6-12-2007

Sensitivity Analysis for Hierarchical Decision Models

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SENSITIVITY ANALYSIS FOR HIERARCHICAL DECISION MODELS

by

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A dissertation submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY
in
SYSTEMS SCIENCE: ENGINEERING MANAGEMENT

Portland State University
2007

DISSERTATION APPROVAL

The abstract and dissertation of Hongyi Chen for the Doctor of Philosophy in Systems Science: Engineering Management were presented June 12, 2007, and accepted by the dissertation committee and the doctoral program.

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ABSTRACT

An abstract of the dissertation of Hongyi Chen for the Doctor of Philosophy in Systems Science: Engineering Management presented June 12, 2007.

Title: Sensitivity Analysis for Hierarchical Decision Models

In this dissertation, a comprehensive algorithm is developed to analyze the sensitivity of hierarchical decision models (HDM), which include the well-known analytic hierarchy process (AHP) and its variants, to single and multiple changes in the local contribution matrices at any level of the decision hierarchy. The algorithm is applicable to all HDM that use an additive function to derive the overall contribution vector. It is independent of pairwise comparison scales, judgment quantification techniques and group opinion combining methods. The *direct impact of changes to a local contribution value on decision alternatives' overall contributions, allowable range/region of perturbations*, *contribution tolerance*, *operating point sensitivity coefficient*, *total sensitivity coefficient* and *the most critical decision element at a certain level* are defined by five groups of theorems and corollaries and two groups of propositions in the HDM SA algorithm. Two examples are presented to demonstrate the applications of the HDM SA algorithm on technology evaluation and energy portfolio forecast. Significant insights gained by the two applications demonstrate the contributions of the algorithm. Theorems and corollaries in the HDM SA algorithm were verified and validated by data from the two application models.

DEDICATION

To my father and mother

Prof. Yunxiang Chen and Lihua Cao

ACKNOWLEDGEMENTS

The writing of a dissertation can be a long and stressful journey. But thanks to all the personal, practical, financial and spiritual supports from my advisor, my family, my department, my college, my friends, and my Christian brothers and sisters, this journey to me has been joyfully and wonderfully unforgettable.

My sincere gratitude goes to my esteemed advisor, Dr. Dundar F. Kocaoglu, who insightfully originated this research topic and enthusiastically guided me throughout the work. This dissertation would not have been possible without his expertise and guidance. Yet all the great support, encouragement, love, and care that Dr. Kocaoglu has shown to me during my Ph.D. life has made him more than an advisor, but also a mentor, a colleague, a friend, and a father to me.

My sincere gratitude also goes to my Systems Science advisor, Dr. Wayne Wakeland, who continuously challenged me to reach higher standards and encouraged me to do better jobs; to my other committee members, Dr. Barry Anderson, Dr. Tim Anderson, and Dr. Tugrul Daim, who provided very helpful suggestions and caught up all the details that make a perfect work; to Ms. Ann White, who professionally edited every paper of mine during my five-year Ph.D. study and helped me improve my written English; to the wonderful friends that I have made during the past five years inside and outside of the ETM family, such as Nathasit Gerdsri, Audrey Alvear, Iwan Sudrajat, Jonathan Ho, Pisek Gerdsri, Ying Lin, Lin Fu, Helen Wang, Liwei Ning and Vern Bissell, just to name a few, for all the help, encouragement, friendship, and laughter.

Because of the Maseeh Fellowship founded by Dr. Fariborz Maseeh and awarded by Maseeh College of Engineering and Computer Science at Portland State University, I was able to use time efficiently and concentrate on the research for this dissertation. This general financial support in years 2004-2006 is very much appreciated.

I am grateful to my husband, Hua Tang, who has gone beyond the role as a husband not only to take care of my everyday living, but also provide valuable suggestions on my dissertation, remind me when I got slowed down on research progress, and encourage me to work hard for publications. I thank my parents, who have spared no effort to support my further education and encouraged me to be perfect in everything. My thanks also go to all my family members and friends in China because without their continuous encouragement and support in love, I would not have gone this far in my life.

Finally, thank God for all the blessings, wisdom and strength that I have received in order to finish my dissertation and successfully complete the Ph.D.

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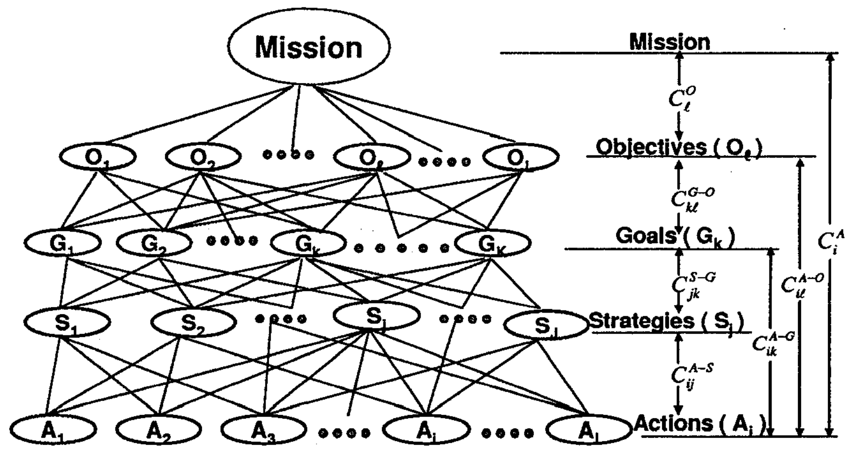
GLOSSARY



O_ℓ: The ℓ^{th} objective, ℓ=1, 2…L

L: Number of objectives

G_k: The k^{th} goal, k=1, 2…K

K: Number of goals

S_j: The j^{th} strategy, j=1, 2…J

J: Number of strategies

A_i: The i^{th} action, i=1, 2…I

I: Number of actions

C_i^A : Overall contribution of the i^{th} action A_i to the mission

r: The rank of i. A_r ranks before A_{r+n}, which indicates $C_r^A > C_{r+n}^A$

$C_{i\ell}^{A-O}$: Contribution of the i^{th} action A_i to the ℓ^{th} objective O_ℓ

C_{ik}^{A-G} : Contribution of the i^{th} action A_i to the k^{th} goal G_k

C_{ij}^{A-S} : Contribution of the i^{th} action A_i to the j^{th} strategy S_j

C_j^S : Overall contribution of the j^{th} strategy to the mission

C_{jk}^{S-G} : Contribution of the j^{th} strategy to the k^{th} goal

$C_{k\ell}^{G-O}$: Contribution of the k^{th} goal to the ℓ^{th} objective

C_ℓ^O : Contribution of the ℓ^{th} objective to the mission

ACRONYMS

SA: Sensitivity Analysis

HDM: Hierarchical Decision Model

AHP: Analytic Hierarchy Process

MCDM: Multi-Criteria Decision Making

RHSV: Right Hand Side Value

MOGSA: Mission-Objectives-Goals-Strategies-Actions

R&D: Research and Development

1. INTRODUCTION

As the world has become complex, decision problems have followed suit, and must contend with increasingly complex relationships and interactions among the decision elements. To assist decision makers and analysts, such problems can be decomposed into hierarchical levels, and formulated as Hierarchical Decision Models (HDM) containing several levels of decision elements which are correlated horizontally and vertically. Among those methods to construct and utilize HDM, AHP (Analytic Hierarchy Process) developed by Saaty [97] is the best known. Many other methods that use the basic concept of AHP to deal with multiple decision levels have been developed by other researchers. These methods offer a variety of ways to derive the contribution matrices using different pairwise comparison ratio scales and judgment quantification techniques (e.g., [9] [12] [24] [50] [55] [56] [59] [60] [61] [67]).

In HDM, the local contributions of decision elements at one level to decision elements on the next higher level are supplied as intermediate input data to the model. The decision obtained by evaluating the final ranking of alternatives is based on the value of these contributions. Since the value of the contributions is seldom known at one hundred percent confidence level, the solution of a problem is not complete with the mere determination of a rank order. It is always helpful to have *Sensitivity Analysis* (SA) as a supplement accompanying the current model solution. Sometimes, SA even gives information more significant and useful than simply knowing the rank order of the alternatives. SA can: 1) help visualize the impact of changes at policy and strategy levels on decisions at the operational level; 2) indicate how robust a particular decision is under different conditions; 3) provide scenarios of possible rankings of decision alternatives under different conditions; and 4) offer answers to "what if" questions.

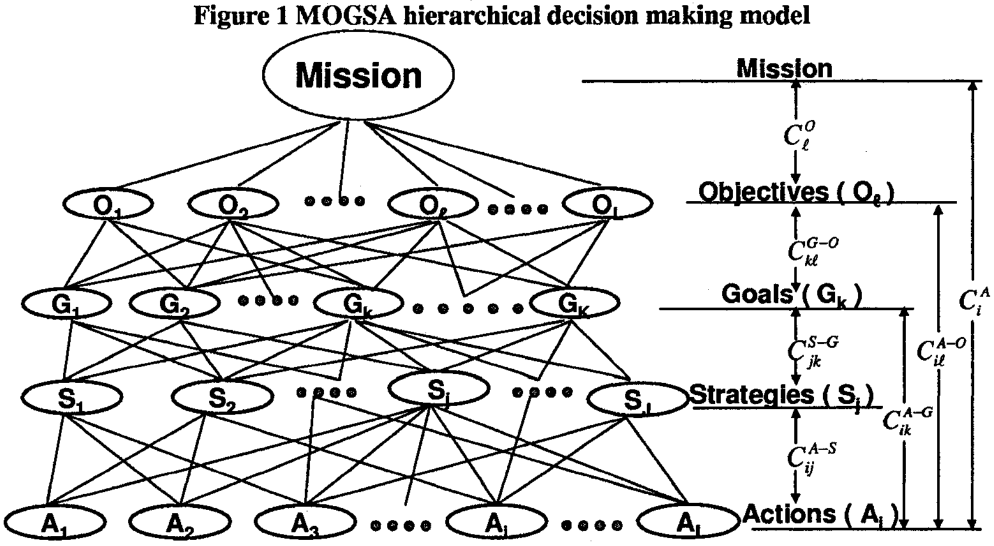
However, although researchers frequently understand the importance of sensitivity analysis and considerable literature reports the application of various methods (e.g. [15][18][42][52][64][85][106][113]) and addresses issues such as ranking irregularities with AHP (e.g. [12][13]), no study has been done to develop an accurate, comprehensive and general SA algorithm for all HDM. To close this literature gap, a comprehensive Hierarchical Decision Models Sensitivity Analysis (HDM SA) is proposed in this dissertation.

This HDM SA algorithm is independent of pair-wise comparison ratio scales and judgment quantification methods by which the local contribution matrices of decision elements at each level of the hierarchy are derived and aggregated, thus making it applicable to all types of HDM. It addresses the situations when variations in the contribution values at all levels of the decision hierarchy occur either one at a time or simultaneously. Three conditions are considered in each kind of variation analysis, namely: 1) when the rank order of a pair of decision alternatives is of concern; 2) when the rank orders of all the decision alternatives are of concern; and 3) when only the first-ranked alternative is of concern.

Since mathematical deduction in symbolic form is utilized to develop this algorithm, a MOGSA model [25] is used to represent a typical HDM model structure. Thus, notations in the MOGSA model are employed throughout the dissertation, defining the *direct impact* of one unit variation of a local contribution to decision alternatives' overall contributions, the *allowable range/region of perturbations*, the *tolerance* of contribution values, *total sensitivity coefficient*, *operating point sensitivity coefficient*, *probability of rank changing*, and the *critical decision elements*. The MOGSA model structure and its notations are presented in details in the following section.

1.1 BASIC HDM MODEL STRUCTURE AND NOTATIONS

Figure 1 MOGSA hierarchical decision making model



O_ℓ: The ℓ^{th} objective, ℓ=1, 2…L

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C_j^S: Overall contribution of the j[th] strategy to the mission

C_{jk}^{S-G}: Contribution of the j[th] strategy to the k[th] goal

$C_{k\ell}^{G-O}$: Contribution of the k[th] goal to the ℓ[th] objective

C_ℓ^O: Contribution of the ℓ[th] objective to the mission

The terms "criteria weights" [93][107][113][114], "priority" [1][6][53][69][98][97] and "performance values" [107] used in previous literature are called "contributions" in this study because they are actually measurements of the contribution of an alternative or sub-criterion to a criterion at a higher level. Saaty [97] used "local priority" and "global priority" to differentiate contributions of decision elements at one level to decision elements at the next higher level and overall contributions of decision alternatives to the mission. In this dissertation, the terms "local contributions" and "overall contributions" are used instead.

The MOGSA model structure was first used by Cleland and Kocaoglu [25]. The model consists of five levels of decision elements labeled Mission, Objectives, Goals, Strategies, and Actions, as shown in Figure 1. On the lowest level, Actions are the decision alternatives under evaluation: they are ranked according to their overall contribution to the Mission, denoted as C_i^A.

"C_i^A" is calculated by combining the local contribution vector and matrices, which are vector $\mathbf{C}_\ell^\mathbf{O}$, and matrices $\mathbf{C}_{\mathbf{k}\ell}^{\mathbf{G-O}}$, $\mathbf{C}_{\mathbf{jk}}^{\mathbf{S-G}}$, and $\mathbf{C}_{\mathbf{ij}}^{\mathbf{A-S}}$ in the MOGSA model, between successive levels of the hierarchy M-O, O-G, G-S, and S-A, into a 1×I overall contribution vector $\mathbf{C}_\mathbf{i}^\mathbf{A}$:

$$C_i^A = \sum_{\ell=1}^{L} C_{i\ell}^{A-O} \times C_\ell^O = \sum_{\ell=1}^{L}\sum_{k=1}^{K} C_{ik}^{A-G} \times C_{k\ell}^{G-O} \times C_\ell^O = \sum_{\ell=1}^{L}\sum_{k=1}^{K}\sum_{j=1}^{J} C_{ij}^{A-S} \times C_{jk}^{S-G} \times C_{k\ell}^{G-O} \times C_\ell^O \quad (1.1)$$

All the values in the matrices are normalized so that contributions of lower level decision elements to each decision element on the next higher level sum up to 1:

$$\sum_{\ell=1}^{L} C_\ell^o = 1, \ \sum_{k=1}^{K} C_{k\ell}^{G-O} = 1, \ \sum_{j=1}^{J} C_{jk}^{S-G} = 1, \text{ and } \sum_{i=1}^{I} C_{ij}^{A-S} = 1. \quad (1.2)$$

In the application of the MOGSA model, levels of the hierarchy can be extended or reduced according to the needs of specific problems.

1.2 RESEARCH OBJECTIVES

In HDM, local contributions of decision elements at one level to the decision elements on the next higher level are supplied as intermediate input data to the model. The decision obtained by evaluating the final ranking of alternatives is based on the values of these local contributions. However, the values of these contributions are seldom known at one hundred percent confidence level. Besides, the social and economic environment of certain decision problems solved by HDM is usually fast changing, and as a result, causes changes to the local contributions' values over time. Therefore, the solution of a practical problem is not complete with the mere determination of a rank order.

Variations of the local contribution values may or may not change the recommended decision. In order to develop an overall strategy to meet the various contingencies, one has to study how the results change due to changes in the local contributions. In Operations Research, Sensitivity Analysis is called "post-optimality analysis" because the analysis is conducted after an optimal solution is obtained [86]. The HDM SA proposed in this dissertation bears the same characteristic: after a conclusion has been reached, HDM SA studies the robustness of the recommended decision with respect to changes in the intermediate input—local contributions.

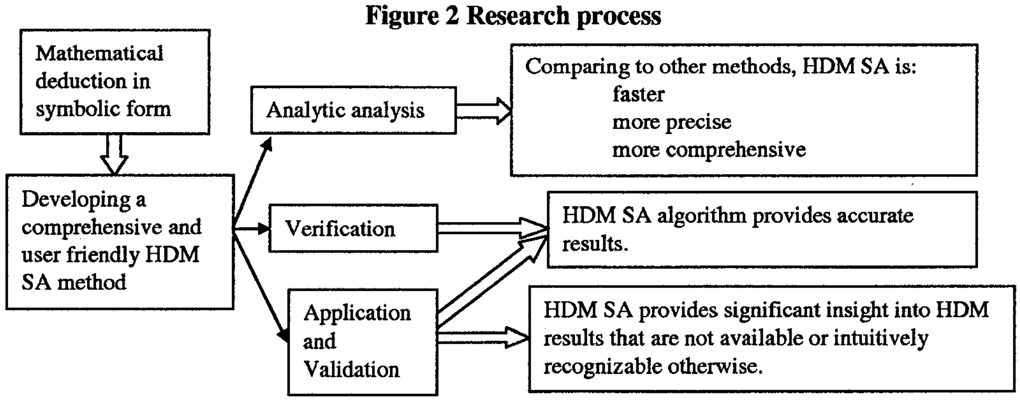
A literature survey was conducted on the current state of knowledge regarding sensitivity analysis in the field of MCDM (Multi-Criteria Decision Making) with a focus on hierarchical models. Limitations and gaps were identified in the literature. A comprehensive HDM SA algorithm is proposed in this dissertation to fill the gaps by studying sensitivity of HDM results in response to single or multiple perturbations induced at any level of the decision hierarchy. Three situations with which decision makers are concerned are addressed: 1) the rank order of a certain pair of decision alternatives, 2) the rank order of all decision alternatives, and 3) the choice of the best alternative.

In addition to theoretical algorithm development, practical applications of the algorithm are also explored. The applications of HDM SA algorithm to previously reported hierarchical models is done not only to verify and validate the algorithm itself, but also demonstrate significant insights provided by performing HDM SA. The two applications shown in this dissertation present the usefulness of HDM SA in assisting strategic technology planning process and in analyzing energy portfolios.

1.3 OVERVIEW OF RESEARCH METHOD

Since the relationship between local contributions and overall contributions are known, situations in which the rank order of decision alternatives need to be reserved are formulated as mathematical expressions. The mathematical expressions, thus lead to the theorems of the proposed HDM SA algorithm. Then mathematical deduction is used in proving the theorems. Using analytic methods, HDM SA algorithm is evaluated by comparing it with other SA methods reported in HDM literature about their comprehensiveness, accuracy, informativeness, and computational complexity. The algorithm is verified and validated by applying it to two previously built models. Significant insights provided by such applications further demonstrate the contributions of HDM SA in different fields, among which technology planning and energy portfolio analysis are specifically discussed in this dissertation. Figure 2 depicts the entire research followed in this dissertation.

Figure 2 Research process



1.4 ORGANIZATION OF THIS DISSERTATION

The rest of the dissertation is organized as follows. Section 2 summarizes literature in the areas of Hierarchical Decision Models, SA for general MCDM problems including Linear Programming, specific SA for all HDM methods, and other related studies in AHP. The purpose of doing SA and the methods employed in doing SA are also summarized. Limitations and gaps identified in literature are listed.

To close the SA literature gap, research questions are presented at the beginning of section 3. Then a systems approach is used to develop the HDM SA algorithm. *Direct impact* of one unit change to a local contribution value on the decision alternatives' overall contributions is first evaluated and presented by theorems 1.1 through 1.3. Then theorems 2 through 4 and their corollaries define the *allowable range/region of perturbations* and *tolerance* of local contribution values that preserve: 1) the current rank order of a pair of decision alternatives, 2) the rank of all decision alternatives, and 3) that leave the rank order of the best alternative unchanged, when single and multiple changes occur in a local contribution vector or matrix at any level of the decision hierarchy. As an extension of these theorems and corollaries, theorems 5.1 through 5.3 define the *allowable region of perturbations* simultaneously induced in different levels of the decision hierarchy. Then, a group of propositions clarify the *total sensitivity coefficient* and *operating point sensitivity coefficient* that measure the robustness of model results to variations in the local contribution values and *probability of rank changing* when certain contribution values vary uniformly within the feasible region. *Critical decision elements* make up the last proposition.

Section 4 verifies and illustrates the usefulness of the proposed HMD SA algorithm by applying it to a hierarchical technology evaluation model reported in a recent Ph.D. dissertation [52]. Based on a review of literature on planning, which is summarized in section 4.1 since it is specific to the first case, a strategic technology planning framework is proposed. HDM SA is employed as a critical step in the proposed framework in order to link synoptic and adaptive planning modes. Significant insights regarding scenario analysis and change management are demonstrated.

In section 5, the HDM SA algorithm is applied to an AHP model built in 1980's to forecast a desired energy mix in year 2000 [43]. This application further verifies the theorems and corollaries of the algorithm. In addition, objective data acquired from US government website are used to compare model results. Through HDM SA, the actual ranking of energy consumption in year 2000 was replicated by the model. More insights, including the rank reversal problem associated with AHP, emerge during the analysis.

In section 6, theoretical and empirical contributions, limitations and future work of this dissertation conclude the dissertation. Mathematical details for the deduction of theorems and corollaries are included in the Appendix.

2. LITERATURE SURVEY

2.1 INTRODUCTION

A literature search was conducted to identify how sensitivity analysis has been performed in previous studies with a focus on deterministic MCDM (Multi-Criteria Decision Making) problems using hierarchical models.

Among the many MCDM models with hierarchical model structure, the Analytic Hierarchy Process (AHP) developed by Saaty [98][97] is the best known and most widely used model. AHP has been used to solve problems in many areas including social, economical, technological, etc. The types of the problems addressed by AHP include selection, evaluation, resource allocation, benchmarking, quality management, public policy, health care and strategic planning, as summarized by Forman and Gass [37]. Several other methods that can be viewed as the variants of AHP were developed concurrent with and shortly after the introduction of AHP.

"Hierarchical Decision Models" is a general term used in this dissertation to describe all the MCDM models that use hierarchical structure, including AHP and its variants. The basic procedures for hierarchical decision modeling are the same:

1) Decompose the problem into decision elements that are correlated horizontally and vertically and build a hierarchical model
2) Compare decision elements at the same level regarding their contributions to certain decision element at the next higher level of the decision model

3) Calculate the local contribution vector and matrices between successive levels of hierarchy
4) Aggregate all local contribution vector and matrices to an overall contribution vector
5) Rank decision alternatives according to their overall contributions to the top level decision element, the Mission.

In utilizing HDM, the difference between the methods lies in the middle three steps. Each method has a different computational approach to quantify judgments in determining the contributions of decision elements to the next higher level.

For pair-wise comparisons at the second step, two types of ratio scale methods are used: One is the constant sum measurement developed by Comrey [28] and Guilford [46] and refined by Kocaoglu [59], and the other one is the 1-9 scale with verbal representation developed by Saaty [97] and used in AHP and some of its variants.

To derive local contribution matrices from pair-wise comparison results at the third step, more than fourteen methods are used by different researchers. Ra summarized these judgment quantification techniques into three basic groups [93]:

1) Column-row orientation methods
2) Eigenvector-based methods
3) Least distance approximation methods

Different approaches are also used to synthesize individuals' opinions if it is a group of people who make the judgments, either before or after the local contribution matrices are derived. Those approaches can be categorized into three basic groups

[36]: 1) Mathematical aggregation, such as simple or weighted arithmetic/geometric mean of different judgments [36][9][52]; 2) Behavioral aggregation with requires discussion and agreement upon a value by the group, such as consensus [98], majority rule [48], etc; and 3) A mixture of the previous two, such as *Delphi* developed by Norman Dalkey, *et al.*, and "nominal group technique," or NGT, investigated by Andre Delbecq, *et al.* [36].

For aggregating local contribution matrices to an overall contribution vector at the fourth step, all of the methods, except the "row geometric mean method" developed by Barzilai, *et al.* [9] and refined by Lootsma [67] that assumes a multiplicative relationship among local contributions, use additive formulas to calculate the overall contributions [12][59][60][61][98][93]. The HDM SA algorithm developed in this dissertation is based on the more generally used additive relationship.

No matter what ratio scales are used to compare decision elements, what judgment quantification methods are used to calculate the local contribution matrices, or what methods are used to aggregate local contribution matrices into an overall contribution vector, as long as the judgments involve subjectivity, the input data will be uncertain. A sensitivity analysis is therefore a necessity to accompany the results derived by the model inputs. A good SA should be general enough to be applicable to all HDM with different underlying algorithms as mentioned above. Besides, different methods of combining group opinions may result in different outcomes even though the initial inputs are the same. A sensitivity analysis is then helpful to test the robustness of current conclusion to different opinion combining methods.

To deal with the uncertainty involved in the decisions, researchers have replaced some of the values in the local contribution matrix with probability distributions [49][19]. Since non-deterministic models employ statistical methods to analyze results, one might expect that SA for non-deterministic models would also be based on statistical analysis. The HDM SA algorithm developed in this dissertation is focused on deterministic HDM, but is also applicable to non-deterministic HDM to a certain degree.

The literature survey focuses on deterministic hierarchical models, but also reviews SA for other MCDM models. The following section summarizes the purpose of SA and the methods employed to perform SA. Limitations of the prior research are discussed and interpreted as gaps or opportunities.

2.2 PURPOSE OF SENSITIVITY ANALYSIS

Literature survey focuses on Sensitivity Analysis in deterministic MCDM problems solved by hierarchical models, while some SA studies on other MCDM models including Linear Programming and Computer Simulations were also reviewed. Based on the literature, the purpose of conducting sensitivity analysis is summarized as follows.

2.2.1 SA in General MCDM Models

Test the robustness of the decision model and its imperviousness to extraneous factors [2][57]

Assess the stability of an optimal solution under changes in parameters [30]

Assess the positive changes of the optimal solution under changes in input data [86]

Identify the most influential variables with respect to the rank ordering of the alternatives [51]

Measure how important a particular input is to the output and determine the need for precise estimation of parameter values under uncertainty [30][86][57][47][100]

Determine which input variables to model stochastically [57][96]

Assess the impact of the lack of controllability of certain parameters [30]

Assess the influence of an assumption on the validity of a model [47]

Communicate the distinction between plan alternatives and decision-relevant factors [2]

Assess the significance of the difference between competing proposals regarding final result [2]

Determine redundant or underweighted variables [2]

2.2.2 SA for HDM

Determine the range in which the value of a paired comparison can change without altering the resultant ordinal ranking [93]

Determine how changes in the local priority matrices affect the global priority vector with respect to the overall goal [53][69]

Identify the "determinant attribute" that strongly contribute to the choice among alternatives [6]

Help come up with a possible ranking of decision alternatives at certain confidence level when the relationship between a pair of entities can only be described with probability distributions [49][19]

Help people make better decisions by determining how critical each criterion is or how sensitive the actual ranking of the alternative is to changes on the current weights of the decision criteria [107]

Characterize scenarios that could affect a change in the ranking of the alternatives [96]

Help group experts reach consensus by demonstrating trends in the preference weight of criteria when the comparison matrix is changed [114]

Test the robustness of the model resulting from opinion changes among the experts [52]

In this group, some of the purposes of conducting SA for HDM are related specifically to the HDM process, and some of them can be viewed as the subset of the purposes mentioned in the previous section.

2.3 METHODS UTILIZED TO CONDUCT SA FOR HDM

2.3.1 Numerical Incremental Analysis

In considerable literature where HDM, especially AHP developed by Saaty [97], were employed to help solve problems, a basic SA was conducted by manually changing the parameters' values to test corresponding changes in the rank orders of the decision alternatives. The trend of changes is then shown with graphic representations [85][15][114][113][52]. The method used in those studies is numerical incremental analysis, which is an iteration-based and data dependent process. The analysis starts with a current parameter value, increasing or decreasing it by a certain unit at each iteration. The process stops when the current rank order of the decision alternatives changes, and the units being changed by then is defined as the threshold of changes on the parameter value that preserves the current rank order.

The numerical incremental analysis is usually used when no closed form expression can be found to describe the relationship between inputs and outputs or when the closed form expression is over-complicated. Each time when an incremental change is made, the output of the model is recalculated, thus making the analysis slower then other methods. Of course with the assistance of computer programs, the time this method takes is no longer a problem. However, as the requirement of precision increases, the computational complexity also increases.

Expert Choice, a software package based on AHP, has a basic sensitivity analysis function which recalculates the global priorities of the decision alternatives when changes occur in the local priorities. Using its "dynamics sensitivity" and

"Performance sensitivity" functions, the users can alter the weights of the second level criteria by dragging the value bars and see graphically how the global priorities (C_i^A in MOGSA model) of the decision alternatives change. However, the function is limited to one change at a time in the first level contribution vector ($\mathbf{C}_\ell^{\mathbf{O}}$) of the decision hierarchy. It does not offer users the option to change contributions at other levels of the hierarchy, nor does it allow multiple changes at the same time. In addition, it does not compare the impacts of changes at input levels on the output, thus does not help people to understand the sensitivity of current decision to changes in different contributions.

2.3.2 Simulations Approach

Hauser and Tadikamalla used a simulation approach to address the situation when the pair-wise comparison result cannot be described by a point estimate but rather an interval or a distribution [49]. In their model, distributions based on people's judgments are embedded in the AHP reciprocal matrix. After simulating the decision process for a few hundred runs, statistical analysis is used to obtain the probabilities of possible rank orders. Expected weight is defined as the normalized sum product of the possible weights and the corresponding probabilities. Then the expected rank is defined based on the expected weights.

In the example illustrated in their study, a uniform distribution of the point estimator obtained from people's judgment $\pm 25\%$ is used to substitute a value in a contribution matrix. After 500 runs of the simulation, a group of conclusions in the form like "the third alternative ranks first 90% of the time" are drawn. Then the expected weights and expected ranks are calculated for all the decision alternatives.

Butler, Jia and Dyer conducted a similar study. However, instead of using distributions based on people's judgments, random numbers were generated by the simulation program to be the weights of criteria, and the decision alternatives' performance values were fixed in the simulations [19]. The result suggests the decision alternatives' ranking under all conditions. The alternatives that are ranked last in all conditions are suggested to be removed from the analysis.

In these two studies, the probabilistic input of the model introduces stochasticity to the output, thus making the model a nondeterministic HDM, which is outside the focus of this dissertation. The HDM SA algorithm developed in this dissertation addresses the issues raised in the common use of hierarchical models with point estimates. Since HDM is fundamentally a deterministic model, HDM SA is thus applied to the analysis of deterministic inputs. Simulation with probabilistic inputs is used to verify the HDM SA results, but not to conduct the SA.

In the Hauser and Tadikamalla [49] study, since the interval of the contribution value is given, it is possible to come up with different rankings when the contribution value varies within such interval. Therefore, it is argued in this dissertation that a statement such as "the third alternative will rank first when the contribution value is within ... range, and the first alternative will rank first when the contribution value is with ... range" gives more information than the expected ranks. Elimination of the dominated decision alternatives beforehand is helpful in the Butler, Jia and Dyer [19] study. However, to show experts possible results before they make any judgments is not necessarily helpful: the experts may assign greater weights than they would have to certain criteria in order to have their preferred alternative selected. In addition, the approach is more useful in problems where one level of contribution matrix is totally unknown while the others are given at very high confidence level. Otherwise, it makes less sense to simulate only one contribution matrix while keeping others fixed.

2.3.3 Mathematical Deduction

The third group of methods used in HDM SA is mathematical deduction, either based on numerical values or using symbolic expressions to represent the relationship between inputs and outputs. The mathematical deduction in symbolic form is independent of the numerical values in specific problems. Using this method, an algorithm is developed to determine the sensitivity of decisions to changes in parameter values through a series of mathematical deductions. The HDM SA algorithm proposed in this dissertation belongs to this category of methods. In the literature surveyed in this section, except the study by Ra [93] which used numerical values in the deduction, all the other studies used mathematical deduction in symbolic form.

Armacost and Hosseini studied the *determinant attributes* in an AHP decision hierarchy. Their analysis is concerned with the attribute that differentiate the final ranking of the alternatives most. The "determinance score" proposed in their study is defined as $d_i = a_i[\frac{1}{m} - (\prod_{j=1}^{m} p_{ij})^{\frac{1}{m}}]$, where a_i is the normalized priority of the i^{th} attribute, and p_{ij} is the normalized priority of the j^{th} alternative with respect to the i^{th} attribute. The greater the d_i is, the more difference the attribute makes in the final priorities of the alternatives. [6]

Masuda proposed a *sensitivity coefficient* as a measurement of the likelihood that the ranks will change. He defined the coefficient as the standard deviation of the "extreme vector" in the judgment matrix. The closer to 0 the coefficient is, the less likely that a rank reversal among alternatives will occur. [69]

Huang showed Masuda's work was invalid in certain situations. Based on Masuda's work, he proposed a different sensitivity coefficient, also as a measurement of the likelihood of rank changes [53].

Triantaphyllou and Sanchez defined the sensitivity coefficient of a certain criterion C_k as:

$$\text{sens}(C_k) = \frac{1}{\min_{1 \le i \le j \le M}\{|\delta'_{k,i,j}|\}} \qquad \delta'_{k,i,j} = \frac{\delta_{k,i,j}}{W_k} \times 100 \tag{2.1}$$

$_{k,i,j}$ is the threshold of changes on W_k (the weight of C_k, or the contribution of C_k to decision element on the next higher level) to keep the rank order of decision alternatives A_i and A_j. [107]

In the literature on sensitivity analysis for hierarchical decision problems, [107] is by far the most comprehensive study. However, similar to the *Expert Choice* software, this analysis is limited to a single change in the first level contribution vector ($\mathbf{C}_\ell^{\mathrm{O}}$) in a decision hierarchy. Further, the authors first induced the perturbation $\delta_{k,i,j}$ on W_k, then normalized the new value of W_k ($W_k^* = W_k -$ $_{k,i,j}$), and determined the threshold, which is the smallest value of $\delta_{k,i,j}$ to alter the rank order of A_i and A_j. Because the new normalized value W_k^* is different from the original value, the threshold deducted as a value of $\delta_{k,i,j}$ no longer applies to the new set.

For example, in their deduction, Triantaphyllou and Sanchez assumed the perturbation was induced on W_1 to alter the rank order of A_1 and A_2, which makes $W_1^* = W_1 - \delta_{1,1,2}$, then to preserve the property that the sum of all weights equals to 1, weights are normalized as follows, with W_i'denoting the normalized value:

$$W_1' = \frac{W_1^*}{W_1^* + W_2 + \ldots + W_n} \quad (2.2a)$$

$$W_2' = \frac{W_2}{W_1^* + W_2 + \ldots + W_n} \quad (2.2b)$$

•

•

•

$$W_n' = \frac{W_n}{W_1^* + W_2 + \ldots + W_n} \quad (2.2c)$$

If we use $\delta^*_{1,1,2}$ to represent the actual threshold instead of the un-normalized threshold $\delta_{1,1,2}$, we have:

$$W_1 - \delta^*_{1,1,2} = W_1' = \frac{W_1^*}{W_1^* + W_2 + \ldots + W_n} = \frac{W_1 - \delta_{1,1,2}}{W_1 - \delta_{1,1,2} + W_2 + \ldots + W_n} = \frac{W_1 - \delta_{1,1,2}}{\sum_{i=1}^{n} W_i - \delta_{1,1,2}} \quad (2.2d)$$

$$(W_1 - \delta^*_{1,1,2}) \times (\sum_{i=1}^{n} W_i - \delta_{1,1,2}) = W_1 - \delta_{1,1,2} \quad (2.2e)$$

$$W_1 \times (\sum_{i=1}^{n} W_i - \delta_{1,1,2}) - \delta^*_{1,1,2} \times (\sum_{i=1}^{n} W_i - \delta_{1,1,2}) = W_1 - \delta_{1,1,2} \quad (2.2f)$$

$$\delta^*_{1,1,2} = \frac{W_1 \times (\sum_{i=1}^{n} W_i - \delta_{1,1,2}) - W_1 + \delta_{1,1,2}}{\sum_{i=1}^{n} W_i - \delta_{1,1,2}} = W_1 - \frac{W_1 - \delta_{1,1,2}}{\sum_{i=1}^{n} W_i - \delta_{1,1,2}} \quad (2.2g)$$

As we can see, the actual threshold, $\delta^*_{k,i,j}$ (shown as $\delta^*_{1,1,2}$ in the above expression) is a value different from the threshold, $\delta_{k,i,j}$ (shown as $\delta_{1,1,2}$ in the above expression), in their study. $\delta^*_{k,i,j}$ is a function of $\delta_{k,i,j}$ but not equal to it. For example, if the contribution values W_i's are 0.4, 0.3, 0.2 and 0.1, and the $\delta_{1,i,j}$ is 0.1, it means before normalization W_1 can go down to 0.3 and keep the rank ordering of the decision alternatives. However, after normalization, W_1 can only go down to 0.33, and the other contribution values are changed to 0.33, 0.22, and 0.11. The actual threshold of changes on W_1 is $0.07 = (0.4 - \frac{0.4-0.1}{1-0.1})$ instead of 0.1.

Aguaron and Moreno-Jimenez [1] studied the tolerance of local priorities, which is called the "local stability interval" in [1], to preserve the ranking of current local priority weights. Based on the tolerance, a stability index is proposed. This study is not closely related to the research questions in this dissertation. It can be viewed as the interface between HDM SA for ranking problems and the AHP interval studies listed in section 2.4.2. However, their deductive logic is similar to what is used in [107] and in this dissertation, except for the way the perturbations are represented.

The application of their method is limited to judgment quantification models utilizing geometric mean method developed by Barzilai and Lootsma [9][67].

Another related study of sensitivity analysis in HDM using mathematical deduction is by Ra [93]. He pointed out that sometimes the decision makers are uncertain of some comparisons, thus, conducted a basic and simple sensitivity analysis on the pair-wise comparison values for HDM. The analysis determines the range in which the scores of a paired comparisons can change without altering the ranking of the elements when Column-Row Sums method and Logarithmic Least Squares method are used to derive the weights from pair-wise comparison results. His method is mathematical deduction based on numerical values.

