restoring the point x_1 to zero, we get

$$I^{(2)} \sim \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{k-l} \left(\frac{d}{2}-k-2\right)_k}{\Gamma(k+2)} \left(x_3^2\right)^{\frac{d}{2}-2-l} \left(x_{23}^2\right)^l G(1,2) \tag{3.51}$$

In practice, these sums must be evaluated up to a given value of k, but since we have an expression for any k this cutoff is arbitrary. As an example, we can set $k=1$. Additionally, setting $d = 4 - 2\epsilon$ and expanding to order $\mathcal{O}\left(\epsilon^0\right)$, we obtain:

$$I^{(1)} \sim \left(1+\frac{Y}{2}\right)\frac{1}{\epsilon} + \left(2-\gamma_e-\log u-\log x_3^2\right) + Y\left(1-\frac{\gamma_e}{2}-\frac{1}{2}\log u-\frac{1}{2}\log x_3^2\right) + \mathcal{O}\left(Y^2\right)$$
$$I^{(2)} \sim -\left(1+\frac{Y}{2}\right)\frac{1}{\epsilon} + \left(\gamma_e+\log x_3^2\right) + Y\left(-\frac{1}{2}+\frac{\gamma_e}{2}+\frac{1}{2}\log x_3^2\right) + \mathcal{O}\left(Y^2\right)$$

where $Y = 1 - v$ and γ_e is Euler's constant. It is now apparent that when we sum over the two regions, the divergences in ϵ cancel out. The final result is:

$$I \sim 2 + \frac{Y}{2} - \left(1+\frac{Y}{2}\right)\log u + \mathcal{O}\left(Y^2\right) \tag{3.52}$$

The integral I is actually known analytically [11]:

$$I = \Phi^{(1)}(u,v)\,, \tag{3.53}$$

where the functions $\Phi^{(L)}(u,v)$ are the multi-ladder functions [12], which can be written in terms of $z, \bar{z}$ as

$$\Phi^{(L)}(u,v) = \frac{1}{z-\bar{z}} f^{(L)}(z,\bar{z}) \tag{3.54}$$

where

$$f^{(L)}(x,\bar{x}) = \sum_{k=0}^{L} \frac{(-1)^k (2L-k)!}{k!(L-k)!L!} \log^k(x\bar{x}) \left(\mathrm{Li}_{2L-k}(x) - \mathrm{Li}_{2L-k}(\bar{x})\right) \tag{3.55}$$

Our result agrees with the analytical result up to the given order.

3.6 An example at two loops

To calculate correlation functions up to two loops, we need to consider integrals with $l = 2$ in (3.1). In this section we give an example of one such integral:

$$I_2 = \frac{x_3^2}{\pi^d} \int \frac{d^d x_5 d^d x_6}{x_5^2 x_{25}^2 x_{56}^2 x_6^2 x_{36}^2} \tag{3.56}$$

At two loops, there are four asymptotic regions:

$$\begin{aligned} &R_1 : x_5^2, x_6^2 \ll x_3^2; \qquad &&R_2 : x_5^2 \ll x_3^2,\ x_6^2 \gg x_2^2 \\ &R_3 : x_5^2 \gg x_2^2,\ x_6^2 \ll x_3^2; \qquad &&R_4 : x_5^2, x_6^2 \gg x_2^2 \end{aligned}$$

This allows us to express the integral I_2 as a sum over regions:

$$I_2 = I_2^{(1)} + I_2^{(2)} + I_2^{(3)} + I_2^{(4)} \tag{3.57}$$

where

$$
\begin{gathered}
I_2^{(1)} = \frac{1}{\pi^d} \sum_{k=0}^{\infty} \frac{1}{\left(x_3^2\right)^k} \int \frac{d^d x_5 d^d x_6 \ \left(2x_3 \cdot x_6 - x_6^2\right)^k}{x_5^2 x_6^2 x_{25}^2 x_{56}^2} \\
I_2^{(2)} = \frac{x_3^2}{\pi^d} \sum_{k=0}^{\infty} \int \frac{d^d x_5 d^d x_6 \left(2x_6 \cdot x_5 - x_5^2\right)^k}{x_5^2 x_{25}^2 \left(x_6^2\right)^{k+2} x_{36}^2} \\
I_2^{(3)} = 0 \\
I_2^{(4)} = \frac{x_3^2}{\pi^d} \sum_{k=0}^{\infty} \int \frac{d^d x_5 d^d x_6 \left(2x_5 \cdot x_2 - x_2^2\right)^k}{\left(x_5^2\right)^{k+2} x_{56}^2 x_6^2 x_{36}^2}
\end{gathered}
$$

The asymptotic expansion in region R_3 yields only scaleless inegrals, so $I_3 = 0$. To leading order in x_2^2:

$$
\begin{gathered}
I_2^{(1)} \sim \frac{1}{\pi^d} \sum_{k=0}^{\infty} \frac{1}{\left(x_3^2\right)^k} \int \frac{d^d x_5 d^d x_6 \ \left(2x_3 \cdot x_6\right)^k}{x_5^2 x_6^2 x_{25}^2 x_{56}^2} \\
I_2^{(2)} \sim \frac{x_3^2}{\pi^d} \sum_{k=0}^{\infty} \int \frac{d^d x_5 d^d x_6 \left(2x_5 \cdot x_6\right)^k}{x_5^2 \left(x_6^2\right)^{k+2} x_{25}^2 x_{36}^2} \\
I_2^{(4)} \sim \frac{x_3^2}{\pi^d} \sum_{k=0}^{\infty} \int \frac{d^d x_5 d^d x_6 \left(2x_2 \cdot x_5\right)^k}{\left(x_5^2\right)^{k+2} x_6^2 x_{36}^2 x_{56}^2}
\end{gathered}
$$

This case is more complicated than the one-loop one. However, some tensor reduction formulas can be deduced for general k if we neglect traces. We tested the following patterns up to $k = 10$ in Mathematica:

$$
\int \frac{d^d x_5 d^d x_6 \ \left(x_3 \cdot x_6\right)^k}{x_5^2 x_6^2 x_{25}^2 x_{56}^2} \sim \frac{\left(x_2 \cdot x_3\right)^k (d-3)(d-2)\left(\frac{d}{2}\right)_{k-1}}{x_2^2 (d-4)(d+k-3)\left(\frac{3d}{2}-3\right)_{k-1}} \int \frac{d^d x_5 d^d x_6}{x_6^2 x_{25}^2 x_{56}^2} \tag{3.58}
$$

$$
\int \frac{d^d x_5 d^d x_6 \left(x_5 \cdot x_6\right)^k}{x_5^2 \left(x_6^2\right)^{k+2} x_{25}^2 x_{36}^2} \sim \frac{(d-6)(d-3)\left(4-\frac{d}{2}\right)_{k-1}\left(\frac{d}{2}\right)_{k-1}\left(x_2 \cdot x_3\right)^k}{8\,(3)_{k-1}(d-1)_{k-1}\left(x_3^2\right)^{k+1}} \int \frac{d^d x_5 d^d x_6}{x_5^2 x_6^2 x_{25}^2 x_{36}^2} \tag{3.59}
$$

$$\int \frac{d^d x_5 d^d x_6 \left(x_2 \cdot x_5\right)^k}{\left(x_5^2\right)^{k+2} x_6^2 x_{36}^2 x_{56}^2} \sim \frac{(d-5)(d-3)(3d-10)(3d-8)(6-d)_{k-1}\left(4-\frac{d}{2}\right)_{k-1}\left(x_2 \cdot x_3\right)^k}{2(d-8)(d-4)(3)_{k-1}\left(5-\frac{d}{2}\right)_{k-1}\left(x_3^2\right)^{k+2}} \int \frac{d^d x_5 d^d x_6}{x_6^2 x_{35}^2 x_{56}^2} \tag{3.60}$$

