# Non-minimum tensor rank Gabidulin codes ${ }^{\text {Th }}$ 

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#### Abstract

The tensor rank of some Gabidulin codes of small dimension is investigated. In particular, we determine the tensor rank of any rank metric code equivalent to an 8-dimensional $\mathbb{F}_{q}$-linear generalized Gabidulin code in $\mathbb{F}_{q}^{4 \times 4}$. This shows that such a code is never minimum tensor rank. In this way, we detect the first infinite family of Gabidulin codes which are not minimum tensor rank.


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## 1. Introduction

Rank metric codes were introduced by Delsarte [10] in 1978 and have been used in several contexts, such as crisscross error correction [24], cryptography [15], and network

[^0]coding [27]. Because of their ubiquitous applications, they attracted increasing attention in the last years; see e.g. [16,23,25].

Very recently, rank-metric codes have been investigated through their tensor rank; see $[6,4,5]$. Indeed, a rank-metric code $\mathcal{C}$ in $\mathbb{F}_{q}^{n \times m}$ can be seen as the slice space of an associated generator 3 -tensor, similarly to the case of linear codes in the Hamming metric, where a code can be described as the row space of a generator matrix. Therefore, after Byrne, Neri, Ravagnani and Sheekey [6], the tensor rank of $\mathcal{C}$ is defined as the tensor rank of a generator tensor of $\mathcal{C}$. Determining the tensor rank of a certain rank-metric code is a hard problem in general and the exact value is known only for specific classes of codes; indeed the problem of computing the rank of a 3 -tensor is NP-complete over any finite field [17]. Several lower and upper bounds for the tensor rank of a rank-metric code were presented in [6] and [4]. In particular, as a consequence of Kruskal's bound [12], the tensor rank of an $h$-dimensional $\mathbb{F}_{q}$-linear rank-metric code $\mathcal{C}$ in $\mathbb{F}_{q}^{n \times m}$ of minimum distance $d$ is lower bounded by $h+d-1$. The code $\mathcal{C}$ is said to be minimum tensor rank (MTR for short) if its tensor rank is exactly $h+d-1$. The interest for rank-metric codes with a low tensor rank is due to the following fact: the smaller the tensor rank of the generating tensors, the more efficient the encoding. Via the correspondence in [9] between full rank codes and semifields, the notion of tensor rank for rank-metric codes extends the same notion for semifields, which was used as an invariant by Lavrauw in [18]. Moreover, some criteria by Kruskal [12, Section 4] use the rank of a tensor to assure its identifiability, i.e. the uniqueness of the pure tensors appearing in its decomposition, which is of interest for the numerical applications within statistics; see [7] and [1, Section $2]$.

A family of particular interest among rank-metric codes is the one of square Gabidulin codes $\mathcal{G}_{k, s}$ in $\mathbb{F}_{q}^{n \times n}$, as they are maximum rank distance, and indeed they have been deeply investigated. However, their tensor rank is not known in general; exact results have been provided in [6] and [4] when $k \in\{1, n-1\}$ and in few other cases. Interestingly, when $q$ is large enough, Gabidulin codes with $k \in\{1, n-1\}$ turn out to be MTR codes.

In this paper we are interested in determining the tensor rank of those codes which are equivalent to an $\mathbb{F}_{q}$-linear 8-dimensional Gabidulin code in $\mathbb{F}_{q}^{4 \times 4}$. The strategy that we apply makes use of [6, Proposition 3.4], which involves rank-one matrices. The framework of our arguments is the one of linearized polynomials, where rank-one matrices correspond to trace functions of the shape $\alpha \operatorname{Tr}(\beta x)$ for some nonzero $\alpha, \beta \in \mathbb{F}_{q^{4}}$ (see [22, Theorem 2.24]), where $\operatorname{Tr}: \mathbb{F}_{q^{4}} \rightarrow \mathbb{F}_{q}$ and $\operatorname{Tr}(x)=x+x^{q}+x^{q^{2}}+x^{q^{3}}$. Our main result is the following.

Theorem 1.1. Let $q$ be a prime power, and $\mathcal{C}$ be a code which is equivalent to an $\mathbb{F}_{q}$-linear 8-dimensional generalized Gabidulin code in $\mathbb{F}_{q}^{4 \times 4}$. Then the tensor rank of $\mathcal{C}$ is 11 if $q \geq 3$, and 12 if $q=2$. In particular, $\mathcal{C}$ is not MTR.

The paper is organized as follows. Section 2 contains preliminary notions on rankmetric codes and on the correspondence with linearized polynomials in the case of square
codes. Section 3 describes basic definitions and known results about tensors and the tensor rank of square generalized Gabidulin codes. Section 4 is devoted to the proof of Theorem 1.1: Section 4.1 shows that $\mathcal{C}$ is not MTR, while in Section 4.2 we determine the tensor rank of $\mathcal{C}$ for $q \geq 5$. The remaining small values of $q$, are worked out computationally in Section 5, as well as other Gabidulin codes in $\mathbb{F}_{q}^{n \times n}$ with small values of $q$ and $n$. Finally, the Appendix contains two auxiliary results which are needed in Section 4.1, whose proof are quite technical.

## 2. Rank metric codes and linearized polynomials

The set $\mathbb{F}_{q}^{n \times m}$ of matrices can be equipped with the rank-metric, defined as

$$
d(A, B)=\operatorname{rk}(A-B), \quad \text { for } A, B \in \mathbb{F}_{q}^{n \times m}
$$

A rank-metric code is a subset $\mathcal{C}$ of $\mathbb{F}_{q}^{n \times m}$ endowed with the rank-metric and its minimum rank distance is defined as

$$
d=d(\mathcal{C})=\min \{d(A, B): A, B \in \mathcal{C}, A \neq B\}
$$

Two $\mathbb{F}_{q}$-linear rank-metric codes $\mathcal{C}$ and $\mathcal{C}^{\prime}$ in $\mathbb{F}_{q}^{n \times m}$ are linearly equivalent if and only if there exist $X \in \operatorname{GL}(n, q)$ and $Y \in \operatorname{GL}(m, q)$ such that

$$
\mathcal{C}^{\prime}=\{X C Y: C \in \mathcal{C}\}
$$

or, if $m=n$,

$$
\mathcal{C}^{\prime}=\left\{X C^{\top} Y: C \in \mathcal{C}\right\}
$$

where $C^{\top}$ denote the transpose of $C$. Since in this paper we will only consider linear equivalence, we will refer to it simply as equivalence.

Delsarte showed in [10] that the parameters of a rank-metric code $\mathcal{C}$ satisfy a Singletonlike bound, namely

$$
|\mathcal{C}| \leq q^{\max \{m, n\}(\min \{m, n\}-d+1)}
$$

When equality holds, we call $\mathcal{C}$ a maximum rank distance (MRD for short) code.
In this paper we are interested only in the square case $m=n$, and in this case rank-metric codes can be described in terms of linearized polynomials. Indeed, consider the $\mathbb{F}_{q}$-linearized (or simply linearized) polynomials of degree less than $q^{n}$ over $\mathbb{F}_{q^{n}}$, i.e. elements of the form

$$
f(x)=\sum_{i=0}^{n-1} f_{i} x^{q^{i}}, \quad f_{i} \in \mathbb{F}_{q^{n}}
$$

The set of linearized polynomials is an $\mathbb{F}_{q}$-algebra $\mathcal{L}_{n, q}$ with the usual addition, scalar multiplication by elements of $\mathbb{F}_{q}$ and composition modulo $x^{q^{n}}-x$. It is well-known that the $\mathbb{F}_{q^{-}}$-algebras $\mathcal{L}_{n, q}$ and $\operatorname{End}_{\mathbb{F}_{q}}\left(\mathbb{F}_{q^{n}}\right)$ are isomorphic, via the correspondence between the linearized polynomial $f(x)$ and the $\mathbb{F}_{q^{-}}$-endomorphism

$$
\alpha \longmapsto \sum_{i=0}^{n-1} f_{i} \alpha^{q^{i}}
$$

of $\mathbb{F}_{q^{n}}$. Hence, $\mathcal{L}_{n, q}$ is also isomorphic to the $\mathbb{F}_{q^{-}}$-algebra $\mathbb{F}_{q}^{n \times n}$ of $n \times n$ matrices over $\mathbb{F}_{q}$. In this correspondence, the rank of a matrix in $\mathbb{F}_{q}^{n \times n}$ equals the rank of the corresponding linearized polynomial in $\mathcal{L}_{n, q}$ as an $\mathbb{F}_{q^{-}}$-endomorphism of $\mathbb{F}_{q^{n}}$. Therefore, rank-metric codes in $\mathbb{F}_{q}^{n \times n}$ can be seen as sets of linearized polynomials in $\mathcal{L}_{n, q}$, so that we can speak of rank-metric codes in $\mathcal{L}_{n, q}$. Notice that the set of matrices of rank 1 in $\mathbb{F}_{q}^{n \times n}$ corresponds to the set of elements of $\mathcal{L}_{n, q}$ of the shape $\alpha \operatorname{Tr}(\beta x)$ for some $\alpha, \beta \in \mathbb{F}_{q^{n}}^{*}$, where $\operatorname{Tr}(z)=z+z^{q}+\cdots+z^{q^{n-1}}$; see [22, Theorem 2.24]. For a reference on linearized polynomials see [28].

The first class of square MRD codes was the one of generalized Gabidulin codes, namely the $\mathbb{F}_{q^{n}}$-subspaces

$$
\mathcal{G}_{k, s}=\left\langle x, x^{q^{s}}, \ldots, x^{q^{s(k-1)}}\right\rangle_{\mathbb{F}_{q^{n}}}
$$

of $\mathcal{L}_{n, q}$, where $1 \leq k \leq n$ and $\operatorname{gcd}(s, n)=1$; they are MRD codes with $\mathbb{F}_{q}$-dimension $k n$ and minimum distance $n-k+1$. Gabidulin codes were first introduced by Delsarte in [10] and later by Gabidulin in [14] in the case $s=1$, and by Gabidulin and Kshevetskiy in [13] in the general case.

