TWO INTEGRAL REPRESENTATIONS FOR APÉRY CONSTANT AND ITS APPLICATIONS TO MULTIPLE ZETA VALUES (Analytic Number Theory and Related Topics)

KOBAYASHI, MASATO

TWO INTEGRAL REPRESENTATIONS FOR APÉRY CONSTANT AND ITS APPLICATIONS TO MULTIPLE ZETA VALUES

MASATO KOBAYASHI

ABSTRACT. We generalize the proof of Basel problem by Boo Rim Choe (1987) to obtain two integral representations for Apéry constant. As applications, we also show integral representations for multiple values $\zeta(3, 2, \ldots, 2)$ and $t(3, 2, \ldots, 2)$.

CONTENTS

This article is based on the author's talk "Two integral representations for Apéry constant" for Analytic Number Theory and Related Topics, Kyoto RIMS (online) on October 12, 2021. There are certain overlaps with [9].

Date: December 20, 2021.
2020 *Mathematics Subject Classification.* Primary:11M32.
Key words and phrases. Apéry constant, arcsin, central binomial sums, multiple zeta values, Riemann zeta function, Wallis integral.

1. Introduction

1.1. **Basel Problem: arcsin and $\zeta(2)$.** Let ζ denote the Riemann zeta function. Boo Rim Choe (1987) [2] gave evaluation of the integral

$$\frac{\pi^2}{8} = \int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \frac{3}{4}\zeta(2) \tag{1}$$

which provides another proof of *Basel problem* $\zeta(2) = \frac{\pi^2}{6}$ (dating back to Euler around 1735). Actually, there is a counterpart of this:

$$\frac{2}{\pi}\int_0^1 \frac{\arcsin^2 x}{2!}\frac{dx}{\sqrt{1-x^2}} = \frac{1}{4}\zeta(2) \tag{2}$$

as the even part of $\zeta(2)$; we will explain why this equation involes $2/\pi$ and $\arcsin^2 x/2!$ later on. The aim of this article is to show analogous integral evaluation

$$\int_0^1 \frac{\arcsin x \arccos x}{x}dx = \frac{7}{8}\zeta(3), \tag{3}$$

$$\frac{2}{\pi}\int_0^1 \frac{\arcsin^2 x}{2!}\frac{\arccos x}{x}dx = \frac{1}{8}\zeta(3) \tag{4}$$

and discuss its applications to central binomial series, multiple values shown as Theorems 2.8, 2.13. We also make two Conjectures 2.19 and 3.3.

1.2. **Central binomial sums.** One of important topics in number theory is *central binomial sums*. Informally speaking, it is an infinite series involving $\binom{2n}{n}$. Lehmer [10] discussed two types of such sums

$$\text{I. } \sum_{n=0}^{+\infty} a_n \binom{2n}{n}, \quad \text{II. } \sum_{n=0}^{+\infty} a_n \binom{2n}{n}^{-1}.$$

Some examples are

$$\sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{8^n} = \sqrt{2}, \quad \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{10^n} = \sqrt{\frac{5}{3}}.$$

He also presented connection between such series and Maclaurin series of $\arcsin x$ and $\arcsin^2 x$. Other examples are

$$\sum_{n=1}^{\infty} \frac{1}{n^2\binom{2n}{n}} = \frac{1}{3}\zeta(2), \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3\binom{2n}{n}} = \frac{2}{5}\zeta(3),$$

as they arise in the work of Apèry [1] and van der Poorten [14] to prove irrationality of $\zeta(3)$.

Remark 1.1. For a nonnegative integer n, let

$$(2n-1)!! = (2n-1)(2n-3)\cdots 3\cdot 1,$$
$$(2n)!! = 2n(2n-2)\cdots 4\cdot 2.$$

We understand $(-1)!! = 0!! = 1$. Notice that the following relation holds.

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{(2n-1)!!}{(2n)!!}.$$

2. Main results

2.1. **Arcsin and $\zeta(3)$.** Toward the proofs of (3), (4), we first setup some notation for convenience.