2.4 OTHER RELATED STUDIES

2.4.1 General sensitivity analysis

Evans [31] pointed out that SA was a fundamental concept in the effective use and implementation of quantitative decision models. He studied the sensitivity of parametric changes in probability values and optimal decisions in classical decision-theoretic problems by utilizing SA concepts in linear programming. According to his analysis, if the current parametric value is located near the center of P* (allowable space) then the decision is robust.

Alexander [2] demonstrated the importance and the different roles of sensitivity analysis in decision models. He pointed out that as the decision models became more and more complex; they were not transparent enough for easy sensitivity analysis. Five sensitivity indicators as measurements of the degree to which a change in a variable affect outcomes were proposed in his study.

Saltelli defined the "global sensitivity analysis" as the study of "how the uncertainty in the output of a model can be apportioned to different sources of uncertainty in the model input." [100] This definition points out that it is the existence of uncertainty in the input factor(s) that calls for the study of SA. In his study, Saltelli presented several "model-free" methods, of which the application does not rely on special assumptions on the behavior of the model, used generally in sensitivity analysis. He also pointed out that, according to the European Commission 2002 handbook for extended impact assessment, a good SA should conduct analysis over the full range of plausible values of key parameters and their interactions, to assess how impacts change in response to changes in key parameters.

2.4.2 Stable Interval Study for AHP

There is a group of studies which are related to AHP and its sensitivity to judgment values given in the pairwise comparison. Literature in this group is initiated by Saaty in his AHP book where he studied the sensitivity of slight changes in the judgment values to the priorities given by the eigenvector components in AHP [97]. Later, as the rank reversal problem was brought up by Belton and Gear [12], more studies on the stable interval of local priorities were conducted.

Albel and Vargas studied the effect of interval judgments on the AHP and found that within these ranges, there exist relatively optimal point estimates which best categorize the intervals. [5]

Moreno-Jimenez and Vargas studied the problem of determining the most probable ranking of alternatives that one should infer when decision makers use interval judgments rather than point estimates in the AHP [80]. However, the study is limited to single reciprocal matrix, while many problem structures are of complex hierarchies (e.g., [49][25][98][52][113][114]).

Sugihara and Tanaka proposed a model using linear programming to calculate the interval of local priority weights under the condition of inconsistent judgment. By comparing the intervals derived by the proposed model, one can know the possible ranking of the local priority weights and thus determine whether the judgments satisfy "strong transitivity" (cardinal consistency), "weak transitivity" (ordinal consistency) or neither [105].

Farkas, Gyorgy, and Rozsa, studied the spectrum of symmetrically reciprocal perturbations of transitive matrices of nonzero complex entries. By a series of complex mathematical deductions, closed forms for all eigenvalues and their pair-wise comparison matrices are given and explicit intervals for the ranges of possible rank reversals between decision alternatives are established in their study [33].

This group of literature is relevant in that they studied the ranges which certain values in the pair-wise comparison matrix can be placed without reversing the rank of the decision alternatives. However, the rank reversal problem addressed in these studies is specifically related to AHP. No ranking irregularity problem has been reported for other hierarchical decision modeling algorithm.

2.5 SUMMARY

Sensitivity analysis has been regarded as a fundamental concept in the effective use and implementation of quantitative decision models [31]. It has several important roles, and serves different purposes in the MCDM process (e.g., [2][6]). Although the SA for AHP and its variants have been studied by several researchers (e.g., [85][15][69][107][53][19][49]), none of them has offered a fast and accurate way to do a comprehensive SA for HDM.

The methods employed in previous literature to conduct SA for HDM are categorized into: 1) Numerical incremental analysis, 2) simulation approach, and 3) mathematical deduction. Limitations and the special conditions applicable to numerical incremental analysis and simulation approach are discussed. Mathematical deduction is identified as an effective solution that offers a closed-form expression to describe how variations in the input affect the output of the model. However, previous studies in this category either proposed a sensitivity coefficient as a likelihood of rank change or only studied a single change in the first level contribution vector without normalizing the threshold value. An accurate and comprehensive HDM SA algorithm in symbolic form is needed to close this research gap. Table 1 summarizes the comparisons of those methods/studies.

Table 1 Comparison of major SA methods and studies, including the present research (last column)

	Numerical Incremental Analysis	Simulation Approach	Mathematical deduction in symbolic form			
			Triantaphyllou and Sanchez (1997)	Masuda (1990)	Huang (2002)	HDM SA
Judgment quantification method independent	Yes	Yes	Yes	No	No	Yes
"Brute force" not needed	No	Yes	Yes	Yes	Yes	Yes
Applicable to different levels	Possible	Possible	No	Yes	Yes	Yes
Applicable to multiple changes simultaneously	Possible	Yes	No	Yes	Yes	Yes
Amount of Information given	Low	Low	Medium	Very low	Very low	High
Precision / Accuracy	Low	Low	Medium	Low	Low	High
Computational complexity	High	High	Low	Low	Low	Low

3. HDM SA ALGORITHM

3.1 OBJECTIVES OF THE STUDY

The HDM SA algorithm developed in this dissertation addresses four major questions that have been asked in most of the literature:

1. What is the allowable range/region of perturbations induced on local contribution(s) or what is the tolerance of a local contribution to keep the decision alternatives' ranking unchanged?
2. How to measure the sensitivity/robustness of model result to variations of different decision elements?
3. Which is the most critical decision element at a given level of the decision hierarchy?
4. How do changes to the local contributions' values impact the overall contributions and priorities of specific decision alternatives?

Although as mentioned before, there exist more than fourteen different judgment quantification techniques that yield the contribution matrices between successive levels of the hierarchy, sensitivity analysis developed in this dissertation is independent of those methods, and can be applied to any deterministic hierarchical decision making model, as long as the contribution matrices are given.

While answering the above research questions, this study considers three conditions: 1) the current rank order of any two decision alternatives are concerned, 2) the current rank orders of all the decision alternatives are concerned, and 3) the current top ranked decision alternative are concerned.

3.2 RESEARCH METHOD—A SYSTEMS APPROACH

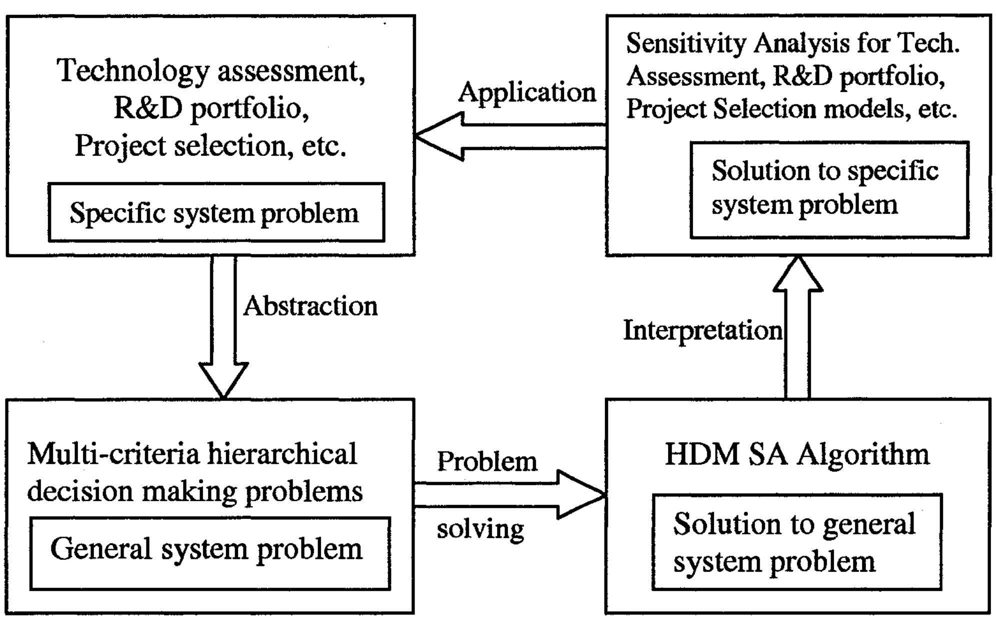
One way to analyze causal influences and their effects is by using traditional deductive logic to carefully deduce an outcome from assumptions made at the beginning [98]. Due to the linear relationships and the deterministic characteristics of HDM, the sensitivity analysis algorithm in this dissertation is developed by logical mathematical deduction. The details of the deduction process are included in the Appendix.

The method proposed in this dissertation is a systems approach since the subject it deals with is a system of correlated decision elements (mission, objectives, goals, strategies and alternatives), and more importantly, it deals with the algorithm itself, which is abstract, rather than the empirical data from specific decision problems. This satisfies the definition of the systems problem given by Klir [58]. The SA algorithm developed in this dissertation is independent of data and the way people obtain data. Operating at this higher and novice abstract level gives the method broad and cross-disciplinary applicability regardless of the judgments people make for the priority weights, and regardless of the method used to generate the contribution matrices (either eigenvelue based AHP [98], or the constant sum method [25], or any other method discussed in the literature review). As long as the decision hierarchy is given, the algorithm is able to answer the sensitivity analysis questions asked in this dissertation.

Another characteristic of the systems approach is that the method developed under one framework can usually be converted to solve comparable systems problems under another framework [58]. Since the sensitivity analysis methods under other frameworks such as Operations Research are more developed, the same deductive logic behind such methods are used to convert the methods within our framework. The two applications of the algorithm in section 4 and 5 also show that HDM SA algorithm can be applied to solve problems in different fields.

In addition, by developing this HDM Sensitivity Analysis algorithm as a solution to general system problem, we can interpret it to solve specific system problems. This process is illustrated in Figure 3.

Figure 3 A systematic approach for HDM Sensitivity Analysis



3.3 ASSUMPTIONS

In this study, all the assumptions that apply to the hierarchical decision making models are applicable, including:

No dependencies among the decision elements on the same level

No feedback loops in the decision process

Linear, additive relationship among the decision elements

In addition, in order to normalize the new values in the contribution matrices before any sensitivity analysis is conducted, it is assumed that, when perturbations are induced on any of the contributions, the values of other related contributions will be changed passively according to their original ratio scale relationships. "Related contributions" are the contributions of other decision elements at the same level to the same decision element at the next higher level as the contribution(s) undergoing changes. For example, if a perturbation $P_{\ell*}^{O}$ is induced on a $C_{\ell*}^{O}$, the new value of that $C_{\ell*}^{O}$ will be:

$$C_{\ell*}^{O}(new) = C_{\ell*}^{O} + P_{\ell*}^{O} \tag{3.1a}$$

The new values of other C_{ℓ}^{O}'s will be:

$$C_{\ell}^{O}(new) = C_{\ell}^{O} + P_{\ell}^{O}, \text{ with } P_{\ell}^{O} = -\frac{P_{\ell*}^{O} \times C_{\ell}^{O}}{\sum_{\ell=1,\ell\neq\ell*}^{L} C_{\ell}^{O}} \tag{3.1b}$$

with $P_{\ell*}^{O}$ representing the induced perturbation, and P_{ℓ}^{O}'s being the changes occurred passively on the related contributions according to their original ratio scales. This is the same assumption as the one used in the sensitivity analysis function of software *Experts Choice* [32].

For another example, when M perturbations $P_{k_m^*\ell^*}^{G-O}$ (m=1, 2…M) are induced on contributions of M goals, $G_{k_m^*}$'s, to a specific objective, $O_{\ell*}$, the new values of $C_{k_m^*\ell^*}^{G-O}$'s will be:

$$C_{k_m^*\ell^*}^{G-O}(new) = C_{k_m^*\ell^*}^{G-O} + P_{k_m^*\ell^*}^{G-O} \tag{3.2a}$$

The new values of other $C_{k\ell^*}^{G-O}$'s will be:

$$C_{k\ell^*}^{G-O}(new) = C_{k\ell^*}^{G-O} + P_{k\ell^*}^{G-O}, \text{ with } P_{k\ell^*}^{G-O} = -\sum_{m=1}^{M} P_{k_m^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum\limits_{\substack{k=1 \\ k\neq k_1^*\ldots k_M^*}}^{K} C_{k\ell^*}^{G-O}} \tag{3.2b}$$

(* indicates that perturbation(s) are induced on contribution(s) related to that specific decision element)

3.4 MEASURING DIRECT IMPACT

First we define the direct impact of changes to a local contribution value on the decision alternatives' overall contributions. This measures the amount of change brought to the decision alternatives' overall contributions as a result of the changes to a local contribution value. Three theorems are presented as:

Theorem 1.1 Let $P_{\ell^*}^{O}$ $(-C_{\ell^*}^{O} \leq P_{\ell^*}^{O} \leq 1-C_{\ell^*}^{O})$ denote the perturbation induced on one of the C_{ℓ}^{O}'s, which is $C_{\ell^*}^{O}$; this will result in a change to the values of C_i^A equaling to: $P_{\ell^*}^{O} \times (C_{i\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{i\ell}^{A-O} \times \frac{C_{\ell}^{O}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_{\ell}^{O}})$ (3.3)

Theorem 1.2 Let $P_{k^*\ell^*}^{G-O}$ $(-C_{k^*\ell^*}^{G-O} < P_{k^*\ell^*}^{G-O} < 1-C_{k^*\ell^*}^{G-O})$ denote a perturbation induced on one of the $C_{k\ell}^{G-O}$'s, which is $C_{k^*\ell^*}^{G-O}$ (contribution of a specific goal G_{k^*} to a specific objective O_{ℓ^*}); this will result in a change to the values of C_i^A equaling to:

$$P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^{O} \times (C_{ik^*}^{A-G} - \sum_{\substack{k=1\\k\neq k^*}}^{K} C_{ik}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}) \quad (3.4)$$

Theorem 1.3 Let $P_{i^*j^*}^{A-S}$ $(-C_{i^*j^*}^{A-S} \leq P_{i^*j^*}^{A-S} \leq 1-C_{i^*j^*}^{A-S})$ denote the perturbation induced on one of the C_{ij}^{A-S}'s, which is $C_{i^*j^*}^{A-S}$ (contribution of a specific action A_{i^*} to a specific strategy S_{j^*}); this will result in a change to the values of C_i^A equaling to

$C_{j^*}^{S} \times P_{i^*j^*}^{A-S}$ (when $P_{i^*j^*}^{A-S}$ is induced on C_{ij}^{A-S}, so $i = i^*$) (3.5a)

or $-C_{j^*}^{S} \times P_{i^*j^*}^{A-S} \times \frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij^*}^{A-S}}$ (when $P_{i^*j^*}^{A-S}$ is not induced on C_{ij}^{A-S}, so $i \neq i^*$) (3.5b)

3.5 TOLERANCE ANALYSIS

Tolerance is defined as the allowable range in which a contribution value can vary without changing the rank order of decision alternatives. To determine the tolerance of each contribution, the *allowable range of perturbations* on the contribution is calculated first. The allowable range of perturbations corresponds to the "slack" or "allowable increase and decrease," as used in the sensitivity analysis of linear programming [86][82].

In an effort to offer a comprehensive algorithm, three groups of theorems and corollaries are presented in the following subsections, covering situations when multiple and single perturbations are induced to any local contribution vector/matrix from the top to the bottom level of the decision hierarchy. Tolerance of the local contributions at each level is also defined.

The logic behind the deduction of the allowable range of perturbations is: Suppose originally A_r ranks before A_t, indicating ($C_r^A > C_t^A$); the rank order of A_r and A_t will be preserved if the new value of C_r^A is still greater than or equal to the new value of C_t^A. Therefore, the relationships between the perturbation(s) and the contributions can be found by representing the new values of C_r^A and C_t^A with an expression containing the original contributions and the induced perturbation(s). For details of the mathematical deductions, please refer to the Appendix.

Based on this logic, several theorems and corollaries are deduced in the following sections regarding sensitivity analysis at different levels of the decision hierarchy. As noted in the literature [107][1], decision makers may be interested in either the ranking of all decision alternatives or only the top choice in different cases. Therefore, three situations are considered to preserve the current rank order of: (i) a pair of decision alternatives, (ii) all decision alternatives, and (iii) the best alternative.

3.5.1 First Level Contribution Vector

Theorem 2 Let $P_{\ell_m^*}^O$ ($-C_{\ell_m^*}^O \le P_{\ell_m^*}^O \le 1-C_{\ell_m^*}^O, \sum_{m=1}^{M} P_{\ell_m^*}^O \le 1-\sum_{m=1}^{M} C_{\ell_m^*}^O$, m=1,2...M) denote M perturbations induced on M of the C_ℓ^O's, which are $C_{\ell_m^*}^O$; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \ge P_{\ell_1^*}^O \times \lambda_{\ell 1}^O + P_{\ell_2^*}^O \times \lambda_{\ell 2}^O + \ldots P_{\ell_m^*}^O \times \lambda_{\ell m}^O + \ldots + P_{\ell_M^*}^O \times \lambda_{\ell M}^O \tag{3.6a}$$

Where $\lambda = C_r^A - C_{r+n}^A$ (3.6b)

$$\lambda_{\ell m}^O = C_{r+n,\ell_m^*}^{A-O} - C_{r\ell_m^*}^{A-O} - \sum_{\substack{\ell=1 \\ \ell \ne \ell_1^* \ldots \ell_M^*}}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\substack{\ell=1 \\ \ell \ne \ell_1^* \ldots \ell_M^*}}^{L} C_\ell^O} + \sum_{\substack{\ell=1 \\ \ell \ne \ell_1^* \ldots \ell_M^*}}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\substack{\ell=1 \\ \ell \ne \ell_1^* \ldots \ell_M^*}}^{L} C_\ell^O} \tag{3.6c}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2...I-1, and n=1.

Theorem 2 defines an M dimensional allowable region for M perturbations induced in the first level contribution vector $\mathbf{C}_\ell^{\mathbf{O}}$. As long as the values of the perturbations fall into this allowable region, current rank orders will remain unchanged. When (M = 1), which means only one C_ℓ^O value is perturbed, the threshold of the perturbation can be determined by corollary 2.1.

Corollary 2.1 Let $P_{\ell^*}^O$ $(-C_{\ell^*}^O \le P_{\ell^*}^O \le 1-C_{\ell^*}^O)$ denote the perturbation induced on one of the C_ℓ^O's, which is $C_{\ell^*}^O$; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \ge P_{\ell^*}^O \lambda^O$$

(3.7a)

Where $\lambda = C_r^A - C_{r+n}^A$ (3.7b)

$$\lambda^O = C_{r+n,\ell^*}^{A-O} - C_{r\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} + \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \quad (3.7c)$$

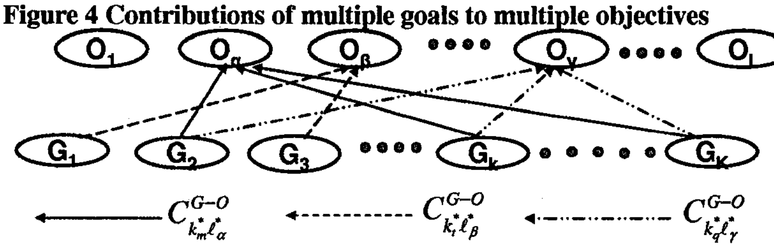
The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2...I-1, and n=1.

Thresholds of the single perturbation $P_{\ell^*}^O$, denoted as $\varepsilon_{\ell-}^O$ (negative) and $\varepsilon_{\ell+}^O$ (positive), to preserve current ranking of interested A_i's can be calculated from (3.7a) to (3.7c). Combining the feasibility constraint ($-C_{\ell^*}^O \le P_{\ell^*}^O \le 1-C_{\ell^*}^O$), which protects any C_ℓ^O value from going below zero or above one, the allowable range of perturbations on $C_{\ell^*}^O$, denoted as [$\delta_{\ell-}^O, \delta_{\ell+}^O$], can be derived as [$Max\{-C_{\ell^*}^O, \varepsilon_{\ell-}^O\}, Min\{1-C_{\ell^*}^O, \varepsilon_{\ell+}^O\}$]. Then, the *tolerance* of the corresponding contribution $C_{\ell^*}^O$ is $[\delta_{\ell-}^O + C_{\ell^*}^O, \delta_{\ell+}^O + C_{\ell^*}^O]$. As long as the value of $C_{\ell^*}^O$ is within this tolerance range, the final ranking of A_i's under consideration will remain unchanged. To derive the allowable range of perturbations or the tolerance of a C_ℓ^O, I inequalities need to be satisfied in both cases: to either preserve the top-ranked alternative only or to preserve the rank order for all A_i's. I is the number of decision alternatives.

3.5.2 Middle Levels of the Decision Hierarchy

While Theorem 2 and corollary 2.1 deal with perturbation(s) induced in local contribution vector at the top level of the decision hierarchy, $\mathbf{C}_\ell^\mathbf{O}$ in the MOGSA model, the following theorem and corollaries are applicable to perturbation(s) induced in contribution matrices at the middle levels, such as $\mathbf{C}_{\mathbf{k}\ell}^{\mathbf{G-O}}$ and $\mathbf{C}_{\mathbf{jk}}^{\mathbf{S-G}}$ in the MOGSA model. Since the same analysis applies to all the middle levels, notations used in Theorem 3 and its corollaries are from $\mathbf{C}_{\mathbf{k}\ell}^{\mathbf{G-O}}$ matrix.

Figure 4 Contributions of multiple goals to multiple objectives



Theorem 3 Let $P_{k_m^*\ell_\alpha^*}^{G-O}$ $(-C_{k_m^*\ell_\alpha^*}^{G-O} \le P_{k_m^*\ell_\alpha^*}^{G-O} \le 1-C_{k_m^*\ell_\alpha^*}^{G-O}$, $\sum_{m=1}^{M} P_{k_m^*\ell_\alpha^*}^{G-O} \le 1-\sum_{m=1}^{M} C_{k_m^*\ell_\alpha^*}^{G-O}$, m = 1, 2...M) denote M perturbations induced on M of the $C_{k\ell_\alpha}^{G-O}$'s (contributions of M goals $G_{k_m^*}$ to the α^{th} changing objective $O_{\ell_\alpha^*}$), $P_{k_i^*\ell_\beta^*}^{G-O}$ $(-C_{k_i^*\ell_\beta^*}^{G-O} \le P_{k_i^*\ell_\beta^*}^{G-O} \le 1-C_{k_i^*\ell_\beta^*}^{G-O}$,

$\sum_{t=1}^{T} P_{k_t^* \ell_\beta^*}^{G-O} \le 1 - \sum_{t=1}^{T} C_{k_t^* \ell_\beta^*}^{G-O}$, t = 1, 2...T) denote T perturbations induced on T of the $C_{k\ell_\beta}^{G-O}$'s (contributions of T goals $G_{k_t^*}$ to the β^{th} changing objective $O_{\ell_\beta^*}$), $P_{k_q^* \ell_\gamma^*}^{G-O}$ ($-C_{k_q^* \ell_\gamma^*}^{G-O} \le P_{k_q^* \ell_\gamma^*}^{G-O} \le 1 - C_{k_q^* \ell_\gamma^*}^{G-O}$, $\sum_{q=1}^{Q} P_{k_q^* \ell_\gamma^*}^{G-O} \le 1 - \sum_{q=1}^{Q} C_{k_q^* \ell_\gamma^*}^{G-O}$, q = 1, 2...Q) denote Q perturbations induced on Q of the $C_{k\ell_\gamma}^{G-O}$'s, (contributions of Q goals $G_{k_q^*}$ to the γ^{th} changing objective $O_{\ell_\gamma^*}$); (see Figure 4) the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \ge \sum_{m=1}^{M} (P_{k_m^* \ell_\alpha^*}^{G-O} \times \lambda_{k_m^* \ell_\alpha^*}^{G-O}) + \sum_{t=1}^{T} (P_{k_t^* \ell_\beta^*}^{G-O} \times \lambda_{k_t^* \ell_\beta^*}^{G-O}) + \sum_{q=1}^{Q} (P_{k_q^* \ell_\gamma^*}^{G-O} \times \lambda_{k_t^* \ell_\beta^*}^{G-O})$$

(3.8a)

Where $\lambda = C_r^A - C_{r+n}^A$

(3.8b)

$$\lambda_{k_m^* \ell_\alpha^*}^{G-O} = C_{\ell_\alpha^*}^{O} \times [C_{r+n,k_m^*}^{A-G} - C_{rk_m^*}^{A-G} + \sum_{\substack{k=1 \\ k \ne k_m^*}}^{K} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \times \frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1 \\ k \ne k_m^*}}^{K} C_{k\ell_\alpha^*}^{G-O}}]$$

(3.8c)

$$\lambda_{k_t^* \ell_\beta^*}^{G-O} = C_{\ell_\beta^*}^{O} \times [C_{r+n,k_t^*}^{A-G} - C_{rk_t^*}^{A-G} + \sum_{\substack{k=1 \\ k \ne k_t^*}}^{K} C_{\ell_\beta^*}^{O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k \ne k_t^*}}^{K} C_{k\ell_\beta^*}^{G-O}}]$$

(3.8d)

$$\lambda^{G-O}_{k_q^*\ell_\gamma^*} = C^{O}_{\ell_\gamma^*} \times [C^{A-G}_{r+n,k_q^*} - C^{A-G}_{rk_q^*} + \sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C^{O}_{\ell_\gamma^*}(C^{A-G}_{rk} - C^{A-G}_{r+n,k}) \times \frac{C^{G-O}_{k\ell_\gamma^*}}{\sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C^{G-O}_{k\ell_\gamma^*}}]$$

(3.8e)

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2…I-1, and n=1.

Theorem 3 deals with a general situation when different numbers (M, T, Q) of the local contributions to three objectives ($O_{\ell_\alpha^*}, O_{\ell_\beta^*}$ and $O_{\ell_\gamma^*}$) are perturbed (see Figure 4). It defines a (M+T+Q) dimensional allowable region for the (M+T+Q) perturbations induced in the local contribution matrix $\mathbf{C}^{G-O}_{k\ell}$. When contributions to more than three objectives need to be changed, (3.8a) can be extended by adding more $\sum_{x=1}^{X}(P^{G-O}_{k_x^*\ell_\theta^*} \times \lambda^{G-O}_{k_x^*\ell_\theta^*})$' s following the same pattern, using x to represent the number of perturbations induced for each $C^{G-O}_{k\ell_\theta^*}$ and θ to differentiate the new O_{ℓ_θ} to which the x contributions will be perturbed. When there is only one $C^{G-O}_{k\ell}$ being changed, the threshold of such change can be determined by corollary 3.1.

Corollary 3.1 Let $P^{G-O}_{k^*\ell^*}$ ($-C^{G-O}_{k^*\ell^*} < P^{G-O}_{k^*\ell^*} < 1 - C^{G-O}_{k^*\ell^*}$) denote a perturbation induced on one of the $C^{G-O}_{k\ell}$'s, which is $C^{G-O}_{k^*\ell^*}$ (contribution of a specific goal G_{k^*} to a specific objective O_{ℓ^*}); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P^{G-O}_{k^*\ell^*} \times \lambda^{G-O}_{k_1\ell} \qquad (3.9a)$$

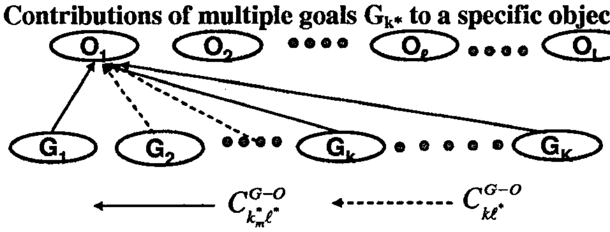
Where $\lambda = C_r^A - C_{r+n}^A$ (3.9b)

$$\lambda_{k_1\ell}^{G\text{-}O} = C_{\ell^*}^{O} \times [C_{r+n,k^*}^{A-G} - C_{rk^*}^{A-G} + (\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{rk}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{r+n,k}^{A-G}) \times \frac{C_{k\ell^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{k\ell^*}^{G-O}}] \quad (3.9c)$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2...I-1, and n=1.

The thresholds of $P_{k^*\ell^*}^{G-O}$ in both directions, denoted as $\varepsilon_{k\ell-}^{G-O}$ and $\varepsilon_{k\ell+}^{G-O}$, can be derived from (3.9a) to (3.9c). Then, the allowable range of perturbations on $C_{k^*\ell^*}^{G-O}$ is $[\delta_{k\ell-}^{G-O}, \delta_{k\ell+}^{G-O}]$, where ($\delta_{k\ell-}^{G-O} = Max\{-C_{k^*\ell^*}^{G-O}, \varepsilon_{k\ell-}^{G-O}\}$) and ($\delta_{k\ell+}^{G-O} = Min\{1 - C_{k^*\ell^*}^{G-O}, \varepsilon_{k\ell+}^{G-O}\}$). The *tolerance* of contribution $C_{k^*\ell^*}^{G-O}$ is $[\delta_{k\ell-}^{G-O} + C_{k^*\ell^*}^{G-O}, \delta_{k\ell+}^{G-O} + C_{k^*\ell^*}^{G-O}]$.

Figure 5 Contributions of multiple goals G_{k^*} to a specific objective O_{ℓ^*}



Corollary 3.2 Let $P_{k_m^*\ell^*}^{G-O}$ ($-C_{k_m^*\ell^*}^{G-O} < P_{k_m^*\ell^*}^{G-O} < 1 - C_{k_m^*\ell^*}^{G-O}$, $\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell^*}^{G-O} - 1 < \sum_{m=1}^{M} P_{k_m^*\ell^*}^{G-O}$ $< \sum_{k=1,k\neq k_m^*}^{K} C_{k\ell^*}^{G-O}$, m=1, 2...M) denote M perturbations induced on M of the $C_{k\ell}^{G-O}$'s, which are $C_{k_m^*\ell^*}^{G-O}$ (contributions of M specific goals G_{k^*}'s to a specific objective O_{ℓ^*}, see Figure 5); the original ranking of A_r and A_{r+n} will not reverse if:

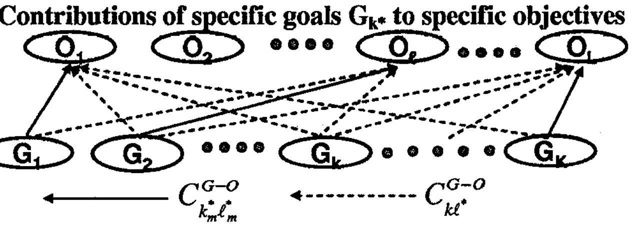
$$\lambda \geq P^{G-O}_{k_1^* \ell^*} \times \lambda^{G-O}_{k_1 \ell} + P^{G-O}_{k_2^* \ell^*} \times \lambda^{G-O}_{k_2 \ell} + \ldots + P^{G-O}_{k_m^* \ell^*} \times \lambda^{G-O}_{k_m \ell} + \ldots + P^{G-O}_{k_M^* \ell^*} \times \lambda^{G-O}_{k_M \ell} \quad (3.10a)$$

Where $\lambda = C^A_r - C^A_{r+n}$ (3.10b)

$$\lambda^{G-O}_{k_m \ell} = C^O_{\ell^*} \times [C^{A-G}_{r+n,k_m^*} - C^{A-G}_{rk_m^*} + \sum_{\substack{k=1 \\ k \neq k_1^* \ldots k_M^*}}^{K} (C^{A-G}_{rk} - C^{A-G}_{r+n,k}) \times \frac{C^{G-O}_{k\ell^*}}{\sum_{\substack{k=1 \\ k \neq k_1^* \ldots k_M^*}}^{K} C^{G-O}_{k\ell^*}}] \quad (3.10c)$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2…I-1, and n=1.

Figure 6 Contributions of specific goals G_{k^*} to specific objectives O_{ℓ^*}



Corollary 3.3 Let $P^{G-O}_{k_m^* \ell_m^*}$ ($-C^{G-O}_{k_m^* \ell_m^*} < P^{G-O}_{k_m^* \ell_m^*} < 1 - C^{G-O}_{k_m^* \ell_m^*}$, m=1, 2…M) denote M perturbations induced on M of the $C^{G-O}_{k\ell}$'s, which are $C^{G-O}_{k_m^* \ell_m^*}$ (contributions of M specific goals G_{k^*}'s to M specific objectives O_{ℓ^*}'s, see Figure 6); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P^{G-O}_{k_1^* \ell_1^*} \times \lambda^{G-O}_{k_1 \ell_1} + P^{G-O}_{k_2^* \ell_2^*} \times \lambda^{G-O}_{k_2 \ell_2} + \ldots + P^{G-O}_{k_m^* \ell_m^*} \times \lambda^{G-O}_{k_m \ell_m} + \ldots + P^{G-O}_{k_M^* \ell_M^*} \times \lambda^{G-O}_{k_M \ell_M} \quad (3.11a)$$

Where $\lambda = C^A_r - C^A_{r+n}$ (3.11b)

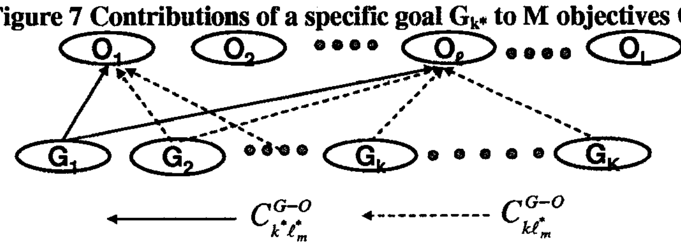
$$\lambda^{G-O}_{k_m \ell_m} = C^{O}_{\ell_m^*} \times \left(C^{A-G}_{r+n,k_m^*} - C^{A-G}_{rk_m^*} + \sum_{\substack{k=1 \\ k \neq k_m^*}}^{K} C^{A-G}_{rk} \times \frac{C^{G-O}_{k\ell_m^*}}{\sum_{k=1,k\neq k_m^*}^{K} C^{G-O}_{k\ell_m^*}} - \sum_{\substack{k=1 \\ k \neq k_m^*}}^{K} C^{A-G}_{r+n,k} \times \frac{C^{G-O}_{k\ell_m^*}}{\sum_{k=1,k\neq k_m^*}^{K} C^{G-O}_{k\ell_m^*}}\right)$$

(3.11c)

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1.

The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2…I-1, and n=1.

Figure 7 Contributions of a specific goal G_{k^*} to M objectives O_{ℓ^*}



Corollary 3.4 Let $P^{G-O}_{k^*\ell_m^*}$ ($-C^{G-O}_{k^*\ell_m^*} < P^{G-O}_{k^*\ell_m^*} < 1 - C^{G-O}_{k^*\ell_m^*}$, m=1, 2…M) denote M perturbations induced on M of the $C^{G-O}_{k\ell}$'s, which are $C^{G-O}_{k^*\ell_m^*}$ (contributions of a specific goal G_{k^*}'s to M objectives O_{ℓ^*}'s, see Figure 7); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P^{G-O}_{k^*\ell_1^*} \times \lambda^{G-O}_{k_1\ell_1} + P^{G-O}_{k^*\ell_2^*} \times \lambda^{G-O}_{k_2\ell_2} + \ldots + P^{G-O}_{k^*\ell_m^*} \times \lambda^{G-O}_{k_m\ell_m} + \ldots + P^{G-O}_{k^*\ell_M^*} \times \lambda^{G-O}_{k_M\ell_M} \qquad (3.12a)$$

Where $\lambda = C^{A}_{r} - C^{A}_{r+n}$ (3.12b)

$$\lambda_{k_m \ell_m}^{G-O} = C_{\ell_m^*}^{O} \left(C_{r+n,k^*}^{A-G} - C_{rk^*}^{A-G} + \sum_{\substack{k=1 \\ k \neq k^*}}^{K} C_{rk}^{A-G} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_m^*}^{G-O}} - \sum_{\substack{k=1 \\ k \neq k^*}}^{K} C_{r+n,k}^{A-G} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_m^*}^{G-O}} \right)$$

(3.12c)

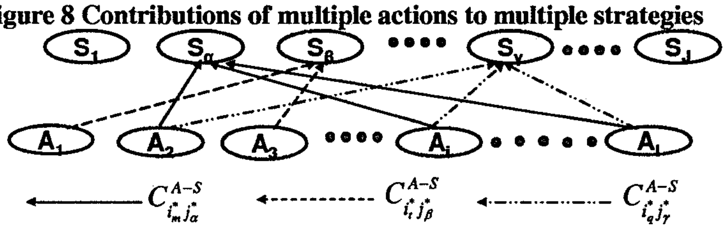
The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1.

The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2...I-1, and n=1.

3.5.3 Bottom Level of the Decision Hierarchy

Theorem 4 and its corollaries deal with perturbations induced in matrix $\mathbf{C}_{ij}^{A-S}$, which is the bottom level of the decision hierarchy. Since the decision alternatives' level is involved in the analysis, it should be noted whether perturbations are induced on the pair of decision alternatives being compared. Consequently, more complex theorem and corollaries are developed to address various situations.

