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Complete directed minors and chromatic number

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Abstract

The *dichromatic number* $\vec{\chi}(D)$ of a digraph D is the smallest k for which it admits a k-coloring where every color class induces an acyclic subgraph. Inspired by Hadwiger's conjecture for undirected graphs, several groups of authors have recently studied the containment of complete directed minors in digraphs with a given dichromatic number. In this note we exhibit a relation of these problems to Hadwiger's conjecture. Exploiting this relation, we show that every directed graph excluding the complete digraph $\overleftrightarrow{K}_t$ of order t as a *strong minor* or as a *butterfly minor* is $O(t(\log\log t)^6)$-colorable. This answers a question by Axenovich, Girão, Snyder, and Weber, who proved an upper bound of $t4^t$ for the same problem. A further consequence of our results is that every digraph of dichromatic number $22n$ contains a subdivision of every n-vertex subcubic digraph, which makes progress on a set of problems raised by Aboulker, Cohen, Havet, Lochet, Moura, and Thomassé.

KEYWORDS
chromatic number, directed graphs, graph minors, Hadwiger's conjecture

1 | INTRODUCTION

For a given integer $t \geq 1$ let $m_\chi(t)$ be the smallest integer for which it is true that every graph with chromatic number at least $m_\chi(t)$ contains a K_t-minor. Hadwiger's conjecture [8], which is one of the most important open problems in graph theory, states that $m_\chi(t) = t$ for all $t \geq 1$. The conjecture remains unsolved for $t \geq 7$. For many years the best general upper bound on $m_\chi(t)$ was due to Kostochka [13,14] and Thomason [27], who independently proved that every graph of average degree at least $O(t\sqrt{\log t})$ contains a K_t-minor, implying that $m_\chi(t) = O(t\sqrt{\log t})$. Recently, however, there has been progress. First, Norin, Postle, and Song [22] showed that $m_\chi(t) = O(t(\log t)^\beta)$ (for any $\beta > \frac{1}{4}$), which was then further improved by Postle [23] to $m_\chi(t) = O(t(\log \log t)^6)$. For more details about Hadwiger's conjecture the interested reader may consult the recent survey by Seymour [26].

This famous conjecture has influenced many researchers and different variations of it have been studied in various frameworks, one of which is directed graphs.

The *chromatic number of a digraph* was introduced by Neumann-Lara [21] in 1982 as the smallest number of acyclic subsets that cover the vertex set of the digraph. The dichromatic number has received increasing attention since 2000 and has been an extremely active research topic in recent years, we refer to [3,4,9,10] as examples of important results on the topic.

In the case of digraphs there are multiple ways to define a minor. Here we consider three popular variants: *strong minors*, *butterfly minors*, and *topological minors*. The containment of these different minors in dense digraphs as well as their relation to the dichromatic number has already been studied in several previous works, see, for example, [2,12,15] for strong minors, [5,11,16,20] for butterfly minors, and [1,6,7,17–19,25] for topological minors. Given digraphs D and H, we say that D is a *strong H-minor model* if $V(D)$ can be partitioned into nonempty sets $\{X_v : v \in V(H)\}$ (called *branch sets*) such that the digraph induced by X_v is strongly connected for all $v \in V(H)$, and for every arc (u, v) in H there is an arc in D from X_u to X_v. More generally, we also say that *D contains H as a strong minor* and write $D \succcurlyeq_s H$ if a subdigraph of D is a strong H-minor model. Pause to note that strong minor containment defines a transitive relation on digraphs, that is, if $D_1 \succcurlyeq_s D_2$ and $D_2 \succcurlyeq_s D_3$ for digraphs D_1, D_2, D_3, then $D_1 \succcurlyeq_s D_3$.

At some places in the manuscript, we will use the following notation: If D is a strong H-minor model witnessed by the partition $\{X_v : v \in V(H)\}$ into branch sets, for an arc $e = (u_1, u_2) \in A(H)$ we denote by $v(e, u_1)$ and $v(e, u_2)$ the endpoints of an arc in D which connects X_{u_1} to X_{u_2}, where $v(e, u_1)$ is the tail of the arc in X_{u_1} and $v(e, u_2)$ is the head of the arc in X_{u_2}.

