ON THE SUM OF DIGITS IN THE BINARY EXPANSION OF PRODUCTS OF INTEGERS

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1. Introduction

Let n be a nonnegative integer. We denote the sum of digits in the binary expansion of n by $s(n)$. For instance, $s(27) = s((11011)_2) = 4$. We investigate integers a and b whose products ab have few nonzero digits, measured by the value $s(ab)$. More precisely, we discuss the Diophantine system with odd integer variables a, b,

$$s(ab) = k, \quad s(a) = \ell, \quad s(b) = m, \tag{1.1}$$

where $k, \ell, m \geq 2$ are fixed integers. In this paper we introduce the main results and the key idea of the paper [5].

The motivation of (1.1) is to investigate the finiteness of the odd positive integers n with

$$s(n) = s(n^2) = k, \tag{1.2}$$

where k is a fixed positive integer. Note that (1.1) is a generalization of (1.2). We first review some results related to (1.2). When we consider the relation between the sum of digits $s(P(n))$ and $s(n)$, where $P(X) \in \mathbb{Z}[X]$ is a polynomial with $P(\mathbb{N}) \subset \mathbb{N}$. One might ask to which extent the quantities $s(n)$ and $s(P(n))$ are independent. Stolarsky [7] investigated the extremal asymptotic behavior of $s(n^d)/s(n)$ $(n = 1, 2, \ldots)$, where d is an integer greater than 1. In particular, he showed that if $d = 2$, then

$$\liminf_{n\to\infty} \frac{s(n^2)}{s(n)} = 0. \tag{1.3}$$

Stolarsky conjectured that (1.3) can be generalized for general $d \geq 2$. Hare, Laishram, and Stoll [4] solved Stolarsky's conjecture as follows: Let $P(X) = a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ with $d \geq 2$ and $a_d > 0$. Then

$$\liminf_{n\to\infty} \frac{s(P(n))}{s(n)} = 0. \tag{1.4}$$

In particular, if d is an integer greater than 1, then $s(n^d)/s(n)$ takes arbitrarily small values. We note that Madritsch and Stoll [6] improved (1.4) as follows: Let $P_1(X), P_2(X) \in \mathbb{Z}[X]$ be non-constant polynomials with $P_1(\mathbb{N}), P_2(\mathbb{N}) \subset \mathbb{N}$. Suppose that $P_1(X)$ and $P_2(X)$ have distinct degrees. Then

$$\frac{s(P_1(n))}{s(P_2(n))} \quad (n = 1, 2, \ldots)$$

is dense in $[0, \infty)$.

Next, let us briefly discuss results on the positive integers n such that $s(n^d)$ takes fixed values. For any positive integer k, let

$$S(k) := \{n \geq 1 \mid n \text{ is odd}, \ s(n^2) = s(n) = k\}.$$

Hare, Laishram, and Stoll [3] proved that if $k \leq 8$, then $S(k)$ is a finite set. Moreover, they showed that if $k \geq 16$ or $k \in \{12, 13\}$, then $S(k)$ is an infinite set.

It is easily seen that if an odd integer $n \geq 1$ satisfies $s(n^2) = 2$, then $n = 3$. Szalay [8] showed that odd solutions $n \geq 1$ of $s(n^2) = 3$ are $n = 2^\ell + 1$ $(\ell \geq 1)$, $n = 7$, and $n = 23$. Corvaja and Zannier [2] showed that there exist only finitely many perfect powers with four nonzero binary digits. More precisely, there exist only finitely many pairs (n, d) with $n \geq 1$ and $d \geq 2$ with

$$s(n^d) = 4. \tag{1.5}$$

Bennett, Bugeaud and Mignotte [1] proved that if (1.5) holds, then $d \leq 4$.

In Section 2, we discuss the finiteness of the odd solutions of (1.1). In Section 3, we construct infinite families of solutions in the case of $k \geq 4$. The proofs of the theorems in Section 2 require technical combinatorial arguments. In this paper we introduce the main idea of the proofs in Section 4.

2. Main results

In this section we introduce and describe the main results of the paper [5]. First we introduce two finiteness results for products of integers with 2 or 3 binary digits.

