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Distributed Approximating Functional Approach to Burgers' Equation using Element Differential Quadrature Method

[1*]VAGHEFI, M; [2]RAHIDEH, H; [3] GOLBAHAR HAGHIGHI, M R; [4]MANSHAD, A K

[1]*Civil Engineering Department., Faculty of Engineering, Persian Gulf University, Bushehr, Iran.*
Email: Vaghefi@pgu.ac.ir
[2] *Chemical Engineering Department., Faculty of Engineering, Persian Gulf University, Bushehr, Iran.*
Email: Rahideh@pgu.ac.ir
[3] *Mechanical Engineering Department., Faculty of Engineering, Persian Gulf University, Bushehr, Iran.*
Email: Golbahar@pgu.ac.ir
[4]*Chemical Engineering Department.,Faculty of Engineering, Persian Gulf University, Bushehr, Iran.;*
Email: Khaksar@pgu.ac.ir

** Corresponding Author, Tel: + 98 771 4222150, Fax: + 98 771 4540376*
Email: Vaghefi@pgu.ac.ir

ABSTRACT: This paper presents a computationally efficient and an accurate methodology in differential quadrature element method (EDQM) analysis of the nonlinear one-dimensional Burgers' equation. Based on this approach, the total spatial and temporal domain is divided into a set of sub-domain and in each sub-domain, the DQ rule is employed to discretize the spatial and temporal domain derivatives. This equation is similar to, but simpler than, the Navier-Stokes equation in fluid dynamics. To verify this advantage through some comparison studies, an exact series solution are also obtained. In addition, the presented scheme has numerically stable behavior. After demonstrating the convergence and accuracy of the method, the effects of velocity parameters on the viscosity distribution are studied. It is found that element differential quadrature method provides highly accurate an exact series solution for Burgers, equation, while a small number of grid points is needed. @JASEM

Key words: Burgers' Equation, Differential quadrature method, Exact Series

Nomenclature

A^i	the first derivative of normalized weighting function $(i = x,t)$	N_x^e	number of grid points along the x-direction for the eth element
B^x	the second derivative of normalized weighting function	t	the non-dimensional time
L_x	length of the x-axis	u	velocity *in x*-axis direction
L_x^e	length of the x-axis for the eth element	x	the non-dimensional *x*-axis
N_I	number of time interval		
N_t^I	number of grid points in the t-directions for the Ith time interval		
N_e	number of elements in x direction	υ	kinematics viscosity

In this paper, the equation Burger as Hyperbolic equation is considered. To solve the equations of the Differential quadrature method is used. One of the most important tasks in fluid dynamics is to predict physical quantities such as pressure, velocity and temperature of a given flow [Wei et al., 1998]. The fundamental equations for the transport of these properties are the equations of continuity motion and energy balance. For a compressible viscous fluid, the equation of motion is the Navier-Stocks equation [Suka and Dag, 2008].

In order to solve the Burgers' equation numerically, Varoglu and Finn (1980) used a new finite element method based on a weighted residual formulation, Caldwell and Smith (1982) finite difference and cubic spline finite element methods, Evans and Abdullah (1984) alternating group explicit methods, Kakuda and Tosaka (1990) the generalized boundary element approach, Ali et al.(1992) a cubic B-spline finite element method based on a collocation formulation, Nguyen and Rynen (1982) a linear

** Corresponding Author, Tel: + 98 771 4222150, Fax: + 98 771 4540376*
Email: Vaghefi@pgu.ac.ir

space-time finite element method based on a least-squares approach, Mittal and Singhal (1993) a technique of finitely reproducing nonlinearities to get a system of nonlinear differential equations, which are solved by a Runge–Kutta–Chebyshev method. Gardner et al.(1996) used a Petrov–Galerkin method by a quadratic B-spline spatial finite elements, and they also used a least-squares technique using linear space-time finite elements [Gardner et al., 1997]. Kutluay et al.(1999) proposed the exact-explicit finite difference method to the Burgers-like problems to obtain numerical solutions of adequate accuracy. Abd-el-Malek and El-Mansi (2000) have used the group-theoretic methods for calculating the solution of Burgers' equation with appropriate boundary and initial conditions. More recently, Wenyuan Liao (2008) have used An implicit fourth-order compact finite difference scheme for solution of one-dimensional Burgers' equation. Bulent Saka and Idris Dag (2008) are researched by making comparison with results which are found with the direct application of the proposed method to the Burgers' equation and the modified Burgers 'equation.
In this paper, we have applied a EDQM algorithm is employed for the nonlinear one-dimensional Burgers' equation and make a comparison of numerical solutions with exact series.