## 3. Tensor rank of generalized Gabidulin codes

The tensors we will investigate in this paper are 3-tensors in $\mathbb{F}_{q}^{h} \otimes \mathbb{F}_{q}^{n} \otimes \mathbb{F}_{q}^{m}$. If $\left\{u_{1}, \ldots, u_{h}\right\},\left\{v_{1}, \ldots, v_{n}\right\}$, and $\left\{w_{1}, \ldots, w_{n}\right\}$ are bases of $\mathbb{F}_{q}^{h}, \mathbb{F}_{q}^{n}$, and $\mathbb{F}_{q}^{m}$ respectively, then an $\mathbb{F}_{q}$-basis of $\mathbb{F}_{q}^{h} \otimes \mathbb{F}_{q}^{n} \otimes \mathbb{F}_{q}^{m}$ is given by

$$
\left\{u_{l} \otimes v_{i} \otimes w_{j}: 1 \leq l \leq h, 1 \leq i \leq n, 1 \leq j \leq m\right\}
$$

The tensors of the form $u \otimes v \otimes w$, with $u \in \mathbb{F}_{q}^{h}, v \in \mathbb{F}_{q}^{n}$ and $w \in \mathbb{F}_{q}^{m}$, are called simple (or pure) tensors. The tensor rank of a tensor $X \in \mathbb{F}_{q}^{h} \otimes \mathbb{F}_{q}^{n} \otimes \mathbb{F}_{q}^{m}$ is defined as

$$
\operatorname{trk}(X)=\min \left\{R \in \mathbb{N}_{0}: X=\sum_{i=1}^{R} u_{i} \otimes v_{i} \otimes w_{i}, u_{i} \in \mathbb{F}_{q}^{h}, v_{i} \in \mathbb{F}_{q}^{n}, w_{i} \in \mathbb{F}_{q}^{m}\right\}
$$

Let $[i]=\{1, \ldots, i\}$. A 3-tensor $X \in \mathbb{F}_{q}^{h} \otimes \mathbb{F}_{q}^{n} \otimes \mathbb{F}_{q}^{m}$ can be represented as a map $X:[h] \times$ $[n] \times[m] \rightarrow \mathbb{F}_{q}$ given by $X=\left(X_{l i j}: 1 \leq l \leq h, 1 \leq i \leq n, 1 \leq j \leq m\right)$. Therefore
$\mathbb{F}_{q}^{h} \otimes \mathbb{F}_{q}^{n} \otimes \mathbb{F}_{q}^{m}$ can be identified with the space $\mathbb{F}_{q}^{h \times n \times m}$, and the tensor $X$ can be written as $X=\left(X_{1}, \ldots, X_{h}\right)$ with $X_{i} \in \mathbb{F}_{q}^{n \times m}$. The first slice space of $X$ (also known as first contraction space), denoted by $\mathrm{ss}_{1}(X)$, is the $\mathbb{F}_{q}$-subspace of $\mathbb{F}_{q}^{n \times m}$ generated by $X_{1}, \ldots, X_{h}$. If $\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathrm{ss}_{1}(X)\right)=h$, we say that $X$ is 1-nondegenerate.

The following result will be a key tool in our investigation.
Proposition 3.1. (see [6, Proposition 3.4] and [3, Proposition 14.45]) Let $X \in \mathbb{F}_{q}^{h \times n \times m}$ and $R$ be a positive integer. The following are equivalent:

1. $\operatorname{trk}(X) \leq R$;
2. there exist $A_{1}, \ldots, A_{R} \in \mathbb{F}_{q}^{n \times m}$ of rank 1 such that $\mathrm{ss}_{1}(X) \subseteq\left\langle A_{1}, \ldots, A_{R}\right\rangle_{\mathbb{F}_{q}}$.

In particular, $\operatorname{trk}(X)=R$ if and only if $R$ is the minimum integer such that there exist $A_{1}, \ldots, A_{R} \in \mathbb{F}_{q}^{n \times m}$ of rank 1 satisfying $\operatorname{ss}_{1}(X) \subseteq\left\langle A_{1}, \ldots, A_{R}\right\rangle_{\mathbb{F}_{q}}$.

Kruskal in [12] bounded the tensor rank of a 3-tensor, using the following map:

$$
m_{1}: \mathbb{F}_{q}^{s \times h} \times \mathbb{F}_{q}^{h \times n \times m} \rightarrow \mathbb{F}_{q}^{s \times n \times m}, \quad\left(A, \sum_{i} u_{i} \otimes v_{i} \otimes w_{i}\right) \mapsto \sum_{i}\left(A u_{i}\right) \otimes v_{i} \otimes w_{i}
$$

Theorem 3.2. (see [12, Corollary 1]) Let $X \in \mathbb{F}_{q}^{h \times n \times m}$ be 1-nondegenerate, then

$$
\operatorname{trk}(X) \geq h+\min \left\{\operatorname{trk}\left(m_{1}(u, X)\right): u \in \mathbb{F}_{q}^{h} \backslash\{0\}\right\}-1
$$

Tensors are related to rank-metric codes as follows. Let $\mathcal{C}$ be an $\mathbb{F}_{q}$-linear code in $\mathbb{F}_{q}^{n \times m}$ of dimension $h$ and minimum distance $d$. A generator tensor for $\mathcal{C}$ is a 3-tensor $X \in \mathbb{F}_{q}^{h \times n \times m}$ such that $\mathrm{ss}_{1}(X)=\mathcal{C}$. Note that

$$
d=\min \left\{\operatorname{trk}\left(m_{1}(u, X)\right): u \in \mathbb{F}_{q}^{h} \backslash\{0\}\right\}
$$

As proved in [6, Proposition 4.2], two generator tensors of the same rank-metric code $\mathcal{C}$ have the same tensor rank. Therefore, we can define the tensor rank $\operatorname{trk}(\mathcal{C})$ of $\mathcal{C}$ as the tensor rank of any generator tensor of $\mathcal{C}$.

Proposition 3.3. (see [6, Proposition 4.5]) If $\mathcal{C}, \mathcal{C}^{\prime}$ are equivalent codes, then $\operatorname{trk}(\mathcal{C})=$ $\operatorname{trk}\left(\mathcal{C}^{\prime}\right)$.

By Theorem 3.2,

$$
\begin{equation*}
\operatorname{trk}(\mathcal{C}) \geq h+d-1 \tag{3.1}
\end{equation*}
$$

If $\mathcal{C}$ attains equality in (3.1), it is called a minimum tensor rank (MTR for short) code.

Although Gabidulin codes form the most studied family of rank-metric codes, the complete determination of their tensor rank is still missing. We now describe the known results on the tensor rank of square Gabidulin codes $\mathcal{G}_{k, s} \subset \mathcal{L}_{n, q}$. Since in this case $d=n-k+1$, the bound (3.1) reads as follows.

Theorem 3.4. For every $k \leq n$, we have $\operatorname{trk}\left(\mathcal{G}_{k, s}\right) \geq(k+1) n-k$.

The tensor rank of $\mathcal{G}_{1, s}$ coincides with the tensor rank of the field $\mathbb{F}_{q^{n}}$ (see [11] and [21] where semifields were described for the first time in terms of tensors). By [3, Propositions 14.47 and 14.48] and a link with a well-studied tensor pointed out in [6, Lemma 5.13], it follows that $\operatorname{trk}\left(\mathcal{G}_{1, s}\right)=2 n-1$ if $q \geq 2 n-1$, and $\operatorname{trk}\left(\mathcal{G}_{1, s}\right)>2 n-1$ if $q \leq 2 n-2$. For $n=3, \operatorname{trk}\left(\mathcal{G}_{1, s}\right)=6$ if $q \in\{2,3\}$ (see [20, Lemma 15] and also [19]). For $n=4$, $\operatorname{trk}\left(\mathcal{G}_{1, s}\right)=9$ if $q=2$ (as proved by Chudnovsky-Chudnovsky [8], see also [6, Example $6.4]$ ), or $q=3$ (see [20, Theorem 4]), while $\operatorname{trk}\left(\mathcal{G}_{1, s}\right)$ is unknown for $n=4$ and $q \in\{4,5\}$.

Further bounds and asymptotic results for the tensor rank of $\mathbb{F}_{q^{n}}$ are known, see e.g. [2].

The following upper bound follows from the tensor rank of $\mathcal{G}_{1, s}$.

Theorem 3.5. (see [6, Proposition 5.15]) Let $q \geq 2 n-2$. For every $k \leq n$, we have $\operatorname{trk}\left(\mathcal{G}_{1, s}\right) \leq \min \left\{n^{2}, k(2 n-1)\right\}$.

A partial result is known also in the case of Gabidulin codes $\mathcal{G}_{n-1, s}$.
Theorem 3.6. (see [4, Theorem 5.15]) Let $q \geq n$. Then $\operatorname{trk}\left(\mathcal{G}_{n-1, s}\right)=n^{2}-n+1$.

The tensor rank of Gabidulin codes $\mathcal{G}_{k, s}$ with $k \notin\{1, n-1\}$ is not known. In this paper we study the first open case, namely $k=2$ and $n=4$. In Section 5 we will investigate the remaining open cases when $n \leq 4$.

## 4. The tensor rank of $\mathcal{G}_{2,1} \subset \mathcal{L}_{4, q}$

The two 8-dimensional generalized Gabidulin codes $\mathcal{G}_{2,1}$ and $\mathcal{G}_{2,3}$ in $\mathcal{L}_{4, q}$ are easily seen to be equivalent. Therefore, by Proposition 3.3, in order to prove Theorem 1.1 it is enough to prove it for the Gabidulin code $\mathcal{G}=\mathcal{G}_{2,1}=\left\langle x, x^{q}\right\rangle_{\mathbb{F}_{q^{4}}}$. In Section 4.1 we show that the tensor rank of $\mathcal{G}$ is not 10 for any $q$. In Section 4.2 we prove that the tensor rank of $\mathcal{G}$ is 11 if $q \geq 5$. We complete the proof in Section 5, where we determine the tensor rank of some Gabidulin codes for some values of $q$.

### 4.1. The tensor rank of $\mathcal{G}$ is larger than 10

This section is devoted to the proof of the following theorem.

Theorem 4.1. For any prime power $q$, we have $\operatorname{trk}(\mathcal{G}) \geq 11$. Thus, $\mathcal{G}$ is not an $M T R$ code.

By Proposition 3.1 and Section $2, \operatorname{trk}(\mathcal{G})=10$ if and only if there exist 10 trace functions $\alpha_{i} \operatorname{Tr}\left(\beta_{i} x\right)$ such that $\mathcal{G} \subseteq\left\langle\alpha_{1} \operatorname{Tr}\left(\beta_{1} x\right), \ldots, \alpha_{10} \operatorname{Tr}\left(\beta_{10} x\right)\right\rangle_{\mathbb{F}_{q}}$ which are $\mathbb{F}_{q}$-linearly independent. Note that this happens if and only if the two sets $\left\{\alpha_{i}: 1 \leq i \leq 10\right\}$ and $\left\{\beta_{i}: 1 \leq i \leq 10\right\}$ are $\mathbb{F}_{q}$-linearly independent. Moreover, $\alpha_{i} \operatorname{Tr}\left(\beta_{i} x\right)$ and $\alpha_{j} \operatorname{Tr}\left(\beta_{j} x\right)$ are $\mathbb{F}_{q}$-linear dependent if and only if $\left(\alpha_{i}, \beta_{i}\right)=\rho\left(\alpha_{j}, \beta_{j}\right)$, for some $\rho \in \mathbb{F}_{q}$. This is equivalent to say that there exist $\alpha_{1} \operatorname{Tr}\left(\beta_{1} x\right), \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right)$ such that there exists an $\mathbb{F}_{q}$-basis of

$$
\left\langle x, x^{q}\right\rangle_{\mathbb{F}_{q^{4}}} \oplus\left\langle\alpha_{1} \operatorname{Tr}\left(\beta_{1} x\right), \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right)\right\rangle_{\mathbb{F}_{q}}
$$

only composed of traces. So, consider $\alpha_{1}, \alpha_{2}, \beta_{1}, \beta_{2} \in \mathbb{F}_{q^{4}}$ such that $H=\left\langle x, x^{q}\right\rangle_{\mathbb{F}_{q^{4}}} \oplus$ $\left\langle\alpha_{1} \operatorname{Tr}\left(\beta_{1} x\right), \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right)\right\rangle_{\mathbb{F}_{q}}$ has dimension 10 over $\mathbb{F}_{q}$. Since the stabilizer of $\mathcal{G}$ acts transitively on rank one elements, ${ }^{1}$ we may assume that $\alpha_{1}=\beta_{1}=1$.