Figure 8 Contributions of multiple actions to multiple strategies



Theorem 4. Let $P_{i_m^* j_\alpha^*}^{A-S}$ ($-C_{i_m^* j_\alpha^*}^{A-S} \leq P_{i_m^* j_\alpha^*}^{A-S} \leq 1 - C_{i_m^* j_\alpha^*}^{A-S}$, $\sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \leq 1 - \sum_{m=1}^{M} C_{i_m^* j_\alpha^*}^{A-S}$, m=1, 2...M) denote M perturbations induced in M of the $C_{ij_\alpha}^{A-S}$'s (contributions of M actions $A_{i_m^*}$ to the α^{th} changing strategy $S_{j_\alpha^*}$), $P_{i_t^* j_\beta^*}^{A-S}$ ($-C_{i_t^* j_\beta^*}^{A-S} \leq P_{i_t^* j_\beta^*}^{A-S} \leq 1 - C_{i_t^* j_\beta^*}^{A-S}$,

$\sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \leq 1-\sum_{t=1}^{T} C_{i_t^* j_\beta^*}^{A-S}$, t = 1, 2...T) denote T perturbations induced in T of the $C_{ij_\beta}^{A-S}$'s (contributions of T actions $A_{i_t^*}$ to the β^{th} changing strategy $S_{j_\beta^*}$), $P_{i_q^* j_\gamma^*}^{A-S}$ ($-C_{i_q^* j_\gamma^*}^{A-S} \leq P_{i_q^* j_\gamma^*}^{A-S} \leq 1-C_{i_q^* j_\gamma^*}^{A-S}$, $\sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \leq 1-\sum_{q=1}^{Q} C_{i_q^* j_\gamma^*}^{A-S}$, q = 1, 2...Q) denote Q perturbations induced in Q of the $C_{k\ell_\gamma}^{G-O}$'s (contributions of Q actions $A_{i_q^*}$ to the γ^{th} changing strategy $S_{j_\gamma^*}$), see Figure 8; the original ranking of A_r and A_{r+n} will not reverse if:

$$C_{r^*}^{A} - C_{r+n}^{A} \geq -C_{j_\alpha^*}^{S} \left(P_{r_m^* j_\alpha^*}^{A-S} + \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r+n, j_\alpha^*}^{A-S}}{\sum_{i=1, i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}}\right) + C_{j_\beta^*}^{S} \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n, j_\beta^*}^{A-S}}{\sum_{i=1, i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}$$

$$+ C_{j_\gamma^*}^{S} \times \sum_{q=1}^{Q} P_{r_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n, j_\gamma^*}^{A-S}}{\sum_{i=1, i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \qquad (3.13a)$$

(when some perturbations are induced on C_{rj}^{A-S}'s **but** not on $C_{r+n,j}^{A-S}$'s)

$$C_{r}^{A} - C_{r+n^*}^{A} \geq C_{j_\alpha^*}^{S} \left(\sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{rj_\alpha^*}^{A-S}}{\sum_{i=1, i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + P_{r+n^* j_\alpha^*}^{A-S}\right) + C_{j_\beta^*}^{S} \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n, j_\beta^*}^{A-S}}{\sum_{i=1, i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}$$

$$+ C_{j_\gamma^*}^{S} \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n, j_\gamma^*}^{A-S}}{\sum_{i=1, i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \qquad (3.13b)$$

(when some perturbations are induced on $C_{r+n,j}^{A-S}$'s but not on C_{rj}^{A-S}'s)

$$C_r^A - C_{r+n}^A \geq C_{j_\alpha^*}^S \times \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{rj_\alpha^*}^{A-S} - C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}$$

$$+ C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \tag{3.13c}$$

(when no perturbation is induced on C_{rj}^{A-S} nor $C_{r+n,j}^{A-S}$)

$$C_{r^*}^A - C_{r+n^*}^A \geq C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}}$$

$$+ C_{j_\alpha^*}^S (P_{r+n_m^* j_\alpha^*}^{A-S} - P_{r_m^* j_\alpha^*}^{A-S}) \tag{3.13d}$$

(when two perturbations are induced on both C_{rj}^{A-S} and $C_{r+n,j}^{A-S}$ to the same $S_{j_\alpha^*}$)

$$C_{r^*}^A - C_{r+n^*}^A \geq P_{r+n_m^* j_\beta^*}^{A-S} C_{j_\beta^*}^S \left(1 + \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}\right) - P_{r_m^* j_\alpha^*}^{A-S} C_{j_\alpha^*}^S \left(1 + \frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}}\right) - C_{j_\alpha^*}^S \sum_{\substack{i_m^*=i_1^*,\\ i_m^*\neq r_m^*}}^{M} P_{i_m^* j_\alpha^*}^{A-S}$$

$$\times \frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{\substack{i_t^*=i_1^*,\\ i_t^*\neq r+n_m^*}}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}}$$

(when two perturbations are induced on $C_{rj_\alpha^*}^{A-S}$ and $C_{r+n,j_\beta^*}^{A-S}$) (3.13e)

The top choice will remain at the top rank if all the above conditions, (3.13a) to (3.13e), are satisfied for all r=1 and n=1, 2...I-1. The original ranking for all A_i's will remain unchanged if all the above conditions, (3.13a) to (3.13e), are satisfied for all r=1, 2...I-1, and n=1.

Theorem 4 deals with a general situation when different numbers (M, T, Q) of the local contributions to three strategies ($S_{j_\alpha^*}$, $S_{j_\beta^*}$ and $S_{j_\gamma^*}$) are perturbed (see Figure 8).

When contributions to more than three strategies need to be changed, (3.13a) to (3.13c) can be extended by adding more ($C_{j_\theta^*}^{S} \times \sum_{x=1}^{X} P_{i_x^* j_\theta^*}^{A-S} \times \frac{C_{ij_\theta^*}^{A-S}}{\sum_{i=1, i \neq i_x^*}^{I} C_{ij_\theta^*}^{A-S}}$) following the same pattern, using x to represent the number of perturbations induced for each $C_{ij_\theta^*}^{A-S}$ and θ to differentiate the new $S_{j_\theta^*}$ to which the x contributions will be perturbed. When only one C_{ij}^{A-S} value is perturbed, the threshold of such a perturbation can be determined based on corollary 4.1.

Corollary 4.1 Let $P_{i^*j^*}^{A-S}$ ($-C_{i^*j^*}^{A-S} \leq P_{i^*j^*}^{A-S} \leq 1 - C_{i^*j^*}^{A-S}$) denote the perturbation induced on one of the C_{ij}^{A-S}'s, which is $C_{i^*j^*}^{A-S}$ (contribution of a specific action A_{i*} to a specific strategy S_{j*}); the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{i^*j^*}^{A-S} \lambda_{ij}^{A-S} \tag{3.14a}$$

Where $\lambda = C_r^A - C_{r+n}^A$ (3.14b)

$$\lambda_{ij}^{A-S} = \frac{C_{j^*}^{S}(C_{rj^*}^{A-S} - C_{r+n,j^*}^{A-S})}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}} \tag{3.14c}$$

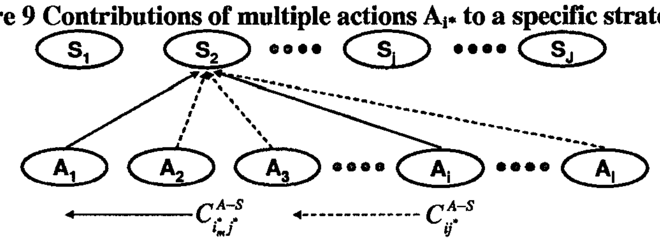
(if $P_{i^*j^*}^{A-S}$ is induced on neither C_{rj}^{A-S} nor $C_{r+n,j}^{A-S}$)

or $\lambda_{ij}^{A-S} = C_{j^*}^{S} \times (1 + \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq r+n}^{I} C_{ij^*}^{A-S}})$ (if $P_{i^*j^*}^{A-S}$ is induced on $C_{r+n,j}^{A-S}$) (3.14d) or $\quad \lambda_{ij}^{A-S} = -C_{j^*}^{S} \times (1+\dfrac{C_{r+n,j^*}^{A-S}}{\sum_{i=1,i\neq r}^{I} C_{ij^*}^{A-S}})$ (if $P_{i^*j^*}^{A-S}$ is induced on C_{rj}^{A-S}) (3.14e)

The top choice will remain at the top rank if all the above conditions are satisfied for all r=1 and n=1, 2…I-1. The original ranking for all A_i's will remain unchanged if all the above conditions, (3.14a) to (3.14e), are satisfied for all r=1, 2…I-1, and n=1.

The thresholds of $P_{i^*j^*}^{A-S}$ in both directions, denoted as ε_{ij-}^{A-S} and ε_{ij+}^{A-S} , can be derived from (3.14a) to (3.14e). The allowable range of perturbations on C_{ij}^{A-S} is $[\delta_{ij-}^{A-S}, \delta_{ij+}^{A-S}]$, where ($\delta_{ij-}^{A-S} = Max\{-C_{i^*j^*}^{A-S}, \varepsilon_{ij-}^{A-S}\}$) and ($\delta_{ij+}^{A-S} = Min\{1-C_{i^*j^*}^{A-S}, \varepsilon_{ij+}^{A-S}\}$). The *tolerance* of contribution $C_{i^*j^*}^{A-S}$ is $[\delta_{ij-}^{A-S} + C_{i^*j^*}^{A-S}, \delta_{ij+}^{A-S} + C_{i^*j^*}^{A-S}]$.

Figure 9 Contributions of multiple actions A_{i^*} to a specific strategy S_{j^*}



Corollary 4.2 Let $P_{i_m^*j^*}^{A-S}$ ($-C_{i_m^*j^*}^{A-S} < P_{i_m^*j^*}^{A-S} < 1 - C_{i_m^*j^*}^{A-S}$, $\sum_{i=1,i\neq i_m^*}^{I} C_{ij^*}^{A-S} - 1 < \sum_{m=1}^{M} P_{i_m^*j^*}^{A-S} < \sum_{i=1,i\neq i_m^*}^{I} C_{ij^*}^{A-S}$, m=1, 2…M) denote M perturbations induced on M of the C_{ij}^{A-S}'s, which are $C_{i_m^*j^*}^{A-S}$ (contributions of M specific actions A_{i^*}'s to a specific strategy S_{j^*}, see Figure 9); the original ranking of A_r and A_{r+n} will not reverse if:

$$C^A_r - C^A_{r+n} \ge \sum_{m=1}^{M} P^{A-S}_{i^*_m j^*} \times C^S_{j^*} \times \frac{C^{A-S}_{rj^*} - C^{A-S}_{r+n,j^*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij^*}} \tag{3.15a}$$

(if $P^{A-S}_{i^*_m j^*}$'s are induced on neither C^{A-S}_{rj} nor $C^{A-S}_{r+n,j}$)

or $$C^A_{r^*} - C^A_{(r+n)^*} \ge (P^{A-S}_{(r+n)^*_m j^*} - P^{A-S}_{r^*_m j^*}) \times C^S_{j^*} \tag{3.15b}$$

(if $P^{A-S}_{i^*_m j^*}$'s are induced on both C^{A-S}_{rj} and $C^{A-S}_{r+n,j}$)

or $$C^A_r - C^A_{(r+n)^*} \ge \sum_{\substack{m=1 \\ i\neq r+n}}^{M} P^{A-S}_{i^*_m j^*} \times C^S_{j^*} \times \frac{C^{A-S}_{rj^*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij^*}} + P^{A-S}_{(r+n)^*_m j^*} \times C^S_{j^*} \times (1 + \frac{C^{A-S}_{rj^*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij^*}}) \tag{3.15c}$$

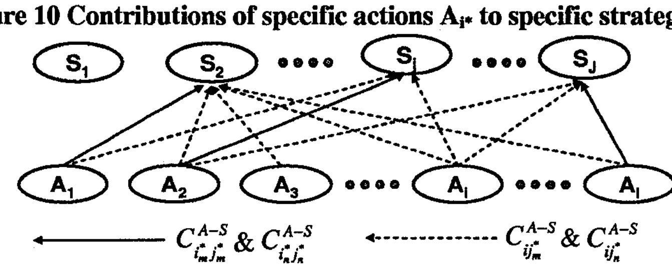
(if one of the $P^{A-S}_{i^*_m j^*}$'s, which is $P^{A-S}_{(r+n)^*_m j^*}$ in this case, is induced on $C^{A-S}_{r+n,j}$)

or $$C^A_{r^*} - C^A_{r+n} \ge - \sum_{m=1,i\neq r}^{M} P^{A-S}_{i^*_m j^*} \times C^S_{j^*} \times \frac{C^{A-S}_{r+n,j^*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij^*}} - P^{A-S}_{r^*_m j^*} \times C^S_{j^*} \times (1 + \frac{C^{A-S}_{r+n,j^*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij^*}}) \tag{3.15d}$$

(if one of the $P^{A-S}_{i^*_m j^*}$'s, which is $P^{A-S}_{r^*_m j^*}$ in this case, is induced on C^{A-S}_{rj})

The top choice will remain at the top rank if (3.15a) to (3.15d) are satisfied for all r=1 and n=1, 2…I-1. The original ranking for all A_i's will remain unchanged if all the above conditions, (3.15a) to (3.15d), are satisfied for all r=1, 2…I-1, and n=1.

Figure 10 Contributions of specific actions A_{i*} to specific strategies S_{j*}



Corollary 4.3 Let $P^{A-S}_{i^*_m j^*_m}$ ($-C^{A-S}_{i^*_m j^*_m} < P^{A-S}_{i^*_m j^*_m} < 1 - C^{A-S}_{i^*_m j^*_m}$, m=1, 2...M) denote M perturbations induced on M of the C^{A-S}_{ij}'s (contributions of M specific actions A_{i^*}'s to M specific strategies S_{j^*}'s, see Figure 10); the original ranking of A_r and A_t (r<t) will not reverse if:

$$C^A_r - C^A_t \ge \sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_m} \times C^S_{j^*_m} \times \frac{C^{A-S}_{rj^*_m} - C^{A-S}_{tj^*_m}}{\sum_{i=1, i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}} \qquad (3.16a)$$

(when perturbations are induced on neither C^{A-S}_{rj} nor C^{A-S}_{tj})

or $$C^A_{r^*_m} - C^A_{t^*_p} \ge C^S_{j^*_p} \times P^{A-S}_{t^*_p j^*_p} \times \left(1 + \frac{C^{A-S}_{r^*_m j^*_q}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}\right) - C^S_{j^*_m} \times P^{A-S}_{r^*_m j^*_m} \times \left(1 + \frac{C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}\right) + \sum_{q=1, q\ne m, q\ne p}^{M} C^S_{j^*_q} \times P^{A-S}_{i^*_q j^*_q} \times \frac{C^{A-S}_{r^*_m j^*_q} - C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} \qquad (3.16b)$$

(when perturbations are induced on both C^{A-S}_{rj} and C^{A-S}_{tj})

or $$C^A_r - C^A_{t^*_m} \ge \sum_{q=1, q\ne m}^{M} C^S_{j^*_q} \times P^{A-S}_{i^*_q j^*_q} \times \frac{C^{A-S}_{rj^*_q} - C^{A-S}_{t^*_m j^*_q}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} + C^S_{j^*_m} \times P^{A-S}_{t^*_m j^*_m} \times \left(1 + \frac{C^{A-S}_{rj^*_m}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_m}}\right)$$

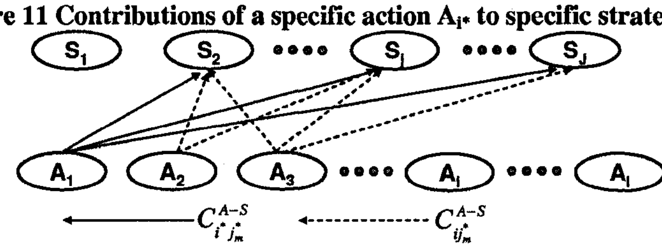
(when the mth perturbation is induced on C^{A-S}_{tj}) (3.16c)

or $$C^A_{r^*_m} - C^A_t \ge \sum_{q=1, q\ne m}^{M} C^S_{j^*_q} \times P^{A-S}_{i^*_q j^*_q} \times \frac{C^{A-S}_{r^*_m j^*_q} - C^{A-S}_{tj^*_q}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} - C^S_{j^*_m} \times P^{A-S}_{r^*_m j^*_m} \times \left(1 + \frac{C^{A-S}_{tj^*_m}}{\sum_{i=1, i\ne i^*_q}^{I} C^{A-S}_{ij^*_m}}\right)$$

(when the mth perturbation is induced on C^{A-S}_{rj}) (3.16d)

The original ranking for all A_i's will remain unchanged if all the above conditions, (3.16a) to (3.16d), are satisfied for all r = 1, 2...I-1 and t = r+1. The top choice will remain at the top rank if (3.16a) to (3.16d) are satisfied for all r = 1 and t = r+1, r+2...r+I-1.

Figure 11 Contributions of a specific action A_{i^*} to specific strategies S_{j^*}



Corollary 4.4 Let $P_{i^*j_m^*}^{A-S}$ ($-C_{i^*j_m^*}^{A-S} < P_{i^*j_m^*}^{A-S} < 1 - C_{i^*j_m^*}^{A-S}$, m=1, 2...M) denote M perturbations induced on M of the C_{ij}^{A-S}'s (contributions of a specific action A_{i^*} to M strategies S_{j^*}'s, see Figure 11); the original ranking of A_r and A_{r+n} will not reverse if:

$$C_r^A - C_{r+n}^A \ge \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^*j_m^*}^{A-S} \times \frac{C_{rj_m^*}^{A-S} - C_{r+n,j_m^*}^{A-S}}{\sum_{i=1}^{I} C_{ij_m^*}^{A-S}} \qquad (3.17a)$$

(when perturbations are induced on neither C_{rj}^{A-S}'s nor $C_{r+n,j}^{A-S}$'s)

$$\text{or } C_r^A - C_{r+n^*}^A \ge \sum_{m=1}^{M} C_{j_m^*}^S \times P_{r+n^*j_m^*}^{A-S} \times \Big(1 + \frac{C_{rj_m^*}^{A-S}}{\sum_{i=1,i\neq t}^{I} C_{ij_m^*}^{A-S}}\Big) \qquad (3.17b)$$

(when perturbations are induced on $C_{r+n,j}^{A-S}$'s)

$$\text{or } C_{r^*}^A - C_{r+n}^A \ge -\sum_{m=1}^{M} C_{j_m^*}^S \times P_{r^*j_m^*}^{A-S} \times \Big(1 + \frac{C_{r+n,j_m^*}^{A-S}}{\sum_{i=1,i\neq r}^{I} C_{ij_m^*}^{A-S}}\Big) \qquad (3.17c)$$

(when perturbations are induced on C_{rj}^{A-S}'s)

The top choice will remain at the top rank if (3.17a) to (3.17c) are satisfied for all r = 1 and t = r+1, r+2...r+I-1. The original ranking for all A_i's will remain unchanged if all the above conditions, (3.17a) to (3.17c), are satisfied for all r = 1, 2...I-1 and t = r+1.

3.5.4 Summary

The above three groups of theorems and corollaries define the allowable region of perturbations and tolerance of contributions at any level of an additive decision hierarchy. Table 2 summarizes the level(s) of the contribution vector/matrix and the number of induced perturbations that each theorem or corollary deals with. The number of inequalities that have to be satisfied in each situation is also specified.

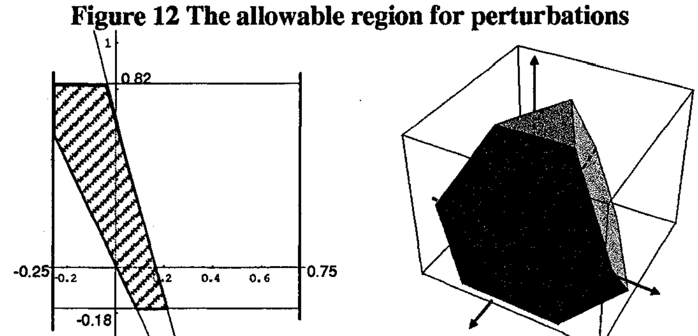
Table 2 Summary of theorems and corollaries 2 to 4.4

Theorems(Th) & Corollaries(Co)	Level(s) in HDM	Number of perturbations	Number of inequalities *	
			Condition 1	Condition 2 & 3
Th 2	Top	M	2+M	I+M
Co 2.1	Top	1	2	I
Th 3 (Figure 4)	Middles	M+T+Q	M+T+Q+4	I+M+T+Q+2
Co 3.1	Middles	1	2	I
Co 3.2 (Figure 5)	Middles	M	2+M	I+M
Co 3.3 (Figure 6)	Middles	M	1+M	I+M-1
Co 3.4 (Figure 7)	Middles	M	1+M	I+M-1
Th 4 (Figure 8)	Bottom	M+T+Q	M+T+Q+4	I+M+T+Q+2
Co 4.1	Bottom	1	2	I
Co 4.2 (Figure 9)	Bottom	M	2+M	I+M
Co 4.3 (Figure 10)	Bottom	M	1+M	I+M-1
Co 4.4 (Figure 11)	Bottom	M	1+M	I+M-1

* Condition 1: rank order of a pair of decision alternative is of concern
Condition 2: rank order of all the decision alternatives is of concern
Condition 3: rank order of the top choice is of concern
M, T, and Q are the numbers of perturbations, and I is the number of decision alternatives.

When the perturbation number equals two, a two-dimensional allowable region for the two perturbations is defined by the inequalities. When it increases to three, the allowable region for the three perturbations is a three-dimensional polyhedron, as shown in Figure 12, with its hyperplanes defined by the inequalities. The origin, where the values of the three perturbations are all zero, represents no changes.

Figure 12 The allowable region for perturbations



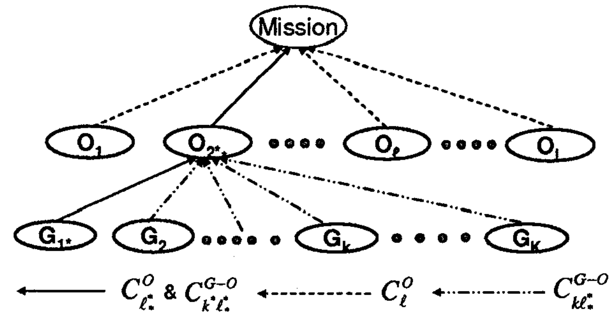
Two-dimensional allowable region   Three-dimensional allowable region

3.6 ALLOWABLE REGION OF MULTI-LEVEL CHANGES

Theorems and corollaries in section 3.5 cover situations when single and multiple perturbations are induced to one specific local contribution vector/matrix while keeping other local contribution matrices unchanged. While there exist a large number of possible combinations in which multiple perturbations are induced simultaneously to different levels of a decision hierarchy, only three cases are discussed here. Unlike the analysis for single level changes addressed by Theorems 2 through 4 and their Corollaries that aims at comprehensively covering all situations, the purpose of this section is to demonstrate how to determine the allowable region of perturbations that are induced simultaneously at different levels of the decision hierarchy.

The first case analyzed is when perturbations are induced in the top level contribution vector and the second level contribution matrix while keeping the lower levels unchanged. Theorem 5.1 defines the allowable region of perturbations induced on the contribution of an objective to the mission, denoted as $C^{O}_{2^{*}}$ and represented by the solid arrow at the upper level in Figure 13, and the contribution of a goal to the same objective, denoted as $C^{G-O}_{1^{*}2^{*}}$ and represented by the solid arrow at the lower level in Figure 13.

Figure 13 Multi-level change case 1: simultaneously change the contributions of an objective to the mission, and a goal to the same objective



Theorem 5.1 Let $P^{O}_{\ell^*}$ $(-C^{O}_{\ell^*} < P^{O}_{\ell^*} < 1 - C^{O}_{\ell^*})$ denote the perturbation induced on one of the C^{O}_{ℓ}'s, which is $C^{O}_{\ell^*}$ (contribution of an objective O_{ℓ^*} to mission); and let $P^{G-O}_{k^*\ell^*}$ $(-C^{G-O}_{k^*\ell^*} < P^{G-O}_{k^*\ell^*} < 1 - C^{G-O}_{k^*\ell^*})$ denote the perturbation induced on one of the $C^{G-O}_{k\ell^*}$'s, which is $C^{G-O}_{k^*\ell^*}$ (contribution of a goal G_{k^*} to the same objective O_{ℓ^*}); see Figure 13; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P^{O}_{\ell^*}\lambda_1 + P^{G-O}_{k^*\ell^*}\lambda_2 + P^{O}_{\ell^*}P^{G-O}_{k^*\ell^*}\lambda_3 \tag{3.18a}$$

Where $\lambda = C^{A}_{r} - C^{A}_{r+n}$ (3.18b)

$$\lambda_1 = \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K} C^{G-O}_{k\ell}\frac{C^{O}_{\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}(C^{A-G}_{rk} - C^{A-G}_{r+n,k}) - \sum_{k=1}^{K} C^{G-O}_{k\ell^*}(C^{A-G}_{rk} - C^{A-G}_{r+n,k}) \tag{3.18c}$$

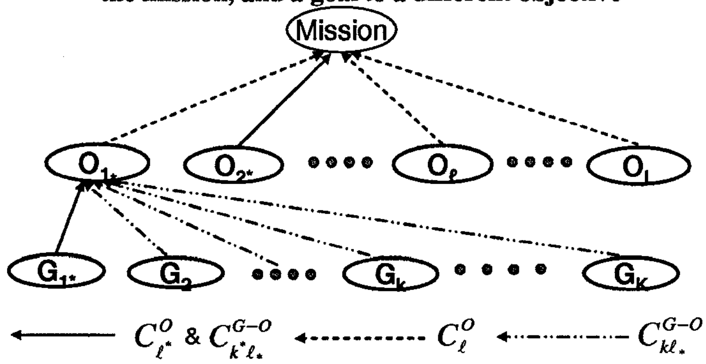
$$\lambda_2 = \sum_{k=1,\,k\neq k^*}^{K}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell^*}}(C^{A-G}_{rk} - C^{A-G}_{r+n,k}) - (C^{A-G}_{rk^*} - C^{A-G}_{r+n,k^*})]C^{O}_{\ell^*} \tag{3.18d}$$

$$\lambda_3 = \sum_{k=1,\,k\neq k^*}^{K}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell^*}}(C^{A-G}_{rk} - C^{A-G}_{r+n,k}) - (C^{A-G}_{rk^*} - C^{A-G}_{r+n,k^*}) \tag{3.18e}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2...I-1, and n=1.

Similarly, Theorem 5.2 defines the allowable region of perturbations induced on the contribution of an objective to the mission, denoted as $C^{O}_{2^*}$ and represented by the solid arrow at the upper level in Figure 14, and the contribution of a goal to a different objective, denoted as $C^{G-O}_{1^*1_*}$ and represented by the solid arrow at the lower level in Figure 14.

Figure 14 Multi-level change case 2: simultaneously change the contributions of an objective to the mission, and a goal to a different objective



Theorem 5.2 Let $P^{O}_{\ell^*}$ $(-C^{O}_{\ell^*} < P^{O}_{\ell^*} < 1 - C^{O}_{\ell^*})$ denote the perturbation induced on one of the C^{O}_{ℓ}'s, which is $C^{O}_{\ell^*}$ (contribution of an objective O_{ℓ^*} to mission); and let $P^{G-O}_{k^*\ell_*}$ $(-C^{G-O}_{k^*\ell_*} < P^{G-O}_{k^*\ell_*} < 1 - C^{G-O}_{k^*\ell_*})$ denote the perturbation induced on one of the $C^{G-O}_{k\ell}$'s, which is $C^{G-O}_{k^*\ell_*}$ (contribution of a goal G_{k^*} to a different objective O_{ℓ_*}); see Figure 14; the original ranking of A_r and A_{r+n} will not reverse if:

$$\lambda \geq P_{\ell^*}^{O}\lambda_1 + P_{k^*\ell_*}^{G-O}\lambda_2 + P_{\ell^*}^{O}P_{k^*\ell_*}^{G-O}\lambda_3 \tag{3.19a}$$

Where $\lambda = C_r^A - C_{r+n}^A$ (3.19b)

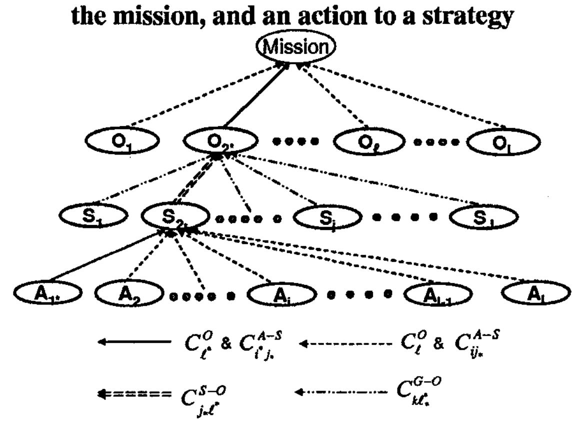
$$\lambda_1 = \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{k\ell}^{G-O}(C_{rk}^{A-G}-C_{r+n,k}^{A-G}) - \sum_{k=1}^{K}C_{k\ell^*}^{G-O}(C_{rk}^{A-G}-C_{r+n,k}^{A-G}) \tag{3.19c}$$

$$\lambda_2 = [\sum_{\substack{k=1,\\ k\neq k^*}}^{K}\frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K}C_{k\ell_*}^{G-O}}(C_{rk}^{A-G}-C_{r+n,k}^{A-G}) - (C_{rk^*}^{A-G}-C_{r+n,k^*}^{A-G})]C_{\ell_*}^{O} \tag{3.19d}$$

$$\lambda_3 = -\frac{\lambda_2}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O} \tag{3.19e}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1. The rank order of all A_i's will remain unchanged if the above condition is satisfied for all r=1, 2…I-1, and n=1.

Figure 15 Multi-level change case 3: simultaneously change the contributions of an objective to the mission, and an action to a strategy



The third case deals with perturbations that are induced in the top level contribution vector and the bottom level contribution matrix while keeping the middle levels unchanged. Theorem 5.3 defines the allowable region of perturbations induced on the contribution of an objective to the mission, denoted as $C^{O}_{2^*}$ and represented by the solid arrow at the upper level in Figure 15, and the contribution of an action to a strategy, denoted as $C^{A-S}_{1^*2_*}$ and represented by the solid arrow at the lower level in Figure 15:

Theorem 5.3 Let $P^{A-S}_{i^*j_*}$ ($-C^{A-S}_{i^*j_*} < P^{A-S}_{i^*j_*} < 1 - C^{A-S}_{i^*j_*}$) denote the perturbation induced on one of the C^{A-S}_{ij}'s, which is $C^{A-S}_{i^*j_*}$ (contribution of an action A_{i^*} to a strategy S_{j_*}); and let $P^{O}_{\ell^*}$ ($-C^{O}_{\ell^*} < P^{O}_{\ell^*} < 1 - C^{O}_{\ell^*}$) denote the perturbation induced on one of the C^{O}_{ℓ}'s, which is $C^{O}_{\ell^*}$ (contribution of an objective O_{ℓ^*} to mission); see Figure 15; the original ranking of A_r and A_t will not reverse if:

$$\lambda \geq P^{O}_{\ell^*}\lambda_1 + P^{A-S}_{r^*j_*}\lambda_2 + P^{O}_{\ell^*}P^{A-S}_{r^*j_*}\lambda_3 \qquad (3.20\text{-}1a)$$

Where $\lambda = C^{A}_{r^*} - C^{A}_{t}$ (3.20-1b)

$$\lambda_1 = \sum_{j=1}^{J} C^{S-O}_{j\ell^*}(C^{A-S}_{ij} - C^{A-S}_{r^*j}) - \sum_{\ell=1,\,\ell\neq\ell^*}^{L}\sum_{j=1}^{J} \frac{C^{O}_{\ell}C^{S-O}_{j\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}(C^{A-S}_{ij} - C^{A-S}_{r^*j}) \qquad (3.20\text{-}1c)$$

$$\lambda_2 = -\left(\frac{C^{A-S}_{tj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + 1\right)\left(\sum_{\ell=1}^{L} C^{O}_{\ell}C^{S-O}_{j_*\ell}\right) \qquad (3.20\text{-}1d)$$

$$\lambda_3 = \left(\sum_{\ell=1,\,\ell\neq\ell^*}^{L} \frac{C^{O}_{\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*}\right)\left(\frac{C^{A-S}_{tj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + 1\right) \qquad (3.20\text{-}1e)$$

Or

$$\lambda \geq P^{O}_{\ell^*}\lambda_1 + P^{A-S}_{t^*j_*}\lambda_2 + P^{O}_{\ell^*}P^{A-S}_{t^*j_*}\lambda_3 \quad (3.20\text{-}2a)$$

Where $\lambda = C^{A}_{r} - C^{A}_{t^*}$ (3.20-2b)

$$\lambda_1 = \sum_{j=1}^{J} C^{S-O}_{j\ell^*}(C^{A-S}_{t^*j} - C^{A-S}_{rj}) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C^{O}_{\ell}C^{S-O}_{j\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}(C^{A-S}_{t^*j} - C^{A-S}_{rj}) \quad (3.20\text{-}2c)$$

$$\lambda_2 = \sum_{\ell=1}^{L} C^{O}_{\ell}C^{S-O}_{j_*\ell}\left(\frac{C^{A-S}_{rj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + 1\right) \quad (3.20\text{-}2d)$$

$$\lambda_3 = \left(C^{S-O}_{j_*\ell^*} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C^{O}_{\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}C^{S-O}_{j_*\ell}\right)\left(\frac{C^{A-S}_{rj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + 1\right) \quad (3.20\text{-}2e)$$

Or

$$\lambda \geq P^{O}_{\ell^*}\lambda_1 + P^{A-S}_{i^*j_*}\lambda_2 + P^{O}_{\ell^*}P^{A-S}_{i^*j_*}\lambda_3 \quad (3.20\text{-}3a)$$

Where $\lambda = C^{A}_{r} - C^{A}_{t}$ (3.20-3b)

$$\lambda_1 = \sum_{j=1}^{J} C^{S-O}_{j\ell^*}(C^{A-S}_{tj} - C^{A-S}_{rj}) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C^{O}_{\ell}C^{S-O}_{j\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}(C^{A-S}_{tj} - C^{A-S}_{rj}) \quad (3.20\text{-}3c)$$

$$\lambda_2 = \sum_{\ell=1}^{L} C^{O}_{\ell}C^{S-O}_{j_*\ell}\frac{C^{A-S}_{rj_*} - C^{A-S}_{tj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} \quad (3.20\text{-}3d)$$

$$\lambda_3 = \left(C^{S-O}_{j_*\ell^*} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C^{O}_{\ell}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^{O}_{\ell}}C^{S-O}_{j_*\ell}\right)\frac{C^{A-S}_{rj_*} - C^{A-S}_{tj_*}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} \quad (3.20\text{-}3e)$$

The top choice will remain at the top rank if (3.20-1a) to (3.20-3e) are satisfied for all r = 1 and t = r+1, r+2…r+I-1. The original ranking for all A_i's will remain unchanged if all the above conditions, (3.20-1a) to (3.20-3e), are satisfied for all r = 1, 2...I-1 and t = r+1.

3.7 SENSITIVITY COEFFICIENT

Different *sensitivity coefficients* (SC) for HDM have been proposed in the literature [69][107][53]. Masuda defined the SC as the standard deviation of the "extreme vector" of an AHP model [69]. Huang showed that Masuda's definition was invalid in certain situations and defined another SC based on Masuda's work, also as a measurement of the likelihood of range changes: the bigger the value of the coefficient is, the more possible rank reversal among alternatives will occur [53]. This kind of sensitivity coefficient does not give information about how sensitive the decision is to changes in the contribution matrices. The SC proposed by Triantaphyllou and Sanchez is the reciprocal of the smallest percentage by which the contribution must change to reverse the alternatives' ranking [107], which gives limited information. Similar to the sensitivity coefficient concept, a local stability index is defined by Aguaron and Moreno-Jimenez as the reciprocal of the local stability interval in multiplicative AHP [1].

To give as complete information as possible, two sensitivity coefficients are proposed here: the *operating point sensitivity coefficient* (OPSC) and the *total sensitivity coefficient* (TSC). The OPSC is defined as the shortest distance from the current contribution value to the edges of its tolerance. It is dependent on the contribution's current value (the operating point) and directions of the change (increasing or decreasing). TSC specifies that the shorter the tolerances of a decision element's contributions are, the more sensitive the final decision is to variations of that decision element. As noted by Evans that if the current parametric value is located near the center of P* (allowable region), then the decision is robust [31], the OPSC proposed in this dissertation can be viewed as an indicator of the robustness of the current decision. On the other hand, TSC reveals more about how flexible the input values can be without changing the decision. OPSC and TSC complement each other to give different but equally important information and thus should be used together.

In addition, since the length of both allowable and feasible range of perturbations induced on a certain local contribution can be calculated, length of the allowable range as a percentage of the total feasible range can be regarded as the probability of rank remaining unchanged when the corresponding contribution values vary uniformly within the feasible range. For example, if the base value of C_1^O is 0.2 and the allowable range of perturbations on C_1^O is 0 to 0.2, that means only when C_1^O changes between 0.2 and 0.4, the ranking of decision alternatives will be preserved. Therefore, when C_1^O changes uniformly between its feasible range, which is 0 to 1, there is a 20 percent chance that the ranking will remain unchanged and an 80 percent chance the ranking will be changed (when C_1^O is between 0.2 to 1).

Since the length of the allowable range of perturbations is measured by *total sensitivity coefficient*, the probability of the rank remaining unchanged can be calculated as TSC divided by the length of the feasible range.