Using these results and the expression (3.42) and (3.43) for the 2-loop master integrals, we get the results

$$I_2^{(1)} \sim \sum_{k=0}^{\infty} \frac{2^k \left(x_2 \cdot x_3\right)^k (d-3)(d-2)\left(\frac{d}{2}\right)_{k-1}\left(x_2\right)^{d-4}}{\left(x_3^2\right)^k (d-4)(d+k-3)\left(\frac{3d}{2}-3\right)_{k-1}} G(1,1)\, G\left(2-\frac{d}{2},1\right)$$

$$I_2^{(2)} \sim \sum_{k=0}^{\infty} \frac{2^k (d-6)(d-3)\left(4-\frac{d}{2}\right)_{k-1}\left(\frac{d}{2}\right)_{k-1}\left(x_2^2\right)^{d/2-2}\left(x_2 \cdot x_3\right)^k}{8(3)_{k-1}(d-1)_{k-1}\left(x_3^2\right)^{k-d/2+2}} G(1,1)^2$$

$$I_2^{(4)} \sim \sum_{k=0}^{\infty} \frac{2^k (d-5)(d-3)(3d-10)(3d-8)(6-d)_{k-1}\left(4-\frac{d}{2}\right)_{k-1}\left(x_2 \cdot x_3\right)^k}{2(d-8)(d-4)(3)_{k-1}\left(5-\frac{d}{2}\right)_{k-1}\left(x_3^2\right)^{k-d+4}} G(1,1) G\left(2-\frac{d}{2},1\right)$$

These sums can now be calculated up to a given value of k. Plugging in $d = 4 - 2\epsilon$ and taking the limit $\epsilon \to 0$, we get, up to $k = 1$:

$$I_2 \sim 6 + \frac{3Y}{8} - \left(3 + \frac{3Y}{8}\right) \log u + \left(\frac{1}{2} + \frac{Y}{8}\right) \log^2 u + \mathcal{O}\left(Y^2\right) \tag{3.61}$$

The integral I_2 is also known analytically:

$$I_2 = \frac{1}{x_{12}^2 x_{34}^2} \Phi^{(2)}(u, v) \tag{3.62}$$

This result agrees with ours up to the given order.

3.7 A five-point example

Some of the integrals we will have to calculate will depend on five external points. In this section we will give an example of one such integral:

$$I = \frac{1}{\pi^d} \int \frac{d^d x_6 \; d^d x_7 \; x_{36}^2}{x_{16}^2 x_{26}^2 x_{27}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{56}^2 x_{67}^2} \tag{3.63}$$

Again, we can simplify the integral by sending a point to infinity. In this case, we choose x_5:

$$I = \frac{1}{\pi^d} \int \frac{d^d x_6 \; d^d x_7 \; x_{36}^2}{x_{16}^2 x_{26}^2 x_{27}^2 x_{37}^2 x_{46}^2 x_{47}^2 x_{67}^2} \tag{3.64}$$

We will start by doing an asymptotic expansion in the limit $x_{12}^2 \to 0$:

$$I = I^{(1)} + I^{(2)} + I^{(3)} + I^{(4)}, \tag{3.65}$$

where, to leading order in x_{12}^2:

$$I^{(1)} \sim \frac{1}{\pi^d} \sum_{k_i} \int \frac{d^d x_5 \; d^d x_6 \; \left(-2x_{13} \cdot x_{16}\right)^{k_1} \left(2x_{17} \cdot x_{13}\right)^{k_2} \left(2x_{17} \cdot x_{14}\right)^{k_3}}{\left(x_{13}^2\right)^{k_1+k_2} \left(x_{14}^2\right)^{k_3+2} x_{16}^2 x_{26}^2 x_{27}^2 x_{67}^2} \tag{3.66}$$

$$I^{(2)} \sim \frac{1}{\pi^d} \sum_{k_i} \int \frac{d^d x_6 \; d^d x_7 \; \left(-2x_{13} \cdot x_{16}\right)^{k_1} \left(2x_{17} \cdot x_{12}\right)^{k_2} \left(2x_{17} \cdot x_{16}\right)^{k_3}}{\left(x_{13}^2\right)^{k_1-1} x_{14}^2 x_{16}^2 \left(x_{17}^2\right)^{k_2+k_3+2} x_{26}^2 x_{37}^2 x_{47}^2} \tag{3.67}$$

$$I^{(3)} = 0 \tag{3.68}$$

$$I^{(4)} \sim \frac{1}{\pi^d} \sum_{k_i} \int \frac{d^d x_6 \; d^d x_7 \; x_{36}^2 \left(2x_{16} \cdot x_{12}\right)^{k_1} \left(2x_{17} \cdot x_{12}\right)^{k_2}}{\left(x_{16}^2\right)^{k_1+2} \left(x_{17}^2\right)^{k_2+1} x_{37}^2 x_{46}^2 x_{47}^2 x_{67}^2} \tag{3.69}$$

Now we will expand these leading integrals in the limit $x_{34}^2 \to 0$. We will not need to expand in region 1, because the integral in that region already depends on only two points. We must, however, take the limit $x_{34}^2 \to 0$ in that region at the end of the calculation as well, for consistency. The expansion for these integrals is

$$I^{(i)} = I^{(i,1)} + I^{(i,2)} + I^{(i,3)} + I^{(i,4)}, \quad i = 2, 3, 4 \tag{3.70}$$

The integrals $I^{(i,j)}$ are, to leading order in x_{12}^2 and x_{34}^2:

$$I^{(2,3)} \sim \sum_{k_i} \frac{\beta^{(2,3)}_{k_i}}{\pi^d} \int \frac{d^dx_6\ d^dx_7\ \left(-x_{13}\cdot x_{16}\right)^{k_1}\left(x_{17}\cdot x_{12}\right)^{k_2}\left(x_{17}\cdot x_{16}\right)^{k_3}\left(x_{37}\cdot x_{13}\right)^{k_4}}{x_{14}^2\left(x_{13}^2\right)^{k_1+k_2+k_3+k_4+1} x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2}$$

$$I^{(2,4)} \sim \sum_{k_i} \int \frac{d^dx_6\ d^dx_7\ \left(-x_{13}\cdot x_{16}\right)^{k_1}\left(x_{17}\cdot x_{12}\right)^{k_2}\left(x_{17}\cdot x_{16}\right)^{k_3}\left(-x_{34}\cdot x_{37}\right)^{k_4}}{\left(x_{13}^2\right)^{k_1-1} x_{14}^2 x_{16}^2 x_{26}^2 \left(x_{17}^2\right)^{k_2+k_3+2}\left(x_{37}^2\right)^{k_4+2}}$$

$$I^{(4,1)} \sim \sum_{k_i} \frac{\beta^{(4,1)}_{k_i}}{\pi^d} \int \frac{d^dx_6\ d^dx_7\ x_{36}^2\left(x_{16}\cdot x_{12}\right)^{k_1}\left(x_{17}\cdot x_{12}\right)^{k_2}\left(x_{13}\cdot x_{36}\right)^{k_3}\left(x_{13}\cdot x_{37}\right)^{k_4}}{\left(x_{13}^2\right)^{k_1+k_2+k_3+k_4+3} x_{37}^2 x_{46}^2 x_{47}^2 x_{67}^2}$$

$$I^{(4,3)} \sim \sum_{k_i} \frac{\beta^{(4,3)}_{k_i}}{\pi^d} \int \frac{d^dx_6\ d^dx_7\ \left(x_{16}\cdot x_{12}\right)^{k_1}\left(x_{17}\cdot x_{12}\right)^{k_2}\left(x_{13}\cdot x_{37}\right)^{k_3}\left(x_{36}\cdot x_{34}\right)^{k_4}\left(x_{36}\cdot x_{37}\right)^{k_5}}{\left(x_{16}^2\right)^{k_1+2}\left(x_{13}^2\right)^{k_2+k_3+1} x_{37}^2 \left(x_{36}^2\right)^{k_4+k_5+1} x_{47}^2}$$