BASIC FORMULATION AND SOLUTION PROCEDURE

The one dimensional governing transient Burgers' equation may be written as:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2} \qquad (1)$$

where ν is the kinematics viscosity. It is assumed that the kinematics viscosity of the domain under consideration is constant.

Due to nonlinear terms in Eq. (1), if it is not impossible to solve it analytically, it is very difficult to obtain such a solution. Hence the approximate method should be used to solve it. As a first attempt, the differential quadrature method as an efficient and accurate numerical tool is employed to solve it under arbitrary boundary and initial conditions[Malekzadeh and Rahideh, 2007].

The differential quadrature method (DQM) is an alternative discretization approach for directly solving the governing equations in mathematics and engineering applications. It has been successfully employed for different structural, heat transfer, and fluid mechanics problems [Malekzadeh and Rahideh, 2007; Golbahar Haghighi et al., 2009; Malekzadeh and Rahideh, 2009; Malekzadeh et al., 2010; Rahideh et al.,2012].

Here, the DQM is used to discretize both the spatial and temporal domain. According to this method, the domain is discretized into a set of N_x^e and N_t^I discrete grid points in the *x*- and t-direction, respectively. Then at a given grid point (x_i, t_j), the first and second order derivatives of an arbitrary function $f(x,t)$ can be approximated as

$$\left.\frac{\partial f(x,t)}{\partial x}\right|_{(x_i,t_j)}=\sum_{m=1}^{N_x^e}A_{im}^x f(x_m,t_j)=\sum_{m=1}^{N_x^e}A_{im}^x f_{mj}\,,\left.\frac{\partial^2 f(x,t)}{\partial x^2}\right|_{(x_i,t_j)}=\sum_{m=1}^{N_x^e}B_{im}^x f_{mj} \qquad (2)$$

$$\left.\frac{\partial f(x,t)}{\partial t}\right|_{(x_i,t_j)}=\sum_{n=1}^{N_t^I}A_{jn}^t f(x_i,t_n)=\sum_{n=1}^{N_t^I}A_{jn}^t f_{in} \qquad (3)$$

where i=1,2,…, N_x^e, j=1,2,…, N_t^I .

In order to determine the weighting coefficients, a set of test functions should be used in Eq. (2) and (3). For polynomial basis functions DQ, a set of Lagrange polynomials is employed as the test functions. The normalized weighting coefficients for the first and second order derivatives in the *x*-direction are thus determined, respectively, as

$$A_{ij}^{x}=\begin{cases}\dfrac{1}{L_x^e}\dfrac{M(x_i)}{(x_i-x_j)M(x_j)} & i\neq j\\ -\displaystyle\sum_{k=1,k\neq i}^{N_x^e}A_{ik}^{x} & i=j\end{cases}\qquad i,j=1,2\ldots,N_x^e \tag{4}$$

and

$$B_{ij}^{x}=\begin{cases}2[A_{ii}^{x}A_{ij}^{x}-\dfrac{A_{ij}^{x}}{x_i-x_j}] & i\neq j\\ -\displaystyle\sum_{k=1,k\neq i}^{N_x^e}B_{ik}^{x} & i=j\end{cases}\qquad i,j=1,2\ldots,N_x^e \tag{5}$$

where,

$$M(x_i)=\prod_{j=1,i\neq j}^{N_x^e}(x_i-x_j)$$

In numerical evaluation, Chebyshev-Gauss-Lobatto quadrature points are used, that is,

$$x_i=\frac{L_x^e}{2}\left\{1-\cos\left[\frac{(i-1)\pi}{(N_x^e-1)}\right]\right\}\qquad i=1,2,\ldots,N_x^e \tag{6}$$