Given an undirected graph G we denote by $\overleftrightarrow{G}$ the directed graph on the same vertex set where for every edge $uv \in E(G)$ the vertices u and v are connected in $\overleftrightarrow{G}$ by an arc in each direction. We are particularly interested in forcing strong $\overleftrightarrow{K_t}$-minors, as those also yield a strong H-minor for every digraph H on at most t vertices. Analogously to the undirected case, one can ask how large the dichromatic number of a digraph should be to guarantee that it contains a strong $\overleftrightarrow{K_t}$-minor. More precisely, we consider the function $sm_{\vec{\chi}}(t)$, which is the smallest integer for which it is true that every digraph D with $\vec{\chi}(D) \geq sm_{\vec{\chi}}(t)$ satisfies $D \succcurlyeq_s \overleftrightarrow{K_t}$. In a recent work, Axenovich, Girão, Snyder, and Weber [2] investigated the function $sm_{\vec{\chi}}(t)$. They showed that $sm_{\vec{\chi}}(t)$ exists for every $t \geq 2$ and proved the bounds

$$t + 1 \leq sm_{\vec{\chi}}(t) \leq t4^t.$$

They then raised the problem of improving in particular the upper bound and expressed that they think that $sm_{\vec{\chi}}(t)$ should be much closer to the lower than to the upper bound. Here we confirm this belief by improving their upper bound substantially as follows.

Theorem 1. *For every $t \geq 1$ we have*

$$sm_{\vec{\chi}}(t) \leq 2m_\chi(t) - 1 = O(t(\log\,\log t)^6).$$

Now let us turn to butterfly minors. Given a digraph D and an arc $(u, v) \in A(D)$, this arc is called *(butterfly-)contractible* if v is the only out-neighbor of u or if u is the only in-neighbor of v in D. Given such a contractible arc e, the digraph D/e is obtained from D by merging u and v into a new vertex and joining their in- and out-neighborhoods, ignoring parallel arcs. A *butterfly minor* of a digraph D is any digraph that can be obtained by repeatedly deleting arcs, deleting vertices or contracting arcs.

In [20], inspired by Hadwiger's conjecture, Millani, Steiner, and Wiederrecht raised the following question: For a given integer $k \geq 1$, what is the largest butterfly minor-closed class $\mathcal{D}_k$ of k-colorable digraphs? They gave a precise characterization of $\mathcal{D}_2$ as the *noneven digraphs*. The question concerning a characterization of $\mathcal{D}_k$ for $k \geq 3$ is closely related to the question of forcing complete butterfly minors in digraphs. For an integer $t \geq 1$, let us define $bm_{\vec{\chi}}(t)$ as the smallest integer such that every digraph D with $\vec{\chi}(D) \geq bm_{\vec{\chi}}(t)$ contains $\overleftrightarrow{K_t}$ as a butterfly minor, and put

$$b(x) := \max\{t \geq 1 \mid bm_{\vec{\chi}}(t) \leq x\}$$

for the integer inverse function of $bm_{\vec{\chi}}(\cdot)$. Let us further denote by $\mathcal{K}_t$ the class of all digraphs with no $\overleftrightarrow{K_t}$ as a butterfly minor. Then, on the one hand, every digraph excluding $\overleftrightarrow{K}_{b(k+1)}$ as a butterfly minor is colorable with $bm_{\vec{\chi}}(b(k + 1)) - 1 \leq k$ colors. On the other hand, every digraph in $\mathcal{D}_k$ must exclude $\overleftrightarrow{K}_{k+1}$ as a butterfly minor, since its dichromatic number exceeds k. Therefore, for every k we have

$$\mathcal{K}_{b(k+1)} \subseteq \mathcal{D}_k \subseteq \mathcal{K}_{k+1}.$$

To see how tight the above inclusions are one needs to obtain good lower bounds on $b(k + 1)$, or equivalently good upper bounds on $bm_{\vec{\chi}}(t)$. In this direction, as an application of Theorem 1 we prove the following corollary. The previously best-known upper bound on $bm_{\vec{\chi}}(t)$ mentioned in [20] was $4^{t^2-t}(t - 1) + 1$ and followed from the work of Aboulker et al. [1].