Proposition 1.1 If the allowable range of perturbations on C_ℓ^O is [$\delta_{\ell-}^O, \delta_{\ell+}^O$] to preserve the final ranking of A_i's, the OPSC and TSC of C_ℓ^O and O_ℓ are:

$$\text{OPSC}\ (O_\ell) = \text{OPSC}\ (C_\ell^O) = Min\{|\delta_{\ell-}^O|, |\delta_{\ell+}^O|\} \tag{3.21a}$$

$$\text{TSC}\ (O_\ell) = \text{TSC}\ (C_\ell^O) = |\delta_{\ell+}^O - \delta_{\ell-}^O| \tag{3.21b}$$

The probability of A_i's rank remaining unchanged when a C_ℓ^O value varies uniformly between zero and one is:

$$\text{TSC}\ (C_\ell^O)\ /\ \text{length of feasible range} = \text{TSC}\ (C_\ell^O) \tag{3.22}$$

Proposition 1.2 If the allowable range of perturbations on $C_{k\ell}^{G-O}$ is [$\delta_{k\ell-}^{G-O}, \delta_{k\ell+}^{G-O}$] to preserve the final ranking of A_i's, the OPSC and TSC of $C_{k\ell}^{G-O}$ and G_k are:

$$\text{OPSC}\ (C_{k\ell}^{G-O}) = Min\{|\delta_{k\ell-}^{G-O}|, |\delta_{k\ell+}^{G-O}|\} \quad (3.23a) \qquad \text{TSC}\ (C_{k\ell}^{G-O}) = |\delta_{k\ell+}^{G-O} - \delta_{k\ell-}^{G-O}| \quad (3.23b)$$

$$\text{OPSC}\ (G_k) = \underset{1\le\ell\le L}{Min}\{|\delta_{k\ell-}^{G-O}|, |\delta_{k\ell+}^{G-O}|\} \quad (3.23c) \qquad \text{TSC}\ (G_k) = \underset{1\le\ell\le L}{Min}\{|\delta_{k\ell+}^{G-O} - \delta_{k\ell-}^{G-O}|\} \quad (3.23d)$$

The probability of A_i's rank remaining unchanged when a $C_{k\ell}^{G-O}$ value varies uniformly between zero and one is:

$$\text{TSC}\ (C_{k\ell}^{G-O})\ /\ \text{length of feasible range} = \text{TSC}\ (C_{k\ell}^{G-O}) \tag{3.24}$$

Proposition 1.3 If the allowable range of perturbations on C_{ij}^{A-S} is [$\delta_{ij-}^{A-S}, \delta_{ij+}^{A-S}$] to preserve the final ranking of A_i's, the OPSC and TSC of C_{ij}^{A-S} and A_i are:

OPSC (C_{ij}^{A-S}) = $Min\{|\delta_{ij-}^{A-S}|, |\delta_{ij+}^{A-S}|\}$ (3.25a) TSC (C_{ij}^{A-S}) = $|\delta_{ij+}^{A-S} - \delta_{ij-}^{A-S}|$ (3.25b)

OPSC (A_i) = $\underset{1 \le j \le J}{Min}\{|\delta_{ij-}^{A-S}|, |\delta_{ij+}^{A-S}|\}$ (3.25c) TSC (A_i) = $\underset{1 \le j \le J}{Min}\{|\delta_{ij+}^{A-S} - \delta_{ij-}^{A-S}|\}$ (3.25d)

The probability of A_i's rank remaining unchanged when a C_{ij}^{A-S} value varies uniformly between zero and one is:

TSC (C_{ij}^{A-S}) / length of feasible range = TSC(C_{ij}^{A-S}) (3.26)

The smaller the sensitivity coefficients of a decision element are, the less robust the decision is to variations of that element. If the TSC of a decision element is one, meaning the tolerance is from zero to one, the decision is not sensitive at all to changes that occur to the contributions of this element.

The above analysis is based on a "one-way SA" in which the influence of a single input to the decision is analyzed while keeping other inputs at their base values [26]. Extending the analysis to multiple simultaneous changes, we can study the sensitivity of a certain decision level in the hierarchy. Recall that in *tolerance analysis* section, an M-dimensional allowable region is defined for M perturbations induced on any local contribution vector/matrix to preserve the final ranking of A_i's. Based on the same logic, the shortest distance from the origin to all hyperplanes of the M-dimensional polyhedron and the polyhedron's volume determine the robustness of the current model regarding changes to the M contribution values. Just like the analysis in the one-dimensional case, the volume of the M-dimensional allowable region (polyhedron) as a percentage of volume of the M-dimensional feasible region is also the probability of keeping A_i's rank orders unchanged when the M contributions vary uniformly from zero to one. When M equals two, the probability of rank remaining unchanged is the area of the allowable region divided by the area of the feasible region. In every case, TSC measures the length/area/volume of the allowable range/area/region for single/double/multiple perturbations.

3.8 CRITICAL DECISION ELEMENTS

In several previous studies, researchers tried to identify the most influential variables with respect to the rank ordering of the alternatives [51] or "determinant attribute" that strongly contribute to the choice among alternatives [6].

As it is mentioned in [107], one may be misled by the name "criticality" and think the weights or contribution values determine the criticality of a decision element. Therefore, it should be noted that the criticality of a decision element to preserve the current ranking is different from the importance of a decision element to the higher level decision elements. Criticality is determined by how sensitive the final ranking is to changes in a decision element's contributions, and the importance is determined by how much a decision element contributes to the higher level decision elements.

In this dissertation, the most critical decision element is defined as the one whose influence on the final decision is most sensitive to perturbations, as defined by Triantaphyllou and Sanchez [107]. Extending their definition to multiple levels of the decision hierarchy, we get:

Proposition 2. The most critical decision element at a given level of the decision hierarchy in terms of preserving the current ranking of A_i's is the decision element corresponding to the smallest TSC and OPSC at that level.

In situations when the smallest TSC and OPSC do not occur on the same decision element, there can be two different decision elements, and each one can be considered the most critical in different situations. Additional analysis can also be carried out to determine which one is more critical.

3.9 ADDING NEW DECISION ALTERNATIVES

There are situations where new decision elements need to be added after a hierarchical decision model has been built. Adding new decision elements to the middle levels of the decision hierarchy will change all the contribution matrices. In this case, it is suggested that a new decision hierarchy be constructed and the overall contribution vector be recalculated. However, introducing new decision alternatives only changes the bottom level of the decision hierarchy; and SA can be applied to that special case.

With the assistance of Corollary 4.4, the impact of adding a new decision alternative can be studied by adding the new A_i to the model and assuming its current contributions are zero and its new contributions are $P^{A-S}_{i^* j^*_m}$, where (i = I+1) and (m = 1, 2...J). The currently top-ranked decision alternative will remain unchanged as long as inequality (3.17b) is satisfied for (r = 1) and (n = I). The current ranking of all decision alternatives will remain unchanged if (3.17b) is satisfied for (r =1, 2...I) and (r+n = I+1), with the new decision alternative ranked last. Based on the same logic, adding multiple new decision alternatives can be analyzed using Theorem 4. The entire decision hierarchy does not have to be re-calculated.

4. APPLICATION 1—APPLYING HDM SA TO STRATEGIC TECHNOLOGY PLANNING

4.1 GENERAL FRAMEWORK FOR TECHNOLOGY PLANNING

Technology plays a critical role in business. It creates and maintains a firm's core competences to outperform its competitors and enables business success [14][40][77]. Technology also alters the rules of competition by changing the business environment. [52] Having realized the importance of technologies, companies are striving to adopt technologies and put them in their business processes. However, technologies must be properly deployed before their economic benefits can be obtained. The fit between technology and business operation should be understood to ensure a successful technology implementation [20][27][52][93]. Companies must be able to adapt to the changes brought by emerging technologies. Therefore, it is critical for management to understand the implications of the technology changes, and to assure that technology change is driven by business strategy, and not the other way around [14].

To ensure long-term business survival, a firm's technology strategy should be integrated with its business strategy [14][34], and linkages must be established between business goals and the technologies needed to achieve such a goal [70]. Therefore, a formal set of technology planning procedures, which effectively facilitates the integration [14] and strives for a good match between the company's external environment and internal structures and processes [38], should be an integral part of the business strategy and planning. Benefits of a deliberate technology plan also include the identification of strategic opportunities to combine technology push, and the coordination of all related activities across the company to build on success and to avoid redundancies [35].

Although various procedures have been proposed for development of a technology plan (i.e. [76][34][70]), two critical steps are common to most of them: assessment of the current state of technology, and forecasting of possible changes and future needs. The assessment helps a company evaluate the degrees to which the technology alternatives support the business goals, understand the influences of new technologies on its strategies, and prioritize the technology investment options [63]; while the forecasting anticipates changes in the core and pacing technologies of the firm, taking into consideration the enterprise evolution, new or improved capabilities of products, production and marketing, etc. [76] Technology scenarios can then be generated, focusing on technological opportunities and their impact on market needs and business opportunities, in a "what-if" mode.

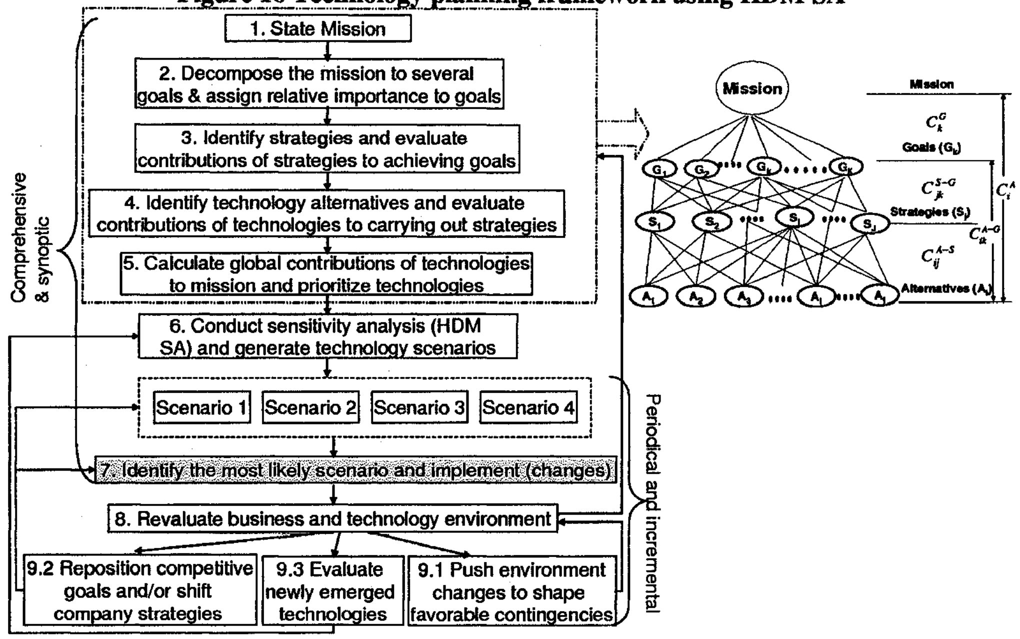
Researchers have employed many different methods to evaluate technologies in the assessment step, but few of them have offered a systematic approach to link the technology alternatives to business goals through the alignment of company strategies [34][52]. To better integrate technology strategies with the business strategy, a strategic hierarchical technology assessment model has been developed by Ho in his dissertation [52]. Ho's model investigates the impacts of emerging technologies on technology strategies, competitive goals, and overall success of a target industry [52] and links these decision elements into a four-level decision hierarchy using the concept of HDM.

Considering the two types of models pervading the strategic planning literature, Ho's model would be considered as a *synoptic planning* type, since it establishes the overall mission, assesses the internal and external environment, evaluates alterative actions (technologies), and develops a plan to achieve the mission [98-101]. However, synoptic planning mode is noted to be relatively ineffective for organizations in an unstable environment [38][107][79]. Due to difficulties to foresee and assess future changes, it is hard to develop a comprehensive plan that covers all possible future moves. Besides, the high uncertainty involved throughout the technology/product development life cycle [42][101], especially when the technology is rapidly changing [81], may cause disagreement among the experts involved to provide judgment; this is likely to alter the analysis. In addition, the evolution of technological trends may not follow experts' forecasts in either direction or pace. Since technological change can be obscure, it is critical for technology managers to understand the impact of changes to industry policy or technology performance on business [76]. To make the technology planning process as complete and effective as possible, especially when a relatively long time period is concerned, adaptive planning proposed by the other group of planning literature [38][16][65][66][76][92][79] should follow as the next step.

While researchers have noted that the *synoptic planning mode* and the *adaptive planning mode* fall on polar ends of a planning continuum [38][102], a technology planning framework is proposed in this dissertation to include both planning modes by linking them with HDM SA. As a middle but critical step in the framework, conducting HDM SA not only improves comprehensiveness of the initial assessment, the most basic feature of the synoptic planning mode [38], but also improves the forecasts of possible future scenarios. It addresses and incorporates uncertain and unstable factors in the decision model to guide future adaptive changes: Scenarios forecasted by HDM SA provide a base against which future business and technology environmental changes will be periodically compared. By paying close attention to internal and external changes and frequently reevaluating critical decision elements identified by HDM SA, companies can respond quickly in the unstable environment, repositioning their strategies, implementing incremental or radical changes, and continuously improving the technological competitiveness on a timely basis. Figure 16 shows the proposed framework for technology planning using HDM SA.

Ho's model aggregated the first five steps in the framework into a hierarchical technology evaluation model. An expert panel was formed to give pair-wise comparisons regarding the contributions of decision elements (mission, goals, strategies and technology alternatives) at on level to those at a higher level, and technology alternatives are evaluated according to their overall contributions to the mission. Ideally, the sensitivity would be tested on changes to the pair-wise comparison judgment. However, varying only certain pairs of the comparisons may result in inconsistency problems and make the process too complex. Since the major purpose of doing a SA here is to forecast different technology scenarios, the HDM SA algorithm developed in this dissertation to test model robustness to changing local contributions is employed as the sixth step of the technology planning process.

Figure 16 Technology planning framework using HDM SA



Step seven of the overall framework can be viewed as the final output of the technology plan since it is at this step that the final choice(s) are determined and implemented. Based on their available resources, companies can invest in the top-ranked technology alternative or in a portfolio based on the technology scenarios. However, as indicated before, the adaptive planning mode should follow step seven to address and incorporate internal and external changes into the plan on a timely basis. Thus, an iterative process including steps seven to nine continues until the need for a radical change is identified, in which case the hierarchical model should be revisited: companies may need to restate their missions and rebuild the decision hierarchy.

It should be noted that the ninth step in the proposed framework is divided into three strategic actions: push environment changes to shape favorable contingencies, reposition competitive goals and/or shift company strategies, and evaluate newly emerged technologies. As noted in the real options logic for technology investment, if a company is able to initiate endogenous changes, it may seek to shape contingencies in its favor [71]. Performing HDM SA in the sixth step enables a company to be clear about situations that favor its current strategic action. Therefore, if the company is able to influence its external environment and push for favorable industry progress, its current investment can be further justified and benefits can be continuously achieved. However, if such an option is not available, then a company would want to compare its current state with other scenarios and adjust its competitive goals or technology investments accordingly. Also, since new technologies may evolve during the time frame of the technology plan, in order to stay competitive, companies should pay close attention to and evaluate the impacts of emerging technologies for future adoption decisions.

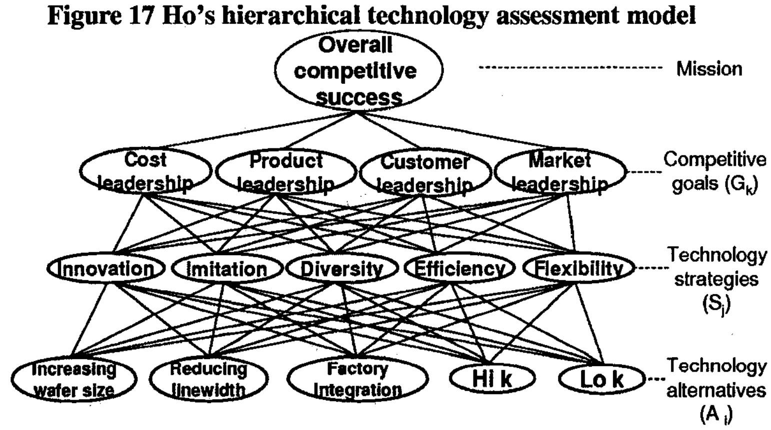
The next section demonstrates the application of the proposed model in detail through a case study on the technology planning for Taiwan's semiconductor foundry companies. The use of HDM SA algorithm in the planning framework is demonstrated in detail and data from an existing technology assessment model in [52] are utilized to verify the HDM SA algorithm.

4.2 CASE STUDY: TECHNOLOGY PLANNING FOR TAIWAN'S SEMICONDUCTOR FOUNDRY INDUSTRY

4.2.1 Ho's Model

Ten experts from industry, research organizations, and government formed the expert panel for Ho's model. The hierarchical technology assessment model, depicted in Figure 17, was presented to the experts. After a series of explanations, question and answers, discussions, and tests, consensus was reached for the model's logic, definitions and measurements of the decision elements and other related issues.

Figure 17 Ho's hierarchical technology assessment model



The finalized elements in each level are summarized below.

Level I: Overall Competitive Success

The overall competitive success is measured by the return on investment (ROI). It is the mission of the foundry business in Taiwan.

Level II: Competitive Goals (G_k, k = 1, 2… 4)

1) Cost Leadership (G_1): Keep overall costs low by reducing cycle time, increasing yield, and utilizing economy of scale.

2) Product Leadership (G_2): Develop cutting edge and proprietary IC process technologies. (For foundry, products are the services of IC manufacturing processes.)

3) Customer Leadership (G_3): Maintain intimate customer relationships to reduce lead time, to improve on-time delivery, and to provide customized processes and services.

4) Market Leadership (G_4): Develop new markets and strengthen the position in the existing market to influence the market and to benefit from scale of scope.

Level III: Technology Strategies (S_j, j = 1, 2... 5)

1) Technology Innovation (S_1): use of advanced technology to develop new products for the market. This strategy leads to developing new technologies and best performance products for the market.

2) Technology Imitation (S_2): quick application of a technology to product development after the product leader has proven the technology successful. This strategy leads to improving products without a heavy investment in technology development.

3) Technology Diversity (S_3): use of technology to support a spectrum of products at different stages of their life cycles. This strategy leads to increasing the variety of products.

4) Technology Efficiency (S_4): use of technology to improve the efficiency of production methods.

5) Technology Flexibility (S_5): use of technology for rapid development of products in response to changing market demands. This strategy leads to developing products with flexibility to serve different market segments and allows for quick adjustments in production volume.

Level IV: Technology Alternatives (A_i , i = 1, 2… 5)

1) A_1 Increasing wafer size to 300 mm and beyond (from the current 200 mm)
2) A_2 Reducing line width to 90 nm and lower
3) A_3 High k gate dielectrics (with k greater than 25 that replaces oxynitride k = 7)
4) A_4 Low k intermetallic dielectrics (with k less than 2.5 that replaces silicate glass)
5) A_5 Factory Integration

For a four-level decision hierarchy, one vector and two matrices of local contributions between successive levels were acquired through judgment quantification instruments sent to the experts. The pair-wise comparison results were collected and calculated to derive the following three local contribution vector and matrices:

Vector $\mathbf{C_k^G}$: The relative importance of competitive goals (G_k) to overall competitive success, as shown in Table 3.

Table 3 Contributions of competitive goals to overall competitive success [52]

C_k^G	Cost Leadership	Product leadership	Customer Leadership	Market Leadership
Overall competitive success	0.36	0.25	0.21	0.18

Matrix $\mathbf{C_{jk}^{S-G}}$: Relative impacts of technology strategies(S_j) on competitive goals (G_k), as shown in Table 4.

Table 4 Contributions of technology strategies to competitive goals [52]

C_{jk}^{S-G}	Innovation	Imitation	Diversity	Efficiency	Flexibility
Cost Leadership	0.02	0.11	0.14	0.43	0.29
Product leadership	0.54	0.14	0.17	0.07	0.08
Customer Leadership	0.11	0.14	0.24	0.27	0.23
Market Leadership	0.16	0.2	0.21	0.18	0.24

Matrix $\mathbf{C_{ij}^{A-S}}$: Contributions of technology alternatives (A_i) to technology strategies (S_j), as shown in Table 5.

Table 5 Contributions of technology alternatives to strategies [52]

C_{ij}^{A-S}	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration
Innovation	0.31	0.19	0.13	0.17	0.2
Imitation	0.29	0.24	0.11	0.17	0.18
Diversity	0.22	0.27	0.15	0.21	0.14
Efficiency	0.22	0.21	0.12	0.18	0.27
Flexibility	0.09	0.29	0.13	0.22	0.27

Aggregating the local contribution matrices $\mathbf{C_k^G}$, $\mathbf{C_{jk}^{S-G}}$ and $\mathbf{C_{ij}^{A-S}}$ into an overall contribution vector, $\mathbf{C_i^A}$ ($C_i^A = \sum_{i=1}^{I}\sum_{j=1}^{J}\sum_{k=1}^{K} C_k^G C_{jk}^{S-G} C_{ij}^{A-S}$), the global contributions of technology alternatives to the overall competitive success are calculated. Technology alternatives are prioritized and ranked based on their C_i^A values, as shown in Table 6. The bold numbers in parenthesis are the ranks of the technology alternative.

Table 6 Overall contribution vector C_i^A

	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory integration
C_i^A	0.2196	0.235	0.132	0.193	0.2204
Ranking	(3)	(1)	(5)	(4)	(2)

4.2.2 Sensitivity Analysis and Scenario Forecast

Based on the assessment results from the previous section, sensitivity analysis is performed to study the influences on the optimal technology portfolio when 1) changes to the economic climate of the industry cause company to shift its emphasis among competitive goals; 2) company developments cause changes in technology strategics to align with altered business strategies; 3) actual technology performance does not reach the expected level or technological advances improve the technology performance dramatically.

Specific questions being answered by performing HDM SA in this section include: 1) What are the critical decision elements in keeping the current assessment result valid? 2) What are the probabilities of priority order changes when a certain decision element varies? 3) What is the optimal technology portfolio or the top investment choice in a most likely scenario with the least risk? 4) What are other technology scenarios in response to future changes?

4.2.2.1 Competitive-Goals Analysis

In the assessment model, four competitive goals have been evaluated according to their relative importance to overall competitive success, which are denoted as C_k^G (k = 1…4). Suppose changes to industry dynamics or the economic climate demand that a company shifts its emphasis to different competitive goals; the company needs to know whether its originally identified investment choice(s) will still remain optimized. To prepare potential solutions before hand, HDM SA is performed in order to test model robustness to varying C_k^G values and to generate technology scenarios corresponding to different C_k^G values. The most critical competitive goal, which would merit special attention, is also identified.

<u>4.2.2.1.1 One-way SA</u>

How variations of C_k^G values impact the rank order of all technology alternatives is first analyzed by a one-way SA. One-way SA determines the influence of changes to a single input by varying that input within its feasible range while keeping other inputs fixed at their base values [5, 25]. Corollary 2.1 in the HDM SA algorithm deals with one-way SA for changes in the top-level contribution vector, and thus is applied here

To use Corollary 2.1, local contribution matrices between competitive goals level and technology alternatives level are integrated and calculated, as summarized in

Table 7. This corresponds to global contribution matrix of $\mathbf{C}_{ik}^{A-G}$ in the HDM SA algorithm.

Table 7 Global contributions of technologies to competitive goals [52]

C_{ik}^{A-G}	Increasing Wafer size	Reducing line width	Hi k	Lo k	Factory Integration
Cost Leadership	0.19	0.24	0.13	0.19	0.24
Product leadership	0.27	0.22	0.13	0.18	0.20
Customer Leadership	0.21	0.24	0.13	0.19	0.22
Market Leadership	0.22	0.24	0.13	0.19	0.21

Based on Corollary 2.1, the allowable range of P_{1*}^{G} is calculated:

$$-C_{k^*}^{G} \le P_{k^*}^{G} \le 1-C_{k^*}^{G} \longrightarrow -0.36 < P_{1*}^{G} < 0.64 \quad (4.1a)$$

when r=1, n=1:

$$\lambda = C_r^A - C_{r+n}^A = C_1^A - C_2^A = 0.235 - 0.2204 = 0.0146$$

(4.1b)

$$\sum_{k=1,k\neq k^*}^{K} C_k^G = C_2^G + C_3^G + C_4^G = 0.25 + 0.21 + 0.18 = 0.64 \quad (4.1c)$$

$$\lambda^G = C_{r+n,k^*}^{A-G} - C_{rk^*}^{A-G} - \sum_{k=1,k\neq k^*}^{K} C_{r+n,k}^{A-G} \times \frac{C_k^G}{\sum_{k=1,k\neq k^*}^{K} C_k^G} + \sum_{k=1,k\neq k^*}^{K} C_{rk}^{A-G} \times \frac{C_k^G}{\sum_{k=1,k\neq k^*}^{K} C_k^G} \quad (4.1d)$$

$$\lambda^G = C_{2,1^*}^{A-G} - C_{11^*}^{A-G} - \sum_{k=2}^{4} C_{2k}^{A-G} \times \frac{C_k^G}{\sum_{k=2}^{4} C_k^G} + \sum_{k=2}^{4} C_{1k}^{A-G} \times \frac{C_k^G}{\sum_{k=2}^{4} C_k^G} \quad (4.1e)$$

$$= 0.24 - 0.24 - (0.22\times0.25/0.64 + 0.24\times0.21/0.64 + 0.24\times0.18/0.64) +$$

$$(0.2\times0.25/0.64 + 0.22\times0.22/0.64 + 0.21\times0.18/0.64) \quad (4.1f)$$

$$= 0.0228$$

$$\frac{\lambda}{\lambda^G} = \frac{0.0146}{0.0228} = 0.64 \qquad\qquad P_{1*}^G \le 0.64 \qquad\qquad (4.1g)$$

Repeating the same steps for n=1 and r=2, 3, 4, we get three other inequalities for the allowable range of P_{1*}^G to keep current ranking of all technology alternatives unchanged. Then the allowable range for perturbations on C_{1*}^G can be determined by combining all inequalities.

$$\left\{\begin{matrix} -0.36 < P_{1*}^G < 0.64 \\ P_{1*}^G \le 0.64 \ (when\ \mathrm{r}=1, \mathrm{n}=1) \\ P_{1*}^G \ge -0.01 \ (when\ \mathrm{r}=2, \mathrm{n}=1) \\ P_{1*}^G \le 0.466 \ (when\ \mathrm{r}=3, \mathrm{n}=1) \\ P_{1*}^G \ge -4.23 \ (when\ \mathrm{r}=4, \mathrm{n}=1) \end{matrix}\right\} \Longrightarrow -0.01 \le P_{1*}^G \le 0.466 \quad (4.1h)$$

Therefore, the allowable range of the perturbations on C_{1*}^G is [-0.01, 0.466] to preserve the current ranking of all technology alternatives. To verify whether this is true, different values beyond and within this range are tried for P_{1*}^G. In each case, the new values of the C_i^A's are recalculated accordingly. Table 8 summarizes the verification results. The new values of the alternatives whose rank orders have been changed are shaded in the table. We can see that the ranking of A_i's are changed only when the value of P_{1*}^G goes beyond the range given above.

According to the definition of tolerance, which is $[\delta_{\ell-}^O + C_{\ell^*}^O, \delta_{\ell+}^O + C_{\ell^*}^O]$, the tolerance of C_1^G to keep current rank order of all the technologies unchanged is [0.35, 0.8256]. Applying Proposition 1.1 here, the sensitivity coefficients of C_1^G are:

OPSC $(G_k) = Min\{|\delta^{O}_{\ell-}|, |\delta^{O}_{\ell+}|\}$=0.01 (4.2a) TSC $(G_k) = |\delta^{O}_{\ell+} - \delta^{O}_{\ell-}|$=0.4756 (4.2b)

Table 8 Data verification for Corollary 2.1

Original Rank	(3)	(1)	(5)	(4)	(2)	
C_i^A / $P_{1\cdot}^G$	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory integration	Rank changes?
-0.011	0.2201	0.2349	0.1321	0.1928	0.2200	Yes
-0.010	0.2200	0.2349	0.1321	0.1928	0.2201	No
0	0.2196	0.2350	0.1321	0.1929	0.2204	No
0.200	0.2103	0.2366	0.1314	0.1951	0.2265	No
0.465	0.1981	0.2386	0.1306	0.1981	0.2346	No
0.470	0.1978	0.2387	0.1306	0.1981	0.2348	Yes

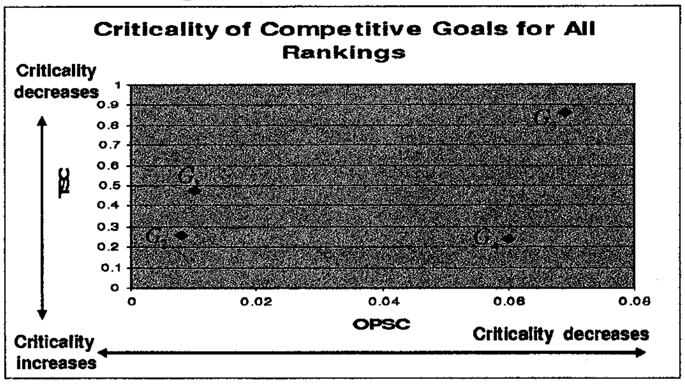
Repeating the same steps for G_2, G_3, and G_4, the allowable ranges of perturbations, tolerances and sensitivity coefficients to keep the ranking of all technology alternatives can be determined. The results are summarized in Table 9. "Base value" is the original value assigned to the corresponding C_k^G in Ho's model results. "Allowable range of perturbations" determines the thresholds of changes to C_k^G values without changing the rank order of technology alternatives, and "tolerance" is the range in which C_k^G value can change without altering the rank order of technology alternatives. "Prob. of rank changes" indicates the probability that technology alternatives' original ranking will change when corresponding C_k^G changes uniformly between zero and one. OPSC and TSC are the operating sensitivity coefficient and total sensitivity coefficient of competitive goals. These definitions are the same for SA at other levels of the hierarchy.

Table 9 HDM SA at competitive-goals level to preserve the ranking of all A_i

	C_1^G (cost)	C_2^G (product)	C_3^G (customer)	C_4^G (market)
Base values	0.36	0.25	0.21	0.18
Allowable ranges of perturbations	[-0.01, 0.47]	[-0.25, 0.01]	[-0.07, 0.79]	[-0.18, 0.06]
Tolerance (C_k^G)	[0.350, 0.826]	[0, 0.258]	[0.141, 1]	[0, 0.240]
Prob. of rank changes	52.4%	74.2%	14.1%	76%
OPSC (G_k)	0.01	0.008	0.069	0.06
TSC (G_k)	0.476	0.258	0.859	0.24

From Table 9 and Figure 18, we can see that "product leadership" and "market leadership" are the two most critical competitive goals in terms of preserving the current ranking of all technology alternatives. They correspond to the smallest allowable change (OPSC) and the shortest tolerance (TSC).

Figure 18 Criticality of competitive goals to keep the ranking of all technologies



Suppose the company forms its technology portfolio based on the current priority order. To keep such choices optimal, "product leadership (G_2)" and "market leadership (G_4)" are the two critical competitive goals worth special attention. For "product leadership," although it is the second most important in terms of its relative importance to overall success (0.25), its relative importance, C_2^G, has the smallest allowable change (0.008) and the second shortest tolerance (0.258) to keep the current technology ranking unchanged. When C_2^G value varies uniformly between zero and one, there is a 74.2% chance that the rank order of the technology alternatives will be changed. Market leadership (G_4) is the least important competitive goal: it contributes the least to the overall competitive success. But its contribution value C_4^G has the shortest tolerance and when C_4^G changes between zero and one with a uniform distribution, there is a 76% chance that the rank order of the technology alternatives will be changed. Therefore, it is the most critical goal for the current rank order of all technology alternatives to remain unchanged.

Cost leadership (G_1), which has the highest impact on overall success, is the second critical competitive goal in terms of OPSC to preserve the rank orders of all technologies: the shortest distance from C_1^G's base value to the edges of its tolerance is 0.01, and C_1^G's tolerance is 0.476 in length. When C_1^G changes uniformly from zero to one, there is a 52.4% chance that the current rank order of the technology alternatives will be changed.

Customer leadership (G_3) is the second least important and the least critical competitive goal. There is only a 14.1% chance that the ranking of technology alternatives will change when its contribution value, C_3^G, varies between zero and one based on a uniform distribution: When C_3^G goes below 0.1413 from its base value, 0.21, the rank order between the second- and the third-ranked technologies will be reversed.

In some situations, the rank orders of all the technology alternatives are of concern, especially when the top choices are close in their scores, as analyzed above. However, in other situations, only the top-ranked technology alternative matters, such as when limited resources restrict a company to focus on only one emerging technology. In this case, the technology manager cares mostly about how robust the current top-ranked technology ("reducing line width" in this case) is at its current rank when the relative importance of competitive goals shifts.

To test the model's robustness in such situation where only the top ranked technology is considered, another calculation is done by applying Corollary 2.1 and having r take the value of 1, and n take the values from 1 to 4 in the calculations. In this case, the allowable ranges for perturbations induced on C_1^G through C_4^G to keep the top choice as "reducing line width" are determined as follows:

$$\left\{\begin{array}{l} -0.36 < P_{1*}^G < 0.64 \\ P_{1*}^G \le 0.64\,(\text{when r} = 1, \text{n} = 1) \\ P_{1*}^G \ge -0.285\,(\text{when r} = 1, \text{n} = 2) \\ P_{1*}^G \le 12.83\,(\text{when r} = 1, \text{n} = 3) \\ P_{1*}^G \ge -9.275\,(\text{when r} = 1, \text{n} = 4) \end{array}\right\} \quad -0.285 \le P_{1*}^G < 0.64 \tag{4.3a}$$

$$\left\{\begin{array}{l} -0.25 < P_{2*}^G < 0.75 \\ P_{2*}^G \ge -2.02\,(\text{when r} = 1, \text{n} = 1) \\ P_{2*}^G \le 0.177\,(\text{when r} = 1, \text{n} = 2) \\ P_{2*}^G \le 15.036\,(\text{when r} = 1, \text{n} = 3) \\ P_{2*}^G \le 5.983\,(\text{when r} = 1, \text{n} = 4) \end{array}\right\} \quad -0.25 < P_{2*}^G \le 0.177 \tag{4.3b}$$

$$\left\{\begin{array}{l} -0.21 < P_{3*}^G < 0.79 \\ P_{3*}^G \ge -2.14\,(\text{when r} = 1, \text{n} = 1) \\ P_{3*}^G \ge -0.833\,(\text{when r} = 1, \text{n} = 2) \\ P_{3*}^G \ge -4.21\,(\text{when r} = 1, \text{n} = 3) \\ P_{3*}^G \ge -28.03\,(\text{when r} = 1, \text{n} = 4) \end{array}\right\} \quad -0.21 < P_{3*}^G < 0.79 \tag{4.3c}$$

$$\left.\begin{cases} -0.18 < P_{4*}^G < 0.82 \\ P_{4*}^G \geq -0.78 \text{ (when r = 1, n = 1)} \\ P_{4*}^G \geq -0.239 \text{ (when r = 1, n = 2)} \\ P_{4*}^G \leq 16.44 \text{ (when r = 1, n = 3)} \\ P_{4*}^G \geq -11.88 \text{ (when r = 1, n = 4)} \end{cases}\right\} \quad -0.18 < P_{4*}^G < 0.82 \qquad (4.3d)$$

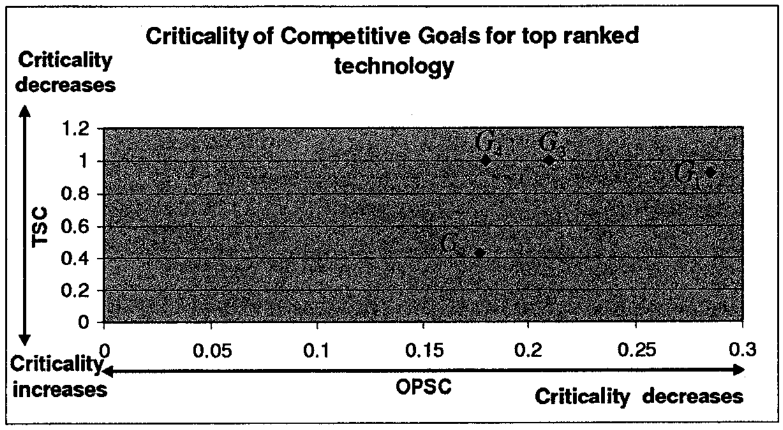
Accordingly, other sensitivity indicators for keeping the top-ranked alternative unchanged are calculated and summarized in Table 10.

Table 10 HDM SA at competitive-goals level to preserve the ranking of the top A_i

	C_1^G (cost)	C_2^G (product)	C_3^G (customer)	C_4^G (market)
Base values	0.360	0.250	0.210	0.180
Allowable ranges of perturbations	[-0.285, 0.640]	[-0.250, 0.177]	[-0.210, 0.790]	[-0.180, 0.820]
Tolerance	[0.075, 1]	[0, 0.427]	[0, 1]	[0, 1]
Prob. of rank changes	7.5%	57.3%	0%	0%
OPSC (G_k)	0.285	0.177	0.210	0.180
TSC (G_k)	0.925	0.427	1	1

As we can see, "reducing line width" technology is very robust at its current top rank. It will not be replaced unless the relative importance of product leadership to overall success increases above 0.427. Changes to the relative importance of the other three competitive goals hardly affect the top-ranked technology: the probabilities of top choice being replaced by other technologies are 7.5%, 0 and 0 in each case. This also makes "product leadership" the most critical competitive goal in keeping "reducing line width" as the top choice.

Figure 19 Criticality of competitive goals to keep the ranking of top technology choice



An in-depth investigation of how and under what conditions the rank of each technology alternative will change generated the following technology ranking scenarios in Table 11. The technology alternatives are ranked from one to five, as shown by the bold numbers in the parentheses, when the corresponding C_k^G value changes from one range to another (the brackets in the second column indicate those ranges). In each scenario, the pair of technology alternatives whose original rank order will be changed is listed in the last column of Table 11.