Also, to accurately and computationally efficient discretization of both the spatial and temporal domain, the elemental EDQM and the incremental IDQM is employed respectively [Malekzadeh and Rahideh, 2007; Rahideh et al., 2012]. Based on this approach, the total spatial and temporal domain is divided into a set of sub-domains N_e, N_I and in each sub-domain, the DQ rule is employed to discretize the spatial and temporal domain derivatives. For each spatial sub-domain a procedure similar to a single domain as stated before could be repeated. The compatibility as well as the nodal equilibrium should be satisfied at the common nodes of the two adjacent elements. Also at the end of each temporal sub-domain, the velocity is used as the initial condition for the next sub-domain and the weighting coefficients determine, as in the x-direction.

Based on the DQ discretization rules, the DQ discretized form of the governing equation (1) for an element can be written as,

$$\left(\sum_{n=1}^{N_t^I}A_{jn}^{t}u_{in}\right)^{e}+\left(u_{ij}\sum_{m=1}^{N_x^e}A_{im}^{x}u_{mj}\right)^{e}=\left(\nu\sum_{m=1}^{N_x^e}B_{im}^{x}u_{mj}\right)^{e} \tag{7}$$

The compatibility conditions and the nodal equilibrium conditions should be satisfied at the common interface of two adjacent elements. For this purpose, at the interface of two adjacent elements ‘e’ and ‘e + 1’ one has,

$$\left(\frac{\partial u_{N_x^e j}}{\partial x}\right)^{e}-\left(\frac{\partial u_{1,j}}{\partial x}\right)^{e+1}=0 \tag{8}$$

$$\left(u_{N_x^e j}\right)^{e}-\left(u_{1j}\right)^{e+1}=0 \tag{9}$$

In a similar manner the boundary conditions can be discretized. Solving the resulting algebraic system of equations in each temporal sub-domain, the velocity distributions at the spatial grid points and at each time step are obtained [Malekzadeh and Rahideh, 2007; Golbahar Haghighi et al., 2009; Malekzadeh and Rahideh, 2009; Malekzadeh et al., 2010; Rahideh et al.,2012].

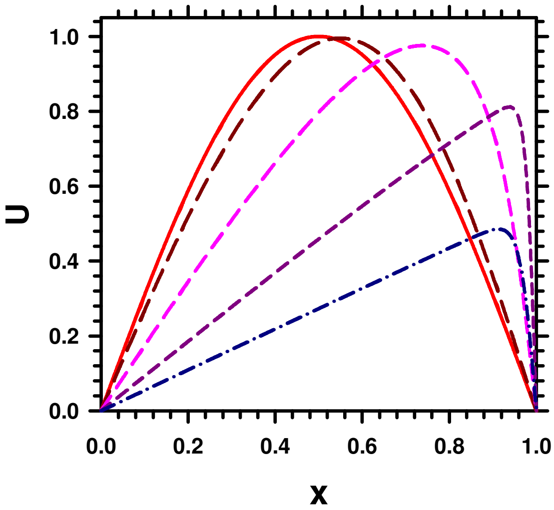
NUMERICAL RESULTS

In order to demonstrate the accuracy of the method, the nonlinear transient one-dimensional Burgers’ equation is considered. The boundary and initial conditions are taken as follows,