Corollary 1. *For $t \geq 1$ we have $bm_{\vec{\chi}}(t) \leq 2m_\chi(2t) - 1 = O(t(\log\,\log t)^6)$.*

For the sake of completeness we remark that a lower bound of $t + 1 \leq bm_{\vec{\chi}}(t)$ follows by taking $D = \overleftrightarrow{G}$ where G is the complete graph on $t + 2$ vertices with a 5-cycle removed. It is a simple exercise to verify that $\overleftrightarrow{\chi}(D) = t$ and that it contains no butterfly $\overleftrightarrow{K_t}$-minor.

Finally, we consider topological minors. Given a digraph H, a *subdivision of H* is any digraph obtained by replacing every arc $(u, v) \in A(H)$ by a directed path from u to v, such that subdivision paths of different arcs are internally vertex-disjoint. Then H is said to be a *topological minor* of some digraph D if D contains a subdivision of H as a subgraph.

Aboulker, Cohen, Havet, Lochet, Moura, and Thomassé [1] initiated the study of the existence of various subdivisions in digraphs of large dichromatic number. For a digraph H they introduced the parameter $\text{mader}_{\vec{\chi}}(H)$, the *dichromatic Mader number of H*, as the smallest integer such that any digraph D with $\vec{\chi}(D) \geq \text{mader}_{\vec{\chi}}(H)$ contains a subdivision of H. In their main result they proved that if H is a digraph with n vertices and m arcs, then

$$n \leq \text{mader}_{\vec{\chi}}(H) \leq 4^m(n - 1) + 1.$$

Gishboliner, Steiner, and Szabó [6] conjectured that $\text{mader}_{\vec{\chi}}(\overleftrightarrow{K_t}) \leq Ct^2$ for some absolute constant C. However, it seems surprisingly hard to find a polynomial upper bound even for quite simple digraphs H. An indication for this increased difficulty compared with the undirected case could be that for digraphs it is not even possible to force a $\overleftrightarrow{K_3}$-subdivision by means of large minimum out- and in-degree (compare Mader [17]).

Gishboliner et al. [6] still managed to identify a wide class of graphs, called octus graphs,[1] for which the lower bound above is tight. Their result means that given a digraph D with $\vec{\chi}(D) \geq n$ it contains the subdivision of every octus graph on at most n vertices.

Here, along the same line of thinking, as a corollary of Theorem 1 we prove a similar result for another class of digraphs. By slightly abusing the terminology, we call a digraph D *subcubic* if D is an orientation of a graph with maximum degree at most three such that the in- and out-degree of any vertex is at most two.

Corollary 2. *For $n \geq 1$ if D is a digraph with $\vec{\chi}(D) \geq 22n$ then it contains a subdivision of every subcubic digraph on at most n vertices.*

1.1 | Notation

For a digraph D and a set $S \subseteq V(D)$ we denote by $D[S]$ the subdigraph spanned by the vertices in S. The set S is called *acyclic* if $D[S]$ is an acyclic digraph. We call D *strongly connected* if for every ordered pair u, v of vertices in D there is a directed path in D from u to v. An in-/out-arborescence is a rooted directed tree where every arc is directed towards/away from the root. For the starting/ending point of an arc we will also use the names tail/head.

[1]We note that this class, in particular, includes orientations of cactus graphs (and hence orientations of cycles), as well as bioriented forests.

A *(proper) coloring* of an undirected graph G with colors in a set A is a map $f : V(G) \to A$ where neighboring vertices are mapped to different colors, or equivalently $f^{-1}(a)$ is an independent set for every $a \in A$. If $|A| = k$ then f is called a k*-coloring*. Analogously, an *(acyclic)* k*-coloring* of a digraph D is a map $f : V(D) \to A$ with $|A| = k$ where $f^{-1}(a)$ is an acyclic set for every $a \in A$. The minimum k for which a k-coloring exists is the *chromatic* (resp., *dichromatic*) *number* of the undirected graph G (resp., digraph D), which we shall denote by $\chi(G)$ (resp., $\vec{\chi}(D)$).

2 | PROOFS

2.1 | Strong minors

The proof of Theorem 1 will be based on the following result.

Theorem 2. *For every digraph D there is an undirected graph G such that*

(i) D is a strong $\overleftrightarrow{G}$-minor model, and

(ii) $\vec{\chi}(D) \le 2\chi(G)$.