Definition 2.1. Let $\mathbf{R}[[x]]$ denote the set of real power series. For $f(x) \in \mathbf{R}[[x]]$, define $W : \mathbf{R}[[x]] \to \mathbf{R}[[x]]$ by

$$Wf(x) = \int_0^1 f(xu)\frac{du}{\sqrt{1-u^2}}.$$

In particular,

$$Wf(x)\Big|_{x=1} = \int_0^1 f(u)\frac{du}{\sqrt{1-u^2}}.$$

Fact 2.2. Recall from calculus that if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ ($a_n \in \mathbf{R}$) is a convergent power series with the radius of convergence R, then so is $\displaystyle\int_0^x f(u)du$ and moreover it is given by termwise integration

$$\int_0^x f(u)\,du = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

In the sequel, we will use this result without mentioning explicitly.

Lemma 2.3. Let $f(x) \in \mathbf{R}[[x]]$.

[1] Moreover, suppose it is odd in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}}\, a_{2k+1}\, x^{2k+1}, \quad a_{2k+1} \in \mathbf{R}.$$

Then

$$Wf(x) = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1}\, x^{2k+1}.$$

[2] Moreover, suppose it is even in the form

$$f(x) = \sum_{k=0}^{\infty} \frac{2^{2k}}{\binom{2k}{k}}\, a_{2k}\, x^{2k}, \quad a_{2k} \in \mathbf{R}.$$

Then

$$Wf(x) = \frac{\pi}{2} \sum_{k=0}^{\infty} a_{2k}\, x^{2k}.$$

Thus, by the opertator W we can "kill" $\frac{\binom{2k}{k}}{2^{2k}}$ or $\frac{2^{2k}}{\binom{2k}{k}}$ from coefficients and instead $\frac{1}{2k+1}$ or $\frac{\pi}{2}$ shows up, respectively.

Proof. To show (1), recall that

$$\int_0^1 \frac{u^{2k+1}}{\sqrt{1-u^2}}\, du = \frac{(2k)!!}{(2k+1)!!} = \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{2k+1}.$$

Then

$$Wf(x) = \int_0^1 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} a_{2k+1} x^{2k+1} \frac{u^{2k+1}}{\sqrt{1-u^2}} du$$

$$= \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} a_{2k+1} x^{2k+1} \int_0^1 \frac{u^{2k+1}}{\sqrt{1-u^2}} du = \sum_{k=0}^{\infty} \frac{a_{2k+1}}{2k+1}\, x^{2k+1}.$$

We can verify (2) likewise with

$$\int_0^1 \frac{u^{2k}}{\sqrt{1-u^2}}\, du = \frac{\pi}{2} \frac{\binom{2k}{k}}{2^{2k}}.$$

□

Lemma 2.4. Let $f(x) \in \mathbf{R}[[x]]$. Suppose moreover $f(0) = 0$. Then

$$W\left(\int_0^x \frac{f(y)}{y}\, dy \right) = \int_0^1 \frac{f(xu)}{u} \arccos u\, du. \tag{5}$$

In particular, for $x = 1$, we have

$$W\left(\int_0^x \frac{f(y)}{y}\, dy \right)\Big|_{x=1} = \int_0^1 \frac{f(u)}{u} \arccos u\, du.$$

Proof. If $x = 0$, then both sides in (5) are 0. Suppose $x \neq 0$. Exchanging order of the double integral (Fubini's Theorem), we have

$$W\left(\int_0^x \frac{f(y)}{y}\, dy \right) = \int_0^1 \int_0^{xu} \frac{f(y)}{y}\, dy \frac{du}{\sqrt{1-u^2}}$$

$$= \int_0^x \int_{y/x}^1 \frac{f(y)}{y} \frac{1}{\sqrt{1-u^2}}\, du dy$$

$$= \int_0^x \frac{f(y)}{y} \arccos \frac{y}{x}\, dy$$

$$= \int_0^1 \frac{f(xu)}{u} \arccos u\, du.$$

□

2.2. **Multiple zeta, t-values.** As a natural generalization of Riemann zeta function, let us introduce the following sums.