$$I^{(4,4)} \sim \sum_{k_i} \int \frac{d^dx_6\ d^dx_7\ \left(x_{16}\cdot x_{12}\right)^{k_1}\left(x_{17}\cdot x_{12}\right)^{k_2}\left(x_{36}\cdot x_{34}\right)^{k_3}\left(x_{37}\cdot x_{34}\right)^{k_4}}{\left(x_{16}^2\right)^{k_1+2}\left(x_{17}^2\right)^{k_2+1}\left(x_{36}^2\right)^{k_3}\left(x_{37}^2\right)^{k_4+2} x_{67}^2}$$

where

$$\beta^{(2,3)}_{k_i} = 2^{\sum_i k_i}\binom{-k_2-k_3-2}{k_4}, \qquad \beta^{(4,1)}_{k_i} = 2^{\sum_i k_i}\binom{-k_1-2}{k_3}\binom{-k_2-1}{k_4}, \qquad \beta^{(4,3)}_{k_i} = 2^{\sum_i k_i}\binom{-k_2-1}{k_3}$$

and all other integrals are zero. Now we need to do tensor reductions on the scalar products, for which we need to derive new expressions, or do them "by hand", contracting with external vectors and solving the linear system of equations. For the purposes of this example, however, we will simply keep the leading term ($k_i = 0$).

$$I^{(1)} \sim \frac{1}{\pi^d\left(x_{14}^2\right)^2} \int \frac{d^dx_5\ d^dx_6}{x_{16}^2 x_{26}^2 x_{27}^2 x_{67}^2}$$

$$I^{(2,3)} \sim \frac{1}{\pi^d x_{14}^2 x_{13}^2} \int \frac{d^dx_6\ d^dx_7}{x_{16}^2 x_{26}^2 x_{37}^2 x_{47}^2}$$

$$I^{(2,4)} \sim \frac{x_{13}^2}{\pi^d x_{14}^2} \int \frac{d^dx_6\ d^dx_7}{x_{16}^2 x_{26}^2\left(x_{17}^2\right)^2\left(x_{37}^2\right)^2}$$

$$I^{(4,1)} \sim \frac{1}{\pi^d\left(x_{13}^2\right)^3} \int \frac{d^dx_6\ d^dx_7\ x_{36}^2}{x_{37}^2 x_{46}^2 x_{47}^2 x_{67}^2}$$

$$I^{(4,3)} \sim \frac{1}{\pi^d x_{13}^2} \int \frac{d^dx_6\ d^dx_7}{\left(x_{16}^2\right)^2 x_{37}^2 x_{36}^2 x_{47}^2}$$

$$I^{(4,4)} \sim \frac{1}{\pi^d} \int \frac{d^dx_6\ d^dx_7}{\left(x_{16}^2\right)^2 x_{17}^2\left(x_{37}^2\right)^2 x_{67}^2}$$

The integrals coming from $I^{(2,3)}$, $I^{(2,4)}$, and $I^{(4,3)}$ are products of one-loop integrals, which we know how to evaluate for general exponents. Using IBP identities, we can express the two-loop integrals as:

$$
\begin{aligned}
I^{(1)} &\sim \frac{(3d-8)}{\pi^d (d-4)\, x_{12}^2 \left(x_{14}^2\right)^2} \int \frac{d^d x_5\ d^d x_6}{x_{17}^2 x_{26}^2 x_{67}^2} \\
I^{(4,1)} &\sim \frac{(d-2)}{\pi^d (d-4) \left(x_{13}^2\right)^3} \int \frac{d^d x_6\ d^d x_7}{x_{37}^2 x_{46}^2 x_{67}^2} \\
I^{(4,4)} &\sim \frac{3(d-5)(d-3)(3d-10)(3d-8)}{\pi^d (d-6)(d-4) \left(x_{13}^2\right)^3} \int \frac{d^d x_6\ d^d x_7}{x_{17}^2 x_{36}^2 x_{67}^2}
\end{aligned}
$$

Evaluating the integrals:

$$
\begin{aligned}
I^{(1)} &\sim \frac{(3d-8)\left(x_{12}^2\right)^{d-4}}{(d-4)\left(x_{14}^2\right)^2} G(1,1)\, G\left(2-\frac{d}{2},1\right) \\
I^{(2,3)} &\sim \frac{\left(x_{12}^2\right)^{\frac{d}{2}-2}\left(x_{34}^2\right)^{\frac{d}{2}-2}}{x_{14}^2 x_{13}^2} G(1,1)^2 \\
I^{(2,4)} &\sim \frac{\left(x_{12}^2\right)^{\frac{d}{2}-2}\left(x_{13}^2\right)^{\frac{d}{2}-3}}{x_{14}^2} G(1,1)\, G(2,2) \\
I^{(4,1)} &\sim \frac{(d-2)\left(x_{34}^2\right)^{d-3}}{(d-4)\left(x_{13}^2\right)^3} G(1,1)\, G\left(2-\frac{d}{2},1\right) \\
I^{(4,3)} &\sim \left(x_{13}^2\right)^{\frac{d}{2}-4}\left(x_{12}^2\right)^{\frac{d}{2}-2} G(1,1)\, G(1,2) \\
I^{(4,4)} &\sim \frac{3(d-5)(d-3)(3d-10)(3d-8)\left(x_{13}^2\right)^{d-6}}{(d-6)(d-4)} G(1,1)\, G\left(2-\frac{d}{2},1\right)
\end{aligned}
$$

Setting $d = 4 - 2\epsilon$ and expanding around $\epsilon = 0$ we get:

$$
x_{13}^2 I \sim 4 - 3\log u_1 + \frac{1}{2}(\log u_1)^2 - 2\log u_3 + \log u_1 \log u_3, \tag{3.71}
$$

to leading order in u_i, $i = 1, 3$ and $1 - u_j$, $j = 2, 4, 5$.

Chapter 4

Extracting CFT data

In this section we will use the previously described methods to calculate the four-point function of half-BPS operators $\mathcal{O}$

$$G_4 = \langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\,\mathcal{O}(x_4)\rangle\,, \tag{4.1}$$

their five-point function

$$G_5 = \langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\,\mathcal{O}(x_4)\,(x_5)\rangle\,, \tag{4.2}$$

and the correlator of four half-BPS operators and one Lagrangian

$$G_{4;1} = \langle \mathcal{O}(x_1)\,\mathcal{O}(x_2)\,\mathcal{O}(x_3)\,\mathcal{O}(x_4)\,\mathcal{L}(x_5)\rangle \tag{4.3}$$

The functions G_4 and G_5 are dual via (1.7) to the null square and pentagon Wilson loops, respectively:

$$\lim_{x_{i,i+1}^2 \to 0} G_4(x_1,x_2,x_3,x_4) \propto W_4(x_1,x_2,x_3,x_4) \tag{4.4}$$

$$\lim_{x_{i,i+1}^2 \to 0} G_5(x_1,x_2,x_3,x_4,x_5) \propto W_5(x_1,x_2,x_3,x_4,x_5) \tag{4.5}$$

These limits are depicted in figure 4.1. The function $G_{4;1}$ is dual to the null square Wilson loop with a Lagrangian insertion, in the limit

$$\lim_{x_{12}^2,x_{23}^2,x_{34}^2,x_{14}^2 \to 0} G_{4;1}(x_1,x_2,x_3,x_4) \propto \frac{\langle W_4(x_1,x_2,x_3,x_4)\,\mathcal{L}(x_5)\rangle}{\langle W_4(x_1,x_2,x_3,x_4)\rangle}, \tag{4.6}$$