Table 11 Scenarios of technology alternatives' ranking regarding different C_k^G values

		Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	Rank reverse
C_1^G (cost)	[0, 0.075]	**(1)**	**(2)**	**(5)**	**(4)**	**(3)**	**(1, 3)**
	[0.076, 0.349]	**(2)**	**(1)**	**(5)**	**(4)**	**(3)**	**(2, 3)**
	[0.350, 0.825]	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	No
	[0.826, 1]	**(4)**	**(1)**	**(5)**	**(3)**	**(2)**	**(3, 4)**
C_2^G (product)	[0, 0.258]	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	No
	[0.259, 0.426]	**(2)**	**(1)**	**(5)**	**(4)**	**(3)**	**(2, 3)**
	[0.427, 1]	**(1)**	**(2)**	**(5)**	**(4)**	**(3)**	**(1, 3)**
C_3^G (customer)	[0, 0.141]	**(2)**	**(1)**	**(5)**	**(4)**	**(3)**	**(2, 3)**
	[0.142, 1]	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	No
C_4^G (market)	[0, 0.240]	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	No
	[0.241, 1]	**(2)**	**(1)**	**(5)**	**(4)**	**(3)**	**(2, 3)**

From an analysis of the scenarios listed in Table 11, we can see that the original top choice, "reducing line width" technology, will drop to the second when C_1^G (relative importance of *cost leadership*) decreases below 0.0752 or when C_2^G (relative importance of *product leadership*) increases above 0.427. "Reducing line width" technology dominates all the other technologies except "increasing wafer size" technology, which originally ranked third. "Factory Integration" takes either second or third rank, dominating both "Hi k dielectrics" and "Lo k dielectrics" technologies.

"Increasing wafer size" will be the top choice when "product leadership" is emphasized or "cost leadership" is deemphasized. However, its ranking is unstable and sensitive to variations of the competitive goals. Especially when the relative importance of cost leadership, C_1^G , increases from zero to one, the ranking of "Increasing wafer size" would drop from the first to the fourth. As analyzed before, this again is related to the huge investment risks associated with developing such a technology. On the other hand, technologies "Hi k dielectrics" and "Lo k dielectrics" are very stable at the fourth and fifth ranks, except that "Lo k dielectrics" may go to the third rank in one of scenarios.

In the semiconductor foundry industry, return on investment highly depends on the equipment utilization rate, and thus is subject to volatile market demands. To better utilize costly equipment investments during low seasons, foundries may shift their emphasis to cost leadership. According to Table 11, if the relative importance of cost leadership to overall success goes up, "reducing line width" and "factory integration" should be the top two technology choices. Contrasting to industry low seasons, when production capacity is short, cost leadership may be considered less important; in which case "increasing wafer size" will become the second- or even the top-ranked technology for Taiwan's semiconductor industry to develop. The SA result also indicates that if there is more than a 17.7% shift of emphasis to product leadership, then "increasing wafer size" technology should be the top technology to be developed.

Another important insight revealed by Table 11 is how to shift company goals accordingly if the company is not able to adopt the most desirable technologies due to different reasons. For example, although "reducing linewidth" is identified to be the most desirable technology for a company to develop if Ho's model accurately represents a certain company's current state, a given company may not be able to adopt this technology due to limited expertise in its personnel to deploy such technology. In such a situation, the company may choose either to recruit necessary human resources, or develop another technology in which the company has made certain initial investment and has sufficient experts, such as "increasing wafer size". Now, by performing HDM SA, the company is aware of situations that maximize the benefits of such investment (corresponds to scenarios in which the rank of "increasing wafer size" is second or first in Table 11). Therefore, senior managers in the company must shift the company goals toward product leadership and avoid competing in low cost product markets. Accordingly, the company image should be built around "high quality" "high tech" product leader, instead of something like "low price everyday."

For verification purpose, a spot check is performed: new values of C_i^A 's are calculated under the condition that an increase of 0.18 (>0.177) is induced on C_2^O, the relative importance of product leadership. Table 12 shows comparisons of the original ranking and the new ranking of the technology alternatives. Increasing wafer size became the top ranked choice as indicated by HDM SA.

Table 12 Comparison of original and new rankings

	C_1^G	C_{2*}^G	C_3^G	C_4^G
Original value	0.36	0.25	0.21	0.18
New value	0.27	0.43	0.16	0.14

	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration
Original ranking	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**
C_i^A	0.2196	0.2350	0.1321	0.1929	0.2204
New ranking	**(1)**	**(2)**	**(5)**	**(4)**	**(3)**
$C_i^A(new)$	0.2317	0.2314	0.1316	0.1898	0.2155

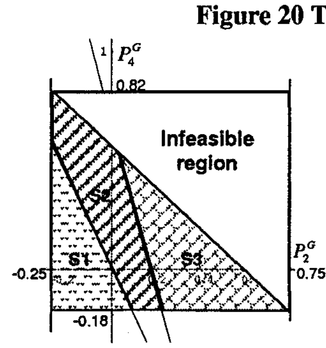
4.2.2.1.2 Two-way SA (two changes)

Among the competitive goals, product leadership (G_2) and market leadership (G_4) represent engineering perspective and market perspective. As the product performance changes rapidly, marketing continues to be a dynamic activity in the high-tech industry [76]. Improvements in technologies within and around the product system will advance the product performance and growth of the market [76]. To forecast and incorporate such progress, a two-way SA is performed accordingly to analyze simultaneous changes to the relative importance of G_2 and G_4.

Theorem 2 in the HDM SA algorithm deals with multiple simultaneous changes in the top- level contribution vector. Based on the theorem, a two-dimensional allowable region is identified for perturbations induced on C_2^G and C_4^G in order to keep the current ranking of alternatives.

As shown in Figure 20, two lines intersect the square feasible region and separate it into three parts representing three technology scenarios when two perturbations, P_2^G and P_4^G, are induced on C_2^G and C_4^G. "Feasible region" for P_2^G and P_4^G is the area in which the two values can change without causing C_2^G and C_4^G values go below zero or above one. Therefore, the four sides of the square are defined as (x $= -C_2^G$=-0.25, x $=1-C_2^G$= 0.75, y = $-C_4^G$= -0.18, y $=1-C_4^G$= 0.82), where the x axis represents P_2^G and y axis represents P_4^G. Origin represents the original judgment when C_2^G and C_4^G are at their base values and P_2^G and P_4^G are zero. Bold numbers in the parentheses again represent the ranking of the technologies in each scenario.

Figure 20 Two-way SA on C_2^G and C_4^G



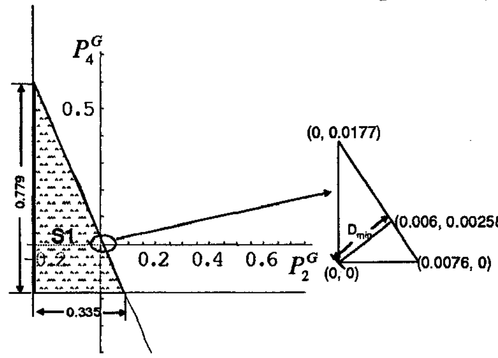
Scenarios of rankings

	Increasing wafer size	Reducing linewidth	Hi k	Lo k	Factory integration
S3	(1)	(2)	(5)	(4)	(3)
S2	(2)	(1)	(5)	(4)	(3)
S1	(3)	(1)	(5)	(4)	(2)

From Figure 20 we can tell that the ranking of "increasing wafer size" will either go up to the first or go down to the third. It is also shown that when the relative importance of product leadership is increased to a certain point, no matter how the relative importance of market leadership is shifted, "increasing wafer size" will be the top technology choice for the semiconductor foundries.

Among the three scenarios, S1 is the allowable region of perturbations introduced on C_2^G and C_4^G to preserve the original ranking of the technology alternatives: As long as the changes to the relative importance of product leadership and market leadership are within this region, the current ranking of all the technologies will remain unchanged. The inequalities defining the sides of S1 are ($0.1053P_2^G + 0.0453P_4^G \leq 0.0008$), ($-0.25 \leq P_2^G \leq 0.085$), and ($-0.18 \leq P_4^G \leq 0.599$).

Figure 21 Allowable region of P_2^G and P_4^G



As shown in Figure 21, the (x, y) coordinates of important points in S1 can be identified through trigonometry since equations of the intersecting lines are known. Therefore, based on the HDM SA algorithm, the two criticality indicators, TSC(G_2&G_4), which is the area of S1, and OPSC(G_2&G_4), which is the shortest distance from the origin to the sides of S1, are calculated:

TSC (G_2&G_4) $= Area\ (S1) = 0.335 \times 0.779 \times 0.5 = 0.13$

OPSC (G_2&G_4) $= D_{min} = \sqrt{0.006^2 + 0.00258^2} = 0.0065$

Since the area of the allowable region (S1 = 0.13) is 26% of the feasible region (S1 + S2 + S3 = 0.5), there is a 74% chance that current ranking of technology alternatives will be changed when the relative importance of product leadership and market leadership simultaneously change in the feasible region based on a uniform distribution.

The economic situation can affect the relative importance of competitive goals for a company. For the semiconductor foundry industry, when the industry is in a situation of oversupply, cost leadership gains the most attention. Conversely, if there are IC applications demanding advanced foundry processes, product leadership may top other competitive goals. It has been noted that the uncertainty in the trade-off between performance and cost in product design may create delays for product introduction and alter product plans [76]. Reducing cost may help companies gain competitiveness for awhile, especially in the early stage of a technology's development when the average cost is high. However, pure cost reduction without continuous product improvement may eventually cause the company to lose its competitiveness. Therefore, the other pair of competitive goals being analyzed together is cost leadership and product leadership.

$$
\begin{cases}
0.0246\, P_{1*}^{G} + 0.0046\, P_{2*}^{G} \le 0.0146 & \text{(4.4a)} \\
-0.0497\, P_{1*}^{G} + 0.0708\, P_{2*}^{G} \le 0.0008 & \text{(4.4b)} \\
0.031\, P_{1*}^{G} - 0.07\, P_{2*}^{G} \le 0.0267 & \text{(4.4c)} \\
-0.0108\, P_{1*}^{G} + 0.0092\, P_{2*}^{G} \le 0.0608 & \text{(4.4d)} \\
-0.36 < P_{1*}^{G} < 0.64 & \text{(4.4e)} \\
-0.25 < P_{2*}^{G} < 0.75 & \text{(4.4f)} \\
-0.61 < P_{2*}^{G} + P_{1*}^{G} < 0.39 & \text{(4.4g)}
\end{cases}
$$

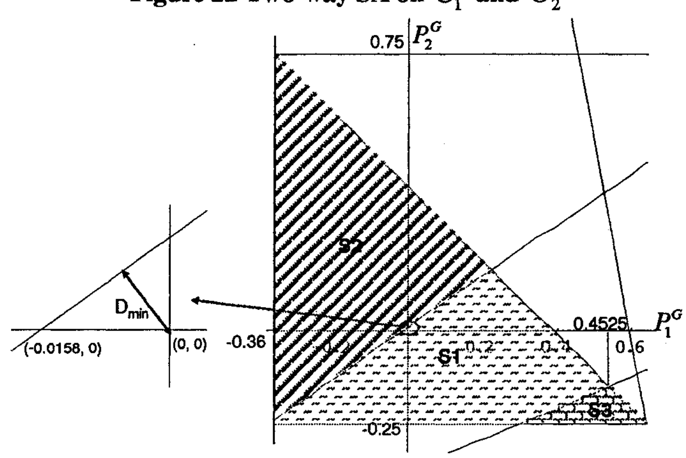
(4.4a)–(4.4d): Allowable Region

(4.4e)–(4.4g): Feasible Region

Based on Corollary 2.1, (I + M = 5+2 = 7) inequalities, (4.4a) through (4.4g), need to be satisfied in order for the original ranking of all alternatives to remain unchanged. Among these inequalities, (4.4b) and (4.4c) define two lines that intersect the feasible region defined by (4.4e) through (4.4g) and divide it into three parts representing three scenarios, as shown in Figure 22.

S1 is the allowable region of P_1^G and P_2^G to keep the original ranking of all technology alternatives unchanged. When the two perturbations fall in the area of S2, the rank order of "(2) factory integration" and "(3) increasing wafer size" will be reversed. This means that if the relative importance of "cost leadership" increases slower or decreases faster than the relative importance of "product leadership" does, then increasing wafer size will be more desirable than factory integration. When the two perturbations fall in the area of S3, the rank order of "(3) increasing wafer size" and "(4) Lo k dielectrics" will be reversed. This indicates that when the importance of "cost" increases to a certain degree while the importance of "product" decreases to a certain degree, "increasing wafer size" will become less desirable and drop to the fourth rank. However, in each case, the top choice remains as reducing linewidth.

Figure 22 Two-way SA on C_1^G and C_2^G



Again, for verification purpose, Theorem 2 is tested when (M = 2). Based on the inequalities, allowable range of P_{2*}^G can be determined when a value is assigned to P_{1*}^G

within the range [-0.36, 0.4525]. Thus, a few values around the specified thresholds for P_{1*}^G and P_{2*}^G are tried to see whether the decision alternatives' rank orders change. As shown in Table 13, when the values of perturbations go beyond the thresholds, rank order of some technology alternatives will be changed. New C_i^A value of the technology alternatives whose rank order has been changed are shaded in Table 13.

Table 13 Data verification for Theorem 2 (M=2)

	C_{1*}^G	C_{2*}^G	C_3^G	C_4^G		
Original values	0.36	0.25	0.21	0.18		
	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original values	0.22	0.24	0.13	0.19	0.22	
Original ranks	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	
Allowable range:		$-0.36 < P_{1*}^G \leq 0.45$				Rank Changes?
	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	
When $P_{1*}^G = -0.3$		$-0.25 < P_{2*}^G \leq -0.197$				
$P_{2*}^G = -0.24$	0.2137	0.2398	0.1350	0.1948	0.2167	No
$P_{2*}^G = -0.18$	0.2170	0.2386	0.1347	0.1939	0.2158	Yes
When $P_{1*}^G = 0.4$		$-0.21 \leq P_{2*}^G \leq -0.01$				
$P_{2*}^G = -0.23$	0.1970	0.2396	0.1312	0.1984	0.2338	Yes
$P_{2*}^G = -0.2$	0.1987	0.2390	0.1310	0.1980	0.2333	No
$P_{2*}^G = -0.01$	0.2092	0.2352	0.1300	0.1952	0.2304	No

4.2.2.2 Technology-Strategies Analysis

Along the technology planning horizon, companies may get new or improved R&D capabilities, production capabilities, capitalization and asset capabilities, and operational capabilities as a result of enterprise evolution [76], or lose some of their research specialties due to critical personnel leaving or company market direction shifts. In addition, the fables design houses, customers of the semiconductor foundries, may develop new IC applications for a variety of markets. These IC chips need a wide range of manufacturing processes. Thus, new IC chips, especially those widely accepted by the market, may cause re-deployment of technologies in a foundry and, as a result, change the technology strategies. In those cases, the relative impact of the technology strategies will be altered. From the perspective of synoptic planning, it is important to anticipate and incorporate the changes into the technology plans. Therefore, HDM SA is also performed to study how variations at the technology-strategies level impact technology choices.

4.2.2.2.1 One-way SA

Again, two conditions are considered when performing the one-way SA: Table 14 summarizes the tolerance of C_{jk}^{S-G}, the relative impact of the j^th^ technology strategy to the k^th^ competitive goal, to keep the original ranking of all technologies unchanged; Table 6 indicates the tolerance of C_{jk}^{S-G} to keep current top choice the same.

Table 14 Allowable range of P_{jk}^{S-G} to preserve the ranking of all A_i

	Innovation		Imitation		Diversity		Efficiency		Flexibility	
	δ_{1k-}^{S-G}	δ_{1k+}^{S-G}	δ_{2k-}^{S-G}	δ_{2k+}^{S-G}	δ_{3k-}^{S-G}	δ_{3k+}^{S-G}	δ_{4k-}^{S-G}	δ_{4k+}^{S-G}	δ_{5k-}^{S-G}	δ_{5k+}^{S-G}
$k=1$	-0.02	0.005	-0.11	0.0046	-0.14	0.0055	-0.355	0.458	-0.0044	0.44
Cost	-	(2, 3)	-	(2, 3)	-	(2, 3)	(2, 3)	(1, 2)	(2, 3)	(3, 4)
$k=2$	-0.54	0.0138	-0.14	0.0259	-0.17	0.1038	-0.009	0.85	-0.0044	0.48
Product	-	(2, 3)	-	(2, 3)	-	(2, 3)	(2, 3)	(1, 2)	(2, 3)	(3, 4)
$k=3$	-0.11	0.011	-0.14	0.01	-0.24	0.012	-0.025	0.73	-0.0063	0.718
Customer	-	(2, 3)	-	(2, 3)	-	(2, 3)	(2, 3)	-	(2, 3)	(3, 4)
$k=4$	-0.16	0.013	-0.22	0.01	-0.21	0.0177	-0.024	0.82	-0.0068	0.76
Market	-	(2, 3)	-	(2, 3)	-	(2, 3)	(2, 3)	-	(2, 3)	-

δ_{jk-}^{S-G} and δ_{jk+}^{S-G} are the lower and upper bounds of P_{jk}^{S-G}'s allowable range. (x, y) is the pair of technologies whose current rank order will be reversed if the P_{jk}^{S-G} value goes beyond δ_{jk-}^{S-G} and/or δ_{jk+}^{S-G}.

Table 14 shows that increasing the relative impact of "innovation," "imitation" and "diversity" or decreasing that of "efficiency" and "flexibility" will reverse the rank order of "factory integration" and "increasing wafer size." "Factory integration" will be the top-ranked technology when the relative impacts of "efficiency" increase; and "increasing wafer size" will become the fourth-ranked technology when "flexibility" is deemphasized. "Increasing wafer size" again has the most unstable rank when C_{jk}^{S-G} values change.

Table 15 Threshold of P_{jk}^{S-G} values to preserve the ranking of the top A_i

	Innovation	Imitation	Diversity	Efficiency	Flexibility
Cost	$\delta_{11+}^{S-G} = 0.3$ **(1, 3)**	$\delta_{21+}^{S-G} = 0.457$ **(1, 3)**	-	$\delta_{31+}^{S-G} = 0.458$ **(1, 2)**	$\delta_{41-}^{S-G} = -0.256$ **(1, 3)**
Product	-	-	-	$\delta_{32+}^{S-G} = 0.85$ **(1, 2)**	-
Customer	$\delta_{13+}^{S-G} = 0.52$ **(1, 3)**	-	-	-	-
Market	$\delta_{14+}^{S-G} = 0.61$ **(1, 3)**	-	-	-	-

δ_{jk-}^{S-G} and δ_{jk+}^{S-G} are the lower and upper bounds of P_{jk}^{S-G}'s allowable range. **(x, y)** is the pair of technologies whose current rank order will be reversed if the P_{jk}^{S-G} value goes beyond δ_{jk-}^{S-G} and/or δ_{jk+}^{S-G}.

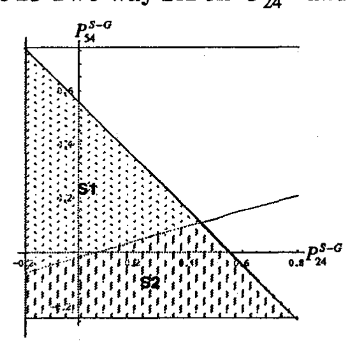
Regarding the current top technology choice, "reducing line width," Table 15 indicates that it is relatively stable at its current rank. The most influential factors at this level to keep "reducing line width" as the top choice are the relative impacts of different strategies, especially flexibility, to the competitive goal of cost leadership.

4.2.2.2.2 Two-way SA

In the technology assessment model, two experts provided judgments regarding the relative impacts of technology strategies on the competitive goals. Then their opinions were averaged to derive the model results. As mentioned before, uncertainties may cause disagreement among the experts. Although averaging experts' judgment is quite effective to aggregate different opinions [7][68], it is noted that the average opinion of a group of experts does not always yield useful results, and sometimes the reality is at one of the extremes of experts' judgments [8]. To address this issue, HDM SA is applied to test the robustness of the model regarding different experts' opinions.

Among the impacts of technology strategies on competitive goals, "flexibility" to "market leadership" (C_{54}^{S-G}) and "imitation" to "market leadership" (C_{24}^{S-G}) received the greatest disagreement from two experts: expert C gave 0.07 to C_{24}^{S-G} and 0.4 to C_{54}^{S-G}, while expert D gave 0.34 to C_{24}^{S-G} and 0.09 to C_{54}^{S-G}.

Figure 23 Two-way SA on C_{24}^{S-G} and C_{54}^{S-G}



Performing a two-way SA on the two contribution values based on Corollary 3.2, two scenarios are generated, as shown in Figure 23. S1 is the allowable region for perturbations induced on C_{24}^{S-G} and C_{54}^{S-G} to keep the current ranking of all the technologies unchanged, and the judgment of expert C falls in S1. However, the judgment of expert D falls in S2, in which the rank order of "increasing wafer size" and "factory integration" will be reversed. But in either case, the top-ranked alternative, "reducing line width" technology, will not be affected.

4.2.2.2.3 Verification

A spot check is performed to verify Corollary 3.1: values within and beyond the allowable range of P_{1*1*}^{S-G} to keep current ranking of all technology alternatives are assigned to P_{1*1*}^{S-G}, and new C_i^A values are calculated. As shown in Table 16, when P_{1*1*}^{S-G} goes beyond the specified value, the rank order of "increasing wafer size" and "factory integration" is reversed, as indicated by HDM SA.

Corollaries 3.2 and 3.3 are tested and verified in the same way that Theorem 2 is verified when (M=2) in previous section. Table 17 and Table 18 summarize the verification results.

Table 16 Data verification for Corollary 3.1 on P_{11}^{S-G}

	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original Rank	(3)	(1)	(5)	(4)	(2)	
P_{1*1*}^{S-G}	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank changes?
< -0.02	$C_{1*1*}^{S-G} < 0$					N/A
-0.02	0.2177	0.2381	0.1276	0.1902	0.2192	No
0	0.2196	0.2350	0.1321	0.1929	0.2204	No
0.005	0.21881	0.2376	0.1276	0.1899	0.21882	No
0.006	0.2189	0.2376	0.1276	0.1899	0.2188	Yes

Table 17 Data verification for Corollary 3.2

	C_{1*1*}^{S-G}	C_{2*1*}^{S-G}	C_{31*}^{S-G}	C_{41*}^{S-G}	C_{51*}^{S-G}	
Original values	0.02	0.11	0.14	0.430	0.290	
	Increasing Wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original values	0.22	0.24	0.13	0.19	0.22	
Original ranks	(3)	(1)	(5)	(4)	(2)	
Allowable range:	$-0.02 < P_{1*1*}^{S-G} < 0.1146$					
	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank Changes?
When $P_{1*1*}^{S-G} = 0.1$	$-0.11 < P_{2*1*}^{S-G} < -0.0954$					
$P_{2*1*}^{S-G} = -0.096$	0.219	0.236	0.128	0.190	0.220	No
$P_{2*1*}^{S-G} = -0.095$	0.220	0.236	0.128	0.190	0.219	Yes
When $P_{1*1*}^{S-G} = -0.01$	$-0.11 < P_{2*1*}^{S-G} \le 0.01456$					
$P_{2*1*}^{S-G} = 0.014$	0.2187	0.238	0.128	0.19	0.2187	No
$P_{2*1*}^{S-G} = 0.015$	0.21873	0.2379	0.128	0.19	0.2187	Yes
$P_{2*1*}^{S-G} = -0.1$	0.2140	0.2382	0.128	0.1911	0.2216	No

Table 18 Data verification for Corollary 3.3

	C_{1*1*}^{S-G}	C_{21*}^{S-G}	C_{31*}^{S-G}	C_{41*}^{S-G}	C_{51*}^{S-G}	
Original values	0.02	0.11	0.14	0.43	0.29	
	C_{12*}^{S-G}	C_{2*2*}^{S-G}	C_{32*}^{S-G}	C_{42*}^{S-G}	C_{52*}^{S-G}	
Original values	0.54	0.14	0.17	0.07	0.08	
A_i	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original values	0.22	0.24	0.13	0.19	0.22	
Original ranks	(3)	(1)	(5)	(4)	(2)	
Allowable range:	$-0.02 < C_{1*1*}^{S-G} < 0.0268$					
	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank Changes?
When $C_{1*1*}^{S-G} = -0.019$	$-0.14 < C_{2*2*}^{S-G} < 0.1224$					
$C_{2*2*}^{S-G} = 0.13$	0.2186	0.2389	0.1269	0.1897	0.2185	Yes
$C_{2*2*}^{S-G} = 0.12$	0.2185	0.2388	0.1269	0.1898	0.2186	No
$C_{2*2*}^{S-G} = -0.13$	0.2169	0.2374	0.1284	0.1906	0.2198	No
When $C_{1*1*}^{S-G} = 0.025$	$-0.14 < C_{2*2*}^{S-G} \le -0.1007$					
$C_{2*2*}^{S-G} = -0.139$	0.2188	0.2364	0.1285	0.1902	0.2192	No
$C_{2*2*}^{S-G} = -0.11$	0.2190	0.2366	0.1283	0.1901	0.2191	No
$C_{2*2*}^{S-G} = -0.1$	0.21904	0.2367	0.1282	0.1901	0.21903	Yes

4.2.2.3 Technology Performance Analysis

As the technologies are being developed, performance may fall short or exceed expectations. Certain technologies may be progressing faster than others. In order for companies to respond quickly in the adaptive planning mode, it is helpful to anticipate different possibilities and incorporate technological advances into the technology plan, especially when the companies are in a fast-changing environment like the semiconductor industry. A good example in this case study is the rapid progress to reduce line width: when the hierarchical technology assessment model was first quantified in year 2004, 90 nm was the state of the art in reducing line width technologies. However, in less than two years, the same technology has reached the point where 65 nm is applied in manufacturing and 35 nm has been achieved in research labs. This advance may alter the contribution matrix $\mathbf{C}_{ij}^{A-S}$ and generally increase the contributions of "reducing line width" technology. Applying the HDM SA algorithm, the impact of technology advances on a semiconductor foundry's technology plan is evaluated.

<u>4.2.2.3.1 One-way SA</u>

The allowable ranges of perturbations induced on C_{ij}^{A-S}, contribution of the ith technology alternative to the jth technology strategy, are calculated and summarized in Table 19. δ_{ij-}^{A-S} and δ_{ij+}^{A-S} indicate the lower and upper bounds of perturbations on C_{ij}^{A-S} to preserve the original ranking of all the technology alternatives.

Table 19 Allowable range of P_{ij}^{A-S} to preserve the ranking of all A_i

	Increasing wafer size		Reducing line width		Hi k dielectrics		Lo k dielectrics		Factory integration	
	δ_{1j-}^{A-S}	δ_{1j+}^{A-S}	δ_{2j-}^{A-S}	δ_{2j+}^{A-S}	δ_{3j-}^{A-S}	δ_{3j+}^{A-S}	δ_{4j-}^{A-S}	δ_{4j+}^{A-S}	δ_{5j-}^{A-S}	δ_{5j+}^{A-S}
$j=1$	-0.118	0.001	-0.011	0.81	-0.012	0.269	-0.011	0.107	-0.001	0.078
Innovation	**(3, 4)**	**(2, 3)**	**(2, 3)**	-	**(2, 3)**	**(4, 5)**	**(2, 3)**	**(3, 4)**	**(2, 3)**	**(1, 2)**
$j=2$	-0.16	0.0016	-0.014	0.76	-0.017	0.364	-0.015	0.147	-0.0015	0.101
Imitation	**(3, 4)**	**(2, 3)**	**(2, 3)**	-	**(2, 3)**	**(4, 5)**	**(2, 3)**	**(3, 4)**	**(2, 3)**	**(1, 2)**
$j=3$	-0.124	0.0014	-0.015	0.73	-0.017	0.275	-0.016	0.123	-0.0013	0.079
Diversity	**(3, 4)**	**(2, 3)**	**(2, 3)**	-	**(2, 3)**	**(4, 5)**	**(2, 3)**	**(3, 4)**	**(2, 3)**	**(1, 2)**
$j=4$	-0.089	0.0008	-0.054	0.018	-0.12	0.02	-0.18	0.019	-0.0009	0.056
Efficiency	**(3, 4)**	**(2, 3)**	**(1, 2)**	**(2, 3)**	-	**(2, 3)**	-	**(2, 3)**	**(2, 3)**	**(1, 2)**
$j=5$	-0.09	0.001	-0.063	0.005	-0.13	0.007	-0.22	0.006	-0.001	0.062
Flexibility	-	**(2, 3)**	**(1, 2)**	**(2, 3)**	-	**(2, 3)**	-	**(2, 3)**	**(2, 3)**	**(1, 2)**

δ_{ij-}^{A-S} and δ_{ij+}^{A-S} are the lower and upper bounds of P_{ij}^{A-S}'s allowable range. (x, y) is the pair of technologies whose current rank order will be reversed if the P_{ij}^{A-S} goes beyond δ_{ij-}^{A-S} and/or δ_{ij+}^{A-S}.

The smallest allowable change to C_{ij}^{A-S} to preserve the original ranking of all technologies happens on C_{14}^{A-S}, the contribution of "increasing wafer size" to "efficiency," making "increasing wafer size" the most critical decision element at the technology-alternatives level to keep the current ranking of technologies. The contribution of "factory integration" to "flexibility" is also very critical because if it decreases by more than 0.0009, it will reverse the rank order of "increasing wafer size" and "factory integration."

By comparing the threshold values in Table 19 to Table 9 and Table 16, we can see that since contribution matrix $\mathbf{C}_{ij}^{A-S}$ is at the bottom level of the decision hierarchy, the ranking of technologies is more sensitive to changes in this matrix. This means that in order to determine whether the technology portfolio is optimal along the planning horizon, advances of technology alternatives are more critical than industry policy changes or company strategy shifts. It is worth doing an in-depth analysis at this level to further investigate technology scenarios when each technology's contribution to each strategy changes.

First analyzed is "reducing line width" technology, the current top choice that advances at the fastest speed. When the advances of "reducing line width" technology cause its contributions to each strategy to increase, the original ranking of all the technologies will remain the same, as shown in Table 20. The rank of "reducing line width" changes only when its contributions decrease from the original value to a certain point where it becomes the second, third or even the fourth rank. "Hi k dielectrics" and "Lo k dielectrics" are again mostly dominated by other technologies. However, judging from the trend of technology advances, the contribution of "reducing line width" will be increasing instead of decreasing.

Then the analysis goes to "increasing wafer size" technology, which is among the top three choices but with the most unstable rank based on the previous analysis. Table 21 shows the scenarios of different technology ranks, as shown by the bold numbers in the parentheses, when C_{1j}^{A-S}, contribution of "increasing wafer size" technology to the jth strategy, changes from one range to another (the brackets in the second column indicate those ranges). The bold number beside each C_{1j}^{A-S} is the base value assigned to it originally.

As Table 21 shows, only "increasing wafer size" and "reducing line width" become the top choice when C_{1j}^{A-S} varies. However, the rank of "increasing wafer size" is again very unstable. "Factory integration" is very stable at rank two or three, and "Hi k dielectrics" and "Lo k dielectrics" are mostly dominated by other technologies.

Applying the same analysis to "factory integration" shows that "reducing line width" technology is the top choice in most cases, but it may drop to the third rank in one scenario when "factory integration" and "increasing wafer size" rank first and second respectively. "Factory integration" can become the top choice when its contributions to strategies are increased to a certain point. "Increasing wafer size" takes either second or third rank. "Lo k dielectrics" ranks fourth in most cases and third a few times, and "Hi k dielectrics" remains as the last choice in every scenario.

Table 20 Technology rankings when contributions of "reducing line width" to strategies change

		Increasing wafer size	Reducing line width	Hi k	Lo k	Factory integration
C_{11}^{A-S} **0.19** (Innovation)	[0.179, 1]	(3)	(1)	(5)	(4)	(2)
	[0.12, 0.178]	(2)	(1)	(5)	(4)	(3)
	[0.113, 0.119]	(1)	(2)	(5)	(4)	(3)
	[0, 0.112]	(1)	(3)	(5)	(4)	(2)
C_{12}^{A-S} **0.24** (Imitation)	[0.226, 1]	(3)	(1)	(5)	(4)	(2)
	[0.144,0.225]	(2)	(1)	(5)	(4)	(3)
	[0.134, 0.143]	(1)	(2)	(5)	(4)	(3)
	[0, 0.133]	(1)	(3)	(5)	(4)	(2)
C_{13}^{A-S} **0.27** (Diversity)	[0.256, 1]	(3)	(1)	(5)	(4)	(2)
	[0.19, 0.255]	(2)	(1)	(5)	(4)	(3)
	[0.184, 0.189]	(1)	(3)	(5)	(4)	(2)
	[0.18, 0.183]	(1)	(2)	(5)	(4)	(3)
	[0.07, 0.179]	(1)	(4)	(5)	(3)	(2)
C_{14}^{A-S} **0.21** (Efficiency)	[0.17, 1]	(3)	(1)	(5)	(4)	(2)
	[0.154, 0.16]	(3)	(2)	(5)	(4)	(1)
	[0.061,0.153]	(2)	(3)	(5)	(4)	(1)
	[0, 0.06]	(2)	(4)	(5)	(3)	(1)
C_{15}^{A-S} **0.29** (Flexibility)	[0.295, 1]	(2)	(1)	(5)	(4)	(3)
	[0.227, 0.294]	(3)	(1)	(5)	(4)	(2)
	[0.211, 0.226]	(3)	(2)	(5)	(4)	(1)
	[0.121, 0.21]	(2)	(3)	(5)	(4)	(1)
	[0, 0.12]	(2)	(4)	(5)	(3)	(1)

Table 21 Ranking of technologies when contributions of "increasing wafer size" change

		Increasing wafer size	Reducing line width	Hi k	Lo k	Factory integration
C_{31}^{A-S} **0.31** (Innovation)	[0.387, 1]	(1)	(2)	(5)	(4)	(3)
	[0.311, 0.386]	(2)	(1)	(5)	(4)	(3)
	[0.193, 0.31]	(3)	(1)	(5)	(4)	(2)
	[0, 0.192]	(4)	(1)	(5)	(3)	(2)
C_{32}^{A-S} **0.29** (Imitation)	[0.39, 1]	(1)	(2)	(4)	(5)	(3)
	[0.292, 0.38]	(2)	(1)	(5)	(4)	(3)
	[0.131, 0.291]	(3)	(1)	(5)	(4)	(2)
	[0, 0.13]	(4)	(1)	(5)	(3)	(2)
C_{33}^{A-S} **0.22** (Diversity)	[0.837, 1]	(2)	(1)	(5)	(4)	(3)
	[0.298, 0.836]	(1)	(2)	(5)	(4)	(3)
	[0.221, 0.297]	(2)	(1)	(5)	(4)	(3)
	[0.097, 0.22]	(3)	(1)	(5)	(4)	(2)
	[0, 0.096]	(4)	(1)	(5)	(3)	(2)
C_{34}^{A-S} **0.22** (Efficiency)	[0.278, 1]	(1)	(2)	(5)	(4)	(3)
	[0.221, 0.277]	(2)	(1)	(5)	(4)	(3)
	[0.131, 0.22]	(3)	(1)	(5)	(4)	(2)
	[0, 0.13]	(4)	(1)	(5)	(3)	(2)
C_{35}^{A-S} **0.09** (Flexibility)	[0.157, 1]	(1)	(2)	(5)	(4)	(3)
	[0.091, 0.156]	(2)	(1)	(5)	(4)	(3)
	[0, 0.09]	(3)	(1)	(5)	(4)	(2)

The analysis on "Hi k dielectrics" and "Lo k dielectrics" reveals that in most cases "reducing line width" ranks first with "increasing wafer size" and "factory integration" ranking second, third or fourth in different scenarios. "Hi k dielectrics" or "Lo k dielectrics" can move up to the first ranking when their contributions to the strategies are increased dramatically.

4.2.2.3.2 Two-way SA

Among all the technology alternatives, "increasing wafer size" and "reducing line width" are the two technology leadership indicators in the industry and they are evolving at a relatively higher speed than the other technology alternatives. Therefore, sensitivity of changes to these two technology alternatives is studied together. Since "efficiency" and "innovation" are the two most important strategies regarding their contributions to overall competitive success, a two-way SA is performed on the contribution of "increasing wafer size" to "efficiency" and the contribution of "reducing line width" to "innovation". As shown in Figure 24, six scenarios are generated as the two perturbations on C_{14}^{A-S} and C_{21}^{A-S} change in the feasible region. From the fact that line width of microprocessors was reduced from 90 to 65 nanometers in October 2005 and the 45 nanometer technology is right on track, while the wafer size stays at 300 mm, it seems that "reducing line width" technology is advancing at a relatively higher speed than "increasing wafer size." Therefore, we conclude that judging from current development trends of the technologies, S2 is the most likely scenario with "reducing line width" technology advancing faster than "increasing wafer size" and remaining as the top-ranked technology alternative under evaluation.

Figure 24 Two-way SA on P_{14}^{A-S} and P_{21}^{A-S}

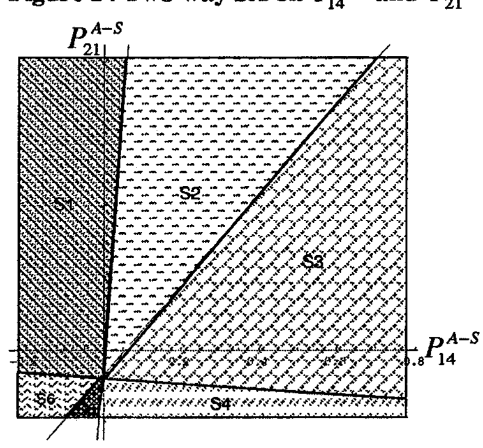


Table 22 Possible ranking scenarios

Scenarios	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration
S1	(3)	(1)	(5)	(4)	(2)
S2	(2)	(1)	(5)	(4)	(3)
S3	(1)	(2)	(5)	(4)	(3)
S4	(1)	(3)	(5)	(4)	(2)
S5	(2)	(3)	(5)	(4)	(1)
S6	(3)	(2)	(5)	(4)	(1)

4.2.2.3.3 Verification

Corollaries 4.1 through 4.3 are verified in the similar way in which the previous corollaries are tested and verified. Tables 23 through 25 summarize the verification results.