Proof. To start with, let us first fix a partition $X_1, X_2, \ldots, X_m$ of $V(D)$ such that for every $i \in \{1, 2, \ldots, m\}$ the set X_i is an inclusionwise maximal subset of $V(D)\setminus(X_1 \cup \cdots \cup X_{i-1})$ with $D[X_i]$ strongly connected and $\vec{\chi}(D[X_i]) \le 2$. Note that the X_i's are well defined since the one vertex-digraph is strongly connected and 2-colorable. Now we define G to be the undirected simple graph with vertex set $\{X_1, \ldots, X_m\}$ and $X_iX_j \in E(G)$ if and only if there are arcs in both directions between X_i and X_j in D. Then, by definition, D is a strong $\overleftrightarrow{G}$-minor model, as one can simply take $X_1, X_2, \ldots, X_m$ as the branch sets.

Therefore, what remains to prove is property (ii). For this let us assume that $\chi(G) = k$ and fix a proper coloring $f_G : V(G) \to \{c_1, c_2, \ldots, c_k\}$ of G. Now, for every $i \le m$ take an arbitrary acyclic two-coloring of $D[X_i]$ (which exists by assumption) with colors $\{c_i', c_i''\}$. The rest of the proof is about showing that by putting these colorings together we obtain an acyclic coloring f_D of D with the $2k$ colors $\{c_1', c_1'', c_2', c_2'', \ldots, c_k', c_k''\}$.

Assume for contradiction that this is not the case, and there is a directed cycle C in D which is monochromatic. We may, without loss of generality, assume that C is a shortest such cycle, in particular, it is an induced cycle. Let i_0 be the smallest index for which C contains a vertex from X_{i_0}. Note that, in particular, $V(C) \subseteq V(D)\setminus(X_1 \cup \cdots \cup X_{i_0-1})$ and, as f_D is a proper coloring on $D[X_{i_0}]$, the cycle C cannot be fully contained in X_{i_0}. Hence, C contains a subsequence $u, w_1, \ldots, w_\ell, v$ of consecutive vertices on C with $(u, w_1), (w_1, w_2), \ldots, (w_\ell, v) \in A(C)$, such that $u, v \in X_{i_0}$ (possibly $u = v$), $w_1, \ldots, w_\ell \in X_{i_0+1} \cup \cdots \cup X_m$, and $\ell > 0$.

Let $s \in \{1, \ldots, \ell\}$ be the smallest index such that w_s has an out-neighbor in X_{i_0}, and denote this out-neighbor by $x \in X_{i_0}$. Note that s is well defined, since $(w_\ell, v) \in A(D)$ and $v \in X_{i_0}$. We claim that w_s has no in-neighbor in D that is contained in X_{i_0}. Suppose towards a contradiction that there exists $y \in X_{i_0}$ such that $(y, w_s) \in A(D)$. Let $j > i_0$ be such that $w_s \in X_j$. Then, because of the arcs $(y, w_s), (w_s, x) \in A(D)$, we have $X_{i_0}X_j \in E(G)$ and hence $f_G(X_{i_0}) \ne f_G(X_j)$. This in turn implies that $f_D(u) \ne f_D(w_s)$ and

$f_D(v) \neq f_D(w_s)$ which contradicts the monochromaticity of C. Hence, we may assume that w_s has no in-neighbor contained in X_{i_0}. In particular, this implies $s \geq 2$. Let us now consider the set

$$X = X_{i_0} \cup \{w_1, ..., w_s\} \subseteq V(D) \setminus (X_1 \cup \cdots \cup X_{i_0-1}).$$

It is clearly strongly connected, as X_{i_0} is so and $u, w_1, ..., w_s, x$ induce a directed path (or cycle in case $u = x$) starting and ending in X_{i_0}. Moreover, any extension of an acyclic $\{1, 2\}$-coloring of $D[X_{i_0}]$ to a $\{1, 2\}$-coloring of $D[X]$ where $w_1, ..., w_{s-1}$ receive color 1 and w_s receives color 2 is acyclic. Indeed, by the definition of s, there are no arcs starting in $\{w_1, ..., w_{s-1}\}$ and ending in X_{i_0}, and by the inducedness of C there are no arcs spanned between nonconsecutive vertices inside $\{w_1, ..., w_{s-1}\}$. Adding the fact that w_s has no in-neighbors in X_{i_0}, these imply that any directed cycle in $D[X]$ is either fully contained in $D[X_{i_0}]$, or contains both w_s and at least one vertex in $\{w_1, ..., w_{s-1}\}$. In any case, it is not monochromatic. However, the existence of the set X then contradicts with the maximality of X_{i_0}, which finishes the proof. □

Now we can easily deduce Theorem 1 from Theorem 2.