The proof strategy relies on two steps:
Step 1: To find explicit necessary and sufficient conditions on $\alpha_{3}, \beta_{3} \in \mathbb{F}_{q^{4}}$ such that $\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right) \in H$.
Step 2: To prove the non-existence of ten $\mathbb{F}_{q}$-linearly independent traces in $H$.
In Steps 1 and 2 we will also need auxiliary results (Theorems A. 1 and A. 2 respectively) which are in the Appendix, due to their technicality.

Proof. Step 1: Let us find explicit conditions on the coefficients of $\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right)$ in such a way that it belongs to $H$. Suppose that $\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right)$ is in $H$. Then there exist $\gamma, \delta \in \mathbb{F}_{q^{4}}$, $c_{1}, c_{2} \in \mathbb{F}_{q}$ such that

$$
\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right)=\gamma x+\delta x^{q}+c_{1} \operatorname{Tr}(x)+c_{2} \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right) .
$$

This polynomial identity implies that $\gamma, \delta, c_{1}$, and $c_{2}$ satisfy the following system:

$$
\begin{cases}\gamma+c_{1}+c_{2} \alpha_{2} \beta_{2} & =\alpha_{3} \beta_{3},  \tag{4.1}\\ \delta+c_{1}+c_{2} \alpha_{2} \beta_{2}^{q} & =\alpha_{3} \beta_{3}^{q}, \\ c_{1}+c_{2} \alpha_{2} \beta_{2}^{q^{2}} & =\alpha_{3} \beta_{3}^{q^{2}}, \\ c_{1}+c_{2} \alpha_{2} \beta_{2}^{q^{3}} & =\alpha_{3} \beta_{3}^{q^{3}}\end{cases}
$$

We will now obtain information on $\alpha_{3}$ and $\beta_{3}$ manipulating System (4.1).
It cannot happen that $\beta_{2}$ and $\alpha_{2}$ are both in $\mathbb{F}_{q}$, otherwise $\alpha_{3} \operatorname{Tr}\left(\beta_{3}\right)$ and $\operatorname{Tr}(x)$ are $\mathbb{F}_{q}$-linearly dependent and hence $\operatorname{dim}_{\mathbb{F}_{q}}(H)<10$.

[^1]Note that $c_{1} c_{2} \neq 0$, since the sum of a map of rank at least 3 (that is $\gamma x+\delta x^{q}$ ) and one of rank one (either $c_{1} \operatorname{Tr}(x)$ or $c_{2} \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right)$ ) cannot have rank one. Therefore, we may assume that $c_{2}=1$.

From now on we always assume $c_{1} \neq 0$. By the last two equations in System (4.1) one gets

$$
\begin{equation*}
\beta_{3}^{q-1}=\frac{c_{1}+\alpha_{2}^{q^{2}} \beta_{2}^{q}}{c_{1}+\alpha_{2}^{q^{2}} \beta_{2}} . \tag{4.2}
\end{equation*}
$$

Note that, since $\beta_{3} \neq 0, c_{1}+\alpha_{2}^{q^{2}} \beta_{2}^{q}=0$ if and only if $c_{1}+\alpha_{2}^{q^{2}} \beta_{2}=0$, that is $1 / \beta_{2}$ and $1 / \alpha_{2}$ both belong to $\mathbb{F}_{q}^{*}$. Again, this is a contradiction to our assumptions.

An element $\beta_{3} \in \mathbb{F}_{q^{4}}$ satisfying Equation (4.2) exists if and only if

$$
\left(\frac{c_{1}+\alpha_{2}^{q^{2}} \beta_{2}^{q}}{c_{1}+\alpha_{2}^{q^{2}} \beta_{2}}\right)^{1+q+q^{2}+q^{3}}=1
$$

that is,

$$
\begin{align*}
& \left(c_{1}+\alpha_{2}^{q^{2}} \beta_{2}^{q}\right)\left(c_{1}+\alpha_{2}^{q^{3}} \beta_{2}^{q^{2}}\right)\left(c_{1}+\alpha_{2} \beta_{2}^{q^{3}}\right)\left(c_{1}+\alpha_{2}^{q} \beta_{2}\right)= \\
& \quad\left(c_{1}+\alpha_{2}^{q^{2}} \beta_{2}\right)\left(c_{1}+\alpha_{2}^{q^{3}} \beta_{2}^{q}\right)\left(c_{1}+\alpha_{2} \beta_{2}^{q^{2}}\right)\left(c_{1}+\alpha_{2}^{q} \beta_{2}^{q^{3}}\right) \tag{4.3}
\end{align*}
$$

We are interested in bounding the number of non- $\mathbb{F}_{q}$-proportional elements $c_{1}$, with $c_{1} \neq 0$. Indeed, this will allow us to determine explicit conditions on $\beta_{3}$. The above polynomial in $c_{1}$ is of degree at most three in $c_{1}$, and its coefficients are as follows:
i) the coefficient of degree 0 is zero;
ii) the coefficient of degree 1 is

$$
\begin{aligned}
& -\alpha_{2}^{q+q^{2}+q^{3}} \beta_{2}^{1+q+q^{3}}+\alpha_{2}^{q+q^{2}+q^{3}} \beta_{2}^{1+q+q^{2}}+\alpha_{2}^{1+q^{2}+q^{3}} \beta_{2}^{q+q^{2}+q^{3}}-\alpha_{2}^{1+q^{2}+q^{3}} \beta_{2}^{1+q+q^{2}} \\
& -\alpha_{2}^{1+q+q^{3}} \beta_{2}^{q+q^{2}+q^{3}}+\alpha_{2}^{1+q+q^{3}} \beta_{2}^{1+q^{2}+q^{3}}-\alpha_{2}^{1+q+q^{2}} \beta_{2}^{1+q^{2}+q^{3}}+\alpha_{2}^{1+q+q^{2}} \beta_{2}^{1+q+q^{3}}
\end{aligned}
$$

iii) the coefficient of degree 2 is

$$
\begin{aligned}
& \alpha_{2}^{q^{2}+q^{3}} \beta_{2}^{q+q^{2}}-\alpha_{2}^{q^{2}+q^{3}} \beta_{2}^{1+q}-\alpha_{2}^{q+q^{3}} \beta_{2}^{q+q^{3}}+\alpha_{2}^{q+q^{3}} \beta_{2}^{1+q^{2}}-\alpha_{2}^{q+q^{2}} \beta_{2}^{1+q^{3}}+\alpha_{2}^{q+q^{2}} \beta_{2}^{1+q} \\
& +\alpha_{2}^{1+q^{3}} \beta_{2}^{q^{2}+q^{3}}-\alpha_{2}^{1+q^{3}} \beta_{2}^{q+q^{2}}+\alpha_{2}^{1+q^{2}} \beta_{2}^{q+q^{3}}-\alpha_{2}^{1+q^{2}} \beta_{2}^{1+q^{2}}-\alpha_{2}^{1+q} \beta_{2}^{q^{2}+q^{3}}+\alpha_{2}^{1+q} \beta_{2}^{1+q^{3}}
\end{aligned}
$$

iv) the coefficient of degree 3 is

$$
\alpha_{2}^{q^{3}} \beta_{2}^{q^{2}}-\alpha_{2}^{q^{3}} \beta_{2}^{q}+\alpha_{2}^{q^{2}} \beta_{2}^{q}-\alpha_{2}^{q^{2}} \beta_{2}-\alpha_{2}^{q} \beta_{2}^{q^{3}}+\alpha_{2}^{q} \beta_{2}+\alpha_{2} \beta_{2}^{q^{3}}-\alpha_{2} \beta_{2}^{q^{2}}
$$

Therefore the number of non- $\mathbb{F}_{q}$-proportional solutions in $c_{1}$ with $c_{1} \neq 0$ is at most 2 , if the polynomial is non-vanishing. Moreover, this polynomial vanishes if and only if

$$
\left\{\begin{array}{l}
Y Z^{q}-Y Z^{q^{2}}+Y^{q} Z^{q^{2}}-Y^{q} Z^{q^{3}}-Y^{q^{2}} Z+Y^{q^{2}} Z^{q^{3}}+Y^{q^{3}} Z-Y^{q^{3}} Z^{q}=0  \tag{4.4}\\
Y^{q+1} Z^{q^{2}+q}-Y^{q+1} Z^{q^{3}+q^{2}}-Y^{q^{2}+1} Z^{q^{2}+1}+Y^{q^{2}+1} Z^{q^{3}+q} \\
+Y^{q^{3}+1} Z^{q+1}-Y^{q^{3}+1} Z^{q^{2}+q}-Y^{q^{2}+q} Z^{q^{3}+1}+Y^{q^{2}+q} Z^{q^{3}+q^{2}}+Y^{q^{3}+q} Z^{q^{2}+1} \\
-Y^{q^{3}+q} Z^{q^{3}+q}-Y^{q^{3}+q^{2}} Z^{q+1}+Y^{q^{3}+q^{2}} Z^{q^{3}+1}=0 \\
Y^{q^{2}+q+1} Z^{q^{3}+q^{2}+1}-Y^{q^{2}+q+1} Z^{q^{3}+q^{2}+q}-Y^{q^{3}+q+1} Z^{q^{2}+q+1}+Y^{q^{3}+q+1} Z^{q^{3}+q^{2}+q} \\
+Y^{q^{3}+q^{2}+1} Z^{q^{2}+q+1}-Y^{q^{3}+q^{2}+1} Z^{q^{3}+q+1}+Y^{q^{3}+q^{2}+q} Z^{q^{3}+q+1}-Y^{q^{3}+q^{2}+q} Z^{q^{3}+q^{2}+1}=0
\end{array}\right.
$$

where $Y=1 / \beta_{2}$ and $Z=1 / \alpha_{2}$. The solutions $(Y, Z) \in \mathbb{F}_{q^{4}}$ of System (4.4) are given in Theorem A.1. From now on we will suppose that $Y=1 / \beta_{2}$ and $Z=1 / \alpha_{2}$ are solutions of System (4.4). In this case, by Equation (4.2), the maximum number of non- $\mathbb{F}_{q^{-}}$-proportional possible values of $\beta_{3} \in \mathbb{F}_{q^{4}}$ is $q-1$ when $c_{1}$ runs in $\mathbb{F}_{q}^{*}$. By System (4.1), to each such value of $\beta_{3}$ there corresponds at most one value of $\alpha_{3} \in \mathbb{F}_{q^{4}}$.