Definition 2.5. For positive integers $i_1, \dots, i_k$ such that $i_1 \geq 2$, define the *multiple zeta value* and *multiple t-value* by

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}},$$

$$t(i_1, i_2, \dots, i_k) = \sum_{\substack{n_1 > n_2 > \cdots > n_k \\ n_j \text{ odd}}} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

Sometimes it is better to interpret $\zeta(i_1, \cdots, i_k)$ as

$$2^{i_1 + \cdots + i_k} \sum_{\substack{m_1 > m_2 > \cdots > m_k \\ m_j \text{ even}}} \frac{1}{(m_1)^{i_1} (m_2)^{i_2} \cdots (m_k)^{i_k}}$$

in contrast to t-values.

Fact 2.6. Let $\{m\}^n$ denote the sequence $(\underbrace{m, m, \dots, m}_{n})$. For a multi-index

$$\mathbf{i} = (a_1 + 1, \{1\}^{b_1 - 1}, a_2 + 1, \{1\}^{b_2 - 1}, \dots, a_k + 1, \{1\}^{b_k - 1}),$$

with integers $k, a_j, b_j \geq 1$, define its *dual*

$$\mathbf{i}^\dagger = (b_k + 1, \{1\}^{a_k - 1}, b_{k-1} + 1, \{1\}^{a_{k-1} - 1}, \dots, b_1 + 1, \{1\}^{a_1 - 1}).$$

Duality formula for multiple zeta values claims that $\zeta(\mathbf{i}) = \zeta(\mathbf{i}^\dagger)$ for all indices such that the first argument is at least 2. Historically, Drinfeld and Kontsevich found *iterated integral expressions* for multiple zeta values and proved the duality in 1990s. Afterward, Kaneko, Hoffman, Zagier and many other researchers developed the theory.

Example 2.7.

[1] the celebrated Euler-Goldbach theorem claims that

$$\zeta(2, 1) = \zeta(3).$$

We can derive this relation from iterated integral expressions

$$\zeta(2, 1) = \int_0^1 \frac{dx_3}{x_3} \int_0^{x_3} \frac{dx_2}{1 - x_2} \int_0^{x_2} \frac{dx_1}{1 - x_1}$$

and

$$\zeta(3) = \int_0^1 \frac{dy_3}{y_3} \int_0^{y_3} \frac{dy_2}{y_2} \int_0^{y_2} \frac{dy_1}{1 - y_1}$$

with changing variables by $y_j = 1 - x_{4-j}$.

[2] Observe that

$$t(2) = \frac{3}{4}\zeta(2), \quad t(3) = \frac{7}{8}\zeta(3)$$

and Hoffman [7, p.4] shows that

$$t(2,1) = -\frac{1}{2}t(3) + t(2)\log 2 \quad (\neq t(3)).$$

2.3. $\zeta(2,\dots,2)$ **and** $t(2,\dots,2)$**.**

Theorem 2.8. For $n \geq 1$,

$$\frac{\zeta(\{2\}^n)}{2^{2n}} = \frac{1}{(2n+1)!}\left(\frac{\pi}{2}\right)^{2n}, \quad t(\{2\}^n) = \frac{1}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}.$$

In fact, we can prove these by equating coefficients of x^{2n} in

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{\pi x}{2}\right)^{2n} = \frac{\sin\frac{\pi x}{2}}{\frac{\pi x}{2}} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(2n)^2}\right)$$

and

$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{\pi x}{2}\right)^{2n} = \cos\frac{\pi x}{2} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(2n-1)^2}\right).$$

However, we give a different proof here because it suggests the application to evaluation of $\zeta(3,2,\dots,2)$ and $t(3,2,\dots,2)$ in the next subsection. For this purpose, we need a lemma.