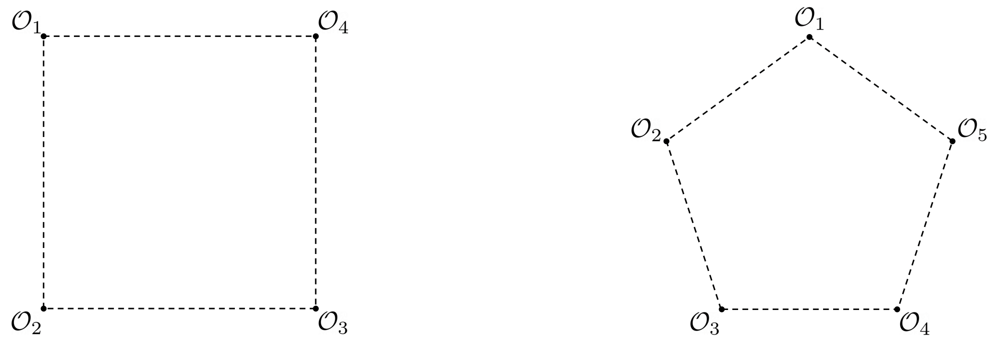

Figure 4.1: A diagram of the null square and null pentagon configurations. Dashed lines symbolize null separations between points.

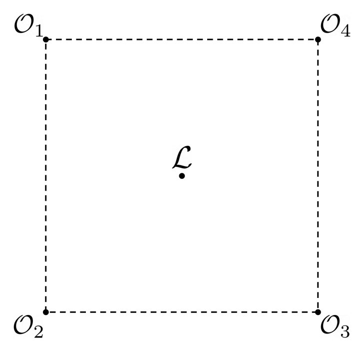

Figure 4.2: A diagram of the null square configuration with a Lagrangian inserted at a point x_5. Dashed lines symbolize null separations between points.

which is depicted in figure 4.2. These functions have expansions in the coupling, and the integrands of some of the loop contributions are known [13, 14]. We will calculate these loop contributions to the correlation functions and, by comparison with the conformal block expansion, we will extract the loop contributions to the CFT data.

4.1 20' operators

We will consider operators of the form

$$\mathcal{O}_{20'}^{IJ} = \mathrm{tr}\left(\Phi^I \Phi^J\right) - \frac{1}{6}\mathrm{tr}\left(\Phi^K \Phi^K\right)\delta^{IJ} \tag{4.7}$$

These operators belong to the 20' representation of $SO(6) \simeq SU(4)$. These operators are interesting because they are protected, meaning both their scaling dimension and the coefficient of their three-point function have no perturbative corrections. We introduce null polarization vectors to take care of the tensor structure:

$$\mathcal{O}(x,y) = y_I y_J \mathcal{O}^{IJ}_{20'}(x) = y_I y_J \mathrm{tr}\left(\Phi^I \Phi^J\right), \tag{4.8}$$

where $y^2 = 0$. We also introduce the following definitions:

$$Z = \Phi_1 + i\Phi_2, \qquad \bar{Z} = \Phi_1 - i\Phi_2 \tag{4.9}$$

$$X = \Phi_3 + i\Phi_4, \qquad \bar{X} = \Phi_3 - i\Phi_4 \tag{4.10}$$

$$Y = \Phi_5 + i\Phi_6, \qquad \bar{Y} = \Phi_5 - i\Phi_6 \tag{4.11}$$

Now consider, for example, the polarization vector

$$y_1 = (1, i, \alpha_1, i\ \alpha_1, 0, 0) \tag{4.12}$$

This is a null vector, since $y_1^2 = 0$. If we choose this polarization for the 20' operator (4.8), we get

$$\mathcal{O}(x_1, y_1) = \mathrm{tr}\left((Z + \alpha_1 X)(Z + \alpha_1 X)\right) \tag{4.13}$$

If we then take a derivative with respect to α_1 and then set $\alpha_1 = 0$, we get

$$\frac{\partial}{\partial \alpha_1}\mathcal{O}(x_1, y_1)\bigg|_{\alpha_1 = 0} = \left(\mathrm{tr}\,(XZ) + \mathrm{tr}\,(ZX)\right) \tag{4.14}$$

In this way, we can select specific external operators that will be useful later. We can now further define

$$y_2 = (1, i, \alpha_2, -i\ \alpha_2, 0, 0), \qquad y_3 = (1, -i, 0, 0, \alpha_3, i\ \alpha_3)$$
$$y_4 = (1, -i, 0, 0, \alpha_4, -i\ \alpha_4)$$

By considering a four-point function of operators with these polarizations and taking a derivative in each of the α_i at $\alpha_i = 0$ we select the operators

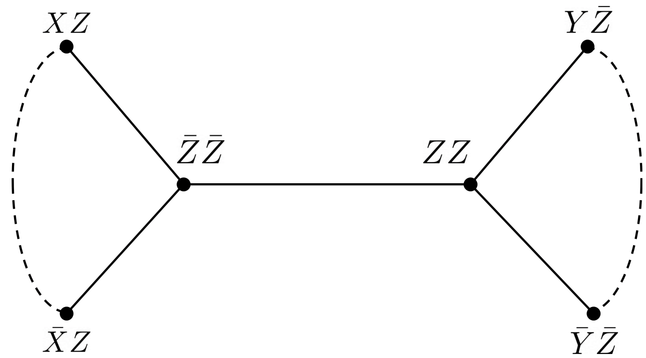

Figure 4.3: Wick contractions between the four operators selected with the polarization vectors. The dashed lines represent contractions between X and Y fields, while solid lines represent contractions between Z fields. In this example the exchanged operators are of the form $\mathrm{tr}(ZZ)$, but the general exchanged operator is more generally given by equation (4.15).

$$\mathcal{O}_1 = \mathrm{tr}\,(XZ) + \mathrm{tr}\,(ZX) \qquad \mathcal{O}_2 = \mathrm{tr}\left(\bar{X}Z\right) + \mathrm{tr}\left(Z\bar{X}\right)$$
$$\mathcal{O}_3 = \mathrm{tr}\left(Y\bar{Z}\right) + \mathrm{tr}\left(\bar{Z}Y\right) \qquad \mathcal{O}_4 = \mathrm{tr}\left(\bar{Y}\bar{Z}\right) + \mathrm{tr}\left(\bar{Z}\bar{Y}\right)$$

Because each of the fields X, Y, Z only has a non-vanishing contraction with its conjugate, the X fields in $\mathcal{O}_1$ and $\mathcal{O}_2$ will be contracted, as will the Y fields in $\mathcal{O}_3$ and $\mathcal{O}_4$. The leftover Z fields will then necessarily contract with the exchanged operator, as shown in figure 4.3.

In this way, by selecting the external operators, we are restricting the operators which appear in the OPE. With this choice, the exchanged operators in the (12) OPE must have the form

$$\mathcal{O}_{J,k}\,(x) = \mathrm{tr}\left(\hat{D}^k \bar{Z} \hat{D}^{J-k} \bar{Z}\right) \tag{4.15}$$

where

$$\hat{D} = z^\mu \partial_\mu, \tag{4.16}$$

and the operators exchanged in the (34) OPE will simply be their conjugates. These operators have twist $\tau = 2$ and spin J. However, they will not, in general, be conformal primaries. The exchanged primary operator for spin J and twist 2 is given by a linear combination[15]

$$\mathcal{O}_J\,(x) = \sum_{k=0}^{J} a_{k,J}\ \mathrm{tr}\left(\hat{D}^k \bar{Z} \hat{D}^{J-k} \bar{Z}\right) \tag{4.17}$$

where the coefficients $a_{k,J}$ are determined by

$$\sum_{k=0}^{J} a_{k,J} x^k y^{J-k} = (x+y)^J C_J^{\frac{d-3}{2}} \left(\frac{x-y}{x+y}\right) \tag{4.18}$$

4.2 Four-point integrands

We start by reviewing some results about the correlation function of four of these operators [13]:

$$G_4(x_i, y_i) = \sum_{l=0}^{\infty} a^l G_4^{(l)}(x_i, y_i) \tag{4.19}$$

In the last equality G_4 is expanded in powers of the t'Hooft coupling $a = g^2 N_c/(4\pi^2)$ and N_c is the number of colors in the theory - or, equivalently, the dimension of the matrices Φ^Iappearing in the Lagrangian.