Define $\lambda=c_{1} \in \mathbb{F}_{q}^{*}$, so that $\beta_{3}=\beta_{3}(\lambda)$ satisfies

$$
\begin{equation*}
\beta_{3}^{q-1}(\lambda)=\frac{\lambda+\alpha_{2}^{q^{2}} \beta_{2}^{q}}{\lambda+\alpha_{2}^{q^{2}} \beta_{2}} \tag{4.5}
\end{equation*}
$$

Now, let $N(\lambda)=\lambda Z^{q^{2}} Y^{q}+1$ and $D(\lambda)=\lambda Z^{q^{2}} Y+1$, so that $\beta_{3}^{q-1}(\lambda)=$ $\beta_{2}^{q-1} N(\lambda) / D(\lambda)$ and, by the third equation of System (4.1),

$$
\begin{equation*}
\alpha_{3} \beta_{3}=c_{2} \alpha_{2} \beta_{2}^{q^{2}} D^{q^{2}}(\lambda) / \beta_{3}^{q^{2}-1}=c_{2} \alpha_{2} \beta_{2} D^{q^{2}+q+1}(\lambda) / N^{q+1}(\lambda) \tag{4.6}
\end{equation*}
$$

and $\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right)$ reads

$$
\begin{aligned}
& \alpha_{3} \beta_{3}\left(x+\beta_{3}^{q-1} x^{q}+\beta_{3}^{q^{2}-1} x^{q^{2}}+\beta_{3}^{q^{3}-1} x^{q^{3}}\right) \\
& =\frac{c_{2} \alpha_{2} \beta_{2} D^{q^{2}+q+1}(\lambda)}{N^{q+1}(\lambda)}\left(x+\beta_{2}^{q-1} \frac{N(\lambda)}{D(\lambda)} x^{q}+\beta_{2}^{q^{2}-1} \frac{N^{q+1}(\lambda)}{D^{q+1}(\lambda)} x^{q^{2}}+\beta_{2}^{q^{3}-1} \frac{N^{q^{2}+q+1}(\lambda)}{D^{q^{2}+q+1}(\lambda)} x^{q^{3}}\right) \\
& =c_{2}\left(\alpha_{2} \beta_{2} \frac{D^{q^{2}+q+1}(\lambda)}{N^{q+1}(\lambda)} x+\alpha_{2} \beta_{2}^{q} \frac{D^{q^{2}+q}(\lambda)}{N^{q}(\lambda)} x^{q}+\alpha_{2} \beta_{2}^{q^{2}} D^{q^{2}}(\lambda) x^{q^{2}}+\alpha_{2} \beta_{2}^{q^{3}} N^{q^{2}}(\lambda) x^{q^{3}}\right) .
\end{aligned}
$$

Step 2: In the previous step, we have determined all the possible expressions of a trace function to be in $H$ depending on a certain value $\lambda$. We now prove that there exist no eight distinct values $\lambda_{1}, \ldots, \lambda_{8} \in \mathbb{F}_{q}^{*}$ such that the eight rank-one linear functions $F_{i}(x)=\alpha_{2} \beta_{2} \frac{D^{q^{2}+q+1}\left(\lambda_{i}\right)}{N^{q+1}\left(\lambda_{i}\right)} x+\alpha_{2} \beta_{2}^{q} \frac{D^{q^{2}+q}\left(\lambda_{i}\right)}{N^{q}\left(\lambda_{i}\right)} x^{q}+\alpha_{2} \beta_{2}^{q^{2}} D^{q^{2}}\left(\lambda_{i}\right) x^{q^{2}}+\alpha_{2} \beta_{2}^{q^{3}} N^{q^{2}}\left(\lambda_{i}\right) x^{q^{3}}$ and $F_{9}(x)=\operatorname{Tr}(x), F_{10}(x)=$ alpha $a_{2} \operatorname{Tr}\left(\beta_{2} x\right)$ are $\mathbb{F}_{q^{\prime}}$-linearly independent. By Proposition 3.1, this will yield that $\operatorname{trk}(\mathcal{G}) \geq 11$.

Equivalently, we prove the existence of $\mu_{1}, \ldots, \mu_{10} \in \mathbb{F}_{q}$ such that

$$
\mu_{1} F_{1}(x)+\cdots+\mu_{10} F_{10}(x)=0
$$

and not all the $\mu_{i}$ 's are zero, so that the ten traces $F_{i}(x), i=1, \ldots, 10$, are $\mathbb{F}_{q}$-linearly dependent. Let $\mu_{1}, \ldots, \mu_{10} \in \mathbb{F}_{q}$ be such that

$$
\begin{equation*}
\mu_{1} F_{1}(x)+\cdots+\mu_{10} F_{10}(x)=0 \tag{4.7}
\end{equation*}
$$

In particular, Equation (4.7) can be seen as a polynomial identity; the coefficients of degree $q^{3}$ and $q^{2}$ yield to

$$
\alpha_{2} \beta_{2}^{q^{3}}\left(\sum_{i=1}^{8} \mu_{i} N^{q^{2}}\left(\lambda_{i}\right)+\mu_{10}\right)+\mu_{9}=0=\alpha_{2} \beta_{2}^{q^{2}}\left(\sum_{i=1}^{8} \mu_{i} D^{q^{2}}\left(\lambda_{i}\right)+\mu_{10}\right)+\mu_{9}
$$

Since $N\left(\lambda_{i}\right)=\lambda_{i} Z^{q^{2}} Y^{q}+1$ and $D\left(\lambda_{i}\right)=\lambda_{i} Z^{q^{2}} Y+1$,

$$
\sum_{i=1}^{8} \mu_{i}+\mu_{10}=-\left(\sum_{i=1}^{8} \mu_{i} \lambda_{i}+\mu_{9}\right) Y^{q^{3}} Z \text { and } \sum_{i=1}^{8} \mu_{i}+\mu_{10}=-\left(\sum_{i=1}^{8} \mu_{i} \lambda_{i}+\mu_{9}\right) Y^{q^{2}} Z
$$

Suppose that $Y^{q^{3}} Z=Y^{q^{2}} Z$, which is equivalent to $Y \in \mathbb{F}_{q}$. Then $N(\lambda)=D(\lambda)$ and $\beta_{3}(\lambda)^{q-1}=\beta_{2}^{q-1}=1$. By System (4.1), this implies $\alpha_{3} \operatorname{Tr}\left(\beta_{3} x\right)=c_{1} \operatorname{Tr}(x)+c_{2} \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right) \in$ $\left\langle\operatorname{Tr}(x), \alpha_{2} \operatorname{Tr}\left(\beta_{2} x\right)\right\rangle_{\mathbb{F}_{q}}$ and hence the $F_{i}{ }^{\prime} \mathrm{s}, i=1, \ldots, 10$, are linearly dependent.

We can then assume that $Y^{q^{3}} Z \neq Y^{q^{2}} Z$, so that

$$
\begin{equation*}
\mu_{9}+\sum_{i=1}^{8} \mu_{i} \lambda_{i}=\sum_{i=1}^{8} \mu_{i}+\mu_{10}=0 . \tag{4.8}
\end{equation*}
$$

Also, by looking at the coefficients of degree 1 and $q$ in Equation (4.7),

$$
\begin{equation*}
\alpha_{2} \beta_{2} \sum_{i=1}^{8} \mu_{i} \frac{D^{q^{2}+q+1}(\lambda)}{N^{q+1}(\lambda)}+\mu_{9}+\mu_{10} \alpha_{2} \beta_{2}=0=\alpha_{2} \beta_{2}^{q} \sum_{i=1}^{8} \mu_{i} \frac{D^{q^{2}+q}(\lambda)}{N^{q}(\lambda)}+\mu_{9}+\mu_{10} \alpha_{2} \beta_{2}^{q} \tag{4.9}
\end{equation*}
$$

Equations (4.9) and their images under the $q$-Frobenius map, together with Equations (4.8), form a homogeneous linear system of ten equations whose matrix is

$$
M=\left(\begin{array}{ccccc}
1 & 1 & \cdots & 0 & 1  \tag{4.10}\\
\lambda_{1} & \lambda_{2} & \cdots & 1 & 0 \\
\frac{D^{q^{2}+q+1}\left(\lambda_{1}\right)}{N^{q+1}\left(\lambda_{1}\right)} & \frac{D^{q^{2}+q+1}\left(\lambda_{2}\right)}{N^{q+1}\left(\lambda_{2}\right)} & \cdots & Y Z & 1 \\
\frac{D^{q^{3}+q^{2}+q}\left(\lambda_{1}\right)}{N^{q^{2}+q}\left(\lambda_{1}\right.} & \frac{D^{q^{3}+q^{2}+q}\left(\lambda_{2}\right)}{N^{q^{2}+q}\left(\lambda_{2}\right)} & \cdots & Y^{q} Z^{q} & 1 \\
\frac{D^{q^{3}+q^{2}+1}\left(\lambda_{1}\right)}{N^{q^{3}+q^{2}\left(\lambda_{1}\right)}} & \frac{D^{q^{3}+q^{2}+1}\left(\lambda_{2}\right)}{N^{q^{3}+q^{2}\left(\lambda_{2}\right)}} & \cdots & Y^{q^{2}} Z^{q^{2}} & 1 \\
\frac{D^{q^{3}+++1}\left(\lambda_{1}\right)}{N^{3}+1\left(\lambda_{1}\right)} & \frac{D^{q^{3}+q+1}\left(\lambda_{2}\right)}{N^{q^{3+1}\left(\lambda_{2}\right)}} & \cdots & Y^{q^{3}} Z^{q^{3}} & 1 \\
\frac{D^{q^{2}+q}\left(\lambda_{1}\right)}{N^{q}\left(\lambda_{1}\right)} & \frac{D^{q^{2}+q}\left(\lambda_{2}\right)}{N^{q}\left(\lambda_{2}\right)} & \cdots & Y^{q} Z & 1 \\
\frac{D^{q^{3}+q^{2}\left(\lambda_{1}\right)}}{N^{q^{2}}\left(\lambda_{1}\right)} & \frac{D^{q^{3}+q^{2}\left(\lambda_{2}\right)}}{N^{q^{2}\left(\lambda_{2}\right)}} & \cdots & Y^{q^{2}} Z^{q} & 1 \\
\frac{D^{3}+1}{N^{q^{3}\left(\lambda_{1}\right)}} & \frac{D^{q^{3}+1}\left(\lambda_{1}\right)}{N^{q^{3}}\left(\lambda_{2}\right)} & \cdots & Y^{q^{3}} Z^{q^{2}} & 1 \\
\frac{D^{q+1}\left(\lambda_{1}\right)}{N\left(\lambda_{1}\right)} & \frac{D^{q+1}\left(\lambda_{2}\right)}{N\left(\lambda_{2}\right)} & \cdots & Y Z^{q^{3}} & 1
\end{array}\right) .
$$

Since the rows of $M$ form orbits under the $q$-Frobenius map, the solutions of the associated system have entries in $\mathbb{F}_{q}$. By Theorem A.2, the rank of $M$ is either 2 or 6 , and hence smaller than 10 . Therefore, there are non-trivial solutions $\left(\mu_{1}, \ldots, \mu_{10}\right) \in \mathbb{F}_{q}^{10}$ of $\mu_{1} F_{1}(x)+\cdots+\mu_{10} F_{10}(x)=0$. Then $\left\langle F_{1}(x), \ldots, F_{10}(x)\right\rangle_{\mathbb{F}_{q}}$ has dimension smaller than 10. This shows that $\operatorname{trk}(\mathcal{G}) \geq 11$. Thus, Theorem 4.1 is proved.