Lemma 2.9. For $n \geq 1$, $|x| \leqslant 1$, we have

$$\frac{\arcsin^{2n} x}{(2n)!} = \sum_{k>m_1>\cdots>m_{n-1}>0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{(2k)^2(2m_1)^2\cdots(2m_{n-1})^2} x^{2k},$$

$$\frac{\arcsin^{2n-1} x}{(2n-1)!} = \sum_{k>m_1>\cdots>m_{n-1}\geq 0} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)(2m_1+1)^2\cdots(2m_{n-1}+1)^2} x^{2k+1}.$$

Proof. This is a rephrasing of J.M.Borwein–Chamberland [4, (1.1)-(1.4)]. □

Proof of Theorem 2.8. Lemma 2.9 asserts that

$$\frac{\arcsin^{2n} x}{(2n)!} = \sum_{k>m_1>\cdots>m_{n-1}>0} \frac{2^{2k}}{\binom{2k}{k}} \frac{1}{(2k)^2(2m_1)^2\cdots(2m_{n-1})^2} x^{2k}$$

so that

$$W\left(\frac{\arcsin^{2n} x}{(2n)!}\right) = \frac{\pi}{2} \sum_{k>m_1>\cdots>m_{n-1}>0} \frac{1}{(2k)^2(2m_1)^2\cdots(2m_{n-1})^2} x^{2k}.$$

Let $x=1$. The left hand side becomes

$$\int_0^1 \frac{\arcsin^{2n} u}{(2n)!} \frac{du}{\sqrt{1-u^2}} = \left[\frac{\arcsin^{2n+1} u}{(2n+1)!} \right]_0^1 = \frac{1}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n+1}$$

while the right hand side turns to be $\frac{\pi}{2}\frac{\zeta(\{2\}^n)}{2^{2n}}$. Hence we proved

$$\frac{\zeta(\{2\}^n)}{2^{2n}} = \frac{1}{(2n+1)!} \left(\frac{\pi}{2}\right)^{2n}.$$

It is quite similar to show $t(\{2\}^n) = \frac{1}{(2n)!}\left(\frac{\pi}{2}\right)^{2n}$ using $\frac{\arcsin^{2n-1} x}{(2n-1)!}$ and the operator W.

2.4. $\zeta(3,2,\ldots,2)$ **and** $t(3,2,\ldots,2)$**.** We just found $\zeta(2,\ldots,2)$ and $t(2,\ldots,2)$ above. A natural subsequence is to evaluate $\zeta(3,2,\ldots,2)$ and $t(3,2,\ldots,2)$ via integrals on powers of arcsin again.

Definition 2.10. For $n \geq 1$, set

$$I\,(n) = \int_0^1 \frac{\arcsin^n x}{x}\, dx.$$

Example 2.11. Observe the first several values.

$$I(1) = \frac{\pi}{2} \log(2) \tag{6}$$

$$I(2) = \frac{\pi^2}{4} \log(2) - \frac{7}{8}\zeta(3) \tag{7}$$

$$I(3) = \frac{\pi^3}{8} \log(2) - \frac{9\pi}{16}\zeta(3) \tag{8}$$

$$I(4) = \frac{\pi^4}{16} \log(2) - \frac{9\pi^2}{16}\zeta(3) + \frac{93}{32}\zeta(5) \tag{9}$$

$$I(5) = \frac{\pi^5}{32} \log(2) - \frac{15\pi^3}{32}\zeta(3) + \frac{225\pi}{64}\zeta(5) \tag{10}$$

$$I(6) = \frac{\pi^6}{64} \log(2) - \frac{45\pi^4}{128}\zeta(3) + \frac{675\pi^2}{128}\zeta(5) - \frac{5715}{256}\zeta(7) \tag{11}$$

Remark 2.12.

[1] Indeed, Wolfram alpha [15] returns the algebraic expressions (6)-(11) for integrals

$$I(n) = \int_0^{\pi/2} y^n \cot y\, dy$$

while she outputs only numerical values for

$$\int_0^1 \frac{\arcsin^n x}{x}\, dx.$$

In fact, there is a precise formula for $I(n)$ giving a $\mathbf{Q}$-linear combination of $\log 2$ and single Riemann zeta values. For the sake of completeness, we discuss it here although we do not need it in the sequel. Let η denote the Dirichlet eta function, that is, $\eta(1) = \log 2$ and $\eta(j) = (1 - 2^{1-j})\,\zeta(j)$ $(j \geq 2)$. Then, there holds

$$I(2n+1) = \frac{(2n+1)!}{2^{2n+1}} \sum_{j=0}^{n} \frac{(-1)^j \pi^{2n+1-2j}}{(2n+1-2j)!}\, \eta(2j+1), \tag{12}$$

$$I(2n) = \frac{(2n)!}{2^{2n}} \left(\sum_{j=0}^{n-1} \frac{(-1)^j \pi^{2n-2j}}{(2n-2j)!}\, \eta(2j+1) + (-1)^n 2(1 - 2^{-2n-1})\,\zeta(2n+1) \right). \tag{13}$$