The tree-level contribution is given by:

$$\begin{aligned} G_4^{(0)}(x_i, y_j) = {} & \frac{\left(N_c^2-1\right)^2}{4\left(4\pi^2\right)^4} \left[\left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{34}^2}{x_{34}^2}\right)^2 + \left(\frac{y_{13}^2}{x_{13}^2}\frac{y_{24}^2}{x_{24}^2}\right)^2 + \left(\frac{y_{41}^2}{x_{41}^2}\frac{y_{23}^2}{x_{23}^2}\right)^2 \right] \\ & + \frac{N_c^2-1}{\left(4\pi^2\right)^4} \left(\frac{y_{12}^2}{x_{12}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{41}^2}{x_{41}^2} + \frac{y_{12}^2}{x_{12}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{34}^2}{x_{34}^2}\frac{y_{13}^2}{x_{13}^2} + \frac{y_{13}^2}{x_{13}^2}\frac{y_{23}^2}{x_{23}^2}\frac{y_{24}^2}{x_{24}^2}\frac{y_{41}^2}{x_{41}^2} \right) \end{aligned} \tag{4.20}$$

Using superconformal symmetry, the loop corrections $G_4^{(l)}(x_i, y_i)$ take the form

$$G^{(l)}(x_i, y_j) = \frac{2(N_c^2-1)}{\left(4\pi^2\right)^4}\ R(x_i, y_j)\ F^{(l)}(x_i) \tag{4.21}$$

where the function $R(x_i, Y_J)$ is given by

$$\begin{aligned} R(x_i, y_j) = {} & \frac{y_{12}^2 y_{23}^2 y_{34}^2 y_{14}^2}{x_{12}^2 x_{23}^2 x_{34}^2 x_{14}^2} \left(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2\right) + \frac{y_{12}^2 y_{13}^2 y_{24}^2 y_{34}^2}{x_{12}^2 x_{13}^2 x_{24}^2 x_{34}^2} \left(x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2\right) \\ & + \frac{y_{13}^2 y_{14}^2 y_{23}^2 y_{24}^2}{x_{13}^2 x_{14}^2 x_{23}^2 x_{24}^2} \left(x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2\right) + \frac{y_{12}^4 y_{34}^4}{x_{12}^2 x_{34}^2} + \frac{y_{13}^4 y_{24}^4}{x_{13}^2 x_{24}^2} + \frac{y_{14}^4 y_{23}^4}{x_{14}^2 x_{23}^2} \end{aligned} \tag{4.22}$$

This allows us to extract all dependence on the polarization vectors Y_J in the multiplicative factor $R(x_i, Y_J)$. Additionally, this function has no dependence on the coupling, so the problem is significantly simplified. It is convenient to write the functions $F^{(l)}$ as

$$F^{(\ell)}\left(x_1,x_2,x_3,x_4\right)=\frac{x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2}{\ell!\left(-4\pi^2\right)^{\ell}}\int d^4x_5\dots d^4x_{4+\ell}f^{(\ell)}\left(x_1,\dots,x_{4+\ell}\right) \tag{4.23}$$

Because $F^{(l)}$ is conformally invariant, the integrand functions $f^{(l)}$ obey a few symmetries. It was shown in [13] that these symmetries are satisfied by functions of the form

$$f^{(\ell)}\left(x_1,\dots,x_{4+l}\right)=\frac{P^{(\ell)}\left(x_1,\dots,x_{4+\ell}\right)}{\prod_{1\le i<j\le 4+\ell}x_{ij}^2}, \tag{4.24}$$

where $P^{(\ell)}$ is a polynomial in x_{ij}^2, which must be symmetric under the permutations of all $4+\ell$ points and have appropriate conformal weights in each one. Using these properties, we can determine the functions $f^{(\ell)}$. The corresponding integrals can then be performed using asymptotic expansions. The polynomials $P^{(\ell)}$ up to three loops are given by

$$P^{(1)}=1 \tag{4.25}$$

$$P^{(2)}=x_{12}^2x_{34}^2x_{56}^2+\text{permutations} \tag{4.26}$$

$$P^{(3)}=\left(x_{12}^2x_{23}^2x_{34}^2x_{45}^2x_{15}^2\right)\left(x_{67}^2\right)^2+\text{permutations} \tag{4.27}$$

We can use these results to calculate the functions $F^{(\ell)}$ up to three loops.

4.3 Four 20' operators

So far, we have discussed how to obtain the integrands for the four-point function, as well as how to calculate the integrals themselves using asymptotic expansions. This gives us an expression for the four-point function in the OPE limit. By comparing with the conformal block expression, we can read off some OPE coefficients and anomalous dimensions. The conformal block expansion of the four-point function is given by

$$G_4=\frac{1}{\left(x_{12}^2x_{34}^2\right)^2}\sum_k c_{\mathcal{O}\mathcal{O}k}^2G_{\Delta_k,J_k}(z,\bar{z}) \tag{4.28}$$

where the conformal blocks G_{Δ_k,J_k} are given by (2.89). The loop dependence in this expression comes from the OPE coefficients $c_{\mathcal{O}\mathcal{O}k}$ and from the dimensions Δ_k of the exchanged operators. These can be expressed as

$$c_{ijk}(a) = \sum_{\ell=0}^{\infty} a^{\ell} c_{ijk}^{(\ell)} \tag{4.29}$$

$$\Delta_k(a) = \Delta_k^{(0)} + \sum_{\ell=1}^{\infty} a^{\ell} \gamma_k^{(\ell)} \tag{4.30}$$

Our goal now is to determine the coefficients $c_{ijk}^{(\ell)}$ and $\gamma_k^{(\ell)}$. Due to the structure of the conformal block expansion, it is more straightforward to determine the loop corrections of some product p_{ijk} of OPE coefficients. In the case of the four-point function, a natural choice would be $p_{\mathcal{O}\mathcal{O}k} = c_{\mathcal{O}\mathcal{O}k}^2$. Using the explicit expression for the four-point conformal block in four dimensions, equation (2.89), we can expand in the coupling a and obtain expressions for the loop corrections $G_4^{(\ell)}$ in terms of the CFT data. If we take the sequential light-cone limits $\bar{z} \to 0$, $z \to 0$, we can determine from the explicit expression that the conformal block behaves as

$$G_{\Delta,J}(z,\bar{z}) \sim \frac{1}{(-2)^J} (z\bar{z})^{\frac{\tau}{2}} z^J \,{}_2F_1\left(\frac{\Delta+J}{2}, \frac{\Delta+J}{2}, \Delta+J, z\right) \tag{4.31}$$

We can see that contributions from higher-twist operators are subleading. Therefore, to leading order in $\bar{z}$, we can consider only the operators with the lowest twist, $\tau = 2$. By equating the expressions obtained this way to the ones we obtained earlier by doing the integrals, we can determine the CFT data.