Proof of Theorem 1. Let D be a digraph with $\vec{\chi}(D) \geq 2m_\chi(t) - 1$. By Theorem 2 there exists an undirected graph G such that $\vec{\chi}(D) \leq 2\chi(G)$ and $D \succcurlyeq_s \overleftrightarrow{G}$. This implies that $\chi(G) \geq m_\chi(t)$, and hence G contains a K_t-minor. Taking the same branch sets in $\overleftrightarrow{G}$ which give a K_t-minor in G shows that $\overleftrightarrow{G} \succcurlyeq_s \overleftrightarrow{K_t}$, and by transitivity $D \succcurlyeq_s \overleftrightarrow{K_t}$. Since D was arbitrarily chosen such that $\vec{\chi}(D) \geq 2m_\chi(t) - 1$, this proves that $sm_{\vec{\chi}}(t) \leq 2m_\chi(t) - 1$, as required. □

We would like to remark that the above proof of Theorem 2 actually yields a slightly stronger conclusion: Let the partition $X_1, ..., X_m$ of $V(D)$ and the graph G be defined as in the proof of Theorem 2. We then claim that for every edge $X_iX_j \in E(G)$ with $i < j$ there are at least *two* arcs in D which go from X_i to X_j, and at least *two* arcs which go from X_j to X_i.

To see this, note that by the definition of G there are edges in both directions spanned between X_i and X_j, which implies that $D[X_i \cup X_j]$ is also a strongly connected digraph. Then suppose that contrary to our claim, there would be at most one edge from X_i to X_j (or at most one edge from X_j to X_i) in D. Let $e \in A(D)$ be such an edge, and note that removing e from $D[X_i \cup X_j]$ destroys the strong connectivity and creates the two strong components X_i and X_j of $D[X_i \cup X_j] - e$. By choice of $X_1, ..., X_m$, we know that there exists an acyclic 2-coloring $f_i : X_i \to \{1, 2\}$ of $D[X_i]$ and an acyclic 2-coloring $f_j : X_j \to \{1, 2\}$ of $D[X_j]$. Possibly after swapping colors 1 and 2 in f_j, we may assume that the vertices $u \in X_i$ and $v \in X_j$ which form the endpoints of e satisfy $f_i(u) \neq f_j(v)$. Now the common extension of f_i and f_j to $D[X_i \cup X_j]$ forms an acyclic 2-coloring, since every directed cycle in $D[X_i \cup X_j]$ intersecting both X_i and X_j must use the edge e and can therefore not be monochromatic. This however means that $X_i \cup X_j$ induces a strongly connected and 2-colorable subgraph of D which properly contains X_i and is disjoint from $X_1 \cup \cdots \cup X_{i-1}$. Finally, this contradicts the maximality of X_i in our choice of the partition of $V(D)$, and proves our above claim. This stronger conclusion can then be used in the proof of Theorem 1 to yield the stronger conclusion that every digraph of dichromatic number at least $2m_\chi(t) - 1$ in fact contains a strong $\overleftrightarrow{K_t}$-minor model in which between every pair of branch sets, at least *two* arcs are spanned in each direction.

2.2 | Butterfly minors

Corollary 1 follows directly from Theorem 1 and the following proposition.

Proposition 1. *Every strong $\overleftrightarrow{K_{2t}}$-minor model contains $\overleftrightarrow{K_t}$ as a butterfly minor.*