To see this, we remark that Buhler-Crandall [5, p.280] stated

$$\int_0^{1/2} x^n \cot(\pi x)\, dx$$

$$= \frac{n!}{2^n} \sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{(-1)^{(k-1)/2}}{\pi^k} \frac{\eta(k)}{(n-k+1)!} + \frac{1}{2}((-1)^n + 1) \frac{4n!(1 - 2^{-n-1})}{(2\pi)^{n+1}} \zeta(n+1).$$

However, the sign $\frac{1}{2}((-1)^n + 1)$ must be $\cos \frac{n\pi}{2}$ (for $n = 2$, the coefficient of $\zeta(3)$ is negative; see (7)). To correct this, set

$$J(n) := \int_0^{1/2} x^n \cot(\pi x)\, dx$$

$$= \frac{n!}{2^n} \left(\sum_{\substack{1 \leq k \leq n \\ k \text{ odd}}} \frac{(-1)^{(k-1)/2}}{\pi^k} \frac{\eta(k)}{(n-k+1)!} \right) + \cos \frac{n\pi}{2} \frac{4n!(1 - 2^{-n-1})}{(2\pi)^{n+1}} \zeta(n+1).$$

Then, with $y = \sin(\pi x)$, we find

$$J(n) = \frac{1}{\pi^{n+1}} \int_0^1 \frac{\arcsin^n(y)}{y}\, dy = \frac{1}{\pi^{n+1}} I(n).$$

Thus, $I(n) = \pi^{n+1} J(n)$. Writing down the cases for the index even and odd with $k = 2j+1$, we get (12), (13) and hence justified (6)-(11).

[2] We can also view $I(n)$ as a *log-sine integral*:

$$I(n) = \int_0^1 \frac{\arcsin^n x}{x} dx = \underbrace{\left[\log x (\arcsin^n x)\right]_0^1}_{0} - n \int_0^1 \log x \frac{\arcsin^{n-1} x}{\sqrt{1-x^2}} dx$$

$$= -n \int_0^{\pi/2} y^{n-1} \log(\sin y) dy.$$

See J.M.Borwein-Broadhurst-Kamnitzer [3] for relation of such integrals and central binomial series, for example.

Theorem 2.13. For $n \geq 0$,

$$\frac{\zeta(3, \{2\}^n)}{2^{2n+3}} = \frac{2}{\pi} \int_0^1 \frac{\arcsin^{2n+2} x}{(2n+2)!} \frac{\arccos x}{x} dx,$$

$$t(3, \{2\}^n) = \int_0^1 \frac{\arcsin^{2n+1} x}{(2n+1)!} \frac{\arccos x}{x} dx.$$

The proof is quite same to the one for Theorem 2.8. Hence we omit it.

Remark 2.14. Recently, Lupu [11], Murakami [12] and Zagier [18] obtained similar results. Each of their proofs is different from ours.

Example 2.15. For $n = 0$, we have

$$\int_0^1 \frac{\arcsin x \arccos x}{x} dx = \frac{7}{8}\zeta(3), \tag{14}$$

$$\frac{2}{\pi} \int_0^1 \frac{\arcsin x}{2!} \frac{\arccos x}{x} dx = \frac{1}{8}\zeta(3) \tag{15}$$

as we mentioned in Introduction. For $n \geq 1$, with $\arccos x = \frac{\pi}{2} - \arcsin x$, we see that