THEOREM 2.1. Let ℓ and m be integers greater than 1. Let a and b be odd integers with $s(a) = \ell$, $s(b) = m$. If $s(ab) = 2$, then we have

$$ab < 2^{-4+2\ell m}.$$

THEOREM 2.2. Let ℓ and m be integers with $\ell, m \geq 2$ and $\max\{\ell, m\} \geq 3$. Let a and b be odd integers with $s(a) = \ell$, $s(b) = m$. If $s(ab) = 3$, then we have

$$ab < 2^{-13+4\ell m}.$$

On the other hand, if $k \geq 4$, then (1.1) has infinitely many odd solutions for arbitrarily large ℓ and m. The following result is a generalization of Theorem 2.3 in [5].

THEOREM 2.3. Let $k \geq 4$ be an integer. For any integer $L \geq 1$ there exist integers $\ell, m \geq L$ such that there exist infinitely many pairs (a, b) of odd integers with

$$s(ab) = k, \quad s(a) = \ell, \quad s(b) = m.$$

By Theorem 2.3, in particular, we cannot give an upper bound for ab when $s(ab) = 4$. On the other hand, we can give an upper bound for $\min\{a, b\}$.

THEOREM 2.4. Let ℓ and m be integers greater than 2. Let a and b be odd integers with $s(a) = \ell$, $s(b) = m$. If $s(ab) = 4$, then we have

$$\min\{a, b\} < 2^{18\ell m}.$$

3. The Diophantine system for $k \geq 4$

In this section we give the proof of Theorem 2.3. The proof relies on the fact that

$$s(2^n - 2^m) = s\bigl(2^m(2^{n-m-1} + 2^{n-m-2} + \cdots + 1)\bigr) = n - m, \tag{3.1}$$

where n, m are integers with $n > m \geq 0$. In this paper, we show Theorem 2.3 in the case of $k = 4$ and $k = 5$, which gives the essential idea and indicates how to proceed for the proof for general k.

Proof in the case of $k = 4$. Let

$$\begin{aligned} f(X) &:= X^2 - X + 1, \\ g(X) &:= (X+1)(X^6 - X^3 + 1) = X^7 + X^6 - X^4 - X^3 + X + 1. \end{aligned}$$

Note that

$$f(X)g(X) = (X+1)(X^2 - X + 1)(X^6 - X^3 + 1) = X^9 + 1. \tag{3.2}$$

Let L be an arbitrary positive integer. For a positive integer n, put

$$a_0 := f(2^n), \quad b_0 := g(2^n).$$

Taking a sufficiently large integer n, we get $s(a_0) \geq L$ and $s(b_0) \geq L$ by (3.1). Let $\ell := s(a_0) \geq L$ and $m := 2s(b_0) \geq L$. For any sufficiently large N, putting

$$a^{(N)} := a_0, \quad b^{(N)} := 2^N b_0 + b_0,$$

we get

$$s(a^{(N)}) = \ell, \quad s(b^{(N)}) = s(2^N b_0) + s(b_0) = 2s(b_0) = m.$$

Moreover, (3.2) implies

$$\begin{aligned} s(a^{(N)}b^{(N)}) &= s((2^N + 1)a_0 b_0) = s((2^N + 1)f(2^n)g(2^n)) \\ &= s(2^{N+9n} + 2^N + 2^{9n} + 1) = 4, \end{aligned}$$

which implies Theorem 2.3 in the case of $k = 4$. □

Proof in the case of $k = 5$. Let

$$\begin{aligned} f(X) &:= X^2 - X + 1, \\ g_1(X) &:= X^6 + X^5 - X^3 + X + 1, \\ g_2(X) &:= X^7 + X^6 - X^4 - X^3 + X + 1. \end{aligned}$$

Then we have

$$f(X)g_1(X) = X^8 + X^4 + 1, \quad f(X)g_2(X) = X^9 + 1. \tag{3.3}$$

For positive integers n and N, put

$$a^{(N)} := f(2^n), \quad b^{(N)} := 2^N g_1(2^n) + g_2(2^n).$$

Using the relation

$$a^{(N)}b^{(N)} = 2^N(2^{8n} + 2^{4n} + 1) + 2^{9n} + 1$$

by (3.3), we can prove Theorem 2.3 in the case of $k = 5$ in the same way as the case of $k = 4$. □

4. Proof of Theorem 2.1

We introduce a key lemma for the proof of Theorem 2.1.

LEMMA 4.1. Let Λ be a nonempty finite set. Let c_n be a nonnegative integer for each $n \in \Lambda$. Assume that

$$s\left(\sum_{n\in\Lambda} 2^{c_n}\right) = 1. \tag{4.1}$$

Then, for all $n, m \in \Lambda$ we have

$$|c_n - c_m| \leq \max\{0, -2 + \mathrm{Card}\,\Lambda\},$$

where Card denotes the cardinality.