Based on Corollary 4.1, when a single perturbation is induced on the contribution of technology alternative "increasing wafer size" to technology strategy "innovation", which is C_{1*1*}^{A-S}, the allowable range of the perturbations is [-0.118, 0.001] to keep the current ranking of all technology alternatives unchanged.

Table 23 Data verification for Corollary 4.1

Original Rank	(3)	(1)	(5)	(4)	(2)	
P_{1*1*}^{A-S}	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank changes?
-0.12	0.1953	0.2442	0.1320	0.1957	0.2256	Yes
-0.118	0.1957	0.2441	0.1319	0.1956	0.2255	No
-0.09	0.2011	0.2426	0.1309	0.1943	0.2240	No
0	0.2186	0.2377	0.1276	0.1900	0.2189	No
0.001	0.2188	0.2377	0.1276	0.1899	0.2188	No
0.0012	0.218829	0.2377	0.1276	0.1899	0.218825	Yes

Table 24 Data verification for Corollary 4.2

	C_{1*1*}^{A-S}	C_{2*1*}^{A-S}	C_{31*}^{A-S}	C_{41*}^{A-S}	C_{51*}^{A-S}	
Original values	0.31	0.19	0.13	0.17	0.2	
	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original values	0.22	0.24	0.13	0.19	0.22	
Original ranks	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	
Allowable range:	$-0.31 < P_{1*1*}^{A-S} \le 0.0228$					
	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank Changes?
When $P_{1*1*}^{A-S} = -0.3$	$0.7489 < P_{2*1*}^{A-S} < 0.8$					
$P_{2*1*}^{A-S} = 0.74$	0.1604	0.3814	0.1054	0.1610	0.1847	Yes
$P_{2*1*}^{A-S} = 0.75$	0.1604	0.3833	0.1049	0.1603	0.1840	No
$P_{2*1*}^{A-S} = 0.8$	0.1604	0.3930	0.1024	0.1570	0.1801	No
When $P_{1*1*}^{A-S} = 0.02$	$-0.0751 \le P_{2*1*}^{A-S} \le -0.0662$					
$P_{2*1*}^{A-S} = -0.076$	0.2225	0.2230	0.1305	0.1937	0.2232	Yes
$P_{2*1*}^{A-S} = -0.07$	0.2225	0.2242	0.1302	0.1933	0.2228	No
$P_{2*1*}^{A-S} = -0.06$	0.2225	0.2261	0.1296	0.1926	0.2220	Yes

When two perturbations are induced on the contributions of technology alternatives "increasing wafer size" and "reducing line width" to technology strategy "innovation," a group of inequalities derived from Corollary 4.2 will define the allowable area of the two perturbations, P_{1*1*}^{A-S} and P_{2*1*}^{A-S}, to keep the current ranking of all technologies unchanged. Different values are assigned to the two perturbations within and beyond their allowable area to test whether rankings of the technologies are changed. Results are summarized in Table 24.

When two perturbations are induced on the contributions of technology alternatives "increasing wafer size" and "reducing line width" to technology strategies "innovation" and "imitation" respectively, a group of inequalities derived from Corollary 4.3 will define the allowable area of the two perturbations, P_{1*1*}^{A-S} and P_{2*2*}^{A-S}, to keep current ranking of all technologies unchanged. Different values are assigned to the two perturbations within and beyond their allowable area to test whether rankings of the technologies are changed. Results are summarized in Table 25.

Table 25 Data verification for Corollary 4.3

	C_{1*1*}^{A-S}	C_{21*}^{A-S}	C_{31*}^{A-S}	C_{41*}^{A-S}	C_{51*}^{A-S}	
Original values	0.31	0.19	0.13	0.17	0.2	
	C_{12*}^{A-S}	C_{2*2*}^{A-S}	C_{32*}^{A-S}	C_{42*}^{A-S}	C_{52*}^{A-S}	
Original values	0.29	0.24	0.11	0.17	0.18	
	Increasing wafer size	Reducing line width	Hi k	Lo k	Factory Integration	
Original values	0.22	0.24	0.13	0.19	0.22	
Original ranks	**(3)**	**(1)**	**(5)**	**(4)**	**(2)**	
Allowable range:	$-0.128 \le P_{1*1*}^{A-S} < 0.065$					
	C_1^A	C_2^A	C_3^A	C_4^A	C_5^A	Rank Changes?
When $P_{1*1*}^{A-S} = -0.12$	$-0.104 < P_{2*2*}^{A-S} < -0.0185$					
$P_{2*2*}^{A-S} = -0.11$	0.2014	0.2284	0.1343	0.1993	0.2294	Yes
$P_{2*2*}^{A-S} = -0.1$	0.2009	0.2298	0.1341	0.1990	0.2291	No
$P_{2*2*}^{A-S} = -0.02$	0.1964	0.2413	0.1324	0.1964	0.2263	No
$P_{2*2*}^{A-S} = -0.015$	0.1961	0.2420	0.1323	0.1962	0.2262	Yes
When $P_{1*1*}^{A-S} = 0.06$	$0.699 \le P_{2*2*}^{A-S} < 0.76$					
$P_{2*2*}^{A-S} = 0.69$	0.1919	0.3336	0.1109	0.1647	0.1917	Yes
$P_{2*2*}^{A-S} = 0.7$	0.19137	0.3351	0.1107	0.1643	0.19139	No
$P_{2*2*}^{A-S} = 0.75$	0.1886	0.3422	0.1096	0.1627	0.1897	No

4.2.2.3 Multi-Level Analysis

4.2.2.3.1 Scenario One

There are times when contributions at different level of the decision hierarchy change simultaneously, such as when a competitive goal of the industry gets emphasized, and at the same time, the technical advancement improves the contribution of certain technology to strategies, or a certain strategy's contribution to a competitive goal in a different aspect, which has been overlooked, is identified and thus increases the overall contribution of that strategy. Theorem 5.1 to 5.3 in HDM SA algorithm that deals with multi-level simultaneous changes can be applied to study effect of such situations.

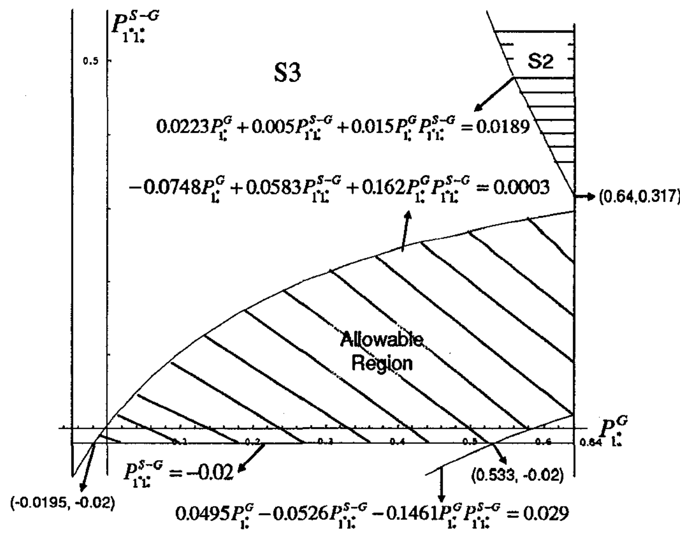
Generally speaking, pursuing the technology strategy of "innovation" implies high capital investment in R&D. This leads to a conclusion that innovation contributes little to "cost leadership." However, on the other hand, if the innovation strategy aims at decreasing manufacturing cost in the long run, the contribution of innovation to cost leadership should be increased. Therefore, in a situation when the technology alternatives are advanced to a certain degree to reach maturity, the industry is going to emphasize more on cost leadership and implement an innovation strategy to lower cost. To help visualize the impact of such kind of situations on Ho's model, Theorem 5.1 is applied to analyze simultaneous changes to the relative importance of "cost leadership" (C_1^G) and the contribution of "innovation" to "cost leadership" (C_{11}^{S-G}). In order to keep the current ranking of all technology alternatives, (I+1=6) inequalities about the two perturbations $P_{1\cdot}^{G}$ and $P_{1'1\cdot}^{S-G}$ need to be satisfied:

$$\begin{cases} \left.\begin{array}{ll} 0.0223P_{1\cdot}^{G}+0.005P_{1'1\cdot}^{S-G}+0.015P_{1\cdot}^{G}P_{1'1\cdot}^{S-G}\le 0.0189 & (4.5a) \\ -0.0748P_{1\cdot}^{G}+0.0583P_{1'1\cdot}^{S-G}+0.162P_{1\cdot}^{G}P_{1'1\cdot}^{S-G}\le 0.0003 & (4.5b) \\ 0.0495P_{1\cdot}^{G}-0.0526P_{1'1\cdot}^{S-G}-0.1461P_{1\cdot}^{G}P_{1'1\cdot}^{S-G}\ge 0.029 & (4.5c) \\ -0.0083P_{1\cdot}^{G}+0.0104P_{1'1\cdot}^{S-G}+0.029P_{1\cdot}^{G}P_{1'1\cdot}^{S-G}\le 0.0624 & (4.5d) \end{array}\right\} \text{Allowable Region} \\ \left.\begin{array}{ll} -0.02\le P_{1'1\cdot}^{S-G}\le 0.98 & (4.5e) \\ -0.36\le P_{1\cdot}^{G}\le 0.64 & (4.5f) \end{array}\right\} \text{Feasible Region} \end{cases}$$

The inequalities define a two-dimensional allowable region for the perturbations to keep current ranking of technology alternatives unchanged, as shown in Figure 25. Inequalities (4.5b) and (4.5c) which prevent the rank reverse of "(2) factory integration" and "(3) increasing wafer size," and "(3) increasing wafer size" and "(4) Lo k dielectrics," are the two inequalities bound the allowable area. This means that if the perturbations go above the defined area, the rank order of "(2) factory integration" and "(3) increasing wafer size" will be reversed; if the perturbations go below this area, "(3) increasing wafer size" will become the fourth ranked technology, and "(4) Lo k dielectrics" will be at the third rank.

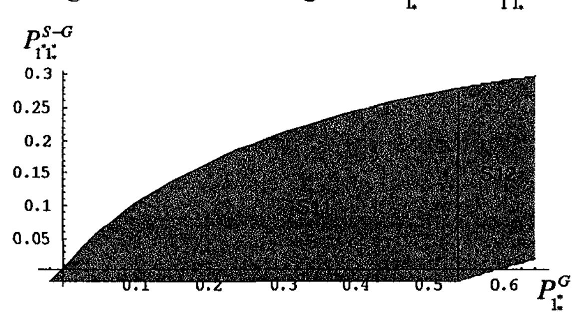
From Figure 25 we can see that when the relative importance of cost leadership increases, only when the change to contribution of "innovation" to "cost leadership" is between the curves defined by (4.5b) and (4.5c), will the current rank order of all technology alternatives be unchanged. When both contributions are increased to extreme values, in which case the two perturbations fall in area S2 in Figure 25, the currently second ranked alternative "factory integration" will be changed to the top rank. However, this is not likely to happen. On the other hand, slightly decreases to the relative importance of cost leadership will result in a rank change no matter how the contribution of "innovation" to "cost leadership" changes.

Figure 25 Two-way SA on $P_{1:}^{G}$ and $P_{1'1:}^{S-G}$



With the inequality functions known, the area of the allowable region can be calculated, which equals to the area of S11 and S12 in Figure 26. Thus, the total sensitivity coefficient of simultaneous changes to these two contributions can be determined.

Figure 26 Allowable region of $P_{1:}^{G}$ and $P_{1'1:}^{S-G}$



TSC (C_1^G & C_{11}^{S-G}) = *Area (Allowable region)* = *Area (S11+S12)*

$$= \int_{-0.0195}^{0.533} \left(\frac{0.0003+0.0748P_{1\cdot}^G}{0.0583+0.162P_{1\cdot}^G}+0.02\right)dP_{1\cdot}^G +$$

$$\int_{0.533}^{0.64} \left(\frac{0.0003+0.0748P_{1\cdot}^G}{0.0583+0.162P_{1\cdot}^G}-\frac{0.029-0.0495P_{1\cdot}^G}{-0.0526-0.1461P_{1\cdot}^G}\right)dP_{1\cdot}^G \qquad = 0.1077+0.0308=0.1385$$

Therefore, when C_1^G and C_{11}^{S-G} change uniformly within their feasible region, there is an 86 percent chance that the ranking of the technology alternatives will be changed.

To verify Theorem 5.1, different pairs of $P_{1\cdot}^G$ and $P_{1'1\cdot}^{S-G}$ values (represented by points "a" through "n" in Table 26) within and beyond the allowable region are tried and new C_i^A values are calculated. Table 26 summarizes the verification results.

As it is shown in Figure 27 and Table 26 show, when the $P_{1\cdot}^G$ and $P_{1'1\cdot}^{S-G}$ values fall into the allowable region, rank order of technology alternatives remains to be the same; however, when the values are beyond the boundary of the allowable region, rank order of technology alternatives will be changed. Shaded cells in the table represent the new C_i^A values of technology alternatives whose rank order is reversed.

The probability of rank remaining unchanged when C_1^G and C_{11}^{S-G} values change uniformly within their feasible region, calculated previously (TSC (C_1^G & C_{11}^{S-G}) = 14%), is verified through Monte Carlo simulation on the original hierarchical decision model.

Figure 27 Data verification for Theorem 5.1

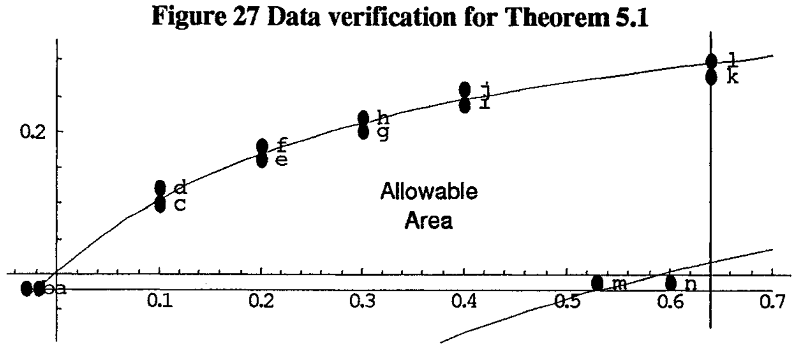


Table 26 Data verification for Theorem 5.1

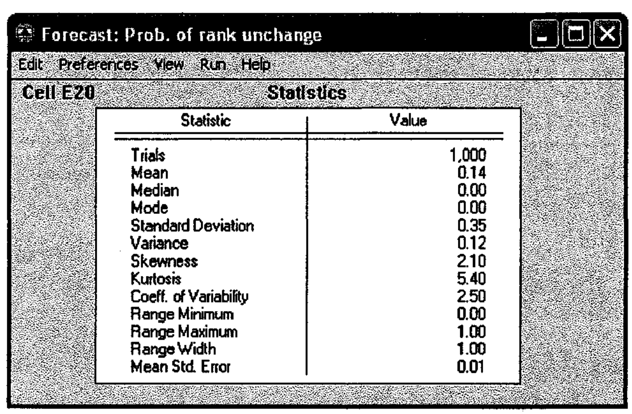
Point #	Original Rank $(P_{1}^{G}, P_{11}^{S-G})$ C_i^A	(3) Increasing wafer size	(1) Reducing line width	(5) Hi k	(4) Lo k	(2) Factory integration	Rank changes?
a	(-0.018, -0.02)	0.219	0.238	0.128	0.190	0.219	No
b	(-0.03, -0.02)	0.219	0.238	0.128	0.190	0.218	Yes
c	(0.1, 0.1)	0.220	0.236	0.127	0.189	0.220	No
d	(0.1, 0.12)	0.221	0.235	0.127	0.189	0.220	Yes
e	(0.2, 0.16)	0.220	0.234	0.127	0.189	0.221	No
f	(0.2, 0.18)	0.222	0.234	0.128	0.188	0.221	Yes
g	(0.3, 0.2)	0.221	0.233	0.127	0.188	0.222	No
h	(0.3, 0.22)	0.223	0.232	0.127	0.188	0.222	Yes
i	(0.4, 0.24)	0.223	0.230	0.127	0.187	0.223	No
j	(0.4, 0.26)	0.224	0.230	0.127	0.187	0.223	Yes
k	(0.64, 0.28)	0.223	0.227	0.126	0.186	0.226	No
l	(0.64,0.3)	0.226	0.226	0.126	0.185	0.225	Yes
M	(0.53, -0.01)	0.194	0.242	0.125	0.192	0.235	No
N	(0.6, -0.01)	0.190	0.243	0.125	0.193	0.237	Yes

To set the simulation, input C_1^G and C_{11}^{S-G} are defined to follow a uniform distribution U(0,1), and the output is binary with "0" representing rank changes and "1" representing no rank changes. Proportions of output being "zero" in the one thousand simulation trials correspond to sample proportion $\hat{p}$, or point estimate for population proportion p, in statistics. To verify the "86% chance of rank changes" calculated before, confidence interval for $\hat{p}$ should include 86%.

Since the sample size, 1000 trials, is sufficiently large, we can assume the sampling distribution of proportion $\hat{p}$ is approximately normal with mean p and standard deviation $\sqrt{p(1-p)n}$. This assumption is verified later by testing whether $nP_L > 5$, $n(1-P_L) > 5$, $nP_U > 5$, and $n(1-P_U) > 5$, where P_L and P_U are the lower and upper limits of the confidence interval. The lower and upper limits, P_L and P_U, are calculated as $\hat{p} \pm Z-multiple \times \sqrt{\hat{p}(1-\hat{p})/n}$, where $\hat{p}$ is the point estimate in the simulation of the true proportion p, and p is the "probability of rank changes" being calculated based on HDM SA algorithm.

As Figure 28 shows, the mean of one thousand simulation trials is 0.14, meaning 14% of the trails output one and the rest 86% of trials output zero. Here the binary output "one" represents rank remains unchanged and "zero" represents rank changes.

Figure 28 Monte Carlo simulations of rank change when C_1^G and C_{11}^{S-G} vary uniformly


Forecast: Prob. of rank unchange
Edit Preferences View Run Help
Cell E20 Statistics

Statistic	Value
Trials	1,000
Mean	0.14
Median	0.00
Mode	0.00
Standard Deviation	0.35
Variance	0.12
Skewness	2.10
Kurtosis	5.40
Coeff. of Variability	2.50
Range Minimum	0.00
Range Maximum	1.00
Range Width	1.00
Mean Std. Error	0.01

At 99.7% confidence level, Z-multiple value is three. Confidence intervals for $\hat{p}$ is calculated as: $0.86 \pm 3 \times \sqrt{0.86(1-0.86)/1000} = 0.86 \pm 3 \times 0.011 = [0.83, 0.89]$. Since $0.86 \in [0.83, 0.89]$, the probability of rank being changed when C_1^G and C_{11}^{S-G} vary uniformly between zero and one is verified by Monte Carlo simulation.

4.2.2.3.2 Scenario Two

As the market evolves, customers will be more attracted to more customized products at lower cost. In such situation, the emphasis of the industry's competitive goals will shift towards cost leadership; in addition, the strategy of "diversity" will become more important to meet the mass customization need and thus contributes more to "product leadership." Therefore, another pair of multi-level perturbations analyzed is the relative importance of "cost leadership" vs. the contribution of "diversity" to "product leadership." Applying Theorem 5.2, the sensitivity of technology alternatives' ranking to perturbations induced on the two contributions C_1^G and C_{32}^{S-G} is analyzed.

The allowable region of the two perturbations to keep the current ranking unchanged is defined by four inequalities that bound the region. Among them, inequalities (4.6a) and (4.6b) define the feasible region to prevent the new C_1^G and C_{32}^{S-G} values go below zero or above one; inequality (4.6c) prevents the rank reverse of "(2) factory integration" and "(3) increasing wafer size;" and inequality (d) prevents the rank reverse of "(3) increasing wafer size" and "(4) Lo k dielectrics."

$$\begin{cases} \left.\begin{array}{ll} -0.36 \le P_{1\cdot}^{G} \le 0.64 & (4.6a) \\ -0.17 \le P_{1\cdot 1\cdot}^{S-G} \le 0.83 & (4.6b) \end{array}\right\} \text{Feasible Region} \\ \left.\begin{array}{ll} -0.0748P_{1\cdot}^{G} + 0.0029P_{3\cdot 2\cdot}^{S-G} - 0.0045P_{1\cdot}^{G}P_{3\cdot 2\cdot}^{S-G} \le 0.0003 & (4.6c) \\ 0.0495P_{1\cdot}^{G} + 0.023P_{3\cdot 2\cdot}^{S-G} - 0.036P_{1\cdot}^{G}P_{3\cdot 2\cdot}^{S-G} \le 0.029 & (4.6d) \end{array}\right\} \text{Allowable Region} \end{cases}$$

Figure 29 Allowable region of $P_{1\cdot}^{G}$ and $P_{3\cdot2\cdot}^{S-G}$

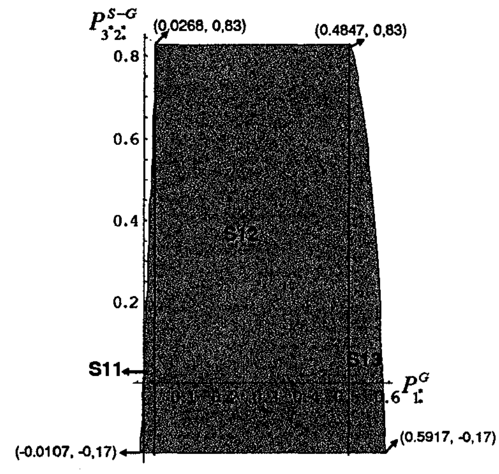


Figure 30 Data points for verification

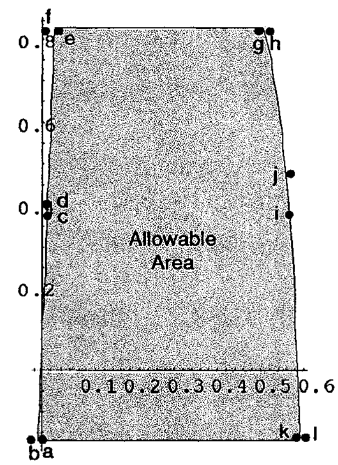


Figure 29 depicts the two-dimensional allowable region of perturbations induced on C_{1}^{G} and C_{32}^{S-G}. As it shows, the model result is very robust to increases to the relative importance of cost leadership: when C_{1}^{G} increases by a value between 0.0268 and 0.4847, no matter how C_{32}^{S-G} changes, the current ranking of the technology alternatives will remain unchanged. However, slight decrease to the relative importance of cost leadership will cause the ranking of "(**2**) factory integration" and "(**3**) increasing wafer size" to reverse, regardless of changes to C_{32}^{S-G}.

Total sensitivity coefficient of simultaneous changes to C_{1}^{G} and C_{32}^{S-G} is the area of the allowable region. With the inequalities known, such area can be calculated:

TSC (C_{1}^{G} & C_{32}^{S-G}) = *Area (Allowable region)* = *Area (S11+S12+S13)*

$$= \int_{-0.0107}^{0.0268} \left(\frac{0.0003 + 0.0748 P_{1\cdot}^{G}}{0.0029 - 0.0045 P_{1\cdot}^{G}} + 0.17\right) dP_{1\cdot}^{G} + (0.505 - 0.0268)$$

$$+ \int_{0.4847}^{0.5917} \left(\frac{0.029 - 0.0495 P_{1\cdot}^{G}}{0.023 - 0.036 P_{1\cdot}^{G}} + 0.17\right) dP_{1\cdot}^{G} = 0.0184 + 0.4579 + 0.0658 = 0.5421 \quad (4.7)$$

Therefore, when C_1^G, relative importance of "cost leadership," and C_{32}^{S-G}, contribution of "diversity" to "product leadership," change uniformly within their feasible region, there is a 45.79 percent chance that the current ranking of technology alternatives will be changed. The most unstable technology alternative is the currently third ranked technology "increasing wafer size." It will become the second ranked technology when "cost leadership" is deemphasized.

To verify Theorem 5.2, different pairs of $P_{1:}^G$ and $P_{3'2:}^{S-G}$ values (represented as points "a" through "l" in Figure 30 and Table 27) within and beyond the allowable region are tried and new C_i^A values are calculated. Table 27 summarizes the verification results.

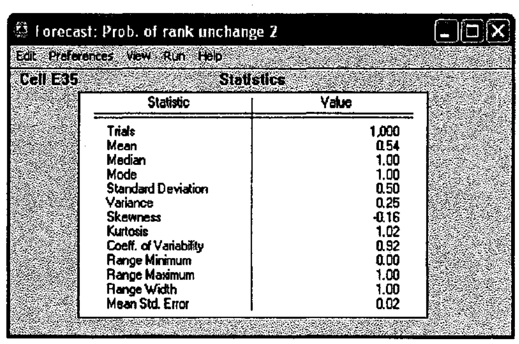
Table 27 Data verification for Theorem 5.2

Point #	Original Rank / ($P_{1:}^G$, $P_{3'2:}^{S-G}$) C_i^A	(3) Increasing wafer size	(1) Reducing line width	(5) Hi k	(4) Lo k	(2) Factory integration	Rank changes?
A	(-0.01, -0.17)	0.222	0.235	0.127	0.188	0.222	no
B	(-0.02, -0.17)	0.222	0.235	0.127	0.188	0.221	yes
C	(0.01, 0.37)	0.213	0.243	0.130	0.193	0.213	no
D	(0.01, 0.4)	0.2126	0.2437	0.1300	0.1934	0.2124	yes
E	(0.03, 0.83)	0.206	0.250	0.132	0.197	0.206	no
F	(0.01, 0.83)	0.206	0.250	0.133	0.197	0.205	yes
G	(0.48, 0.83)	0.194	0.244	0.127	0.194	0.229	no
H	(0.5, 0.83)	0.193	0.244	0.127	0.194	0.231	yes
I	(0.55, 0.41)	0.193	0.243	0.126	0.193	0.234	no
J	(0.55, 0.5)	0.1927	0.2428	0.1258	0.1929	0.2339	yes
K	(0.58, -0.17)	0.193	0.242	0.125	0.192	0.236	no
L	(0.6, -0.17)	0.1916	0.2419	0.1251	0.1924	0.2368	yes

As it shows, when the $P_{1:}^G$ and $P_{3'2:}^{S-G}$ values fall into the allowable region, rank order of technology alternatives remains to be the same; however, when the values are beyond the boundary of the allowable region, rank order of technology alternatives will be changed. Shaded cells in Table 27 represent the new C_i^A value of technology alternatives whose rank order is reversed.

The probability of rank remain unchanged when C_1^G and C_{32}^{S-G} values change uniformly within their feasible region, calculated as (TSC ($C_1^G \& C_{32}^{S-G}$) = 54.21%) previously, is verified through Monte Carlo simulation. As Figure 31 shows, mean of one thousand simulation trials is 0.54, meaning 54% of the trails output one and the rest 46% of trials output zero. Here the binary output "one" represents rank remains unchanged and "zero" represents rank changes. Based on the same reasoning, since 0.5421 is included in the confidence interval of proportion estimate by Monte Carlo simulation at 99.7% confidence level, the probability of rank remain unchanged when C_1^G and C_{32}^{S-G} values vary uniformly between zero and one is verified.

Figure 31 Monte Carlo simulations of rank change probability when $C_1^G \& C_{32}^{S-G}$ vary uniformly



Statistic	Value
Trials	1,000
Mean	0.54
Median	1.00
Mode	1.00
Standard Deviation	0.50
Variance	0.25
Skewness	-0.16
Kurtosis	1.02
Coeff. of Variability	0.92
Range Minimum	0.00
Range Maximum	1.00
Range Width	1.00
Mean Std. Error	0.02

4.2.2.3.3 Scenario Three

The third scenario analyzes sensitivity of changes to the relative importance of "product leadership" and the contribution of "reducing linewidth" to "innovation." As mentioned in competitive goals' analysis, when IC applications demand advanced foundry processes, product leadership may top other competitive goals. This requires an increase from the current 0.25 assigned to the relative importance of product leadership by at least 0.08 to surpass the relative importance of cost leadership. On the other hand, as one of indicators of technology leadership in the semiconductor industry, "reducing linewidth" technology is growing rapidly, owing to the fast advancement of nanotechnologies. The capability to produce transistors of the shorted linewidth has become a symbol of a company's innovation capacity. As a result, contribution of "reducing linewidth" to "innovation" should also be increased.

Based on Theorem 5.3, allowable region of perturbations on C_2^G, relative importance of "product leadership," and C_{11}^{A-S}, contribution of "reducing linewidth" to "innovation," are defined by inequalities (4.8a) to (4.8e):

$$\begin{cases} \left.\begin{array}{ll} -0.25 \le P_{2\cdot}^{G} \le 0.75 & (4.8a) \\ -0.19 \le P_{1\cdot 1\cdot}^{A-S} \le 0.81 & (4.8b) \end{array}\right\} \text{Feasible Region} \\ \left.\begin{array}{ll} -0.0049P_{2\cdot}^{G} - 0.242P_{1\cdot 1\cdot}^{A-S} - 0.575P_{2\cdot}^{G}P_{1\cdot 1\cdot}^{A-S} \le 0.0189 & (4.8c) \\ 0.0944P_{2\cdot}^{G} - 0.0264P_{1\cdot 1\cdot}^{A-S} - 0.0626P_{2\cdot}^{G}P_{1\cdot 1\cdot}^{A-S} \le 0.0003 & (4.8d) \\ -0.0772P_{2\cdot}^{G} + 0.0335P_{1\cdot 1\cdot}^{A-S} + 0.0797P_{2\cdot}^{G}P_{1\cdot 1\cdot}^{A-S} \le 0.0286 & (4.8e) \end{array}\right\} \text{Allowable Region} \end{cases}$$

$-0.0772P_{2\cdot}^{G} + 0.0335P_{1\cdot 1\cdot}^{A-S} + 0.0797P_{2\cdot}^{G}P_{1\cdot 1\cdot}^{A-S} \le 0.0286 \quad (4.8e)$

Figure 52 shows the allowable region of perturbations $P_{2\cdot}^{G}$ and $P_{1\cdot 1\cdot}^{A-S}$ to keep current ranking of technology alternatives unchanged. Total sensitivity coefficient of simultaneous changes to C_1^G and C_{32}^{S-G} is the area of the allowable region. With the inequalities known, this area can be calculated:

TSC (C_2^G & C_{11}^{A-S}) = *Area (Allowable region)* = *Area (S11+S12+S13)*

$$=\int_{-0.25}^{-0.116}\left(\frac{0.0286+0.0772P_{2\cdot}^G}{0.0335+0.0797P_{2\cdot}^G}-\frac{0.0189+0.0049P_{2\cdot}^G}{-0.242-0.575P_{2\cdot}^G}\right)dP_{2\cdot}^G$$

$$+\int_{-0.116}^{-0.0186}\left(0.81-\frac{0.0189+0.0049P_{2\cdot}^G}{-0.242-0.575P_{2\cdot}^G}\right)dP_{2\cdot}^G+\int_{-0.0186}^{0.496}\left(0.81-\frac{0.0003-0.0944P_{2\cdot}^G}{-0.0264-0.0626P_{2\cdot}^G}\right)dP_{2\cdot}^G$$

$$= 0.12 + 0.088 + 0.168 = 0.376 \qquad (4.9)$$

Figure 32 Allowable region of $P_{2\cdot}^G$ and $P_{1\cdot 1\cdot}^{A-S}$

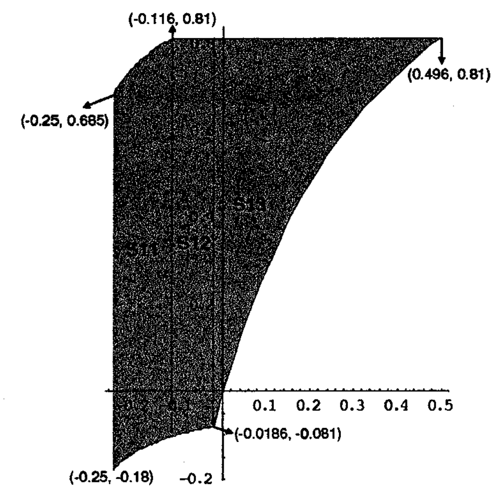
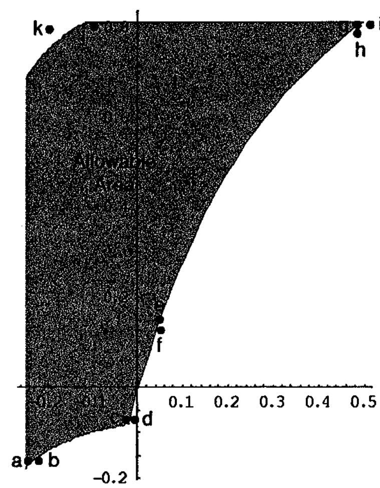


Figure 33 Data points for verification



Therefore, when C_2^G and C_{11}^{A-S} change uniformly within their feasible region, there is a 62.4 percent chance that the current ranking of technology alternatives will be changed.

To verify Theorem 5.3, different pairs of $P_{2\cdot}^G$ and $P_{1\cdot 1\cdot}^{A-S}$ values (represented by points "a" through "k" in Figure 33 and Table 28) within and beyond the allowable region are tried and new C_i^A values are calculated. Table 28 summarizes the verification results.

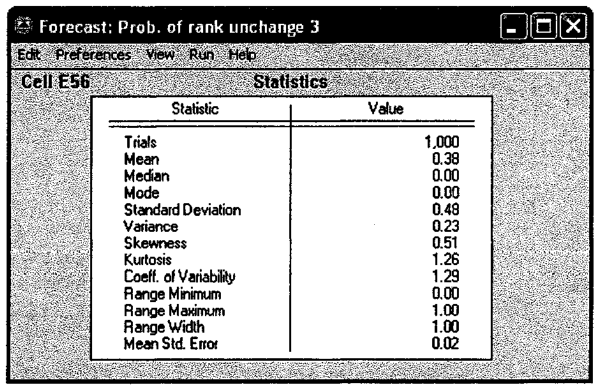
Table 28 Data verification for Theorem 5.3

# Point	Original Rank $(P_{1:}^{G}, P_{1'1:}^{S-G})$ C_i^A	(3) Increasing wafer size	(1) Reducing line width	(5) Hi k	(4) Lo k	(2) Factory integration	Rank changes?
A	(-0.25, -0.17)	0.207	0.230	0.129	0.196	0.229	no
B	(-0.23, -0.17)	0.209	0.228	0.129	0.196	0.229	yes
C	(-0.02, -0.08)	0.2231	0.2234	0.130	0.193	0.2229	no
D	(-0.015, -0.08)	0.2233	0.2231	0.130	0.193	0.2230	yes
E	(0.05, 0.15)	0.209	0.269	0.123	0.183	0.209	no
F	(0.05, 0.14)	0.2103	0.2670	0.1229	0.1830	0.2100	yes
G	(0.49, 0.81)	0.121	0.566	0.075	0.113	0.121	no
H	(0.49, 0.75)	0.130	0.541	0.079	0.118	0.127	yes
I	(0.51, 0.81)	0.1191	0.5734	0.0734	0.1112	0.1185	yes
J	(-0.1, 0.81)	0.166	0.360	0.108	0.166	0.192	no
K	(-0.2, 0.81)	0.174	0.325	0.114	0.175	0.204	yes

As it shows, when the $P_{2:}^{G}$ and $P_{1'1:}^{A-S}$ values fall into the allowable region, rank order of technology alternatives remains to be the same; however, when the values are beyond the boundary of the allowable region, rank order of technology alternatives will be changed. Shaded cells in Table 28 represent the new C_i^A value of technology alternatives whose ranks are changed.

The probability of rank remaining unchanged when C_2^G and C_{11}^{A-S} values change uniformly within their feasible region, calculated as (TSC (C_2^G & C_{11}^{A-S}) = 37.6%) previously, is verified through Monte Carlo simulation. As Figure 34 shows, mean of one thousand simulation trials is 0.38, meaning 38% of the trails output one and the rest 62% of trials output zero. Here the binary output "one" represents rank remains unchanged and "zero" represents rank changes. Based on the same reasoning, since 0.376 is included in the confidence interval of proportion estimate by Monte Carlo simulation at 99.7% confidence level, the probability of rank remain unchanged when C_2^G and C_{11}^{A-S} values vary uniformly between zero and one is verified.

Figure 34 Monte Carlo simulations of rank change when C_2^G & C_{11}^{A-S} vary uniformly



Statistic	Value
Trials	1,000
Mean	0.38
Median	0.00
Mode	0.00
Standard Deviation	0.48
Variance	0.23
Skewness	0.51
Kurtosis	1.26
Coeff. of Variability	1.29
Range Minimum	0.00
Range Maximum	1.00
Range Width	1.00
Mean Std. Error	0.02

4.2.3 Final Steps

Based on results from the hierarchical technology assessment model and its sensitivity analysis, a general conclusion can be reached: "reducing line width" is the top choice to be adopted and should be allocated the most resources; "factory integration" and "increasing wafer size" rank second and third respectively and are very close in their overall contributions to the business success. SA results reveal that "increasing wafer size" gets more chances to become the top choice than "factory integration" does when the local contributions vary; however, its rank is relatively unstable and can drop from the first to the fourth in different cases. Companies with a risk-aversion attitude should consider developing "factory integration" before "increasing wafer size." "Hi k dielectrics" and "Lo k dielectrics" are dominated by other technologies in most cases. Unless their performances can be improved dramatically, resulting in improved contributions to the technology strategies to a certain degree, they will remain as the last choices and should be allocated the least resources.