### 4.2. The tensor rank of $\mathcal{G}$ is 11 for $q \geq 5$

In this section we use the notations of Section 4.1, and assume that $q \geq 5$. By Theo$\operatorname{rem} 4.1, \operatorname{trk}(\mathcal{G}) \geq 11$. We prove the following theorem.

Theorem 4.2. For any prime power $q \geq 5$, we have $\operatorname{trk}(\mathcal{G})=11$.

By Proposition 3.1, it is enough to show the existence of $11 \mathbb{F}_{q}$-linearly independent trace functions whose $\mathbb{F}_{q}$-span contains $\mathcal{G}$. Our key tool is Step 1 in Section 4.1, where we have determined some necessary and sufficient criteria on the coefficients of a trace function for it being in $H$.

Proof. Let $\alpha_{0}, \beta_{0} \in \mathbb{F}_{q^{4}}^{*}$ and $\lambda_{1}, \lambda_{1}^{\prime}, \ldots, \lambda_{4}, \lambda_{4}^{\prime} \in \mathbb{F}_{q}^{*}$ with $\lambda_{i} \neq \lambda_{j}$ and $\lambda_{i}^{\prime} \neq \lambda_{j}^{\prime}$ for $i \neq j$. Let $\alpha, \beta, \alpha^{\prime}, \beta^{\prime} \in \mathbb{F}_{q^{4}}^{*}$ be such that $Y=\beta / \beta_{0}, Z=\alpha / \alpha_{0}, Y^{\prime}=\beta^{\prime} / \beta_{0}$ and $Z^{\prime}=\alpha^{\prime} / \alpha_{0}$ satisfy $Z, Z^{\prime} \notin \mathbb{F}_{q^{2}}$ and $Y=1 / Z^{q^{2}+q}, Y^{\prime}=1 /\left(Z^{\prime}\right)^{q^{2}+q}$. By Theorem A.1, $(Y, Z)$ and $\left(Y^{\prime}, Z^{\prime}\right)$ are solutions of System (4.4). As in the proof of Theorem 4.1, by (4.5) and (4.6) for any $i \in\{1, \ldots, 4\}$ there exist $\alpha_{i}, \beta_{i}, \alpha_{i}^{\prime}, \beta_{i}^{\prime} \in \mathbb{F}_{q^{4}}^{*}$ such that

- $\beta_{i}^{q-1}=\beta_{0}^{q-1} N\left(\lambda_{i}\right) / D\left(\lambda_{i}\right)$;
- $\left(\beta_{i}^{\prime}\right)^{q-1}=\beta_{0}^{q-1} N^{\prime}\left(\lambda_{i}^{\prime}\right) / D^{\prime}\left(\lambda_{i}^{\prime}\right)$;
- $\alpha_{i}=c_{i} \alpha_{0} \beta_{0} D\left(\lambda_{i}\right)^{q^{2}+q+1} / N\left(\lambda_{i}\right)^{q+1}$;
- $\alpha_{i}^{\prime}=c_{i}^{\prime} \alpha_{0} \beta_{0} D^{\prime}\left(\lambda_{i}^{\prime}\right)^{q^{2}+q+1} / N^{\prime}\left(\lambda_{i}^{\prime}\right)^{q+1}$,
where $c_{i}, c_{i}^{\prime} \in \mathbb{F}_{q}^{*}, N\left(\lambda_{i}\right)=\lambda_{i} Z^{q^{2}} Y^{q}+1, N^{\prime}\left(\lambda_{i}^{\prime}\right)=\lambda_{i}^{\prime}\left(Z^{\prime}\right)^{q^{2}}\left(Y^{\prime}\right)^{q}+1, D\left(\lambda_{i}\right)=\lambda_{i} Z^{q^{2}} Y+1$ and $D^{\prime}\left(\lambda_{i}^{\prime}\right)=\lambda_{i}^{\prime}\left(Z^{\prime}\right)^{q^{2}} Y^{\prime}+1$.

Define the rank-one functions $F_{0}(x)=\alpha_{0} \operatorname{Tr}\left(\beta_{0} x\right), F(x)=\alpha \operatorname{Tr}(\beta x), F^{\prime}(x)=$ $\alpha^{\prime} \operatorname{Tr}\left(\beta^{\prime} x\right)$ and, for $i \in\{1, \ldots, 4\}, F_{i}(x)=\frac{1}{c_{i}} \alpha_{i} \operatorname{Tr}\left(\beta_{i} x\right), F_{i}^{\prime}(x)=\frac{1}{c_{i}^{\prime}} \alpha_{i}^{\prime} \operatorname{Tr}\left(\beta_{i}^{\prime} x\right)$. All such functions are elements of the linear $\mathbb{F}_{q^{4} \text {-space }} V=\mathcal{G}+\left\langle F(x), F^{\prime}(x), F_{0}(x)\right\rangle_{\mathbb{F}_{q^{4}}}$ because of Step 1 in Section 4.1. We show that, for some suitable choice of the elements $\lambda_{i}, \lambda_{i}^{\prime}, \alpha_{0}, \beta_{0}, \alpha, \beta, \alpha^{\prime}, \beta^{\prime}$, the 11 elements $F(x), F^{\prime}(x), F_{0}(x), F_{1}(x), \ldots, F_{4}(x)$, $F_{1}^{\prime}(x), \ldots, F_{4}^{\prime}(x)$ are $\mathbb{F}_{q}$-linearly independent, which implies $\operatorname{trk}(\mathcal{G})=11$.

Let $\mu, \mu^{\prime}, \mu_{0}, \mu_{1}, \mu_{1}^{\prime}, \ldots, \mu_{4}, \mu_{4}^{\prime} \in \mathbb{F}_{q}$ be such that

$$
\mu F(x)+\mu^{\prime} F^{\prime}(x)+\mu_{0} F_{0}(x)+\mu_{1} F_{1}(x)+\cdots \mu_{4} F_{4}(x)+\mu_{1}^{\prime} F_{1}^{\prime}(x)+\cdots \mu_{4}^{\prime} F_{4}^{\prime}(x)=0
$$

which can be seen a polynomial identity and hence implies

$$
\begin{cases}\mu \alpha \beta+\mu^{\prime} \alpha^{\prime} \beta^{\prime}+\mu_{0} \alpha_{0} \beta_{0}+\alpha_{0} \beta_{0} \sum_{i=1}^{4}\left(\mu_{i} \frac{D^{q^{2}+q+1}\left(\lambda_{i}\right)}{N^{q+1}\left(\lambda_{i}\right)}+\mu_{i}^{\prime} \frac{\left(D^{\prime}\right)^{q^{2}+q+1}\left(\lambda_{i}\right)}{\left(N^{\prime}\right)^{q+1}\left(\lambda_{i}\right)}\right) & =0  \tag{4.11}\\ \mu \alpha \beta^{q}+\mu^{\prime} \alpha^{\prime}\left(\beta^{\prime}\right)^{q}+\mu_{0} \alpha_{0} \beta_{0}^{q}+\alpha_{0} \beta_{0}^{q} \sum_{i=1}^{4}\left(\mu_{i} \frac{D^{q^{2}+q}\left(\lambda_{i}\right)}{N^{q}\left(\lambda_{i}\right)}+\mu_{i}^{\prime} \frac{\left(D^{\prime}\right)^{q^{2}+q}\left(\lambda_{i}\right)}{\left(N^{\prime}\right)^{q}\left(\lambda_{i}\right)}\right) & =0 \\ \mu \alpha \beta^{q^{2}}+\mu^{\prime} \alpha^{\prime}\left(\beta^{\prime}\right)^{q^{2}}+\mu_{0} \alpha_{0} \beta_{0}^{q^{2}}+\alpha_{0} \beta_{0}^{q^{2}} \sum_{i=1}^{4}\left(\mu_{i} D^{q^{2}}\left(\lambda_{i}\right)+\mu_{i}^{\prime}\left(D^{\prime}\right)^{q^{2}}\left(\lambda_{i}\right)\right) & =0 \\ \mu \alpha \beta^{q^{3}}+\mu^{\prime} \alpha^{\prime}\left(\beta^{\prime}\right)^{q^{3}}+\mu_{0} \alpha_{0} \beta_{0}^{q^{3}}+\alpha_{0} \beta_{0}^{q^{3}} \sum_{i=1}^{4}\left(\mu_{i} N^{q^{2}}\left(\lambda_{i}\right)+\mu_{i}^{\prime}\left(N^{\prime}\right)^{q^{2}}\left(\lambda_{i}\right)\right)=0 .\end{cases}
$$

The four equations in (4.11), together with their images under the $q$-Frobenius map, provide a homogeneous linear system of twelve equations with solutions in $\mathbb{F}_{q}^{11}$, of which $\left(\mu, \mu^{\prime}, \mu_{0}, \mu_{1}, \mu_{1}^{\prime}, \ldots, \mu_{4}, \mu_{4}^{\prime}\right)$ is a solution. The matrix $M$ of such a system is

Since $\left|\mathbb{F}_{q}^{*}\right| \geq 4$, we can choose $\lambda_{1} \neq 0, \lambda_{1}^{2} \neq 1$ and $\lambda_{1}^{3} \neq 1$ and then

- $\lambda_{2}=\lambda_{1}^{2}, \lambda_{3}=\lambda_{1}^{3}$ and $\lambda_{4}=\lambda_{1}^{4}$;
- $\lambda_{i}=\lambda_{i-4}$ for any $i \in\{5,6,7,8\}$.