$$t(3,2) = \int_0^1 \frac{\arcsin^3 x}{3!} \frac{\arccos x}{x} dx = \frac{1}{3!}\left(\frac{\pi}{2} I(3) - I(4)\right) = \frac{1}{64}\left(3\pi^2\zeta(3) - 31\zeta(5)\right),$$

$$\frac{\zeta(3,2)}{2^5} = \frac{2}{\pi}\frac{1}{4!}\left(\frac{\pi}{2} I(4) - I(5)\right) = \frac{1}{64}\left(\pi^2\zeta(3) - 11\zeta(5)\right),$$

$$t(3,2,2) = \frac{1}{5!}\left(\frac{\pi}{2} I(5) - I(6)\right) = \frac{1}{2048}\left(2\pi^4\zeta(3) - 60\pi^2\zeta(5) + 381\zeta(7)\right)$$

and so on.

Corollary 2.16. Let $\mathbf{Q}\left[\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+3)\right]_{2n+3}$ denote the set of all elements of degree $2n+3$ in the rational polynomial ring in $\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+3)$ with grading $\deg \pi = 1$ and $\deg \zeta(2j+1) = 2j+1$. Then

$$\frac{\zeta(3, \{2\}^n)}{2^{2n+3}}, t(3, \{2\}^n) \in \mathbf{Q}\left[\pi, \zeta(3), \zeta(5), \ldots, \zeta(2n+3)\right]_{2n+3}.$$

Remark 2.17. Not all multiple values satisfy such a property. For example, as mentioned before,

$$t(2,1) = \frac{\pi^2}{8}\log 2 - \frac{7}{16}\zeta(3)$$

involves a rational multiple of $\pi^2 \log 2$.

2.5. **Conjecture on central binomial sums.** Keeping iterated integrals in mind, now we discuss powers of $\log x$ as certain operator; at the end, we make one conjecture.

For $f(x) \in \mathbf{R}[[x]]$ such that $f(0) = 0$, it is the technique to consider

$$\int_0^y \frac{f(x)}{x} dx$$

to construct another series as we often encountered.

We can generalize this little more by changing the part "$\int \frac{1}{x}$" with the iterated integral

$$\underbrace{\int \frac{dx}{x} \int \frac{dx}{x} \cdots \int \frac{dx}{x}}_{r+1} = \int \frac{\log^r x}{r!x}.$$

Fact 2.18 ([13, p.1, 57-58])**.** For integers $n \geq 1, r \geq 0$, we have

$$\int_0^1 x^n \frac{\log^r x}{r!x} dx = \frac{(-1)^r}{n^{r+1}}.$$

Let us see what if we apply this idea to arcsin integrals. Again, recall that

$$\arcsin x = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{x^{2k+1}}{2k+1}.$$

We now see another central binomial series

$$-\int_0^1 \arcsin x \frac{\log x}{x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{2k+1} \int_0^1 x^{2k+1} \frac{\log x}{x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^3}$$

and similarly

$$\int_0^1 \arcsin x \frac{\log^2 x}{2!x} dx = \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^4}.$$

Wolfram alpha [15] says that

$$-\int_0^1 \arcsin x \frac{\log x}{x} dx = \frac{1}{48}(\pi^3 + 12\pi \log^2 2).$$

$$\int_0^1 \arcsin x \frac{\log^2 x}{2!x} dx = \frac{1}{48}(6\pi\zeta(3) + 4\pi \log^3 2 + \pi^3 \log 2).$$

This is only computer calculation. Hence let us state it as a conjecture.

Conjecture 2.19.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^3} = \frac{1}{48}(\pi^3 + 12\pi \log^2 2),$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{2^{2k}} \frac{1}{(2k+1)^4} = \frac{1}{48}(6\pi\zeta(3) + 4\pi \log^3 2 + \pi^3 \log 2).$$

Remark 2.20. Jianqiang Zhao pointed out (e-mail) to the author that we can prove these with Xu-Zhao [17, Theorem 8.12] (for $m = 4, k = \emptyset$) without much trouble.

3. Other multiple sums

3.1. **Multiple T, S-values.** Following Xu-Zhao [16], let us introduce more general multiple sums here.