"Product leadership" and "innovation" are the most critical competitive goals and the most critical strategies to keep "reducing line width" as the top choice. Changes to their relative importance have the most impact on technology choices. This provides the foundries with insights into shaping favorable contingencies and directions in which they should push industry environment changes.

When "product leadership" is emphasized and/or "cost leadership" is deemphasized, "increasing wafer size" will move up to be the top-ranked technology, with "reducing line width" and "factory integration" ranking second and third. When the relative importance of "product leadership" increases more than 21%, no matter how the "market leadership" changes, "increasing wafer size" will remain as the top technology choice. However, if the relative importance of competitive goals is shifted only to "cost leadership" to a certain degree, "increasing wafer size" will drop from its current third rank to the fourth rank. "Reducing line width" always dominates "factory integration," "Lo k dielectrics" and "Hi k dielectrics" regardless of how the relative importance of competitive goals changes.

SA at the technology strategies level indicates that "factory integration" will be the top-ranked technology when the relative impacts of "efficiency" on the competitive goals are increased, and "increasing wafer size" will drop to the fourth choice when "flexibility" is deemphasized. "Reducing line width" is insensitive to variations in the impact of "diversity." It is also insensitive to changes to "flexibility" and "imitation" except that decreasing the relative impact of "flexibility" to "cost leadership" or increasing the relative impact of "imitation" to "cost leadership" will reverse the rank order of "increasing wafer size" and "reducing line width." Emphasizing the "innovation" strategy to a certain degree will also make "increasing wafer size" the top choice and "reducing line width" the second. Although the experts had great disagreement evaluating the relative impacts of "flexibility" on "market leadership" and "imitation" on "market leadership," SA results show that they do not affect the top choice, "reducing line width," only the rank order of "factory integration" and "increase wafer size," which are very close anyway.

Factors at every level of the decision hierarchy all influence the technology choices; however, the direction and speed of technological advancement are more critical in determining the optimal technology portfolio than industry policy changes and company strategy shifts. Judging from current trends of technology developments, "reducing line width" and "increasing wafer size" are the two technologies that advance faster. Therefore, in the most likely scenario, the ranking of all the technologies is "reducing line width" (1), "increasing wafer size" (2), "factory integration" (3), "Lo k dielectrics" (4), and "Hi k dielectrics" (5). Resources should be allocated to the top three technologies, with "reducing line width" getting the most. Unless there is dramatic improvement on the "Lo k dielectrics" and "Hi k dielectrics" technologies, these two technologies should be the lowest priority for investments.

If a company is only able to develop or acquire technologies that rank lower than others, or if it wants to justify its previous investments in those technologies, it needs to shift its emphasis among competitive goals, alter its strategies, or push for technological advancement of such technologies based on the HDM SA scenarios to take full advantages of its investment. It is the same for the development of the technologies: as indicated in the general framework, the purpose of periodical reevaluation of the business and technology environment is to redirect company development in a timely fashion for better utilization of its technology investments. The company will also identify whether dramatic changes are needed to redo the whole planning process or if just some strategic modifications based on some HDM SA scenarios will be adequate.

In addition, new technology options frequently become available in the semiconductor foundry industry. These technologies are either planned in the SEMATECH international roadmap or emerge from outside the industry. Emergence of critical new technologies should be incorporated along the way and be evaluated together with the original technology alternatives.

Applying Corollary 4.4 in the HDM SA algorithm, the condition that needs to be satisfied in order for the current top choice, "reducing line width," to remain at its rank is:

$$0.97C_{61}^{A-S} + 0.75C_{62}^{A-S} + 0.97C_{63}^{A-S} + 1.33C_{64}^{A-S} + 1.17C_{65}^{A-S} \leq 1 \tag{4.10}$$

where C_{6j}^{A-S} (j = 1, 2…5) are the contributions of the new technology alternative, A6, to the five technology-strategies. The conditions for the current second- and third-ranked technologies to remain unchanged are:

$$1.06C_{61}^{A-S} + 0.77C_{62}^{A-S} + 0.94C_{63}^{A-S} + 1.52C_{64}^{A-S} + 1.25C_{65}^{A-S} \leq 1 \tag{4.11}$$

$$1.16C_{61}^{A-S} + 0.84C_{62}^{A-S} + C_{63}^{A-S} + 1.46C_{64}^{A-S} + 1.07C_{65}^{A-S} \leq 1 \quad (4.12)$$

From inequalities (4.10)-(4.12), we can see that C_{64}^{A-S}, the new technology's contribution to efficiency, is a relatively more critical value in keeping the current top-, second- and third-ranked technologies unchanged. For example, if the new technology performs on an average value, and thus contributes to all the strategies no more than one sixth, the current top three choices will remain the same. However, it is the combination of all C_{6j}^{A-S} (j=1, 2…5) values that determines whether the new technology should be adopted. With the assistance of HDM SA, the stability of current technology choices can be roughly assessed before determining whether experts' judgments are needed for new pairwise comparisons at the technologies level. For example, System on a Chip (SoC), which is not evaluated in the original model, may be something the semiconductor foundry industry wants to look at as an additional technology alternative in the planning process.

4.3 CONCLUSION

In this chapter, HDM SA algorithm is applied in order to propose a strategic technology planning framework by linking synoptic and adaptive planning modes, which improves the comprehensiveness of the initial technology assessment by HDM, and helps to forecast future changes and possible solutions in different technology scenarios. The planning framework is built upon a previous Ph.D. dissertation by Ho [52] in which emerging technologies in Taiwan's semiconductor foundry industry were evaluated according to their overall contributions to business success through alignment of competitive goals and technology strategies. Theorem 2 through 5 and their corollaries in HDM SA algorithm were verified using data from Ho's model. The application of the proposed strategic technology planning framework as well as the HDM SA algorithm itself were illustrated in detail through the case study on Taiwan's semiconductor foundry industry. Contributions of HDM SA in technology planning were comprehensively demonstrated.

5. APPLICATION 2—APPLYING HDM SA TO ENERGY PORTFOLIO FORECASTING

5.1 INTRODUCTION

Energy resources are increasingly critical issues facing the world. Global warming due to the excessive carbon dioxide emission, and vulnerability to hostile regimes and terrorists resulting from of America's heavy dependence on oil [115], calls for the U.S. to adjust its energy consumption portfolio. As a result, numerous research programs have been launched in different areas to solve the energy problem in every possible way.

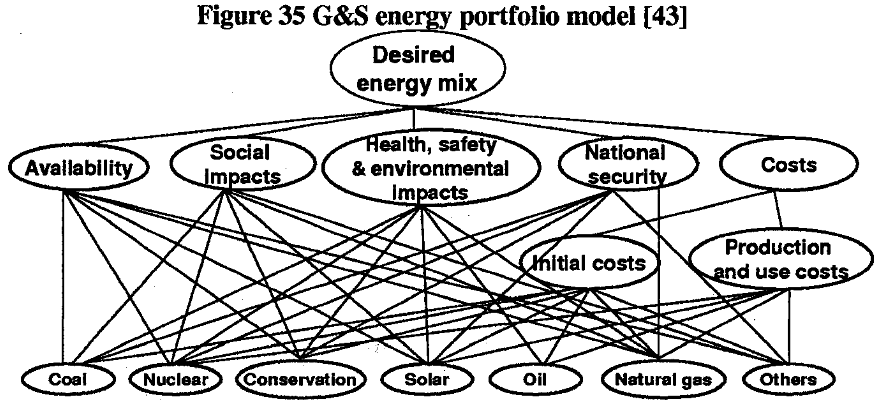
HDM and its sensitivity analysis can be applied to help establish a desired portfolio of energy resources. In fact, energy issues, though not as serious as they are today, have been discussed and modeled in 1980's by academic researchers using AHP concepts in order to suggest a desired energy portfolio for the U.S [43][44][99]. By performing HDM SA on a previously constructed model [43], new insights are provided in decision making techniques in the energy industry. The application of HDM SA on this energy model also verifies the proposed algorithm and demonstrates its usefulness in different fields.

5.2 CASE STUDY: ENERGY PORTFOLIO FORECASTING FOR US

5.2.1 Gholamnezhad and Saaty's Model

In the 1980's, Gholamnezhad and Saaty (G&S) utilized AHP to forecast a desired energy mix for the United States in year 2000 [43]. Their model is depicted in Figure 35.

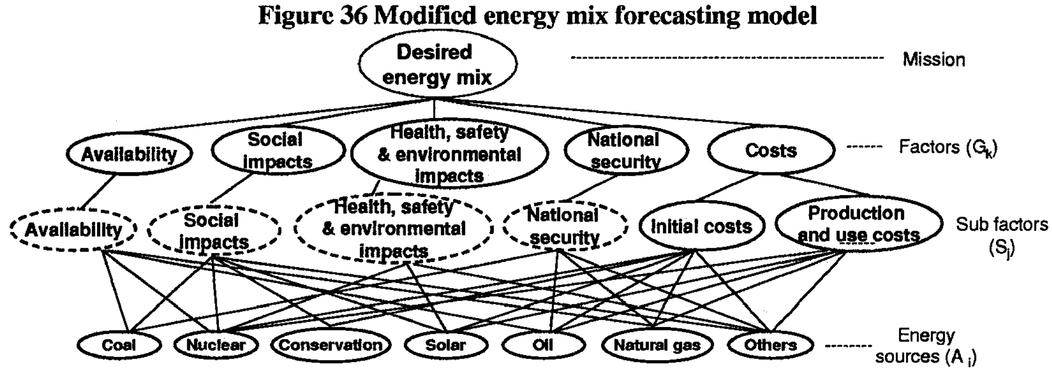
Figure 35 G&S energy portfolio model [43]



The overall mission of this decision problem is to propose a desired energy mix for the U.S. in year 2000. At the second level of the decision hierarchy are criteria that influence the future demand and supply of energy. They are used to evaluate different energy sources. The third level of the decision hierarchy consists of two sub-criteria that contribute only to the fifth criterion, which is "costs." This is said to be an "incomplete" hierarchy since some elements in a given level (the second level) do not function as criteria for all the elements in the level below [98]. At the bottom level of the decision hierarchy, decision alternatives are energy sources including coal, nuclear, solar, oil, natural gas, and other resources. They are pair-wise compared regarding their contributions to the first four criteria and the two sub-criteria. Among the decision alternatives, "conservation" is not really an energy resource. It refers to the reduction in demand for energy through technological and procedural changes without affecting productivity [43].

In order to fit the model in the standard MOGSA structure and make it a complete hierarchy, the first four criteria are replicated in the third level as sub-criteria with 100% contribution to itself and zero contribution to other criteria. Figure 36 shows the new model.

Figure 36 Modified energy mix forecasting model



The finalized elements in each level are summarized below.

Level I: Mission—A desired energy mix for the U.S. in 2000

The overall mission is to forecast a desired energy mix for the U.S. to make a smooth transition to more abundant energy sources in year 2000

Level II: Evaluating Criteria (G_k, k = 1, 2... 5)

5) Availability (G_1): Availability of energy resources and the materials needed for their production and utilization. (For energy conservation, this is the potential for increased efficiency and reduced consumption of energy)

6) Social Impacts (G_2): Influence on American lifestyles, standard of living, and employment rate.

7) Health, Safety, and Environmental Impacts (G_3): Impacts on air, water, land and the inhabitants, including accidents due to the utilization of energy.

8) National Security (G_4): Vulnerability to energy-supply interruptions; nuclear proliferation, and the maintenance of world peace.

9) Costs (G_5): Expenditures involved in the provision of energy.

Level III: Sub Evaluating Criteria (S_j, j = 1, 2... 6)

Sub criteria one through four are the same as the criteria one through four at level II since they are replicated to this level. Sub criterion "Initial Cost" (S_5) measures the expenditures of initial exploration and early development of energy resources, and sub criterion "Cost of Production and Use" (S_6) corresponds to the expenditure of production, distribution and utilization.

Level IV: Energy Sources (A_i, i = 1, 2... 7)

6) A_1 Coal

7) A_2 Nuclear Power: using uranium to generate electric power

8) A_3 Oil

9) A_4 Natural Gas

10) A_5 Solar Energy

11) A_6 Conservation: reduction in demand for energy through technological and procedural changes without affecting productivity.

12) A_7 Other sources: geothermal, oil shale, and tar sands.

For a four-level decision hierarchy, one vector and two matrices of local contributions between successive levels were calculated from the pair-wise comparison results of Gholamnezhad and Saaty.

Vector $\mathbf{C_k^G}$: Weights of evaluating criteria (G_k) in forecasting the desired energy mix for US in 2000, as shown in Table 29.

Table 29 Relative weights of evaluating criteria [43]

C_k^G	Availability	Social impacts	Environmental impacts	National security	Cost
Mission	0.362	0.039	0.076	0.161	0.362

Matrix $\mathbf{C_{jk}^{S-G}}$: Contributions of sub-criteria (S_j) to evaluating criteria (G_k), as shown in Table 30.

Table 30 Contributions of technology strategies to competitive goals [52]

C_{jk}^{S-G}	Availability	Social impacts	Environmental impacts	National security	Cost
Availability	1				
Social impacts		1			
Environmental impacts			1		
National security				1	
Initial cost					0.60
Production and use cost					0.40

Matrix $\mathbf{C_{ij}^{A-S}}$: Contributions of energy sources (A_i) to sub-criteria (S_j), as shown in Table 31.

Table 31 Contributions of energy sources to sub-criteria [43]

C_{ij}^{A-S}	Availability	Social impacts	Environmental impacts	National security	Initial cost	Production and use cost
Coal	0.297	0.03	0.031	0.249	0.421	0.04
Nuclear	0.054	0.019	0.019	0.066	0.049	0.199
Conservation	0.025	0.069	0.252	0.43	0.11	0.526
Solar	0.458	0.44	0.381	0.141	0.028	0.02
Oil	0.032	0.192	0.096	0.02	0.116	0.074
Natural Gas	0.045	0.192	0.17	0.027	0.233	0.11
Others	0.09	0.058	0.051	0.066	0.044	0.031

Table 32 Priority and percentage of energy sources proposed by G&S model [43]

	Coal	Solar	Conservation	Natural Gas	Nuclear	Oil	Others
C_i^A	0.248	0.244	0.2	0.108	0.072	0.065	0.063
Priority	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**	**(7)**
Percentage	31%	30.5%	--	13.5%	9%	8.1%	7.9%

The percentages were calculated by taking the ratio scale of each energy source in the energy mix except conservation in the overall contribution vector $\mathbf{C_i^A}$. For example, the desired percentage of coal in the energy mix is (0.248/(1-0.2)=31%), and the desired percentage of nuclear is (0.072/(1-0.2)=9%). The results predicted a heavy dependence on coal and solar energy in year 2000. Energy sources such as natural gas, nuclear, oil and others only account for 8% to 13.5% each. Conservation was given the third priority after coal and solar energy.

5.2.2 Sensitivity Analysis and Scenario Forecast

Based on the forecasting results from the previous section, sensitivity analysis is performed to study the influences on the desired energy mix when 1) changes to the social, economic and natural environment inside and outside of the U.S. cause the country to shift its emphasis among the evaluating criteria for energy sources; 2) advancement in technology and newly identified energy resources cause the contributions of energy sources to the evaluating criteria to vary.

Specific questions being answered by performing HDM SA in this section include: 1) What are the critical decision elements in keeping the priorities of energy sources determined by the G&S model valid? 2) How do variations of the evaluation criteria's weights and the energy sources' contributions to the criteria impact the proposed percentage of energy sources in the desired energy mix? 3) How would the contributions of specific decision elements need to change in order for the G&S model to replicate the actual energy consumption in year 2000? 4) What is the probability of rank being changed when certain local contribution values vary uniformly within the feasible region?

5.2.2.1 Evaluation Criteria Analysis

In G&S forecasting model, five criteria have been pair-wise compared according to their relative importance to evaluating different energy sources. From the pair-wise comparison result, local contributions of these criteria to the overall mission, denoted as C_k^G (k = 1…5), are derived. As shown in Table 29, "availability" and "cost" are identified as the most important criteria with equal weights in evaluating the energy sources, each accounting for 36.2 percent importance among the five criteria. "National security," "health, safety and environmental impacts" and "social impacts" received 16.1, 7.6 and 3.9 percent respectively as their relative importance in the evaluation. Suppose changes to the social and economic environment demand that a country shifts its emphasis among the evaluating criteria; the country needs to know whether its originally identified priority for energy source development will remain valid. It is also desired to know how the percentage assigned to each energy source changes according to the environmental changes. Thus, HDM SA is performed to answer these questions at the evaluation criteria's level.

5.2.2.1.1 One-way SA

How variations of C_k^G values impact the rank order of all technology alternatives is first analyzed by a one-way SA, which determines the influence of changes to a single input by varying that input within its feasible range while keeping other inputs fixed at their base values [5, 25]. Corollary 2.1 in the HDM SA algorithm that deals with one-way SA for changes in the top-level contribution vector is applied here.

To use Corollary 2.1, local contribution matrices between evaluation criteria level and energy resources level are integrated and calculated, as summarized in Table 33. This corresponds to global contribution matrix of $\mathbf{C}_{ik}^{A-G}$ in HDM SA algorithm.

Table 33 Global contributions of energies to evaluating criteria

C_{ik}^{A-G}	Availability	Social impacts	Environmental impacts	National security	Cost
Coal	0.297	0.03	0.031	0.249	0.268
Solar	0.458	0.44	0.381	0.141	0.025
Conservation	0.025	0.069	0.252	0.43	0.277
Natural Gas	0.045	0.192	0.17	0.027	0.184
Nuclear	0.054	0.019	0.019	0.066	0.109
Oil	0.032	0.192	0.096	0.02	0.099
Others	0.09	0.058	0.051	0.066	0.039

Based on Corollary 2.1 and Proposition 1.1, different sensitivity indicators for the evaluation criteria are calculated and summarized in Table 34. As it shows, both comparisons of TSCs and OPSCs reveals that "social impacts" is the most critical evaluation criterion in keeping the original suggestion valid, since it has the shortest tolerance and smallest allowable change threshold. "Environmental impact" turns out to be the second critical criterion to keep current ranking of energy sources unchanged. "Availability" and "cost," the two most important criteria in the evaluation model, are the least critical judging by TSC, since they have the longest tolerance to keep current ranking. However, the model result is sensitive to decreases to the weight of "cost": as soon as the weight of cost decreases more than 0.013, the current ranking of energy sources will be changed.

When the weight of each criterion G_1 through G_5 changes uniformly between zero and one, rank order of the energy sources will be changed with a probability of 91.1%, 97.4%, 94.3%, 92.6% and 89.4% respectively, since length of the tolerance of G_1 through G_5 is 0.089, 0.026, 0.057, 0.074 and 0.106, as shown in Table 34.

Table 34 HDM SA at evaluation-criteria level to preserve the ranking of all A_i

	Availability C_1^G	Social impacts C_2^G	Environmental impacts C_3^G	National security C_4^G	Cost C_5^G
Base values	0.362	0.039	0.076	0.161	0.362
Allowable ranges of perturbations	[-0.071, 0.018]	[-0.015, 0.011]	[-0.045, 0.012]	[-0.038, 0.036]	[-0.013, 0.093]
Tolerance (C_k^G)	[0.291, 0.38]	[0.024, 0.05]	[0.031, 0.088]	[0.123, 0.197]	[0.349, 0.455]
Prob. of rank changes	91.1%	97.4%	94.3%	92.6%	89.4%
OPSC (G_k)	0.018	0.011	0.012	0.037	0.013
TSC (G_k)	0.089	0.026	0.057	0.074	0.106

To verify the allowable range of perturbations induced on C_k^G (k = 1, 2 ... 5) calculated in Table 34, values within and beyond those ranges are tried on P_k^G and new C_i^A (i = 1, 2 ... 7) values are calculated in each case to see whether rank order of energy sources does change. Table 35 shows the verification results: each time when P_k^G value goes beyond its allowable range, the ranking of A_i will be changed. Shaded cells in Table 35 represent the new C_i^A value of the A_i's whose rank order is reversed.

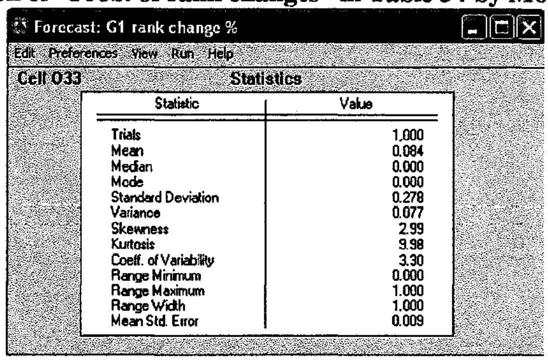
Probability of rank changes when each C_k^G value varies uniformly between zero and one is also verified by Monte Carlo simulation. Five pairs of inputs and outputs are defined for the simulation: input C_k^G follows a uniform distribution U(0,1), and the corresponding output is binary with "0" representing rank changes and "1" representing no rank changes.

Table 35 Data verification of "allowable range of perturbations" in Table 34

Original Rank	(1)	(2)	(3)	(4)	(5)	(6)	(7)	
P_{k*}^{G} \ C_i^A	Coal	Solar	Conser-vation	Natural gas	Nuclear	Oil	Others	Rank changes?
$P_{1*}^{G}=-0.072$	0.243	0.219	0.220	0.115	0.074	0.069	0.060	Yes
$P_{1*}^{G}=-0.07$	0.243	0.220	0.219	0.114	0.074	0.069	0.060	No
$P_{1*}^{G}=0.018$	0.2497	0.2496	0.195	0.106	0.071	0.065	0.064	No
$P_{1*}^{G}=0.019$	0.2497	0.250	0.195	0.106	0.071	0.064	0.064	Yes
$P_{2*}^{G}=-0.016$	0.252	0.240	0.202	0.106	0.073	0.0634	0.0635	Yes
$P_{2*}^{G}=-0.014$	0.251	0.241	0.202	0.106	0.073	0.0636	0.0635	No
$P_{2*}^{G}=0.01$	0.2460	0.2456	0.199	0.108	0.071	0.067	0.063	No
$P_{2*}^{G}=0.012$	0.2456	0.2460	0.199	0.109	0.071	0.067	0.063	Yes
$P_{3*}^{G}=-0.047$	0.259	0.237	0.198	0.104	0.075	0.0639	0.0640	Yes
$P_{3*}^{G}=-0.045$	0.259	0.237	0.198	0.105	0.074	0.064	0.064	No
$P_{3*}^{G}=0.012$	0.245	0.245	0.201	0.108	0.071	0.066	0.063	No
$P_{3*}^{G}=0.013$	0.245	0.246	0.201	0.108	0.071	0.066	0.063	Yes
$P_{4*}^{G}=-0.04$	0.2483	0.2485	0.189	0.111	0.072	0.068	0.063	Yes
$P_{4*}^{G}=-0.038$	0.248	0.248	0.190	0.111	0.072	0.068	0.063	No
$P_{4*}^{G}=0.036$	0.248	0.239	0.210	0.104	0.072	0.064	0.063	No
$P_{4*}^{G}=0.037$	0.248	0.239	0.211	0.104	0.072	0.063	0.064	Yes
$P_{5*}^{G}=-0.013$	0.2479	0.2480	0.199	0.106	0.071	0.065	0.064	Yes
$P_{5*}^{G}=-0.012$	0.2479	0.2477	0.199	0.106	0.071	0.065	0.064	No
$P_{5*}^{G}=0.093$	0.251	0.212	0.211	0.119	0.077	0.070	0.060	No
$P_{5*}^{G}=0.094$	0.251	0.211	0.212	0.119	0.077	0.070	0.060	Yes

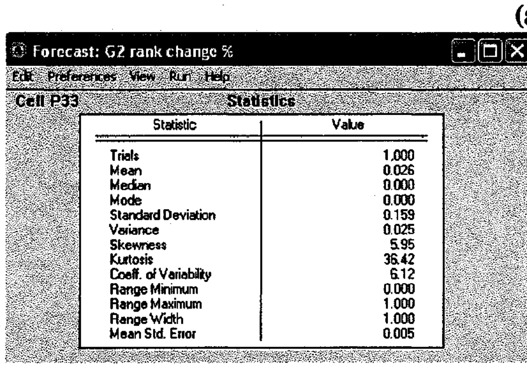
As shown in Figure 37 (a) through (e), the mean of output in each scenario is 0.084, 0.026, 0.055, 0.073, and 0.106 respectively. Since the output can only be zero or one, these values indicate that the proportion of output being one in each case is 8.4%, 2.6%, 5.5%, 7.3% and 10.6%, and the proportion of output being zero is 91.6%, 97.4%, 94.5%, 92.7% and 89.4% respectively. In statistical theory, the proportions resulting from the simulation correspond to sample proportion $\hat{p}$, or point estimate for population proportion p. To verify the calculation of "probability of rank changes" with HDM SA algorithm, confidence interval for the proportion estimate of output being zero should be calculated. Then, as long as the probabilities calculated based on HDM SA algorithm and shown in Table 34 fall in the confidence intervals, the verification purpose will be met.

Figure 37 Verification of "Prob. of rank changes" in Table 34 by Monte Carlo simulation


Forecast: G1 rank change %
Edit Preferences View Run Help
Cell O33 Statistics

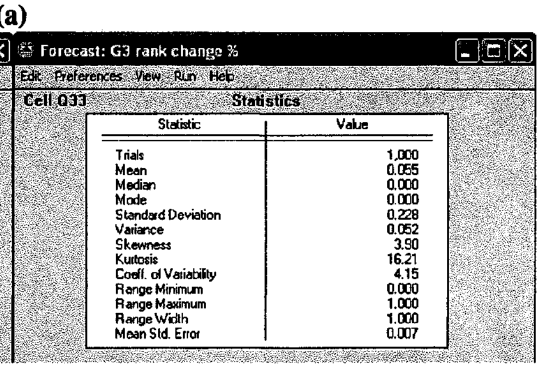
Statistic	Value
Trials	1,000
Mean	0.084
Median	0.000
Mode	0.000
Standard Deviation	0.278
Variance	0.077
Skewness	2.99
Kurtosis	9.98
Coeff. of Variability	3.30
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.009

(a)


Forecast: G2 rank change %
Edit Preferences View Run Help
Cell P33 Statistics

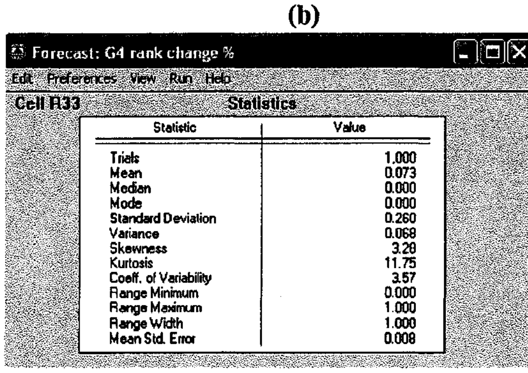
Statistic	Value
Trials	1,000
Mean	0.026
Median	0.000
Mode	0.000
Standard Deviation	0.159
Variance	0.025
Skewness	5.95
Kurtosis	36.42
Coeff. of Variability	6.12
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.005

(b)


Forecast: G3 rank change %
Edit Preferences View Run Help
Cell Q33 Statistics

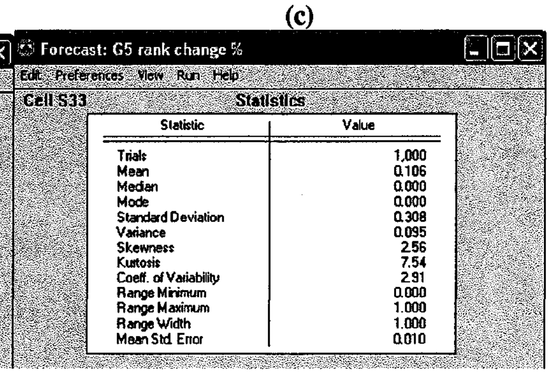
Statistic	Value
Trials	1,000
Mean	0.055
Median	0.000
Mode	0.000
Standard Deviation	0.228
Variance	0.052
Skewness	3.90
Kurtosis	16.21
Coeff. of Variability	4.15
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.007

(c)


Forecast: G4 rank change %
Edit Preferences View Run Help
Cell R33 Statistics

Statistic	Value
Trials	1,000
Mean	0.073
Median	0.000
Mode	0.000
Standard Deviation	0.260
Variance	0.068
Skewness	3.28
Kurtosis	11.75
Coeff. of Variability	3.57
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.008

(d)


Forecast: G5 rank change %
Edit Preferences View Run Help
Cell S33 Statistics

Statistic	Value
Trials	1,000
Mean	0.106
Median	0.000
Mode	0.000
Standard Deviation	0.308
Variance	0.095
Skewness	2.56
Kurtosis	7.54
Coeff. of Variability	2.91
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.010

(e)

Since the sample size, 1000 trials, is sufficiently large, we can assume the sampling distribution of proportion $\hat{p}$ is approximately normal with mean p and standard deviation $\sqrt{p(1-p)/n}$. This assumption is verified later by testing whether

$nP_L > 5$, $n(1-P_L) > 5$, $nP_U > 5$, and $n(1-P_U) > 5$, where P_L and P_U are the lower and upper limits of the confidence interval. The lower and upper limits, P_L and P_U, are calculated as $\hat{p} \pm Z-multiple \times \sqrt{\hat{p}(1-\hat{p})/n}$, where $\hat{p}$ is the proportion calculated in the simulation and a point estimate of p, the true "probability of rank changes."

At 99.7% confidence level, Z-multiple value is three. Confidence intervals for $\hat{p}$ in Figure 37 (a) to (e) are calculated as:

$$0.916 \pm 3 \times \sqrt{0.916(1-0.916)/1000} = 0.916 \pm 3 \times 0.0088 = [0.89, 0.942] \quad (5.1a)$$

$$0.974 \pm 3 \times \sqrt{0.974(1-0.974)/1000} = 0.974 \pm 3 \times 0.005 = [0.959, 0.989] \quad (5.1b)$$

$$0.945 \pm 3 \times \sqrt{0.945(1-0.945)/1000} = 0.945 \pm 3 \times 0.007 = [0.924, 0.966] \quad (5.1c)$$

$$0.927 \pm 3 \times \sqrt{0.927(1-0.927)/1000} = 0.927 \pm 3 \times 0.008 = [0.903, 0.951] \quad (5.1d)$$

$$0.894 \pm 3 \times \sqrt{0.894(1-0.894)/1000} = 0.894 \pm 3 \times 0.01 = [0.864, 0.924] \quad (5.1e)$$

Since $91.1\% \in [0.89, 0.942]$, $97.4\% \in [0.959, 0.989]$, $94.3\% \in [0.924, 0.966]$, $92.6\% \in [0.903, 0.951]$, and $89.4\% \in [0.864, 0.924]$, the simulations successfully verified the probabilities of rank changes calculated in Table 34 as each C_k^G value varies uniformly between zero and one.

A close look at how changes to the relative weights of evaluation criteria impact the percentage of each energy source is taken by applying Theorem 1.1 in HDM SA. Impacts of one unit's increase in each C_k^G value on the C_i^A values are measured, verified and summarized in Table 36

Table 36 Changes to C_i^A values when C_k^G value increases by 10%

C_i^A \ C_k^G	Availability	Social impacts	Environmental impacts	National security	Cost
Coal	0.8%	-2.3%	-2.4%	0.0%	0.3%
Solar	3.4%	2.0%	1.5%	-1.2%	-3.4%
Conservation	-2.7%	-1.4%	0.6%	2.7%	1.2%
Natural Gas	-1.0%	0.9%	0.7%	-1.0%	1.2%
Nuclear	-0.3%	-0.5%	-0.6%	-0.1%	0.6%
Oil	-0.5%	1.3%	0.3%	-0.5%	0.5%
Others	0.4%	-0.1%	-0.1%	0.0%	-0.4%

As it shows in Table 36, "social impacts" and "environmental impacts," the first and second critical criteria have great impact on the overall contribution of "coal"—ten percent's increases in the weights of these two criteria will result in 2.3 percent and 2.4 percent decrease in coal's overall contribution. This means that the rank of "coal" is sensitive to increases of the weights of "social impacts" and "environmental impacts." "Solar" is another energy source that is very sensitive to weight changes among the criteria. Specifically, ten percent's increase to the weight of "availability" will increase solar's overall contribution by 3.4 percent, however, a ten percent increase in the weight of "cost" will result in 3.4 percent decrease of solar' s overall contribution.

The other information revealed by Table 36 is whether the relative contribution of a specific energy source to each evaluation criterion is positive or negative as compared to other energy sources. Take the last column "cost" in Table 36 for example, the percentage changes of "solar" and "others" are negative. This reveals that developing "solar" and "others" takes relatively higher cost; thus, when "cost," as an evaluation criterion, receives more importance in the decision process, the desirability of "solar" and "others" will be decreased. In this situation, the desirability of “coal,” “oil,” and “nuclear” will increase, but not to the degree that the desirability of “conservation” and “natural gas” increases. This information is different from what contribution matrix $\mathbf{C}_{ik}^{A-G}$ tells us. The same analysis can be applied to look at other columns in Table 36 for different energy sources’ relative contributions to each evaluation criterion.

Among the five criteria, “cost” is the criteria that people care about most in reality. More detailed analysis was conducted to generate rank scenarios when the relative importance of “cost” increases from the base value to one, as shown in Table 37.

Table 37 Ranking scenarios as C_5^G increases from 0.362

C_5^G threshold	Coal	Solar	Conservation	Natural Gas	Nuclear	Oil	Others
0.362	(1)	(2)	(3)	(4)	(5)	(6)	(7)
0.456	(1)	(3)	(2)	(4)	(5)	(6)	(7)
0.656	(1)	(4)	(2)	(3)	(5)	(6)	(7)
0.79	(1)	(5)	(2)	(3)	(4)	(6)	(7)
0.812	(1)	(6)	(2)	(3)	(4)	(5)	(7)
0.907	(2)	(6)	(1)	(3)	(4)	(5)	(7)
0.954	(2)	(7)	(1)	(3)	(4)	(5)	(6)

As Table 37 shows, when the relative importance of cost increases from its base value (0.362) to 1, the rank of solar will drop from the second to the seventh. This may be a good reason to explain why solar’s actual consumption in year 2000 was ranked as the last together with other energy sources instead of as the second [116]: in reality, “cost” is usually the only thing that people care about.

5.2.2.1.2 Two-way SA

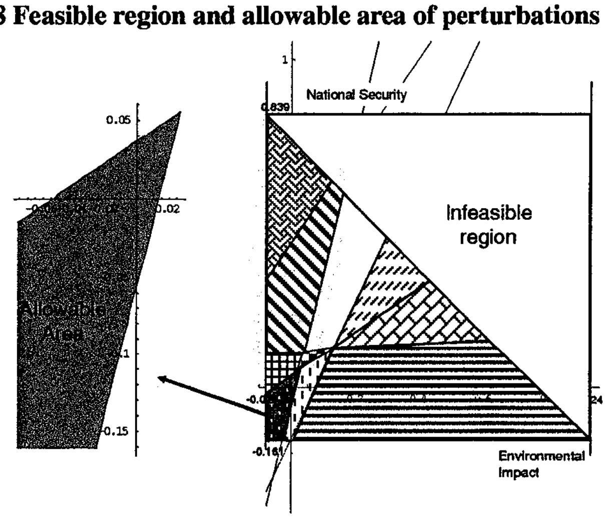
Environmental impacts are receiving more attention currently than twenty years ago when the G&S model was developed. The other evaluation criterion whose importance has been increasing is national security.

Therefore, a two-way sensitivity analysis is conducted to test the model's robustness to simultaneous variations of the weights of "environmental impact" and "national security", namely C_3^G and C_4^G. Applying Corollary 2.2 in HDM SA algorithm, the allowable area for the two perturbations induced on C_3^G and C_4^G is defined by the following inequalities. Only the inequalities that bound the allowable region are listed here.

$$\begin{cases} -0.076 \le P_{3^*}^G \le 0.924 & (5.2a) \\ -0.161 \le P_{4^*}^G \le 0.839 & (5.2b) \\ -0.237 \le P_{3^*}^G + P_{4^*}^G \le 0.763 & (5.2c) \\ 0.368P_{3^*}^G - 0.09P_{4^*}^G \le 0.005 & (5.2d) \\ -0.037P_{3^*}^G + 0.054P_{4^*}^G \le 0.002 & (5.2e) \end{cases}$$

(5.2a)–(5.2c): Feasible Region; (5.2d)–(5.2e): Allowable Region

Figure 38 Feasible region and allowable area of perturbations $P_{3^*}^G$ and $P_{4^*}^G$



As shown in Figure 38, the allowable area of the two perturbations to keep current ranking unchanged is a small part at the lower left corner of the feasible region. This means that only when the weights of "environmental impact" and "national security" decreases will the current ranking remain unchanged; otherwise, increases in one or both weights will result in rank changes among the energy sources. In Figure 38, the feasible region is separated into eleven different parts by five lines, including the two defined by (5.2d) and (5.2e), and three others that are not listed and do not bound the allowable region, that intersect the feasible region. The eleven parts within the feasible region represent eleven different rank scenarios. This also reveals that the original rankings of energy sources are relatively unstable: Each energy source has different chances being at different ranks.