Proof. Let D be a strong $\overleftrightarrow{K_{2t}}$-minor model and let $\{X_1^+, X_1^-, ..., X_t^+, X_t^-\}$ be a corresponding partition of $V(D)$ into $2t$ branch sets. In particular, for every $i \in \{1, ..., t\}$ there exist $r_i^+ \in X_i^+$ and $r_i^- \in X_i^-$ such that $(r_i^-, r_i^+) \in A(D)$. Since $D[X_i^-]$ and $D[X_i^+]$ are strongly connected digraphs, there exist[2] oriented spanning trees $T_i^- \subseteq D[X_i^-]$ and $T_i^+ \subseteq D[X_i^+]$ such that T_i^- is an in-arborescence rooted at r_i^- and T_i^+ is an out-arborescence rooted at r_i^+. Let us consider the spanning subdigraph D' of D consisting of the arcs contained in

$$T := \bigcup_{i=1}^{t} \left(\left\{ \left(r_i^-, r_i^+ \right) \right\} \cup A\left(T_i^+\right) \cup A\left(T_i^-\right) \right),$$

as well as all arcs of D starting in X_i^+ and ending in X_j^- for $i \neq j$. Then every arc of D' contained in T is either the unique arc in D' emanating from its tail or the unique arc in D' entering its head. It follows that all arcs in T are butterfly-contractible. Note that the contraction of an arc does not affect the butterfly-contractibility of other arcs, hence the digraph D'/T, obtained from D' by successively contracting all arcs in T, is a butterfly minor of D. The vertices of D'/T can be labeled $v_1, ..., v_t$, where v_i denotes the vertex corresponding to the contraction of the (weakly) connected component of D' inside $X_i^+ \cup X_i^-$. As D is a strong $\overleftrightarrow{K_{2t}}$-minor model, by definition of D' for every $(i, j) \in \{1, ..., k\}^2$ with $i \neq j$, there exists an arc in D' starting in X_i^+ and ending in X_j^-. Therefore, D'/T is a butterfly minor of D isomorphic to $\overleftrightarrow{K_t}$, concluding the proof. □

2.3 | Topological minors

Finally, we prove Corollary 2.

Proof of Corollary 2. As a first step note that given $n \in \mathbb{N}$, every undirected graph G with a minimum degree at least $10.5n > n + 6.291 \cdot \frac{3}{2}n$ contains every n-vertex subcubic graph as a minor. This follows directly from a result of Reed and Wood [24], who proved that every graph with an average degree at least $n + 6.291m$ contains every graph with n vertices and m edges as a minor.

[2]Such trees can easily be obtained by considering a breadth-first in-search (resp., out-search) starting from r_i^- (resp., r_i^+).

Let now D be any digraph with $\vec{\chi}(D) \geq 22n$, F a subcubic digraph on $n \geq 2$ vertices and H its underlying undirected subcubic graph. By Theorem 2 there exists an undirected graph G such that D is a strong $\overleftrightarrow{G}$-minor model and $\chi(G) \geq 11n$. In particular, G contains a subgraph of minimum degree at least $11n - 1 > 10.5n$ and hence, by our earlier remark, an H-minor. This implies that $\overleftrightarrow{G}$ contains a strong $\overleftrightarrow{H}$-minor and hence D does so. However, as $F \subseteq \overleftrightarrow{H}$, it also follows that D contains a strong F-minor, that is, a subdigraph D' which is a strong F-minor model. Let $\{X_f : f \in V(F)\}$ be a branch set partition of $V(D')$ witnessing this. Recall that, by definition, for every arc $e = (u_1, u_2) \in A(F)$ there exist vertices $v(e, u_1) \in X_{u_1}$ and $v(e, u_2) \in X_{u_2}$ such that $(v(e, u_1), v(e, u_2)) \in A(D') \subseteq A(D)$.

Let next $u \in V(F)$ be an arbitrary vertex with total degree $d = d(u) \in \{0, 1, 2, 3\}$ and let us denote the arcs incident to u by $e_1, \ldots, e_d$. Furthermore, for $i = 1, \ldots, d$ we put $v_i := v(e_i, u)$. We claim that there exists a vertex $b(u) \in X_u$ and for every $i = 1, \ldots, d$ a directed path P_i^u in $D[X_u]$ such that

- $P_1^u, \ldots, P_d^u$ only intersect at $b(u)$;
- if u is the tail of e_i, then P_i^u is a directed path from $b(u)$ to v_i;
- if u is the head of e_i, then P_i^u is a directed path from v_i to $b(u)$.

This claim holds trivially if $d = 0$, and if $d = 1$ then we can simply put $b(u) = v_1$ and let P_1^u be the trivial one-vertex path consisting of v_1.