4.3.1 Tree level

We begin by studying the tree-level part of the correlator, which is given by equation (4.20). Using the special polarization detailed earlier, this expression simplifies to

$$G_4^{(0)} = \frac{\left(N_c^2 - 1\right) u(1+v)}{16\pi^8 \left(x_{12}^2 x_{34}^2\right)^2 v} \tag{4.32}$$

Taking the same limits in the conformal block, we can determine the CFT data at tree-level. Some results are displayed in Table 4.1.

These results fit the pattern

$$p_J^{(0)} = \frac{2^J (J)!^2}{(2J)!} \tag{4.33}$$

which agrees with the results in the literature ([16, 17], for example).

J	$p_J^{(0)}$	J	$p_J^{(0)}$
0	1	8	$\frac{128}{6435}$
2	$\frac{2}{3}$	10	$\frac{256}{46189}$
4	$\frac{8}{35}$	12	$\frac{1024}{676039}$
6	$\frac{16}{231}$	14	$\frac{2048}{5014575}$

Table 4.1: CFT data obtained from the four-point function at tree level. The coefficients p are defined by $p_J = \frac{8\pi^8 c^2_{\mathcal{O}\mathcal{O}J}}{N_c^2-1}$, where $c_{\mathcal{O}\mathcal{O}J}$ is the OPE coefficient associated with the exchange of an operator with twist $\tau = 2$ and spin J.

4.3.2 One loop

The one-loop integrand is given by

$$f^{(1)} = \frac{P^{(1)}(x_1, \ldots, x_5)}{\prod_{1\le i<j\le 5} x_{ij}^2} = \frac{1}{x_{12}^2 x_{13}^2 x_{14}^2 x_{15}^2 x_{23}^2 x_{24}^2 x_{25}^2 x_{34}^2 x_{35}^2 x_{45}^2} \tag{4.34}$$

Therefore, the function $F^{(1)}$ is given by

$$F^{(1)} = -g(1,2,3,4) = \frac{-1}{4\pi^2}\int \frac{d^4x_5}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{4.35}$$

We have already calculated the integral g using asymptotic expansions. Using the explicit form for the function $F^{(1)}$, we obtain the result

$$G^{(1)} = -\frac{N_c^2-1}{8\,(4\pi^2)} R(1,2,3,4)\,\Phi^{(1)}(z,\bar{z}) \tag{4.36}$$

Expanding the analytical result in powers of z and $\bar{z}$ and comparing it with the conformal block expansion, we can extract the OPE data at one loop. This data can be found in Table 4.2 for twist-two operators up to spin 14. In principle, however, since we have an explicit expression, we could determine the data up to arbitrary spin and higher twist using this method.

These results are also in agreement with [16].

4.3.3 Two loops

At two loops, the integrand is given by

J	$p_J^{(1)}$	$\gamma_J^{(1)}$	J	$p_J^{(1)}$	$\gamma_J^{(1)}$
0	0	0	8	$-\frac{115040456}{1127251125}$	$\frac{761}{140}$
2	-2	3	10	$-\frac{53014438928}{1749495609225}$	$\frac{7381}{1260}$
4	$-\frac{410}{441}$	$\frac{25}{6}$	12	$-\frac{40330200644864}{4638368350036575}$	$\frac{86021}{13860}$
6	$-\frac{8848}{27225}$	$\frac{49}{10}$	14	$-\frac{2764313720840484608}{1133832577565190628125}$	$\frac{1171733}{180180}$

Table 4.2: CFT data obtained from the four-point function at one loop. The coefficients p are defined by $p_J = \frac{8\pi^8 c^2_{\mathcal{O}\mathcal{O}J}}{N_c^2-1}$, where $c_{\mathcal{O}\mathcal{O}J}$ is the OPE coefficient associated with the exchange of an operator with twist $\tau = 2$ and spin J.

$$f^{(2)} = \frac{P^{(2)}(x_1,\ldots,x_6)}{\prod_{1\le i<j\le 6} x_{ij}^2} = \frac{1}{x_{13}^2x_{14}^2x_{15}^2x_{16}^2x_{23}^2x_{24}^2x_{25}^2x_{26}^2x_{35}^2x_{36}^2x_{45}^2x_{46}^2} + \text{permutations} \tag{4.37}$$

Using asymptotic expansions, we can obtain the result

$$F^{(2)} = \frac{1}{4^2 x_{13}^2 x_{24}^2}\Big(28 - 16\log u + 3(\log u)^2 + 12\,\zeta(3)\Big) + \mathcal{O}(u) + \mathcal{O}(Y) \tag{4.38}$$

This function can also be determined analytically. We can write it as

$$F^{(2)} = \frac{1}{2}g(1,2,3,4)^2(x_{12}^2x_{34}^2 + x_{13}^2x_{24}^2 + x_{14}^2x_{23}^2) + 2(h(1,2;3,4) + h(1,3;2,4) + h(1,4;2,3)) \tag{4.39}$$

where $g(1,2,3,4)$ was previously defined and

$$h(1,2;3,4) = \frac{x_{34}^2}{(4\pi^2)^2}\int \frac{d^4x_5 d^4x_6}{(x_{15}^2x_{35}^2x_{45}^2)x_{56}^2(x_{26}^2x_{36}^2x_{46}^2)} = \frac{1}{16}\frac{1}{x_{12}^2x_{34}^2}\Phi^{(2)}(u,v) \tag{4.40}$$

Once again, the analytic expression agrees with our result up to the given order. By comparing with the conformal block expansion, we can determine the CFT data displayed in Table 4.3.

This data is also in agreement with the results in [16].

4.4 Five 20' operators

We now turn our attention to the five-point function of 20' operators:

J	$p_J^{(2)}$	$\gamma_J^{(2)}$
0	0	0
2	$7+3\zeta_3$	-3
4	$\frac{76393}{18522}+\frac{10}{7}\zeta_3$	$-\frac{925}{216}$
6	$\frac{880821373}{539055000}+\frac{28}{55}\zeta_3$	$-\frac{45619}{9000}$
8	$\frac{5944825782678337}{10663175654381250}+\frac{12176}{75075}\zeta_3$	$-\frac{138989861}{24696000}$
10	$\frac{17105079356593232665 9}{97189327195286303250 0}+\frac{21472}{440895}\zeta_3$	$-\frac{12120281899}{2000376000}$
12	$\frac{146151793649350085402 44231}{275810510299034275520051250}+\frac{846976}{60063465}\zeta_3$	$-\frac{17061829801679}{2662500456000}$
14	$\frac{566041205925631272638053216892969}{3662399288782125880446712981289062 5}+\frac{299963648}{75293843625}\zeta_3$	$-\frac{39197535449025593}{5849513501832000}$

Table 4.3: CFT data obtained from the four-point function at two loops. The coefficients p are defined by $p_J=\frac{8\pi^8 c^2_{\mathcal{O}\mathcal{O}J}}{N_c^2-1}$, where $c_{\mathcal{O}\mathcal{O}J}$ is the OPE coefficient associated with the exchange of an operator with twist $\tau=2$ and spin J.

$$G=\langle\mathcal{O}\left(x_1,y_1\right)\mathcal{O}\left(x_2,y_2\right)\mathcal{O}\left(x_3,y_3\right)\mathcal{O}\left(x_4,y_4\right)\mathcal{O}\left(x_5,y_5\right)\rangle \tag{4.41}$$

Just like the four-point function, we can decompose this correlator in conformal blocks. For five points, these are given by equation (2.98) with $\Delta_5=2$. Using this result, we can extract OPE coefficients.

4.4.1 Tree level

The tree-level contribution to this correlator is given by

$$G_5^{(0)}\propto\frac{y_{23}^2y_{24}^2y_{34}^2y_{15}^4}{x_{15}^4x_{23}^2x_{24}^2x_{34}^2}+\frac{y_{12}^2y_{23}^2y_{34}^2y_{45}^2y_{15}^2}{x_{12}^2x_{23}^2x_{34}^2x_{45}^2x_{15}^2}+\text{permutations} \tag{4.42}$$

where the permutations being summed over are all the different ways to arrange the points into the configurations displayed in figure 4.4.