We also choose $\alpha_{0}, \beta_{0}, \alpha, \beta, \alpha^{\prime}, \beta^{\prime}$ such that $Z^{q^{2}+1}=1$ and $Z^{\prime}=Z^{q}$. By direct computation with MAGMA,

$$
\begin{aligned}
\operatorname{det}(M)= & \lambda_{1}^{40}\left(\lambda_{1}-1\right)^{12}\left(\lambda_{1}+1\right)^{4}\left(\lambda_{1}^{2}+\lambda_{1}+1\right)^{2}\left(Z^{2}-1\right)^{6 q+6}\left(Z^{q}-Z\right)^{4}\left(Z^{q+1}-1\right)^{4} \\
& \cdot\left(Z^{3 q+2}-Z^{2 q+1}-2 Z^{q+2}+Z^{q}+Z^{3}\right)\left(Z^{3 q+1}-Z^{2 q+2}+Z^{q+3}-2 Z^{q+1}+1\right) \\
& \cdot\left(Z^{3 q}+Z^{2 q+3}-2 Z^{2 q+1}-Z^{q+2}+Z\right)\left(Z^{3 q+3}-2 Z^{2 q+2}+Z^{2 q}-Z^{q+1}+Z^{2}\right)^{2} .
\end{aligned}
$$

For some $Z \in \mathbb{F}_{q^{4}} \backslash \mathbb{F}_{q^{2}}$ satisfying $Z^{q^{2}+1}=1$, we have $\operatorname{det}(M) \neq 0$; for $q \geq 16$ this follows because $q^{2}+1$ is greater than the sum of the degrees of the polynomials in parentheses, while for $q<16$ this follows by direct checking. Therefore, for a suitable choice of $\lambda_{i}, \lambda_{i}^{\prime}, \alpha_{0}, \beta_{0}, \alpha, \beta, \alpha^{\prime}, \beta^{\prime}$, the matrix $M$ has full rank 11 and hence

$$
\left(\mu, \mu^{\prime}, \mu_{0}, \mu_{1}, \mu_{1}^{\prime}, \ldots, \mu_{4}, \mu_{4}^{\prime}\right)=(0, \ldots, 0) .
$$

Thus, $F(x), F^{\prime}(x), F_{0}(x), F_{1}(x), F_{1}^{\prime}(x), \ldots, F_{4}(x), F_{4}^{\prime}(x)$ are $\mathbb{F}_{q}$-linearly independent and $\mathcal{G}$ has tensor rank 11 .

## 5. Tensor rank of $n \times n$ generalized Gabidulin codes for $n \leq 4$

We compute the tensor rank of some generalized Gabidulin code $\mathcal{C} \subseteq \mathcal{L}_{n, q}$ of dimension $k$ over $\mathbb{F}_{q^{n}}$ for $n \leq 4$. Notice that, up to equivalence, $\mathcal{C}=\mathcal{G}_{k, 1}$.

By Section 4.2 and [4, Table 1], the open cases are exactly for $n, k, q$ as in the table below. Since the lower bound on the tensor rank of $\mathcal{G}_{k, 1}$ is $n k+n-k$, we start with the exhaustive search for $t=n-k$ rank-one functions $\alpha_{i} \operatorname{Tr}\left(\beta_{i} x\right) \in \mathbb{F}_{q^{n}}[x]$ such that the rank-one functions in $\mathcal{C}+\left\langle\alpha_{1} \operatorname{Tr}\left(\beta_{1} x\right), \ldots, \alpha_{t} \operatorname{Tr}\left(\beta_{t} x\right)\right\rangle_{\mathbb{F}_{q}}$ generate an $\mathbb{F}_{q}$-space $U$ of dimension $n k+t$. If this succeeds, then we compute explicitly a perfect basis of $U$ (i.e. a basis of pure tensors). Otherwise, we increase $t$ by 1 and perform the same search again. In this way we obtain the tensor rank and a perfect basis $B=\left\{\eta^{i} \operatorname{Tr}\left(\eta^{j} x\right):(i, j) \in I\right\}$ for $\mathcal{G}_{k, 1}$, where $\eta$ is a primitive element of $\mathbb{F}_{q^{n}}$ and $I \subseteq\left\{0, \ldots, q^{n}-2\right\}^{2}$. The precise value of the tensor rank is obtained for all but two cases, namely $n=4, k=1$ and $q \in\{4,5\}$; in these cases, an upper bound is provided by means of a random search.

For instance, in the case $(n, k, q)=(3,2,2)$ Table 1 provides the following perfect basis $B$ for $\mathcal{G}_{2,1} \subseteq \mathcal{L}_{3,2}$, where $\eta \in \mathbb{F}_{2^{3}}$ satisfies $\eta^{3}+\eta+1=0$ :

$$
\begin{aligned}
B= & \left\{\eta^{2} \operatorname{Tr}(x),(\eta+1) \operatorname{Tr}\left(\left(\eta^{2}+\eta+1\right) x\right), \eta \operatorname{Tr}\left(\eta^{2} x\right), \operatorname{Tr}\left(\left(\eta^{2}+\eta\right) x\right)\right. \\
& \left.\left(\eta^{2}+1\right) \operatorname{Tr}\left(\left(\eta^{2}+1\right) x\right),\left(\eta^{2}+\eta+1\right) \operatorname{Tr}(\eta x),\left(\eta^{2}+\eta\right) \operatorname{Tr}((\eta+1) x)\right\}
\end{aligned}
$$

Remark 5.1. The fourth, fifth, and sixth rows of the table complete the proof of Theorem 1.1.

Table 1
Tensor rank of some generalized Gabidulin codes $\mathcal{G}_{k, 1} \subseteq \mathcal{L}_{n, q}$.

| $n$ | $k$ | $q$ | $\operatorname{TR}\left(\mathcal{G}_{k, 1}\right)$ | MTR | $\operatorname{MinPol}(\eta)$ | $I$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 2 | 2 | 7 | yes | $x^{3}+x+1$ | $(2,0),(3,5),(1,2),(0,4),(6,6),(5,1),(4,3)$ |
| 4 | 1 | 4 | 8 | no | $\begin{aligned} & x^{8}+x^{4}+ \\ & x^{3}+x^{2}+1 \end{aligned}$ | $\begin{aligned} & (73,168),(22,202),(0,180),(69,249), \\ & (80,90),(1,96),(33,213),(67,162) \end{aligned}$ |
| 4 | 1 | 5 | 8 | no | $\begin{aligned} & x^{4}+4 x^{2} \\ & +4 x+2 \end{aligned}$ | $\begin{aligned} & (16,432),(135,81),(21,405),(10,132), \\ & (56,593),(24,556),(74,569),(54,268) \end{aligned}$ |
| 4 | 2 | 2 | 12 | no | $x^{4}+x+1$ | $\begin{aligned} & (14,12),(9,14),(4,0),(7,5),(1,13),(5,10), \\ & (6,3),(2,1),(12,7),(3,2),(0,4),(13,6) \end{aligned}$ |
| 4 | 2 | 3 | 11 | no | $x^{4}-x^{3}-1$ | $\begin{aligned} & (23,8),(0,13),(28,14),(32,46),(2,1),(19,26), \\ & (1,18),(6,37),(7,12),(36,28),(21,59) \end{aligned}$ |
| 4 | 2 | 4 | 11 | no | $\begin{aligned} & x^{8}+x^{4}+ \\ & x^{3}+x^{2}+1 \end{aligned}$ | $\begin{aligned} & (13,133),(56,175),(20,71),(30,31),(81,124),(0,51), \\ & (3,88),(76,34),(70,215),(29,132),(9,24) \end{aligned}$ |
| 4 | 3 | 2 | 13 | yes | $x^{4}+x+1$ | $\begin{aligned} & (12,13),(8,6),(4,4),(9,4),(1,5),(3,1),(14,9), \\ & (11,0),(10,2),(0,7),(6,10),(7,8),(5,12) \end{aligned}$ |
| 4 | 3 | 3 | 13 | yes | $x^{4}-x^{3}-1$ | $\begin{aligned} & (31,3),(29,49),(0,56),(25,61),(7,75),(26,18),(22,30), \\ & (20,36),(39,19),(18,2),(13,57),(32,40),(3,47) \end{aligned}$ |

Remark 5.2. The examples provided in Table 1 for the case $n=4, k=1$ and $q \in\{4,5\}$ only prove that $\operatorname{TR}\left(\mathcal{G}_{1,1}\right) \leq 8$. However, in [6, Corollary 5.14] it has been proved that a generator tensor for $\mathcal{G}_{1,1}$ is the 3 -tensor

$$
T_{n, n, n}:(g, h) \in \mathbb{F}_{q}[x]_{<n} \times \mathbb{F}_{q}[x]_{<n} \mapsto g h \quad(\bmod f) \in \mathbb{F}_{q}[x]_{<n},
$$

where $\mathbb{F}_{q}[x]_{<n}$ is the set of all polynomials in $\mathbb{F}_{q}[x]$ with degree less than $n$ and $f$ is a fixed irreducible polynomial in $\mathbb{F}_{q}[x]$ of degree $n$. Therefore, $T_{n, n, n}$ is also the tensor associated with the multiplication in $\mathbb{F}_{q^{n}}$. Applying the main result in [26] (see also [2, Theorem 9.1]) to the case $n=4$ and $q \in\{4,5\}$ we obtain that $\operatorname{trk}\left(T_{4,4,4}\right)=8$ and hence $\operatorname{TR}\left(\mathcal{G}_{1,1}\right)=8$.

Remark 5.3. Notice that, although only one perfect basis is showed in the table, the computations provide a much larger number of perfect bases in each case. Therefore, no generator tensor of such codes is identifiable.

## Declaration of competing interest

There are no conflict of interests.