If $d = 2$ then, without loss of generality, by the symmetry of reversing all arcs in D and F, we may assume that u is the head of e_1. We then can put $b(u) := v_2$, let P_1^u be any directed path in $D[X_u]$ from v_1 to v_2, and take P_2^u to be the trivial one-vertex path consisting only of v_2.

Finally suppose $d = 3$. Since F is subcubic, u either has in-degree one and out-degree two, or vice versa. As before, without loss of generality, by symmetry we may assume that the first case occurs, and it is e_1 that enters u and e_2 and e_3 that emanate from it. Take now P_{12} and P_{13} to be directed paths in $D[X_u]$ starting at v_1 and ending at v_2 and v_3, respectively. We define now $b(u)$ as the first vertex in $V(P_{12})$ that we meet when traversing P_{13} backwards (starting at v_3), P_1^u as the subpath of P_{12} directed from v_1 to $b(u)$, P_2^u as the subpath of P_{12} directed from $b(u)$ to v_2, and P_3^u as the subpath of P_{13} directed from $b(u)$ to v_3. It follows by definition that P_1, P_2, P_3 are internally vertex-disjoint, and hence the claim follows.

To finish the proof, let $S \subseteq D$ be a subdigraph with vertex set

$$V(S) := \bigcup_{u \in V(F)} \left(\bigcup_{i=1}^{d(u)} V\left(P_i^u\right) \right),$$

and arcs

$$A(S) := \{(v(e, u_1), v(e, u_2)) | e = (u_1, u_2) \in A(F)\} \cup \left(\bigcup_{u \in V(F)} \left(\bigcup_{i=1}^{d(u)} A\left(P_i^u\right) \right) \right).$$

S is a digraph isomorphic to a subdivision of F in which a vertex $u \in V(F)$ is represented by the branch-vertex $b(u)$. This concludes the proof. □

3 | CONCLUDING REMARKS

In this note we showed that $sm_{\vec{\chi}}(t) \le 2m_\chi(t) - 1$ and $bm_{\vec{\chi}}(t) \le 2m_\chi(2t) - 1$ for any $t \ge 1$. As far as lower bounds are concerned, it is not hard to see that $m_\chi(t) \le \min\{sm_{\vec{\chi}}(t), bm_{\vec{\chi}}(t)\}$ for every $t \ge 1$. Indeed, for any graph G with $\chi(G) \ge \min\{sm_{\vec{\chi}}(t), bm_{\vec{\chi}}(t)\}$, as $\vec{\chi}(\overleftrightarrow{G}) = \chi(G)$, by definition $\overleftrightarrow{G}$ contains $\overleftrightarrow{K}_t$ either as a strong minor or as a butterfly minor, each of which implies that G contains a K_t-minor. Therefore, our results reduce the question about the asymptotics of $sm_{\vec{\chi}}(t)$ and $bm_{\vec{\chi}}(t)$ to the well-studied undirected version of the problem. Also, as Hadwiger's conjecture is known to be true for small values, for $3 \le t \le 6$ we have

$$t + 1 \le sm_{\vec{\chi}}(t) \le 2t - 1 \quad \text{and} \quad t + 1 \le bm_{\vec{\chi}}(t) \le 4t - 1.$$

We believe that the upper bounds should not be tight. To support this intuition, recall from our remark after the proof of Theorem 1 that a more careful analysis of the proof yields the stronger statement that any digraph D with $\vec{\chi}(D) \ge 2m_\chi(t) - 1$ contains a strong $\overleftrightarrow{K}_t$-minor model in which between any two branch sets, there are at least two arcs spanned in both directions. Under the assumption that Hadwiger's conjecture is true, the bound $2t - 1$ for this stronger property would be sharp, as shown by $\overleftrightarrow{K}_{2t-2}$. This indicates that our proof should not be expected to give a tight bound for the problem of forcing a strong $\overleftrightarrow{K}_t$-minor. Instead it seems plausible that $sm_{\vec{\chi}}(t) = t + 1$ (and maybe $bm_{\vec{\chi}}(t) = t + 1$) for any $t \ge 3$.

Problem 1. Does every digraph D with $\vec{\chi}(D) \ge t + 1$ contain $\overleftrightarrow{K}_t$ as a strong minor (butterfly minor)?

Already resolving the first open case $t = 3$ would be quite interesting.

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