Using the same special polarization as before for y_i, $i=1,\ldots,4$, this simplifies to

$$G_5^{(0)}=\frac{8u_1u_3x_{13}^2}{\left(x_{12}^2x_{34}^2\right)^2x_{15}^2x_{35}^2}\left[\omega_{1,2}\left(1+\frac{1}{u_2u_4}+u_5+\frac{u_5}{u_4}\right)+\omega_{3,4}\left(u_1+\frac{u_1}{u_2}\right)+\omega_{5,6}\left(\frac{u_3u_5}{u_4}+\frac{u_3u_5}{u_2u_4}\right)\right], \tag{4.43}$$

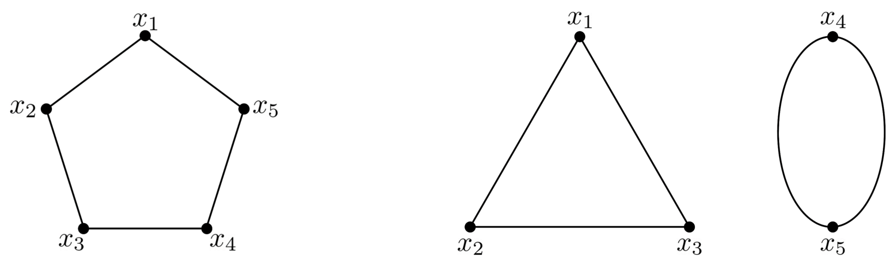

Figure 4.4: The configurations contributing to the tree-level part of the five-point function.

(J_1, J_2, ℓ)	$p^{(0)}_{J_1 J_2 \ell}$	(J_1, J_2, ℓ)	$p^{(0)}_{J_1 J_2 \ell}$
$(0,0,0)$	$\frac{1}{32}$	$(4,2,1)$	$\frac{4}{105}$
$(2,0,0)$	$\frac{1}{48}$	$(4,2,2)$	$\frac{1}{35}$
$(2,2,0)$	$\frac{1}{72}$	$(4,4,0)$	$\frac{2}{1225}$
$(2,2,1)$	$\frac{1}{18}$	$(4,4,1)$	$\frac{32}{1225}$
$(2,2,2)$	$\frac{1}{72}$	$(4,4,2)$	$\frac{72}{1225}$
$(4,0,0)$	$\frac{1}{140}$	$(4,4,3)$	$\frac{32}{1225}$
$(4,2,0)$	$\frac{1}{210}$	$(4,4,4)$	$\frac{2}{1225}$

Table 4.4: OPE coefficients obtained from the five-point function at tree level. The coefficients p are defined by $p_{J_1 J_2 \ell} = c_{\mathcal{O}\mathcal{O}J_1} c_{\mathcal{O}\mathcal{O}J_2} c^{\ell}_{J_1 J_2 \mathcal{O}}$.

where

$$\omega_{i,j} = (y_5^i)^2 + (y_5^j)^2 \tag{4.44}$$

and y_5^i is the i-th component of the polarization vector y_5. Because we will be taking the limits $x_{12}^2, x_{34}^2 \to 0$, we will keep only the leading terms in this limit:

$$G_5^{(0)} = \frac{8\omega_{1,2} u_1 u_3 x_{13}^2}{\left(x_{12}^2 x_{34}^2\right)^2 x_{15}^2 x_{35}^2} \left(1 + \frac{1}{u_2 u_4} + u_5 + \frac{u_5}{u_4}\right) \tag{4.45}$$

Because the dependence in y_5 is contained in the simple prefactor $\omega_{1,2}$, we can study the correlator without specifying a polarization for $\mathcal{O}\,(x_5)$. Using the conformal block decomposition, we can determine the tree-level OPE coefficients. Some of these coefficients are displayed in table 4.4.

We can now try to find a pattern in this sequence of OPE coefficients. In fact, the data we found matches the expression

$$p^{(0)}_{J_1 J_2 \ell} = \frac{2^{J_1+J_2-8}\,(J_1!)^3\,(J_2!)^3}{(\ell!)^2\,(2J_1)!\,(2J_2)!\,(J_1-\ell)!\,(J_2-\ell)!} \tag{4.46}$$

4.4.2 One loop

At one loop, the integrand is given, for example, in [18]:

$$G_5^{(1)} \propto (F_{1234,5} + F_{1324,5} + F_{1243,5}) + (5 \leftrightarrow 1) + (5 \leftrightarrow 2) + (5 \leftrightarrow 3) + (5 \leftrightarrow 4)\,, \tag{4.47}$$

where

$$F_{1234,5} = D_{1234}\left(\frac{y_{15}^2 y_{25}^2 y_{34}^2}{x_{15}^2 x_{25}^2 x_{34}^2} + \frac{y_{12}^2 y_{35}^2 y_{45}^2}{x_{12}^2 x_{35}^2 x_{45}^2}\right) \tag{4.48}$$

$$D_{1234} = \frac{1}{x_{12}^2 x_{34}^2}\Phi^{(1)}(u,v)\left(2u y_{13}^2 y_{24}^2 + (u-1-v)\,\frac{u}{v} y_{14}^2 y_{23}^2 + (v-1-u)\, y_{12}^2 x_{34}^2\right), \tag{4.49}$$

where u and v are the four-point cross-ratios and the remaining terms are obtained by permuting the indices. Note that, when applying the permutations, one must also permute the hidden indices in u and v. For example, when permuting $1 \leftrightarrow 3$:

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \to \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} = v, \tag{4.50}$$

and, similarly, $v \to u$. Because there are many such permutations, the definition of $\Phi^{(1)}$ in terms of $z, \bar{z}$, eq. (3.54) is clearly inconvenient to work with in this case. Although this function has an explicit expression, we can derive a simpler expression from the previous definition if we keep only the leading order in the first argument. First, note that taking the sequential limits $u \to 0, v \to 1$ is equivalent to taking the limits $\bar{z} \to 0, z \to 0$. Now, if we look at eq. (3.54) and take the limit $\bar{z} \to 0$, we get

$$\Phi^{(L)}(u,v) \sim \sum_{k=0}^{L} \frac{(-1)^k (2L-k)!}{k!(L-k)!L!} \log^k (z\bar{z})\, \mathrm{Li}_{2L-k}(z) \tag{4.51}$$

If we take the same limit in the cross-ratios, we get

$$u \to z\bar{z}, \quad v \to 1-z \tag{4.52}$$

(J_1, J_2, ℓ)	$p^{(1)}_{J_1 J_2 \ell}/p^{(0)}_{J_1 J_2 \ell}$	(J_1, J_2, ℓ)	$p^{(1)}_{J_1 J_2 \ell}/p^{(0)}_{J_1 J_2 \ell}$
$(0,0,0)$	0	$(4,2,1)$	$-\frac{11707}{1344}$
$(2,0,0)$	$-\frac{9}{2}$	$(4,2,2)$	$-\frac{1651}{336}$
$(2,2,0)$	-9	$(4,4,0)$	$-\frac{1025}{84}$
$(2,2,1)$	$-\frac{27}{4}$	$(4,4,1)$	$-\frac{14825}{1344}$
$(2,2,2)$	$-\frac{9}{4}$	$(4,4,2)$	$-\frac{1525}{168}$
$(4,0,0)$	$-\frac{1025}{168}$	$(4,4,3)$	$-\frac{3125}{672}$
$(4,2,0)$	$-\frac{1781}{168}$	$(4,4,4)$	$-\frac{17900}{63}$

J	$\gamma^{(1)}_J$
0	0
2	-3
4	$-\frac{925}{216}$

Table 4.5: OPE coefficients obtained from the five-point function at one loop. The coefficients p are defined by $p_{J_1 J_2 \ell} = c_{\mathcal{O}\mathcal{O}J_1} c_{\mathcal{O}\mathcal{O}J_2} c^{\ell}_{J_1 J_2 \mathcal{O}}$.