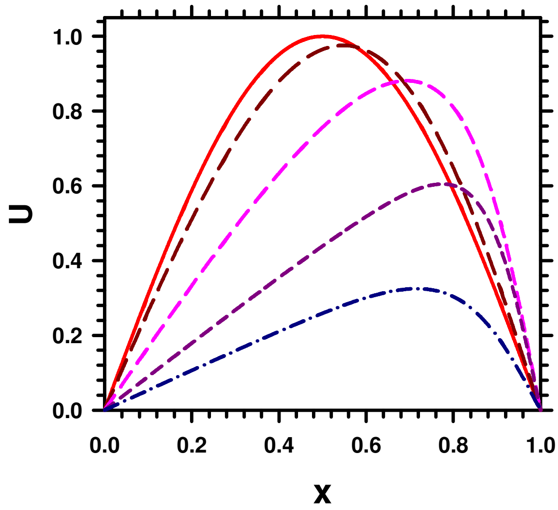
Boundary conditions: $u(0,t)=u(1,t)=0$

Initial conditions: $u(x,0) = \sin(\pi x)$

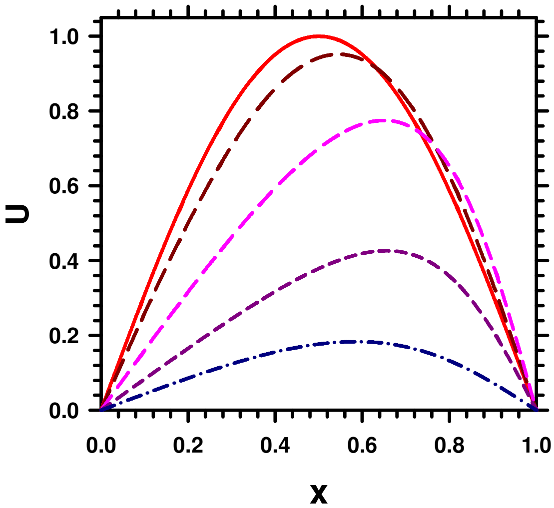
Also, in preparation of the numerical results, the value of the kinematics viscosity is assumed to be unit $(\nu = 0.01)$. The results for the time history of the viscosity at the center of the domain $(x = 0.5)$ and also the viscosity distribution along the domain at the selected time are presented in Fig. 1 and 2, respectively. Excellent agreement between the results of the presented EDQM and the exact solution is obvious.



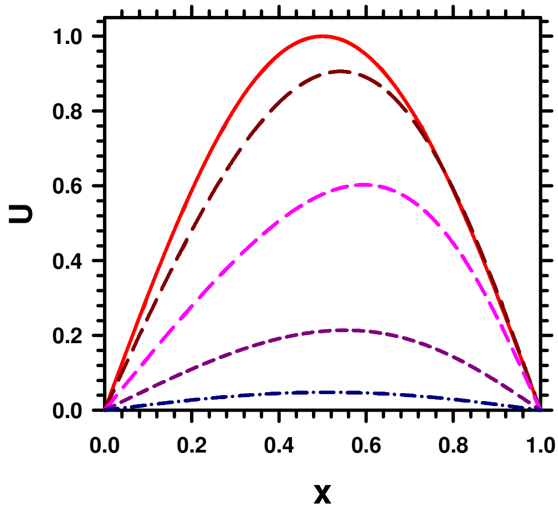
(a) $(\nu = 0.01)$



(b) $(\nu = 0.05)$



(c) $(\nu = 0.1)$



(d) $(\nu = 0.2)$

Fig. 1. (a)-(d). . The distribution of the velocity at different time.

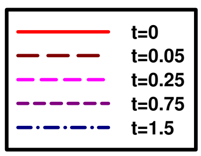


Table 1.Numerical solutions for EDQM and comparison with exact series solution ($\nu = 0.1$)

$N_x^e \times N_x$	$N_t^I \times N_t$	$u(0.5,0.05)$		$u(0.5,0.25)$		$u(0.5,0.75)$		$u(0.5,1.5)$	
		EDQM	*series*	*EDQM*	*series*	*EDQM*	*series*	*EDQM*	*series*
1×3	1×3	*0.96079*	*0.94237*	*0.81897*	*0.70001*	*0.55195*	*0.37892*	*0.30973*	*0.17691*
1×5	1×5	*0.94509*		*0.70759*		*0.37224*		*0.17240*	
1×7	1×7	*0.94250*		*0.69734*		*0.37717*		*0.17670*	
2×7	2×7	*0.94240*		*0.70007*		*0.37898*		*0.17702*	
3×9	2×9	*0.94237*		*0.70001*		*0.37892*		*0.17690*	
3×9	3×9	*0.94237*		*0.70001*		*0.37892*		*0.17691*	