Proof. We may assume that Card $\Lambda \geq 2$. Let

$$c' := \min\{c_n \mid n \in \Lambda\}, \quad c'' := \max\{c_n \mid n \in \Lambda\}.$$

We consider the carry generated in the calculation of $\sum_{n\in\Lambda} 2^{c_n}$. Note that carry propagation goes from the lower to the higher significant digits. By (4.1), the carry generated by $2^{c'}$ is transposed as far as to interact with $2^{c''}$. On the other hand, the number of carries generated by

$$\{2^{c_n} \mid n \in \Lambda, c_n < c''\}$$

is at most $-2 + \mathrm{Card}\,\Lambda$. Thus we obtain Lemma 4.1. □

In what follows, we give the proof of Theorem 2.1. The proofs of Theorems 2.2 and 2.4 follow a similar pattern but the investigation is much more involved since there are many more carries to deal with.

We denote odd integers a and b by

$$a = \sum_{i=0}^{\ell-1} 2^{a_i}, \quad b = \sum_{j=0}^{m-1} 2^{b_j},$$

where $a_{\ell-1} > \cdots > a_1 > a_0 = 0$ and $b_{m-1} > \cdots > b_1 > b_0 = 0$. By $s(ab) = 2$, we put $s(ab) = 2^x + 1$, where x is a positive integer. Setting

$$\Lambda := \{(i,j) \mid 0 \leq i \leq \ell - 1,\ 0 \leq j \leq m-1,\ (i,j) \neq (0,0)\},$$

we see that Card $\Lambda = \ell m - 1 \geq 3$ by $\ell, m \geq 2$ and that

$$\sum_{(i,j)\in\Lambda} 2^{a_i+b_j} = 2^x. \tag{4.2}$$

Considering the carry propagation of the left-hand side of (4.2), we get $a_1 = b_1$. Applying Lemma 4.1 with $c_{(i,j)} = a_i + b_j$, we see that

$$0 < (a_{\ell-1} + b_{m-1}) - (a_1 + 0) \leq -3 + \ell m, \tag{4.3}$$

$$0 \leq (a_{\ell-1} + b_{m-1}) - (a_1 + a_1) \leq -3 + \ell m. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$2a_1 \leq a_{\ell-1} + b_{m-1} \leq a_1 - 3 + \ell m,$$

and so

$$a_1 \leq -3 + \ell m.$$

Using (4.3) again, we obtain

$$a_{\ell-1} + b_{m-1} \leq a_1 - 3 + \ell m \leq -6 + 2\ell m.$$

Finally we deduce by $a < 2^{1+a_{\ell-1}}$ and $b < 2^{1+b_{m-1}}$ that

$$ab < 2^{2+a_{\ell-1}+b_{m-1}} \leq 2^{-4+2\ell m},$$

which implies Theorem 2.1.

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References

[1] M. Bennett, Y. Bugeaud, M. Mignotte, Perfect powers with few binary digits and related Diophantine problems, II, *Math. Proc. Cambridge Phil. Soc.* **153** (2012), 525–540.

[2] P. Corvaja, U. Zannier, Finiteness of odd perfect powers with four nonzero binary digits, *Ann. Inst. Fourier (Grenoble)* **63** (2013), 715–731.

[3] K.G. Hare, S. Laishram, T. Stoll, The sum of digits of n and n^2, *Int. J. Number Theory* **7** (2011), no. 7, 1737–1752.

[4] K.G. Hare, S. Laishram, T. Stoll, Stolarsky's conjecture and the sum of digits of polynomial values, *Proc. Amer. Math. Soc.* **139** (2011), no. 1, 39–49.

[5] H. Kaneko, T. Stoll, Products of integers with few nonzero digits, to appear (2022) in *Unif. Distrib. Theory.*

[6] M. Madritsch, T. Stoll, On simultaneous digital expansions of polynomial values, *Acta Math. Hungar.* **143** (2014), no. 1, 192–200.

[7] K.B. Stolarsky, The binary digits of a power, *Proc. Amer. Math. Soc.* **71** (1978), 1–5.

[8] L. Szalay, The equations $2^n \pm 2^m \pm 2^l = z^2$, *Indag. Math. (N.S.)* **13** (2002), no. 1, 131–142.

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