Therefore, we can rewrite the previous limit as

$$\Phi^{(L)}(u,v) \sim \sum_{k=0}^{L} \frac{(-1)^k (2L-k)!}{k!(L-k)!L!} \log^k(u) \operatorname{Li}_{2L-k}(1-v) \tag{4.53}$$

For the specific case $L=1$, we get

$$\Phi^{(1)}(u,v) \sim 2\operatorname{Li}_2(1-v) - \log(u)\log(1-v) \tag{4.54}$$

This expression is valid to leading order as the first argument approaches zero. Fortunately, the functions $\Phi^{(1)}(x,y)$ we will have to evaluate can all be brought to a form in which this limit is valid using the identities

$$\Phi^{(1)}(x,y) = \Phi^{(1)}(y,x) \tag{4.55}$$

$$\Phi^{(1)}(x,y) = \frac{1}{x}\Phi^{(1)}\left(\frac{1}{x}, \frac{y}{x}\right) \tag{4.56}$$

Now, using these results and the conformal block expression, we can determine some one-loop OPE coefficients and anomalous dimensions. These results are displayed in Table 4.5.

Note that the anomalous dimensions $\gamma^{(1)}_J$ have already been determined from the four-point function. Comparing the results, we can see that they are consistent. The OPE coefficients are in agreement with the closed expression found in [2].

4.5 Four 20' and one Lagrangian

We now move on to the correlator of four 20' operators and one Lagrangian

$$G_{4;1} = \langle \mathcal{O}(x_1, y_1)\, \mathcal{O}(x_2, y_2)\, \mathcal{O}(x_3, y_3)\, \mathcal{O}(x_4, y_4)\, \mathcal{L}(x_5) \rangle \tag{4.57}$$

This correlator is useful because, using the Lagrangian insertion procedure, one can show that:

$$G_4 = \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = \sum_{l=0}^{\infty} g^{2l} G_n^{(l)} \tag{4.58}$$

where

$$G_n^{(l)} = \int d^d x_5 \cdots d^d x_{n+l}\, \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \mathcal{L}_5 \cdots \mathcal{L}_{n+4} \rangle|_{\text{Born}} \tag{4.59}$$

and $\mathcal{O}_i \equiv \mathcal{O}(x_i, y_i)$, $\mathcal{L}_i \equiv \mathcal{L}(x_i)$. The subscript "Born" indicates that we are evaluating the correlator at the lowest nontrivial loop order. The integrands of the correlator $G_{4;1}$ are known for arbitrary positions x_i [14], and we want to calculate the corresponding integrals in the null-square limit. We will also expand the correlator in conformal blocks. The conformal blocks for this correlator are similar to the five-point function:

$$G_{4;1} = \frac{1}{\left(x_{12}^2\right)^2 \left(x_{34}^2\right)^2} \left(\frac{x_{13}^2}{x_{15}^2 x_{35}^2} \right)^2 \sum_{k,j,l} p_{kjl} G_{kjl}(u_i) \tag{4.60}$$

where the conformal blocks are again given by 2.98, but in this case $\Delta_5 = 4$.

4.5.1 Tree level

We now move on to calculating the loop contributions to the correlation function. We start with the tree level, which corresponds to the Born-level approximation in this case. Therefore, the tree-level contribution can be read off directly from the one-loop integrand of the four-point function using (4.58):

$$G_{4;1}^{(0)} = \frac{-2(N_c^2 - 1)}{\left(4\pi^2\right)^5} R(1, 2, 3, 4) \frac{1}{x_{15}^2 x_{25}^2 x_{35}^2 x_{45}^2} \tag{4.61}$$

Comparing this with the conformal block expansion, we obtain the CFT data in table 4.6.

(J_1, J_2, ℓ)	$p^{(0)}_{J_1 J_2 \ell}$	(J_1, J_2, ℓ)	$p^{(0)}_{J_1 J_2 \ell}$
$(0,0,0)$	0	$(2,2,2)$	$18a$
$(2,0,0)$	0	$(4,0,0)$	0
$(2,2,0)$	0	$(4,2,2)$	$5a$
$(2,2,1)$	0	$(4,4,4)$	$250a$

Table 4.6: OPE coefficients obtained from the correlator $G_{4;1}$ at tree level. The coefficients p are defined by $p_{J_1 J_2 \ell} = c_{\mathcal{O}\mathcal{O}J_1} c_{\mathcal{O}\mathcal{O}J_2} c^{\ell}_{J_1 J_2 \mathcal{L}}$. The data is determined up to a normalization constant a.

(J_1, J_2, ℓ)	$p^{(1)}_{J_1 J_2 \ell}$	(J_1, J_2, ℓ)	$p^{(1)}_{J_1 J_2 \ell}$
$(0,0,0)$	0	$(2,2,2)$	$90a$
$(2,0,0)$	0	$(4,0,0)$	0
$(2,2,0)$	0	$(4,2,2)$	$\frac{7855}{252}a$
$(2,2,1)$	0	$(4,4,4)$	$\frac{387725}{252}a$

J	$\gamma^{(1)}_J$
0	0
2	-3
4	$-\frac{925}{216}$

Table 4.7: CFT data obtained from the correlator $G_{4;1}$ at one loop. The coefficients p are defined by $p_{J_1 J_2 \ell} = c_{\mathcal{O}\mathcal{O}J_1} c_{\mathcal{O}\mathcal{O}J_2} c^{\ell}_{J_1 J_2 \mathcal{L}}$. The data is determined up to a normalization constant a.

4.5.2 One loop

At one loop, we only need to evaluate four-point integrals, which we already calculated previously. Evaluating the correlator to leading order in the cross-ratios u_1 and $u_3$and comparing with the conformal block expansion, we can extract the CFT data.

Because these coefficients are different from those in the five-point function, we unable to fix the constant a by comparison.

Chapter 5

Conclusion and outlook

In this thesis, we endeavoured to calculate correlation functions in $\mathcal{N} = 4$ SYM theory, in order to verify the duality with Wilson loops.

We started from the integrands of some of the loop contributions to these functions. We described the method of asymptotic expansions, which can be used to evaluate those integrals in the OPE limit. The integrals we encountered can be calculated exactly, however, and so this method will be more useful for higher loop order, or when we consider more points.

By comparing the correlation functions with their conformal block expansions, we extracted the loop contributions to the OPE coefficients and anomalous dimensions from the four-point function of 20' operators up to two loops, from the five-point function up to one loop, and from the correlator of four 20' and one Lagrangian up to one loop.

To verify the duality (1.7) using this data, we would also need to calculate the relevant Wilson loops, so this work is ongoing. Furthermore, as the procedure used here is easily extended to higher loop orders - as well as different correlators - interesting continuations of this work would be calculating CFT data for the six-point function of half-BPS operators for higher loops, and comparing the data with the dual Wilson loops.

Another possible avenue of research would be to study the properties of the loop contributions to the five- and six-point functions of operators of the form

$$\mathcal{O}_k = Y_{I_1} \cdots Y_{I_k} \mathrm{tr}\left(\Phi^{I_1} \cdots \Phi^{I_k}\right) \tag{5.1}$$

For four points and large k, it was found [19] that imposing that the correlator have certain properties at all loops, it is possible to determine it uniquely. It would be interesting to examine higher-point functions to try to find out if this is possible for more than four points.

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