Table 2.Numerical solutions for EDQM and comparison with exact series solution ($\nu = 0.01$)

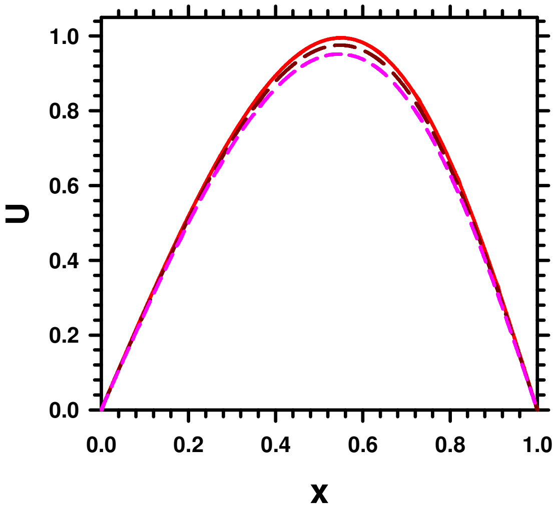
$N_x^e \times N_x$	$N_t^I \times N_t$	$u(0.6,0.4)$		$u(0.6,0.8)$		$u(0.6,1.2)$		$u(0.6,3.0)$	
		EDQM	*series*	*EDQM*	*series*	*EDQM*	*series*	*EDQM*	*series*
5×7	5×5	*0.77451*	*0.77345*	*0.52397*	*0.52401*	*0.39071*	*0.39044*	*0.18025*	*0.18018*
5×7	5×7	*0.77452*		*0.52398*		*0.39070*		*0.18016*	
10×5	5×7	*0.77348*		*0.52400*		*0.39044*		*0.18018*	
10×5	10×5	*0.77348*		*0.52400*		*0.39044*		*0.18019*	
10×7	10×7	*0.77345*		*0.52401*		*0.39044*		*0.18018*	
10×11	10×11	*0.77345*		*0.52401*		*0.39044*		*0.18018*	

The exact solution Presented by Cole for this problem is available as an infinite series for this parameter range $\nu \geq 0.01$ and noted by Miller the infinite series converges too slowly for it to be of practical use for $\nu \prec 0.01$ which is expressed in the Wei et al. paper (1998).

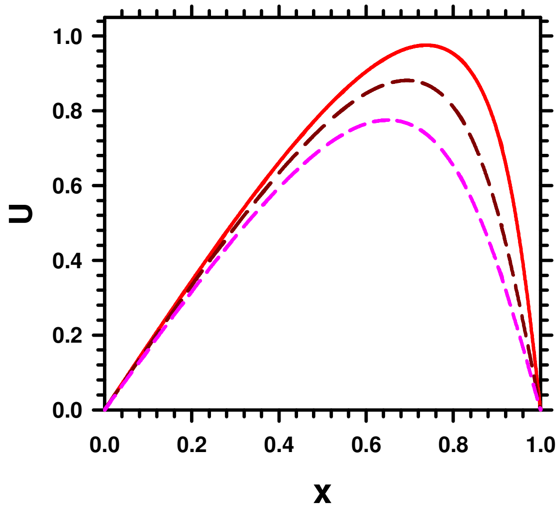
The present work is limited to moderately small ν values (ν = 0.01, 0.05, 0.1 and 0.2). The numerical solution from the EDQM, using number of grid points in time, number of time interval, number of grid points in x direction and number of elements in x direction in different scenarios for $\nu = 0.1$, and the series results [Wei et al., 1998] as shown in Table 1. The present EDQM results are obtain using a distance 0.5 from the non dimensional x axis, the total spatial and temporal domain is divided into a set of sub-domain and in each sub-domain, the EDQ rule is employed to discretize the spatial and temporal domain derivatives.The results indicate that in EDQM, when the number of elements in X direction 3 and the number of grid points in each sub-domain is divided into 9 points and the time axis is used as the meshing, the series solution and EDGM calculations agree to accuracy and are equal.