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## Appendix A

Theorem A.1. Let $Y, Z \in \mathbb{F}_{q^{4}}$. Then $(Y, Z)$ is a solution of System (4.4) if and only if one of the following pairwise mutually exclusive conditions holds:
(C1) $Y \in \mathbb{F}_{q}$ or $Z \in \mathbb{F}_{q}$;
(C2) $Y \notin \mathbb{F}_{q^{2}}$ and $Z=\rho Y^{q+1}$ for some $\rho \in \mathbb{F}_{q}^{*}$;
(C3) $Z \notin \mathbb{F}_{q^{2}}$ and $Y=\rho / Z^{q^{2}+q}$ for some $\rho \in \mathbb{F}_{q}^{*}$.
Proof. For any $i=0,1,2,3$, write $y_{i}=Y^{q^{i}}$ and $z_{i}=Z^{q^{i}}$. Then System (4.4) reads

$$
\left\{\begin{array}{l}
f_{1}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)=0  \tag{A.1}\\
f_{2}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)=0 \\
f_{3}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)=0
\end{array}\right.
$$

where

$$
\begin{aligned}
f_{1}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)= & y_{0} z_{1}-y_{0} z_{2}+y_{1} z_{2}-y_{1} z_{3}-y_{2} z_{0}+y_{2} z_{3}+y_{3} z_{0}-y_{3} z_{1}, \\
f_{2}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)= & y_{0} y_{1} z_{1} z_{2}-y_{0} y_{1} z_{2} z_{3}-y_{0} y_{2} z_{0} z_{2}+y_{0} y_{2} z_{1} z_{3}+y_{0} y_{3} z_{0} z_{1} \\
& -y_{0} y_{3} z_{1} z_{2}-y_{1} y_{2} z_{0} z_{3}+y_{1} y_{2} z_{2} z_{3}+y_{1} y_{3} z_{0} z_{2}-y_{1} y_{3} z_{1} z_{3} \\
& -y_{2} y_{3} z_{0} z_{1}+y_{2} y_{3} z_{0} z_{3} \\
f_{3}\left(y_{0}, y_{1}, y_{2}, y_{3}, z_{0}, z_{1}, z_{2}, z_{3}\right)= & y_{0} y_{1} y_{2} z_{0} z_{2} z_{3}-y_{0} y_{1} y_{2} z_{1} z_{2} z_{3}-y_{0} y_{1} y_{3} z_{0} z_{1} z_{2} \\
& +y_{0} y_{1} y_{3} z_{1} z_{2} z_{3}+y_{0} y_{2} y_{3} z_{0} z_{1} z_{2}-y_{0} y_{2} y_{3} z_{0} z_{1} z_{3} \\
& +y_{1} y_{2} y_{3} z_{0} z_{1} z_{3}-y_{1} y_{2} y_{3} z_{0} z_{2} z_{3} .
\end{aligned}
$$

We denote by $\operatorname{Res}_{x}\left(g_{1}, g_{2}\right)$ the resultant of two (multivariate) polynomials $g_{1}$ and $g_{2}$ with respect to the indeterminate $x$. We have

$$
\begin{aligned}
\operatorname{Res}_{z_{1}}\left(\operatorname{Res}_{z_{0}}\left(f_{3}, f_{1}\right), \operatorname{Res}_{z_{0}}\left(f_{2}, f_{1}\right)\right)= & -\left(z_{2}-z_{3}\right)^{2} y_{3}\left(y_{2}-y_{3}\right)\left(y_{2} z_{3}-y_{3} z_{2}\right) . \\
& \left(y_{1}-y_{2}\right)^{2}\left(y_{0}-y_{3}\right)\left(y_{0}-y_{1}\right)^{2}\left(y_{0} z_{2}-y_{3} z_{3}\right) . \\
& \left(y_{0} z_{2}-y_{2} z_{3}\right)^{2}\left(y_{0} y_{2}-y_{1} y_{3}\right)\left(y_{1}-y_{3}\right) .
\end{aligned}
$$

Thus, every solution $(Y, Z) \in \mathbb{F}_{q^{4}}^{2}$ of System (4.4) with $Y Z \neq 0$ satisfies one of the following conditions.

1. $y_{3}=0$, that is $Y^{q^{3}}=0$, a contradiction.
2. $z_{2}-z_{3}=0$, that is $Z \in \mathbb{F}_{q}$. Indeed $(Y, Z)$ is a solution of System (4.4) whenever $Z \in \mathbb{F}_{q}$. In the following cases we can then assume $Z \notin \mathbb{F}_{q}$.
3. $y_{2}-y_{3}=0$, or $y_{1}-y_{2}=0$, or $y_{0}-y_{3}=0$, or $y_{0}-y_{1}=0$. This is equivalent to $Y \in \mathbb{F}_{q}$, and indeed $(Y, Z)$ is a solution of System (4.4) whenever $Y \in \mathbb{F}_{q}$. In the following cases we can then assume $Y \notin \mathbb{F}_{q}$.
4. $y_{1}-y_{3}=0$, that is $Y \in \mathbb{F}_{q^{2}}$. Then System (4.4) reads

$$
\left\{\begin{array}{l}
Z^{q^{2}+q+1}-Z^{q^{3}+q+1}+Z^{q^{3}+q^{2}+1}-Z^{q^{3}+q^{2}+q}=0 \\
Z-Z^{q}+Z^{q^{2}}-Z^{q^{3}}=0 \\
\left(Y^{q}+Y\right)\left(Z^{q^{2}+1}-Z^{q^{3}+q}\right)=0
\end{array}\right.
$$

The first equation yields $Z^{q^{3}+q^{2}+1}+Z^{q^{2}+q+1} \in \mathbb{F}_{q}$, that is $Z^{q^{2}+1}\left(Z^{q^{2}}+Z\right)^{q} \in \mathbb{F}_{q}$, while the second equation yields $Z^{q^{2}}+Z \in \mathbb{F}_{q}$. Therefore $Z^{q^{2}+1} \in \mathbb{F}_{q}$, and hence the third equation is also satisfied. Now, the two conditions $Z^{q^{2}}+Z \in \mathbb{F}_{q}$ and $Z^{q^{2}+1} \in \mathbb{F}_{q}$ yield $Z^{q^{3}}=Z^{q^{2}}-Z^{q}+Z$ and $Z^{q^{3}}=Z^{q^{2}-q+1}$. This implies $Z\left(Z^{q-1}-1\right)^{q+1}=0$, whence $Z \in \mathbb{F}_{q}$. In the following cases we can then assume $Y \notin \mathbb{F}_{q^{2}}$.
5. $y_{2} z_{3}-y_{3} z_{2}=0$, that is $Y^{q-1}=Z^{q-1}$, and hence $Y=\rho Z$ for some $\rho \in \mathbb{F}_{q}^{*}$. Then $y_{i}=\rho z_{i}$ for any $i=1, \ldots, 3$, and System (A.1) reads

$$
\left\{\begin{array}{l}
g_{1}=z_{0} z_{1}-2 z_{0} z_{2}+z_{0} z_{3}+z_{1} z_{2}-2 z_{1} z_{3}+z_{2} z_{3}=0 \\
g_{2}=z_{0}^{2} z_{1} z_{3}-z_{0}^{2} z_{2}^{2}+z_{0} z_{1}^{2} z_{2}-2 z_{0} z_{1} z_{2} z_{3}+z_{0} z_{2} z_{3}^{2}-z_{1}^{2} z_{3}^{2}+z_{1} z_{2}^{2} z_{3}=0
\end{array}\right.
$$

From $\operatorname{Res}_{z_{1}}\left(g_{1}, g_{2}\right)=0$ it follows that $\left(z_{2}-z_{3}\right)^{2}\left(z_{3}-z_{0}\right)^{2}\left(z_{2}-z_{0}\right)^{2}=0$, which is equivalent to $Z \in \mathbb{F}_{q^{2}}$. Then $Y=\rho Z \in \mathbb{F}_{q^{2}}$.
6. $y_{0} z_{2}-y_{3} z_{3}=0$, that is $Y^{q-1}=1 / Z^{\left(q^{2}+q+1\right)(q-1)}$, and hence $Y=\rho / Z^{q^{2}+q+1}$ for some $\rho \in \mathbb{F}_{q}^{*}$. Then $y_{i}=\rho /\left(z_{i} z_{i+1} z_{i+2}\right)$ for any $i=0, \ldots, 3$ (where the indices are modulo 4) and System (A.1) reads

$$
\left\{\begin{array}{l}
h_{1}=z_{0} z_{1}-2 z_{0} z_{2}+z_{0} z_{3}+z_{1} z_{2}-2 z_{1} z_{3}+z_{2} z_{3}=0 \\
h_{2}=z_{0}^{2} z_{1} z_{3}-z_{0}^{2} z_{2}^{2}+z_{0} z_{1}^{2} z_{2}-2 z_{0} z_{1} z_{2} z_{3}+z_{0} z_{2} z_{3}^{2}-z_{1}^{2} z_{3}^{2}+z_{1} z_{2}^{2} z_{3}=0
\end{array}\right.
$$

From $\operatorname{Res}_{z_{3}}\left(h_{1}, h_{2}\right)=0$ it follows that $z_{2}=z_{0}$. Then $Z \in \mathbb{F}_{q^{2}}$ and hence $Y=$ $\rho /\left(Z^{2} Z^{q}\right)$. By System (4.4), this implies $Z \in \mathbb{F}_{q}$.
7. $y_{0} z_{2}-y_{2} z_{3}=0$, that is $Y Z^{q^{2}}-Y^{q^{2}} Z^{q^{3}}=0$. This is equivalent to $Z^{q-1}=Y^{q^{2}-1}$, and hence to $Z=\rho Y^{q+1}$ for some $\rho \in \mathbb{F}_{q}^{*}$. By direct checking, this is indeed a solution of System (4.4) for any $Y \in \mathbb{F}_{q^{4}}$. If we require $Y \notin \mathbb{F}_{q^{2}}$, this also implies $Z \notin \mathbb{F}_{q}$.
8. $y_{0} y_{2}-y_{1} y_{3}=0$, that is $y_{3}=y_{0} y_{2} / y_{1}$, or equivalently $Y^{q^{3}}=Y^{1+q^{2}} / Y^{q}$. Then System (A.1) reads

$$
\left\{\begin{aligned}
p_{1}= & y_{0} y_{1} z_{1}-y_{0} y_{1} z_{2}+y_{0} y_{2} z_{0}-y_{0} y_{2} z_{1}+y_{1}^{2} z_{2}-y_{1}^{2} z_{3}-y_{1} y_{2} z_{0}+y_{1} y_{2} z_{3}=0 \\
p_{2}= & y_{0} y_{1} z_{0} z_{1} z_{2}-y_{0} y_{1} z_{1} z_{2} z_{3}-y_{0} y_{2} z_{0} z_{1} z_{2}+y_{0} y_{2} z_{0} z_{1} z_{3}-y_{1}^{2} z_{0} z_{2} z_{3} \\
& \quad+y_{1}^{2} z_{1} z_{2} z_{3}-y_{1} y_{2} z_{0} z_{1} z_{3}+y_{1} y_{2} z_{0} z_{2} z_{3}=0 \\
p_{3}= & y_{0}^{2} y_{2} z_{0} z_{1}-y_{0}^{2} y_{2} z_{1} z_{2}+y_{0} y_{1}^{2} z_{1} z_{2}-y_{0} y_{1}^{2} z_{2} z_{3}-y_{0} y_{2}^{2} z_{0} z_{1} \\
& \quad+y_{0} y_{2}^{2} z_{0} z_{3}-y_{1}^{2} y_{2} z_{0} z_{3}+y_{1}^{2} y_{2} z_{2} z_{3}=0
\end{aligned}\right.
$$