Definition 3.1. For a multi-index $(i_k, \ldots, i_1)$ with $i_k \geq 2$, define a *multiple T-value* (MTV) and a *multiple S-value* (MSV) by

$$T(i_k, \ldots, i_1) = 2^k \sum_{\substack{n_k > n_{k-1} > \cdots > n_1 \\ n_j \equiv j \pmod 2}} \frac{1}{n_k^{i_k} n_{k-1}^{i_{k-1}} \ldots n_1^{i_1}},$$

$$S(i_k, \ldots, i_1) = 2^k \sum_{\substack{n_k > n_{k-1} > \cdots > n_1 \\ n_j \equiv j+1 \pmod 2}} \frac{1}{n_k^{i_k} n_{k-1}^{i_{k-1}} \ldots n_1^{i_1}}.$$

We understand that the empty index $\emptyset$ has weight 0 and

$$\zeta(\emptyset) = t(\emptyset) = T(\emptyset) = S(\emptyset) = 1.$$

Table 1. Conjectural dim sequences

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
z_n	1	0	1	1	1	2	2	3	4	5	7	9	12	16	$\cdots$
t_n	1	0	1	2	3	5	8	13	21	34	55	89	144	233	$\cdots$
T_n	1	0	1	1	2	2	4	5	9	10	19	23	42	49	$\cdots$
S_n	1	0	1	2	3	4	6	10	15	22	32	52	76	?	$\cdots$

For $n \geq 0$, let MZV_n, MtV_n, MTV_n, MSV_n denote the set of $\mathbf{Q}$-span of all MZVs, MtVs, MTVs, MSVs of weight n, respectively. Moreover, let

$$z_n = \dim_{\mathbf{Q}} \mathrm{MZV}_n, \quad t_n = \dim_{\mathbf{Q}} \mathrm{MtV}_n,$$

$$T_n = \dim_{\mathbf{Q}} \mathrm{MTV}_n, \quad S_n = \dim_{\mathbf{Q}} \mathrm{MSV}_n.$$

By $(d_n)_{n\geq 0}$ and $(f_n)_{n\geq 0}$, we mean *Padovan* and *Fibonacci* sequences; to be more precise, they are ones satisfying

$$d_0 = 1, d_1 = 0, d_2 = 1 \text{ and } d_n = d_{n-2} + d_{n-3} \text{ for } n \geq 3,$$

$$f_0 = 0, f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2.$$

Conjecture 3.2.

[1] (Zagier) $z_n = d_n$ for $n \geq 1$.
[2] (Hoffman) $t_n = f_n$ for $n \geq 2$.

What about T_n, S_n? Kaneko-Tsumura [8], Xu-Zhao [16] observed that the conjectural sequence $(T_n)_{0\leq n\leq 13}$ satisfies *restricted* Fibonacci relation

$$T_{2k} = T_{2k-1} + T_{2k-2}, \quad 1 \leq k \leq 6.$$

Here we also observe that the conjectural $(S_n)_{0\leq n\leq 13}$ satisfies *restricted* Padovan-like relation

$$S_{2k+1} = S_{2k-1} + 2S_{2k-2}, \quad 1 \leq k \leq 5.$$

Thus, S_{13} is conjecturally equal to

$$52 * 2 \times 32 = 116.$$

There might exist such relations for all $(T_n)_{n\geq 0}, (S_n)_{n\geq 0}$ throughout. We are planning to pursure these details at another opportunity.

3.2. **Conjecture on** $t(\{2\}^{n+1}, 1)$**.** We evaluated $\zeta(3, \{2\}^n), t(3, \{2\}^n)$ together in Theorem 2.13. Notice that

$$(3, \{2\}^n)^\dagger = (\{2\}^{n+1}, 1)$$

so that MZV duality implies

$$\zeta(3, \{2\}^n)^\dagger = \zeta(\{2\}^{n+1}, 1).$$

It is now natural to ask what $t(\{2\}^{n+1}, 1)$ is.

