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# Exact Solutions for Chebyshev Equations by using the Asymptotic Iteration Method

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**ABSTRACT:** The asymptotic iteration method is used in order to solve the Chebyshev differential equations, and to reproduce the Chebyshev polynomials $T_n(x)$, $U_n(x)$ of the first and second kinds respectively. It is shown that the asymptotic iteration method is valid for any degree $n$ .@ JASEM

The Chebyshev polynomials are important in many areas of mathematics, and physics. Particularly in the approximation theory since the roots of the Chebyshev polynomials of the first kind are used in the polynomial interpolation [1]. In the study of differential equations, Chebyshev polynomials arise as the solution to the Chebyshev differential equations

$$(1-x^2)\,y_n''(x) - x\,y_n'(x) + n^2\,y_n(x) = 0, \qquad (1)$$

and

$$\left[(1-x^2)\,y_n''(x) - 3x\,y_n'(x) + n(n+2)\,y_n(x) = 0,\right] \qquad (2)$$

where $n = 0,1,2,3,\ldots$ for the polynomials $T_n(x), U_n(x)$ of the first and second kinds respectively [2,8]. These equations are special case of the Sturm-Liouville equation [3].

Chebyshev polynomials are used virtually in the field of numerical analysis, and it holds particular importance in different subjects including orthogonal polynomials, and polynomial approximation. Ell-gendi [4] has extensively shown how Chebyshev polynomials can be used to solve linear integral equations, integro-differential equations, and ordinary differential equations. Various methods for solving linear and nonlinear ordinary differential equations [5, 6, 7] were devised at about the same time and were based on the discrete orthogonality relationships of the Chebyshev polynomials.

In the literature [2, 8], the Chebyshev differential equations has been solved very heavily using the power series solution method. The reader may face several problems in following the power series solution technique, in which guessing the solution in many cases is very difficult task. Therefore, we applied a new method, the asymptotic iteration method (AIM) [9] to solve this kind of differential equations, where we don't need to use the recurrence relation to find the general solution. This method is very easy to implement in the case of Chebyshev differential equations. The results of this method are very accurate. Moreover, the reader can obtain the solutions without a strong background in mathematics. The paper is organized as follows: in section 2 we will describe the AIM to solve the Chebyshev differential equations. In section 3 our analytical results for the Chebyshev polynomials, and then we conclude and remark therein.

## 2. Formalism of the asymptotic iteration method for the Chebyshev differential equations

The starting point to apply the AIM is to rewrite equations (1) and (2) in the following form:

$$y_n''(x) = k_0(x)\,y_n'(x) + z_0(x)\,y_n(x). \quad (3)$$

Where $k_0(x)$ and $z_0(x)$ are defined for equations (1), and (2) as:

$$k_0)x) = \frac{3ax}{1-x^2}, \qquad z_0(x) = \frac{n(n+2b)}{x^2-1} \quad (4)$$

Note that for equation (1) $a = \frac{1}{3}, b = 0$, and for equation (2) $a = b = 1$.

In order to find a general solution to equation (3) we rely on the symmetric structure of the right -hand side of equation (3). Thus if we differentiate equation (3) with respect to $x$, we obtain [9-12]

$$y'''_n(x) = k_1(x)\,y_n'(x) + z_1(x)\,y_n(x) \quad (5)$$

where

$$k_1(x) = k_0'(x) + z_0(x) + k_0^2(x), \qquad (6)$$
$$z_1(x) = z_0'(x) + z_0(x)\,k_0(x).$$

Likewise, the calculations of the second derivative of equation (3) yield

$$y''''_n(x) = k_2(x)\,y_n'(x) + z_2(x)\,y_n(x) \qquad (7)$$

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where

$$k_2(x) = k_1{}'(x) + z_1(x) + k_0(x)k_1(x), \quad (8)$$
$$z_2(x) = z_1{}'(x) + z_0(x)k_1(x).$$

Thus for $(j+1)$ and $(j+2)^{th}$ derivatives, $j = 1,2,3,\ldots$ we have

$$y_n^{(j+1)}(x) = k_{j-1}(x)y_n{}'(x) + z_{j-1}(x)y_n(x), \quad (9)$$

and

$$y_n^{(j+2)}(x) = k_j(x)y_n{}'(x) + z_j(x)y_n(x), \quad (10)$$

respectively, where

$$k_j(x) = k_{j-1}{}'(x) + z_{j-1}(x) + k_0(x)k_{j-1}(x), \quad (11)$$
$$z_j(x) = z_{j-1}{}'(x) + z_0(x)k_{j-1}(x).$$