From $\operatorname{Res}_{y_{2}}\left(p_{1}, p_{2}\right)=0$ it follows that $\left(z_{0}-z_{2}\right)\left(z_{0} z_{2}-z_{1} z_{3}\right)\left(y_{0} z_{1}-y_{1} z_{3}\right)=0$.
8.1 Suppose $z_{0}-z_{2}=0$, i.e. $Z \in \mathbb{F}_{q^{2}}$, whence also $z_{3}=z_{1}$. Then, by System (A.1), either $Y \in \mathbb{F}_{q}$ or $Z \in \mathbb{F}_{q}$.
8.2 Suppose $z_{0} z_{2}-z_{1} z_{3}=0$, so that $z_{3}=z_{0} z_{2} / z_{1}$. By System (A.1),

$$
\left\{\begin{aligned}
\ell_{1}= & y_{0}^{2} y_{2} z_{0} z_{1}-y_{0}^{2} y_{2} z_{1} z_{2}+y_{0} y_{1}^{2} z_{1} z_{2}-y_{0} y_{1}^{2} z_{2}^{2}-y_{0} y_{2}^{2} z_{0} z_{1} \\
& +y_{0} y_{2}^{2} z_{0} z_{2}-y_{1}^{2} y_{2} z_{0} z_{2}+y_{1}^{2} y_{2} z_{2}^{2}=0 \\
\ell_{2}= & y_{0} z_{0} z_{1}-y_{0} z_{1} z_{2}-y_{1} z_{0} z_{2}+y_{1} z_{1} z_{2}-y_{2} z_{0} z_{1}+y_{2} z_{0} z_{2}=0 \\
\ell_{3}= & y_{0} y_{1} z_{1}-y_{0} y_{1} z_{2}+y_{0} y_{2} z_{0}-y_{0} y_{2} z_{1}-y_{1} y_{2} z_{0}+y_{1} y_{2} z_{2}=0
\end{aligned}\right.
$$

From $\operatorname{Res}_{y_{2}}\left(\ell_{1}, \ell_{3}\right)=0$ it follows $y_{0} z_{1}-y_{1} z_{2}=0$, so that $Y^{q-1}=\left(1 / Z^{q}\right)^{q-1}$. This implies $Y=\rho / Z^{q}$ for some $\rho \in \mathbb{F}_{q}^{*}$, whence $y_{i}=\rho / z_{i+1}$ for any $i=0, \ldots, 3$ (indices modulo 4). Then, by System (A.1),

$$
\left\{\begin{array}{l}
m_{1}=z_{0}^{2} z_{1}-z_{0} z_{1} z_{2}-2 z_{0} z_{1} z_{3}+z_{0} z_{2} z_{3}+z_{1}^{2} z_{3}=0 \\
m_{2}=z_{0}^{2} z_{2}+z_{0} z_{1}^{2}-2 z_{0} z_{1} z_{2}-z_{0} z_{1} z_{3}+z_{1} z_{2} z_{3}=0 \\
m_{3}=z_{0}^{2} z_{1}+z_{0}^{2} z_{2}+z_{0} z_{1}^{2}-3 z_{0} z_{1} z_{2}-3 z_{0} z_{1} z_{3}+z_{0} z_{2} z_{3}+z_{1}^{2} z_{3}+z_{1} z_{2} z_{3}=0 .
\end{array}\right.
$$

From $\operatorname{Res}_{z_{3}}\left(m_{1}, m_{2}\right)=0$ it follows that $Z \in \mathbb{F}_{q}$.
8.3 Suppose $y_{0} z_{1}-y_{1} z_{3}=0$. This implies $Y^{q-1}=\left(1 / Z^{q^{2}+q}\right)^{q-1}$, whence $Y=$ $\rho / Z^{q^{2}+q}$ for some $\rho \in \mathbb{F}_{q}^{*}$. Then $y_{i}=\rho /\left(z_{i+2} z_{i+1}\right)$ for any $i=0, \ldots, 3$ (indices modulo 4), which is indeed a solution of System (A.1) and provide a solution $(Y, Z)$ of System (4.4). Notice that the condition $Y=\rho / Z^{q^{2}+q}$ with $Z \in \mathbb{F}_{q^{4}}$ implies $Y^{q^{3}}=Y^{1+q^{2}} / Y^{q}$. For such a solution, the require $Z \notin \mathbb{F}_{q^{2}}$ is equivalent to $Y \notin \mathbb{F}_{q}$. Also, if $Y, Z \in \mathbb{F}_{q^{4}}^{*}$ are such that $Y=\rho / Z^{q^{2}+q}$ and $Z=\rho^{\prime} Y^{q+1}$ with $\rho, \rho^{\prime} \in \mathbb{F}_{q}$, then $Y^{q^{3}+q^{2}+q+1+q^{2}}=\rho / \rho^{\prime} \in \mathbb{F}_{q}^{*}$, whence $Y \in \mathbb{F}_{q}$.

Theorem A.2. Let $(Y, Z) \in \mathbb{F}_{q^{4}}^{2}$ be a solution of System (4.4), and $M$ be the matrix in (4.10).
(R1) If $Y \in \mathbb{F}_{q}$ or $Z \in \mathbb{F}_{q}$, then $\operatorname{rank}(M)=2$.
(R2) If $Y \notin \mathbb{F}_{q^{2}}$ and $Z=\rho Y^{q+1}$ for some $\rho \in \mathbb{F}_{q}^{*}$, then $\operatorname{rank}(M)=6$.
(R3) If $Z \notin \mathbb{F}_{q^{2}}$ and $Y=\rho / Z^{q^{2}+q}$ for some $\rho \in \mathbb{F}_{q}^{*}$, then $\operatorname{rank}(M)=6$.
Proof. For any $i=1, \ldots, 10$, denote respectively by $M^{(i)}$ and $M_{(i)}$ the $i$-th row and the $i$-th column of $M$. Note that $M_{(9)}$ and $M_{(10)}$ are linearly independent. Note also that,
by construction of $M$, any possible $\mathbb{F}_{q}$-linear combination of the columns of $M$ needs to be checked only on the first, second, third, and seventh rows of $M$.
(R1) Suppose $Y \in \mathbb{F}_{q}$. Then $N(\lambda)=D(\lambda)$, whence $\frac{D^{q^{2}+q+1}(\lambda)}{N^{q+1}(\lambda)}=\frac{D^{q^{2}+q}(\lambda)}{N^{q}(\lambda)}=\lambda Z Y+1$. Therefore $M_{(j)}=\lambda_{j} M_{(9)}+M_{(10)}$ for any $j=1, \ldots, 8$, and $\operatorname{rank}(M)=2$.
Suppose $Z \in \mathbb{F}_{q}$. Similarly, one has $N(\lambda)=D^{q}(\lambda)$ and $M_{(j)}=\lambda_{j} M_{(9)}+M_{(10)}$ for any $j=1, \ldots, 8$, so that $\operatorname{rank}(M)=2$.

For $i=6,7$, denote by $S_{i}$ the $i \times i$ submatrix of $M$ given by the first $i$ rows and the last $i$ columns of $M$. If for any distinct $\lambda_{4}, \ldots, \lambda_{8} \in \mathbb{F}_{q}^{*}$ one has $\operatorname{det}\left(S_{6}\right) \neq 0$ and $\operatorname{det}\left(S_{7}\right)=0$, then this is enough to conclude that $\operatorname{rank}(M)=6$ for any distinct $\lambda_{1}, \ldots, \lambda_{8} \in \mathbb{F}_{q}^{*}$. Indeed, this is due to the fact that the use of the column $M_{(j)}, j \in\{1,2,3\}$, instead of $M_{(4)}$, implies the replacement of $\lambda_{4}$ with $\lambda_{j}$ in $\operatorname{det}\left(S_{7}\right)$, and in this way the fourth column becomes any of the remaining columns. If $S_{7}^{\prime}$ is obtained from $S_{7}$ by replacing $M^{(7)}$ with $M^{(i)}, i \in\{8,9,10\}$, then the elementwise $q^{i-7}$-power $\Phi$ maps $M^{(7)}$ to $M^{(i)}$, while $M^{(1)}$ and $M^{(2)}$ are fixed by $\Phi$, and $M^{(3)}, M^{(4)}, M^{(5)}, M^{(6)}$ are cyclically permuted by $\Phi$. Therefore $\Phi$ maps the rows of $S_{7}$ to the rows of $S_{7}^{\prime}$, so that $\operatorname{det}\left(S_{7}\right)=0$ if and only if $\operatorname{det}\left(S_{7}^{\prime}\right)=0$.
(R2) Suppose $Y \notin \mathbb{F}_{q^{2}}$ and $Z=\rho Y^{q+1}$ for some $\rho \in \mathbb{F}_{q}^{*}$. Then $\operatorname{det}\left(S_{6}\right)$ equals

$$
\begin{array}{r}
\rho^{10} Y^{4\left(q^{3}+q^{2}+q+1\right)}\left(\prod_{i=5}^{8} \lambda_{i}\right)\left(\prod_{5 \leq i<j \leq 8}\left(\lambda_{i}-\lambda_{j}\right)\right) \\
\left(\prod_{i=0}^{3}\left(Y^{q^{i}}-Y^{q^{i+1}}\right)^{2}\right)\left(\prod_{i=0}^{1}\left(Y^{q^{i}}-Y^{q^{i+2}}\right)^{3}\right)
\end{array}
$$

and hence $\operatorname{det}\left(S_{6}\right) \neq 0$ because $Y \notin \mathbb{F}_{q^{2}}$ and the $\lambda_{i}$ 's are nonzero and distinct. Also, $\operatorname{det}\left(S_{7}\right)=0$. Therefore, $\operatorname{rank}(M)=6$.
(R3) Suppose $Z \notin \mathbb{F}_{q^{2}}$ and $Y=\rho / Z^{q^{2}+q}$ for some $\rho \in \mathbb{F}_{q}^{*}$. Then

$$
\begin{array}{r}
\operatorname{det}\left(S_{6}\right)=\rho^{10}\left(\prod_{i=5}^{8} \lambda_{i}\right)\left(\prod_{5 \leq i<j \leq 8}\left(\lambda_{i}-\lambda_{j}\right)\right) \\
\left(\prod_{i=0}^{3}\left(Z^{q^{i}}-Z^{q^{i+1}}\right)^{2}\right)\left(\prod_{i=0}^{1}\left(Z^{q^{i}}-Z^{q^{i+2}}\right)^{3}\right)
\end{array}
$$

is nonzero. Also, $\operatorname{det}\left(S_{7}\right)=0$. Therefore, $\operatorname{rank}(M)=6$.

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[^1]:    ${ }^{1}$ Here we consider the action of $\mathcal{A}=\left\{(g, h): g, h \in \mathcal{L}_{n, q}\right.$ are invertible $\}$ on the $\mathbb{F}_{q}$-subspaces of $\mathcal{L}_{n, q}$ induced by $f \in \mathcal{L}_{n, q} \mapsto g \circ f \circ h \in \mathcal{L}_{n, q}$, where $(g, h) \in \mathcal{A}$.