With (7), (9), (11) and [7, Appendix], we observe that

$$t(2,1) = \frac{1}{2}\left(\frac{\pi^2}{4}\log 2 - \frac{7}{8}\zeta(3)\right) = \frac{I(2)}{2!},$$

$$t(2,2,1) = \frac{1}{24}\left(\frac{\pi^4}{16}\log 2 - \frac{9}{16}\pi^2\zeta(3) + \frac{93}{32}\zeta(5)\right) = \frac{I(4)}{4!},$$

$$t(2,2,2,1) = \frac{1}{720}\left(\frac{\pi^6}{64}\log(2) - \frac{45\pi^4}{128}\zeta(3) + \frac{675\pi^2}{128}\zeta(5) - \frac{5715}{256}\zeta(7)\right) = \frac{I(6)}{6!}.$$

It would be nice if we can generalize this.

Conjecture 3.3 ([6])**.** For $n \geq 0$,

$$t\left(\{2\}^{n+1}, 1\right) = \int_0^1 \frac{\arcsin^{2n+2} x}{(2n+2)!}\frac{dx}{x}.$$

Acknowledgment.

The author thanks Satomi Abe, Shoko Asami, Yuko Takada and Michihito Tobe for sincere support of his study.

References

[1] R. Apéry, "Irrationalité de $\zeta(2)$ et $\zeta(3)$", Astérisque 61 (1979), 11-13.

[2] Boo Rim Choe, An elementary proof of $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, Amer. Math. Monthly 94 (1987), 662-663.

[3] J. M. Borwein-D. Broadhurst-J. Kamnitzer, Central Binomial Sums, Multiple Clausen Values and Zeta Values, Experiment. Math. 10 (2000), 25-34.

[4] J.M. Borwein-M. Chamberland, Integer powers of arcsin. Int. J. Math. Math. Sci., Volume 2007, Article ID 19381, 10 pages.

[5] J. P. Buhler-R. E. Crandall, On the evaluation of Euler sums, Experiment. Math. 3 (1994), 275-285.

[6] S. Chavan-M. Kobayashi-J. Layja, Integral evaluation of odd Euler sums, multiple t-value $t(3,2,\dots,2)$ and multiple zeta zalue $\zeta(3,2,\dots,2)$, `arXiv:2111.07097 [math.NT]`.

[7] M. Hoffman, An odd variant of multiple zeta values, Communications in Number Theory and Physics, Volume 13 (2019), 529-567.

[8] M. Kaneko-H. Tsumura, On multiple zeta values of level two, Tsukuba J. Math. 44 (2) (2021), 213-234.

[9] M. Kobayashi, From Base problem to multiple zeta values, `arXiv:2112.03361 [math.NT]`.

[10] D. H. Lehmer, Interesting Series Involving the Central Binomial Coefficient, Amer. Math. Monthly, 92 (1985), 449-457.

[11] C. Lupu, Another look at Zagier's formula for multiple zeta values involving Hoffman elements, `arXiv:2011.11718v1 [math.NT]`.

[12] T. Murakami, On Hoffman ' s t-values of maximal height and generators of multiple zeta values. Math. Ann., 2021.

[13] C.I. Valean, (Almost) Impossible Integrals, Sums, and Series, Problem Books in Mathematics, Springer International Publishing, 2019.

[14] A. van der Poorten, a proof that Euler missed, Apéry's proof of the irrationality of $\zeta(3)$, an informal report, ALF's reprints, Paper 45 (2005), 1-16.

[15] Wolfram alpha, `https://www.wolframalpha.com`.

[16] C. Xu-J. Zhao, Variants of multiple zeta values with even and odd summation indices, to appear in Math. Z., 2021.

[17] C. Xu-J. Zhao, Apéry-Type Series and Colored Multiple Zeta Values, `arXiv:2111.10998v1 [math.NT]`.

[18] D. Zagier, Evaluation of the multiple zeta values $\zeta(2,\dots,2,3,2,\dots,2)$, Annals of Mathematics 175 (2012), 977-1000.

Masato Kobayashi, Department of Engineering, Kanagawa University, 3-27-1 Rokkaku-bashi, Yokohama 221-8686, Japan.

Email address: `masato210@gmail.com`