When the ν value is less or equal than 0.01, smaller meshing leads to detail answers and series is the same results. The numerical solution from the EDQM, using different meshing in spatial and temporal domain for $\nu = 0.01$, and the series results [Wei et al., 1998] as shown in Table 2. The results indicate that in EDQM, when the number of elements in X direction 10 and the number of grid points in each sub-domain is divided into 11 points and the time axis is used as the meshing, the series solution and EDQM calculations agree to accuracy and both are equal.

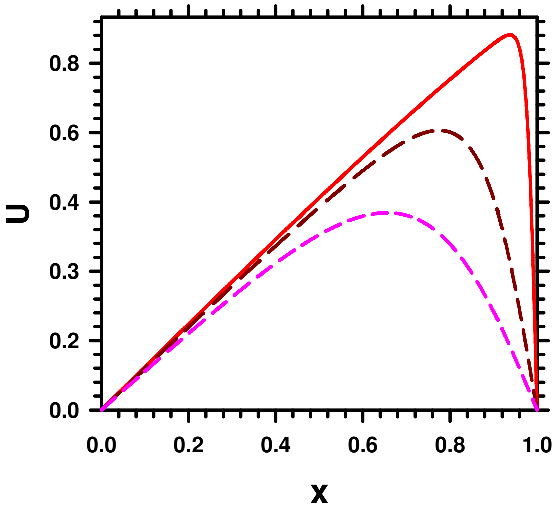
Figure (1) shows the distribution of the velocity versus spatial domain at difference time and different kinematic viscosity. It is clear that in the constant kinematic viscosity, by decreasing the time interval and increasing the number of grid points in time, the velocity distribution is closer to the exact solution (t=0 in series results). Also the accuracy of EDQM with decreasing of kinematic viscosity increases. The trend of variation of velocity distribution at EDQM is represent of the effect of kinematic viscosity on the rate of convergence. Figure (2) shows the distribution of the velocity versus spatial domain at difference kinematic viscosity. It is clear that by decreasing of kinematic viscosity, the velocity of distribution in spatial domain increases. Also by increasing time interval, the velocity distribution for difference kinematic viscosity at Burgers' equation increases.



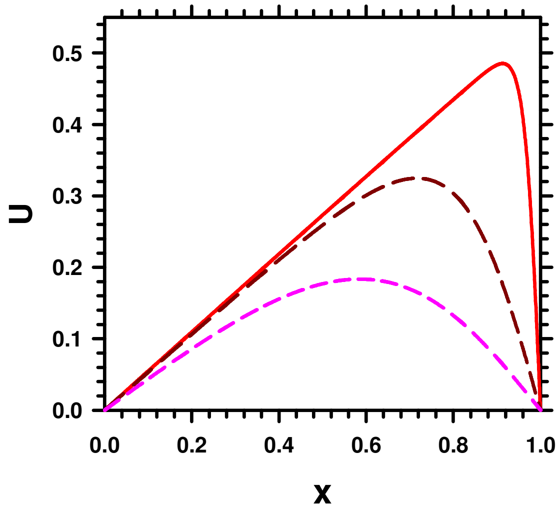
(a) $(t = 0.05)$



(b) $(t = 0.25)$



(c) $(t = 0.75)$



(d) $(t = 1.5)$

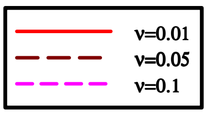


Fig. 2. (a)-(d). The distribution of the velocity for different kinematics viscosity.

*Conclusions***:** As a first attempt, the Element differential quadrature method (EDQM) is employed to study the nonlinear transient one-dimensional Burgers' equation. Both the spatial as well as the temporal domain is discretized using the EDQM. The superior accuracy with fewer degrees of freedom is shown. The convergence and accuracy of the method is demonstrated. The effects of different values of the kinematics viscosity are studied. It is shown that this parameter has significant effect on the velocity distribution of the domain.

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