The ratio of the $(j+1)$ and $(j+2)^{th}$ derivatives can be expressed as

$$\frac{d}{dx}\ln(y_n{}^{j+1}(x)) = \frac{y_n{}^{j+2}(x)}{y_n{}^{j+1}(x)} = \frac{k_j(y_n{}'(x) + \frac{z_j(x)}{k_j(x)}y_n(x))}{k_{j-1}(y_n{}'(x) + \frac{z_{j-1}(x)}{k_{j-1}(x)}y_n(x))} \quad (12)$$

For sufficiently large $j$, we can introduce the" asymptotic" aspect of the method, that is

$$\frac{z_j(x)}{k_j(x)} = \frac{z_{j-1}(x)}{k_{j-1}(x)} = \varphi(x). \quad (13)$$

Thus equation (12) can be reduced to

$$\frac{d}{dx}\ln(y_n{}^{j+1}(x)) = \frac{k_j(x)}{k_{j-1}(x)} \quad (14)$$

which yields

$$y_n^{j+1}(x) = C_{n1}\exp(\int \frac{k_j(x)}{k_{j-1}(x)}dx) = C_{n1}k_{j-1}(x)\exp[\int\{\varphi(x)+k_0(x)\}dx]$$

(15)

Where $C_{n1}$ is the integration constant, and the right-hand of equation (15) follows from equation (11), and the definition of $\varphi(x)$. Substituting equation (15) into equation (9) we obtain a first-order differential equation

$$y_n{}'(x) + \varphi(x)y_n(x) = C_{n1}\exp[\int\{\varphi(x)+k_0(x)\}dx]$$

(16)

This, in turn, yields the general solution to the equation (3)

$$y_n(x) = \exp(-\int\varphi(x)dx)[C_{n2} + C_{n1}\int\exp[\int\{k_0(x)+2\varphi(x)\}dx])dx] \quad (17)$$

Since the problem is exactly solvable, the first part of the above equation is enough to obtain the general solution. Therefore, the general solution for this case is

$$y_n(x) = C_n\exp(-\int\varphi(x)dx) \quad (18)$$

## 3. Analytical results for the Chebyshev polynomials

Within the framework of the asymptotic iteration method mentioned in the above section, the Chebyshev polynomials are calculate by means of equation (18). To obtain the Chebyshev polynomials, the iterations should be terminated depending on the degree of the polynomial to be found as an approximation to equation (18). On the other hand, the integral in equation (18) depends on the variables $C_n$. We obtain exact results for the Chebyshev polynomials of the first kind, which is the solution of equation (1). However, in equation (2) to reproduce the exact solution of the Chebyshev polynomials of the second kind, we noticed that the output of the results using the Maple software producing a constant of the form

$$C_n = \begin{cases} 1, & n = 2k\,;k \in N \cup \{0\} \\ 2^m, & n\,(\text{mod } 2^{m+1}) \equiv 2^m - 1 \end{cases}$$

(19)

Where $N = 1,2,3,\ldots$ and $m \in N$ .

The results of our calculations with different values of $n$ are given, so that the reader may, if so inclined reproduce our results.

Case (1): The first few Chebyshev polynomials of the first kind are

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1. \end{aligned} \quad (20)$$

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Case (2): The first few Chebyshev polynomials of the second kind are

$$\begin{aligned}
U_0(x) &= C_0[1],\\
U_1(x) &= C_1[2x],\\
U_2(x) &= C_2[4x^2 - 1],\\
U_3(x) &= C_3[8x^3 - 4x],\\
U_4(x) &= C_4[16x^4 - 12x^2 + 1],\\
U_5(x) &= C_5[32x^5 - 32x^3 + 6x],\\
U_6(x) &= C_6[64x^6 - 80x^4 + 20x^2 - 1.
\end{aligned} \tag{21}$$

In all cases, we have only considered the sixth order of polynomials of the first and second kinds. This was so to make a clear comparison between the results of this method and the results of [2, 8]. The obtained polynomials are all in excellent agreement with the exact ones.

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