Based on Proposition 1.1, total sensitivity coefficient of C_3^G and C_4^G is calculated as:

$$\text{TSC}\ (C_3^G\ \&\ C_4^G) = \text{Area (Allowable Area)} = \int_{-0.076}^{-0.0266} (0.039 + 0.686P_3^G + 0.161)dP_3^G$$

$$+\int_{-0.0266}^{0.0267} (0.039 + 0.686P_3^G - 4.1P_3^G + 0.052)dP_3^G = 0.0081 + 0.0048 = 0.013 \qquad (5.3)$$

Probability of rank changes = 1 – Area (Allowable Area) / Area (Feasible Region)

= 1 – 0.013/0.5 = 1 – 0.026 = 0.974 (5.4)

Therefore, when C_3^G and C_4^G vary within their feasible region based on uniform distribution, the probability of rank changes is 97.4%.

5.2.2.2 Energy Resources Analysis

As new energy reserves are being discovered, technologies to develop and utilize different energy sources are being advanced, methods to better manage energy consumption waste are being created, matrix $\mathbf{C}_{ij}^{A-S}$, which contains the contributions of different energy sources to the evaluation criteria, will be changed. In this section, Theorem 4 and its corollaries in HDM SA will be applied to analyze impact of change at this level to the decisions.

<u>5.2.2.2.1 One-way SA</u>

The most obvious changes that can be easily evaluated are the contributions of different energy sources to "availability" since they measure the abundance of each energy sources. Therefore, first analyzed is sensitivity of ranking to variations of C_{i1}^{A-S}. Besides "availability", cost is another criterion to which the energy sources' contributions are likely to vary as energy development and utilization technologies are being advanced. To make the analysis simpler, initial costs and costs of production and use are grouped together. As a result, the HDM SA on energy sources' contribution to costs is performed on C_{i5}^{A-G} instead of on C_{i5}^{A-S} and C_{i6}^{A-S} separately. The third criterion to which the contributions of energy sources are analyzed is "environmental impact." The generation and consumption of different energy sources, especially coal, oil, and nuclear, have negative impacts on the health and safety of our environment. As people try to solve the problems, contributions of different energy sources to environmental impact may be increased to different degrees.

Applying Corollary 4.1, a one-way SA is first performed to analyze impact of model results to varying contributions to the three criteria discussed above. Table 38 summarizes the analysis results, where δ_{ij-}^{A-S} and δ_{ij+}^{A-S} are the lower and upper limits of P_{ij}^{A-S}'s allowable range. (x, y) is the pair of technologies whose current rank order will be reversed if the P_{ij}^{A-S} value goes beyond δ_{ij-}^{A-S} and/or δ_{ij+}^{A-S}.

Table 38 Allowable range of P_{ij}^{A-S} to preserve the ranking of all A_i

	Availability		Environmental impact		Cost	
	δ_{i1-}^{A-S}	δ_{i1+}^{A-S}	δ_{i3-}^{A-S}	δ_{i3+}^{A-S}	δ_{i5-}^{A-S}	δ_{i5+}^{A-S}
Coal i = 1	-0.008 (1, 2)	0.195 (2, 3)	-0.045 (1, 2)	0.597 (6, 7)	-0.0128 (1, 2)	0.07 (6, 7)
Solar i = 2	-0.0545 (6, 7)	0.0085 (1, 2)	-0.381	0.0598 (1, 2)	-0.025	0.01 (1, 2)
Conservation i = 3	-0.0799 (1, 2)	0.0816 (2, 3)	-0.1342 (1, 2)	0.3784 (2, 3)	-0.204 (3, 4)	0.039 (1, 2)
Natural gas i = 4	-0.078 (1, 2)	0.249 (3, 4)	-0.1489 (1, 2)	0.512 (6, 7)	-0.087 (4, 5)	0.044 (1, 2)
Nuclear i = 5	-0.017 (5, 6)	0.094 (4, 5)	-0.0757 (5, 6)	0.4 (4, 5)	-0.0157 (5, 6)	0.048 (1, 2)
Oil i = 6	-0.005 (6, 7)	0.0165 (5, 6)	-0.0263 (6, 7)	0.0814 (5, 6)	-0.0056 (6, 7)	0.0156 (5, 6)
Others i = 7	-0.0746 (1, 2)	0.0056 (6, 7)	-0.051	0.025 (6, 7)	-0.039	0.0053 (6, 7)

Performing HDM SA on contributions C_{i3}^{A-S} and C_{i5}^{A-G} also reveals additional information, such as "in order for a certain energy source to be more/the most desirable, how much should its development and utilization costs decrease and how much should its negative impacts on the environment be decreased." For example, in order for "nuclear" to be more desirable, its contribution to "environmental impact," represented by C_{53}^{A-S}, should be increased by more than 0.4. In this case, "nuclear" that currently ranks the fifth, will be moved up to the fourth rank. However, since the threshold of changes to C_{53}^{A-S} to reverse the rank order of "nuclear" and the original top three energy sources are greater than one, increasing C_{53}^{A-S} value alone can only move "nuclear" up to the fourth rank. This is due to the small values that the G&S model assigned to other contributions of "nuclear." This indicates that, if the G&S model had correctly represented the reality, in order for "nuclear" to be more desirable than "coal," its contributions to other criteria such as social impacts, national security and costs also need to be improved.

For another example, "solar" was evaluated as the second most desired energy source in the G&S model. Since the model was built before any commercial solar applications had been seen, assessment of solar's contributions to different evaluation criteria may have been far away from the reality. It would have also made the model difficult to validate. As it turned out later, many aspects of solar did not reach what was expected in the G&S model. In the model, contribution of solar to availability was assigned to be 0.485, making solar the single most available energy among the alternatives. This value is nearly ten times of the contribution of nuclear and natural gas to availability, and more than ten times of the contribution of oil to availability. Theoretically, the sun provides endless resources for solar energy. However, the space required to collect sunlight is huge, and materials used to store solar energy are either extremely expensive or of very low efficiency. This has dramatically limited the production and utilization of solar energy. Even in today, 27 years after the publication of G&S model, the promise of cheap and abundant solar power remains unmet. As "availability" is defined to be not only the availability of energy resources but also the availability of materials (and spaces) to produce and utilize the energy, the value 0.485 assigned to solar's contribution to availability seems in retrospect to be much too large. It only encompasses the theoretical availability of solar, but not the practical availability of solar energy. The impact of decreases in this contribution value, C_{21}^{A-S}, is analyzed by HDM SA. If C_{21}^{A-S} value decreases by 0.11, 0.35, 0.427, or 0.43, solar will be dropped to be the third, fourth, fifth, or sixth ranked energy source.

In fact, this is not the only invalid or inaccurate assessment in the G&S model of contributions of energy sources. Performing only one-way SA is not able to evaluate other simultaneous changes to the model. Therefore, two-way SA and three-way SA are also performed in the following sections.

5.2.2.2.2 Two-way SA

Continuing with the discussion of solar energy from the previous section, two simultaneous changes to the contributions of solar energy can be analyzed in a two-way SA. As discussed above, solar cells have a well-deserved reputation of being too expensive. In the G&S model, in addition to the availability of space and efficient materials to collect and convert light energy into electrical current, the contribution of solar energy to cost is also questionable. Therefore, we analyzed how changes to these two contributions, C_{21}^{A-G} and C_{25}^{A-G}, would affect the model results. Again, for simplicity, global contribution of solar energy to cost is considered, without differentiating between initial cost and the cost of production and use.

Applying Corollary 4.4, a group of inequalities defining the allowable region of two perturbations induced on C_{21}^{A-G} and C_{25}^{A-G} is identified. Two lines defined by two of the inequalities bound the allowable area, thus only these two inequalities are listed beside the inequalities defining the feasible region:

$$\begin{cases} \left.\begin{array}{ll} -0.458 \le P_{21}^{A-G} \le 0.542 & (5.5a) \\ -0.025 \le P_{25}^{A-G} \le 0.975 & (5.5b) \end{array}\right\} \text{Feasible Region} \\ \left.\begin{array}{ll} 0.56P_{21}^{A-G} + 0.462P_{25}^{A-G} \le 0.005 & (5.5c) \\ -0.039P_{21}^{A-G} + 0.0224P_{25}^{A-G} \le 0.002 & (5.5d) \end{array}\right\} \text{Allowable Region} \end{cases}$$

Figure 39 Allowable area of perturbations P_{21}^{A-G} and P_{25}^{A-G}

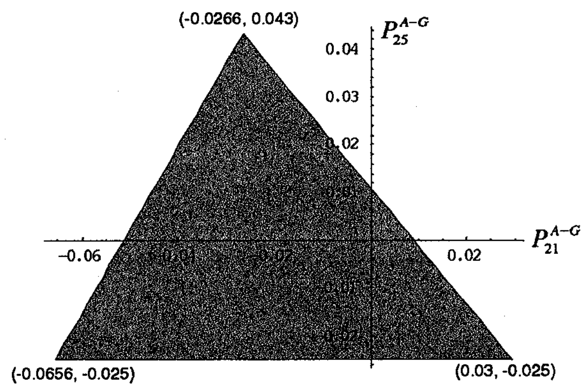


Figure 39 shows the allowable area of perturbations induced on C_{21}^{A-G} and C_{25}^{A-G}, contributions of "solar" to "availability" and "cost," in order to keep current ranking unchanged. As C_{21}^{A-G} decreases from its base value 0.458 to 0.407, no matter how much does C_{25}^{A-G} value decreases, the original ranking will remain the same. However, as C_{21}^{A-G} value continues to decrease, the original rank order of energy sources will be changed inevitably. Based on Proposition 1.3, total sensitivity analysis of C_{21}^{A-G} and

C_{25}^{A-G} is the area of the allowable region. Since coordinates of the three points of the triangular area are identified, TSC of C_{21}^{A-G} and C_{25}^{A-G} can be calculated as:

$$\text{TSC}\ (C_{21}^{A-G} \,\&\, C_{25}^{A-G}) = \frac{1}{2} \times (0.03 + 0.0656) \times (0.043 + 0.025) = 0.003 \qquad (5.6)$$

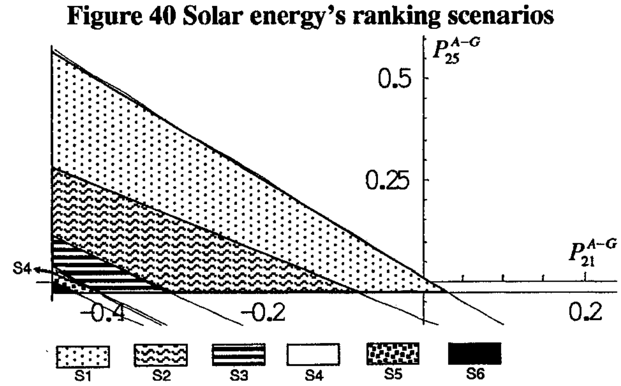
Since the area of feasible region for P_{21}^{A-G} and P_{25}^{A-G} is one, probability of rank being changed when C_{21}^{A-G} and C_{25}^{A-G} vary uniformly between zero and one equals $(1 - 0.003 = 99.7\%)$.

The above analysis looks at the ranking of all energy sources. Since we are particularly interested in seeing how the rank of solar energy changes when its contributions to availability and cost are reduced, we now focus on the inequalities to keep solar as the second-ranked energy source. Bold numbers in the parenthesis following each inequality represent the pair of energy sources whose original rank order will be changed if that specific inequality is not satisfied.

$$\begin{cases} 0.56P_{21}^{A-G} + 0.462P_{25}^{A-G} \le 0.005 & \mathbf{(1, 2)} & (5.7a) \\ -0.379P_{21}^{A-G} - 0.465P_{25}^{A-G} \le 0.043 & \mathbf{(2, 3)} & (5.7b) \\ -0.392P_{21}^{A-G} - 0.43P_{25}^{A-G} \le 0.136 & \mathbf{(2, 4)} & (5.7c) \\ -0.398P_{21}^{A-G} - 0.402P_{25}^{A-G} \le 0.172 & \mathbf{(2, 5)} & (5.7d) \\ -0.383P_{21}^{A-G} - 0.399P_{25}^{A-G} \le 0.178 & \mathbf{(2, 6)} & (5.7e) \\ -0.422P_{21}^{A-G} - 0.376P_{25}^{A-G} \le 0.18 & \mathbf{(2, 7)} & (5.7f) \end{cases}$$

Inequalities (5.7a) and (5.7b) bound the allowable region for the two perturbations in order to keep solar as the second-ranked energy source. Together with inequalities (5.7c) to (5.7f), they define seven possible scenarios for solar energy's ranking. As shown in Figure 40, S1 represents the scenario when solar ranked as the second energy source, and in scenarios S2 through S6, the rank of solar drops to the third, fourth, fifth, sixth and seventh. This shows how ranking of solar changes when both C_{21}^{A-G} and C_{25}^{A-G} decrease. The unspecified area on the upper right side to S1 represents the scenario in which C_{21}^{A-G} value increases and solar becomes the first-rank. However, this is not likely to happen unless abundant high efficiency materials are developed to utilize solar energy.

Figure 40 Solar energy's ranking scenarios



Besides solar energy, contributions of oil and natural gas to availability were also analyzed in detail. In the G&S model, two very small values (0.032 and 0.045) are assigned to the contributions of oil and natural gas to availability, C_{61}^{A-S} and C_{41}^{A-S}. The rationale behind this judgment was that the U.S. proven oil reserve were about 29 billion barrels and "would not last past the decade with the current production rate," and that the U.S. natural gas reserve of 21.8 trillion cubic feet would "last only ten years (from 1980)" [43]. However, comparing the prediction in [43] with official energy statistics from the U.S. government website, we can see great differences: official statistics show that the proven natural gas reserves in year 2000 was 186.51 trillion cubic feet and the U.S. proven reserve of oil was 22 billion barrels in 2000, both are far away from "running out," while consumption percentages of different energy sources was almost the same in the years from 1980 to 2000 [116].

A two-way SA is performed to test the model sensitivity to changes to C_{61}^{A-S} and C_{41}^{A-S} values. Based on Corollary 4.2, a group of inequalities are defined in order for the current ranking to remain unchanged (Bold numbers in the parenthesis following each inequality represent the pair of energy sources whose original rank order will be changed if that specific inequality is not satisfied):

$$
\left\{
\begin{array}{lll}
-0.063P_{61}^{A-S} - 0.063P_{41}^{A-S} \le 0.005 & \mathbf{(1,2)} & (5.8a) \\
0.17P_{61}^{A-S} + 0.17P_{41}^{A-S} \le 0.043 & \mathbf{(2,3)} & (5.8b) \\
0.372P_{41}^{A-S} + 0.01P_{61}^{A-S} \le 0.093 & \mathbf{(3,4)} & (5.8c) \\
-0.383P_{41}^{A-S} - 0.021P_{61}^{A-S} \le 0.036 & \mathbf{(4,5)} & (5.8d) \\
0.021P_{41}^{A-S} + 0.383P_{61}^{A-S} \le 0.006 & \mathbf{(5,6)} & (5.8e) \\
-0.035P_{41}^{A-S} - 0.397P_{61}^{A-S} \le 0.002 & \mathbf{(6,7)} & (5.8f) \\
-0.077 \le P_{41}^{A-S} + P_{61}^{A-S} \le 0.923 & & (5.8g) \\
-0.045 \le P_{41}^{A-S} \le 0.955 & & (5.8h) \\
-0.032 \le P_{41}^{A-S} \le 0.968 & & (5.8i)
\end{array}
\right.
$$

(5.8a)–(5.8f): Allowable Region; (5.8g)–(5.8i): Feasible Region

Figure 41 Allowable area of P_{61}^{A-S} & P_{41}^{A-S}

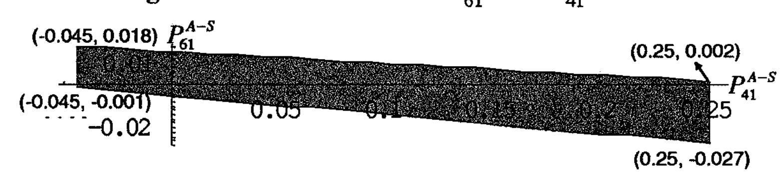
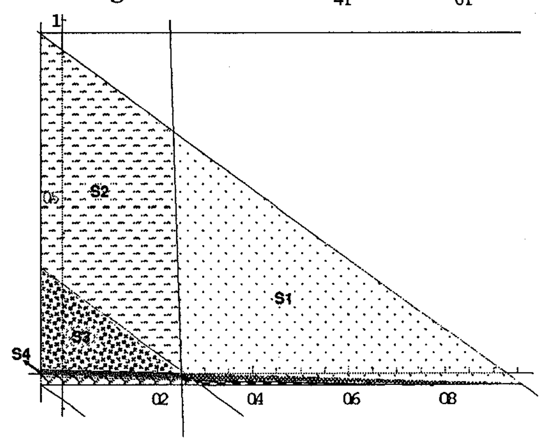


Figure 41 shows the allowable area for P_{61}^{A-S} and P_{41}^{A-S} to keep current ranking of energy sources unchanged. The three lines bounding the allowable area are defined by inequality (5.8c), (5.8e) and (5.8f). Total sensitivity coefficient of C_{61}^{A-S} and C_{41}^{A-S} can be calculated as:

$$\text{TSC}(C_{61}^{A-S} \,\&\, C_{41}^{A-S}) = \int_{-0.045}^{0.25} 0.016 - 0.055P_{41}^{A-S} + 0.005 + 0.089P_{41}^{A-S} = 0.007 \qquad (5.9)$$

Probability of rank being changed as C_{61}^{A-S} and C_{41}^{A-S} vary uniformly between zero and one within their feasible region is thus (1 – 0.007/0.5 = 98.6%). If the contribution of oil to availability (C_{61}^{A-S}) increases by more than 0.018, no matter how C_{41}^{A-S} changes, the original ranking of energy sources will be changed. On the other hand, if the contribution of natural gas to availability (C_{41}^{A-S}) increases by more than 0.25, no matter how C_{61}^{A-S} changes, the original ranking of energy sources will also be changed.

Figure 42 Ranking scenarios when C_{41}^{A-S} and C_{61}^{A-S} change



As Figure 42 shows, lines defined by inequalities (5.8a) to (5.8f) intersect the feasible region and divide it into several parts. S4 is the allowable region for P_{61}^{A-S} and P_{41}^{A-S}, as shown in Figure 41. S1 and the major part of S2 and S3 represent situations when both C_{61}^{A-S} and C_{41}^{A-S} increase. When P_{61}^{A-S} and P_{41}^{A-S} fall into S3, rank order of (5) nuclear and (6) oil will be reversed; when P_{61}^{A-S} and P_{41}^{A-S} fall into S2, in addition to the rank reversal of (5) nuclear and (6) oil, the rank order of (2) solar and (3) reservation will also be reversed; and when P_{61}^{A-S} and P_{41}^{A-S} fall into S1, a third pair of energy sources, (**3**) reservation and (**4**) natural gas, will have reversed rank order in addition to the above ones.

Inequalities (5.8a) through (5.8f) only consider rank order of six pairs of energy sources that rank successively. When values given to P_{61}^{A-S} and P_{41}^{A-S} satisfy the inequality group, the original ranking of all energy sources will remain unchanged. However, if one inequality is not satisfied, not only the rank order of energy sources protected by that specific inequality will be reversed, but also rank order of other energy sources not protected by the other five inequalities may change as well. For example, if inequality (5.8a) is not satisfied, then not only will the first- and second-ranked energy sources be changed, but also the original rank order of (**1**) and (**3**) may be changed. This means that when P_{61}^{A-S} and P_{41}^{A-S} fall into S1, S2 and S3, there may be more than three different ranking scenarios. To represent all the possible ranking scenarios, 21 lines in total defined by 21 (6+5+4+3+2+1=21) inequalities need to intersect the feasible region and divide it into different ranking scenario. A figure to represent this would be too complex to be useful, and would not be meaningful since the original model represented the view of two people over twenty years ago. Instead, an analysis is done with the aim of replicating the ranking of actual consumption of energy sources in year 2000.

Table 39 shows the actual energy consumption in year 2000. Ranking of the different energy sources based on the actual consumption percentage is **(1)** Oil -> **(2)** Natural gas -> **(3)** Coal -> (**4**) Nuclear -> (**5**) Others (including "solar").

Table 39 Actual energy consumption ranking and percentage [116]

	Coal	Natural Gas	Nuclear	Oil	Others				
					Solar	Hydro-electric	Bio-mass	Geo-thermal	Wind
Quadrillion Btu	22.58	23.916	7.862	38.404	0.066	2.811	2.922	0.317	0.057
Ranking	(3)	(2)	(4)	(1)	(5)				
Percentage	22.8%	24.2%	8%	38.8%	0.1%	2.8%	3%	0.3%	0.1%
					6.2%				

The top three energy sources consumed in year 2000 were oil, natural gas and coal. If the G&S model were to represent this actual consumption ranking for the top three energy sources, how should C_{61}^{A-S} and C_{41}^{A-S} values change? This can be analyzed by performing HDM SA on the G&S model.

Since oil, natural gas and coal, which ranked as the sixth, fourth and first in G&S model were the top three energy sources being consumed in year 2000, this means the rank of "(**6**) oil" and "(**4**) natural gas" surpassed all the other energy sources to become the first and second, and the rank order between these two energy sources is also reversed. Suppose that the priority of "conservation" remained to be the same, then the actual rankings of oil, natural gas, conservation and coal would be (**1**) through (**4**). To replicate this rank change by varying C_{61}^{A-S} and C_{41}^{A-S} values, P_{61}^{A-S} and P_{41}^{A-S} values should violate the following inequalities (Bold numbers in the parenthesis following each inequality represent the pair of energy sources whose original rank order will be reversed if that specific inequality is violated.):

$$0.116P_{41}^{A-S} + 0.478P_{61}^{A-S} \leq 0.183 \quad (\mathbf{1,6}) \qquad (5.10a)$$

$$0.179P_{41}^{A-S} + 0.541P_{61}^{A-S} \leq 0.178 \quad (\mathbf{2,6}) \qquad (5.10b)$$

$$0.018P_{41}^{A-S} + 0.38P_{61}^{A-S} \leq 0.135 \quad (\mathbf{3,6}) \qquad (5.10c)$$

$$-0.362P_{41}^{A-S} + 0.362P_{61}^{A-S} \leq 0.042 \quad (\mathbf{4,6}) \qquad (5.10d)$$

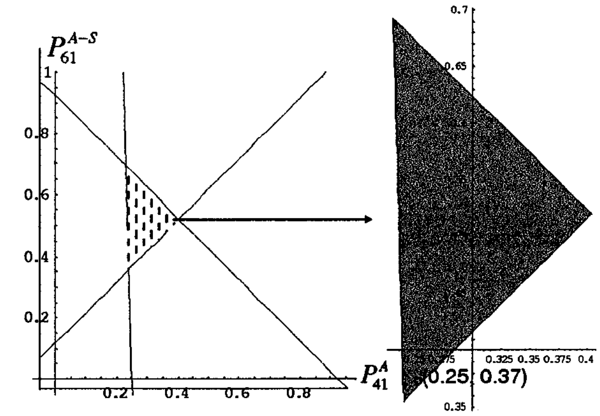
$$0.021_{41}^{A-S} + 0.383P_{61}^{A-S} \leq 0.006 \quad (\mathbf{5,6}) \qquad (5.10e)$$

$$0.478_{41}^{A-S}+0.116P_{61}^{A-S}\le 0.141 \qquad \textbf{(1, 4)} \qquad (5.10f)$$

$$0.541_{41}^{A-S}+0.179P_{61}^{A-S}\le 0.136 \qquad \textbf{(2, 4)} \qquad (5.10g)$$

$$0.372_{41}^{A-S}+0.01P_{61}^{A-S}\le 0.093 \qquad \textbf{(3, 4)} \qquad (5.10h)$$

Figure 43 Area for P_{61}^{A-S} and P_{41}^{A-S} to replicate actual consumption ranking



For inequalities (5.10a) through (5.10h), if (5.10d) and (5.10h) are violated, all the other inequalities will be violated. Thus, if P_{41}^{A-S} and P_{61}^{A-S} values fall into the area bounded by ($-0.362_{41}^{A-S}+0.362P_{61}^{A-S}\ge 0.042$) and ($0.372_{41}^{A-S}+0.01P_{61}^{A-S}\ge 0.093$) within the feasible region, shown as the shaded area in Figure 43, the rank of oil and natural gas will become the first and the second, with conservation and coal being at the third and the fourth rank.

For example, when the two changes are $P_{41}^{A-S}=0.25$ and $P_{61}^{A-S}=0.37$, as shown in Figure 43, the top three ranked energy sources (excepting conservation) become oil, natural gas and coal, which exactly replicates the historical truth. As shown in Table 40.

Table 40 Ranking of energy sources (original vs. new)

	Coal	Solar	Conservation	Natural gas	Nuclear	Oil	Others
Original values	0.248	0.244	0.200	0.108	0.072	0.065	0.063
New values	0.176	0.132	0.194	0.198	0.059	0.199	0.042
Original ranks	**1**	**2**	**3**	**4**	**5**	**6**	**7**
New ranks	4	5	3	2	6	1	7

In this case the contributions of oil and natural gas to availability are increased to 0.402 and 0.295 from the original values of 0.032 and 0.045. If we compare the data given in [43], where the U.S. proven natural gas reserve was claimed to be 21.8 trillion cubic feet in 1980, with official energy statistics from the U.S. Government, where it shows that the proven natural gas reserves in year 2000 was 186.51 trillion cubic fee, it is safe to say that the 0.25 increase in natural gas' contribution to availability is reasonable. On the other hand, although the U.S. proven reserve of oil declined from the 29.8 billion barrels in 1980 to 22 billion barrels in 2000, it is far from "will not last past the decade with the current production rate" as analyzed in [43]. The increases to C_{61}^{A-S} and C_{41}^{A-S} can be justified or validated.

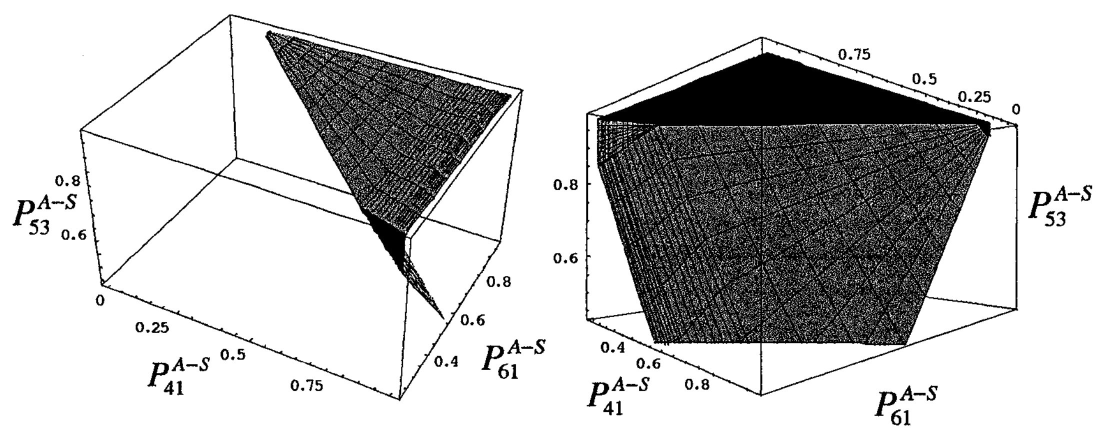
5.2.2.2.3 Three-way SA

Continuing with the analysis of previous section, the next objective is to replicate the actual ranking of energy consumption in year 2000, which is "**(1)** Oil, **(2)** Natural gas, **(3)** Coal, **(4)** Nuclear, **(5)** Solar and others," exactly by inducing three perturbations to the $\mathbf{C_{ij}^{A-S}}$ matrix. In this case, the rank of oil and natural gas should surpass all the energy sources that rank before them, the rank of nuclear should surpass solar and others, and oil should rank before natural gas. Conservation, which is not a real energy source, is assumed to remain at its original rank. In this case, besides changing

C_{61}^{A-S} and C_{41}^{A-S}, as in previous analysis, the contribution of nuclear to environmental impact, C_{53}^{A-S}, is also considered. With such analysis, people will be able to know that in order for nuclear to be preferred over solar and other new energy sources, how much would its impact on the environment need to be changed? Such an improvement could make, possibly, through better nuclear waste management.

Based on the same analysis, inequalities that need to be satisfied in order to create such a scenario are plotted in a three-dimensional space. Figure 44 shows in two different viewpoints the space in which if P_{61}^{A-S}, P_{41}^{A-S} and P_{53}^{A-S} values fall in, the actual energy consumption ranking will be replicated by the model.

Figure 44 Space for P_{61}^{A-S}, P_{41}^{A-S} and P_{53}^{A-S} value to replicate actual ranking of energy sources



One point in the space could be (0.25, 0.5, 0.85). To verify whether the rank changes as indicated, the values 0.25, 0.5 and 0.85 were assigned to P_{41}^{A-S}, P_{61}^{A-S} and P_{53}^{A-S}, and new C_i^A values were calculated. Table 41 shows the original and new C_i^A values and energy sources' ranking.

Table 41 Ranking of energy sources (original vs. new)

	Coal	Conservation	Natural gas	Nuclear	Oil	Solar	Others
Original values	0.248	0.200	0.108	0.072	0.065	0.244	0.063
New values	0.159	0.176	0.187	0.121	0.240	0.084	0.034
						0.118	
Original ranks	(1)	(3)	(4)	(5)	(6)	(2)	(7)
New ranks	(4)	(3)	(2)	(5)	(1)	(6)	
New ranks w/o "conservation"	(3)	-	(2)	(4)	(1)	(5)	

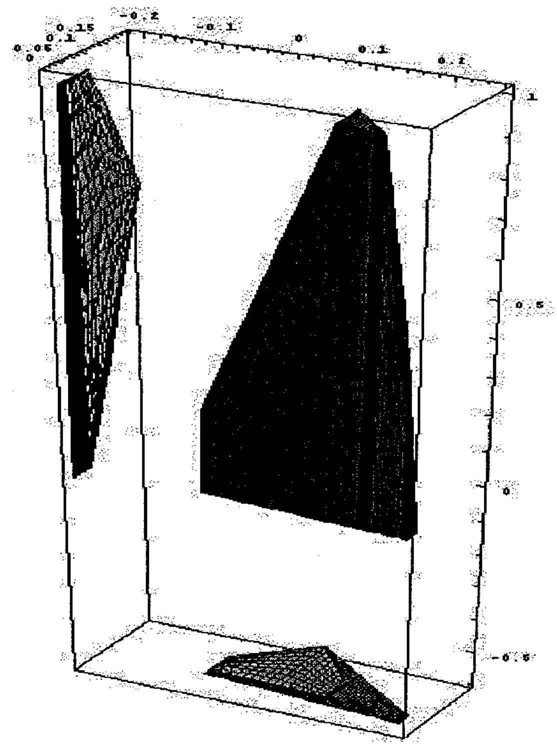
As Table 41 shows, energy sources that rank from the first to the last are oil, natural gas, coal, nuclear, solar and others. This correctly represents the actual rank of energy consumptions in year 2000, as shown in Table 39.

When C_{41}^{A-S}, C_{61}^{A-S} and C_{53}^{A-S} change simultaneously, the original rank order can be preserved only when the following inequalities are satisfied:

$$
\begin{cases}
\left.\begin{array}{ll}
-0.0631P_{41}^{A-S} - 0.0631P_{61}^{A-S} - 0.0305P_{53}^{A-S} \le 0.005 & (5.11a) \\
0.1696P_{41}^{A-S} + 0.1696P_{61}^{A-S} + 0.0111P_{53}^{A-S} \le 0.043 & (5.11b) \\
0.362P_{41}^{A-S} + 0.0098P_{61}^{A-S} + 0.007P_{53}^{A-S} \le 0.093 & (5.11c) \\
-0.3832P_{41}^{A-S} - 0.0212P_{61}^{A-S} + 0.0907P_{53}^{A-S} \le 0.036 & (5.11d) \\
0.0212P_{41}^{A-S} + 0.3832P_{61}^{A-S} - 0.0843P_{53}^{A-S} \le 0.006 & (5.11e)
\end{array}\right\} \text{Allowable Region} \\
-0.0353P_{41}^{A-S} - 0.362P_{61}^{A-S} + 0.0039P_{53}^{A-S} \le 0.002 \quad (5.11f) \\
\left.\begin{array}{ll}
-0.045 \le P_{41}^{A-S} \le 0.955 & (5.11g) \\
-0.032 \le P_{61}^{A-S} \le 0.968 & (5.11h) \\
-0.077 \le P_{41}^{A-S} + P_{61}^{A-S} \le 0.923 & (5.11i) \\
-0.019 \le P_{53}^{A-S} \le 0.981 & (5.11j)
\end{array}\right\} \text{Feasible Region}
\end{cases}
$$

A three dimensional allowable region defined by the above inequalities is shown in Figure 45. Shadow of the region on Z-X and X-Y planes are also shown in the figure. X, y and z axes represent P_{41}^{A-S}, P_{61}^{A-S} and P_{53}^{A-S} respectively.

Figure 45 Allowable space for P_{61}^{A-S}, P_{41}^{A-S} and P_{53}^{A-S} to keep original ranking of energy sources

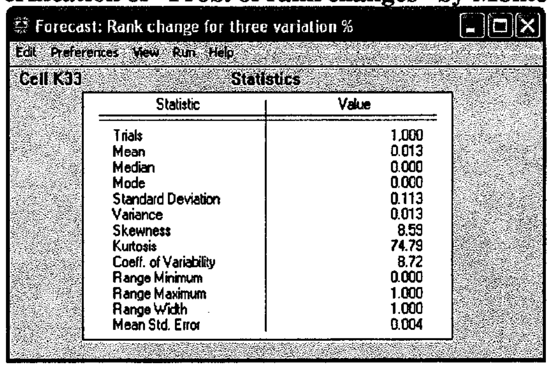


Due to complexity of identifying the intersections of different planes defined by the inequalities, the calculation of TSC and probability of rank changing become more complex in this case. However, thanks to the advanced program offered by software Mathematica® (version 5.2 and above), volume of the three dimensional allowable space can be calculated by integration within the region that satisfies inequalities (5.11a) through (5.11j). The corresponding Mathematica commend is:

```
Integrate[Boole[{-0.0631*x - 0.0631*y - 0.0305*z <= 0.005 && 0.1696*x +
0.1696*y + 0.0111*z <= 0.043 && 0.362*x + 0.0098*y + 0.007*z <= 0.093 && -
0.3832*x - 0.0212*y + 0.0907*z <= 0.036 && 0.0212*x + 0.3832*y - 0.0843*z <=
0.006 && -0.0353*x - 0.362*y + 0.0039*z <= 0.002 && x + y <= 0.923}], {x, -
0.045, 0.955}, {y, -0.032, 0.968}, {z, -0.019, 0.981}]
```

As a result, TSC (P_{61}^{A-S} & P_{41}^{A-S} & P_{53}^{A-S}) = 0.014. This means that when C_{41}^{A-S}, C_{61}^{A-S} and C_{53}^{A-S} vary uniformly between zero and one, there is a 2.8% (=0.014/0.5) chance that the original ranking of energy sources will remain unchanged; in another word, there is a 97.2% chance that the rank will be changed. This percentage is again verified through Monte Carlo simulation: As shown in Figure 46, $\hat{p}$ of the one thousand trials of simulation to estimate the true proportion of rank remaining unchanged (output being "1") is 0.013.

Figure 46 Verification of "Prob. of rank changes" by Monte Carlo simulation


Forecast: Rank change for three variation %

Edit Preferences View Run Help

Cell K33 Statistics

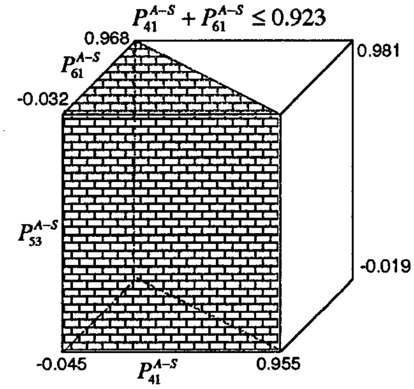
Statistic	Value
Trials	1,000
Mean	0.013
Median	0.000
Mode	0.000
Standard Deviation	0.113
Variance	0.013
Skewness	8.59
Kurtosis	74.79
Coeff. of Variability	8.72
Range Minimum	0.000
Range Maximum	1.000
Range Width	1.000
Mean Std. Error	0.004

Based on the same analysis shown in previous sections, at 99.7% confidence level, the lower and upper limit of confidence interval can be calculated by: $\hat{p} \pm Z-multiple \times \sqrt{\hat{p}(1-\hat{p})/n} = 0.013 \pm 3 \times \sqrt{0.013(1-0.013)/1000}$ = [0.002, 0.024]. 0.014, calculated as TSC ($P_{61}^{A-S} \& P_{41}^{A-S} \& P_{53}^{A-S}$) is within this range.

It should be noted that 1.4% is the probability of rank remaining unchanged when C_{41}^{A-S}, C_{61}^{A-S} and C_{53}^{A-S} vary uniformly between zero and one. However, because of the constraint $P_{41}^{A-S} + P_{61}^{A-S} \leq 0.923$, the feasible region is the shaded half of the cube instead of the whole cube, as shown in Figure 47. As a result, volume of the feasible region is 0.5 instead of 1. Then the probability of rank remaining unchanged when

C_{41}^{A-S}, C_{61}^{A-S} and C_{53}^{A-S} vary uniformly within this feasible region will be (0.014/0.5 = 0.028). Thus probability of rank changing is 97.2%.

Figure 47 Feasible region of P_{61}^{A-S} & P_{41}^{A-S} & P_{53}^{A-S}



5.2.2.3 Multi-level Analysis

5.2.2.3.1 Scenario one

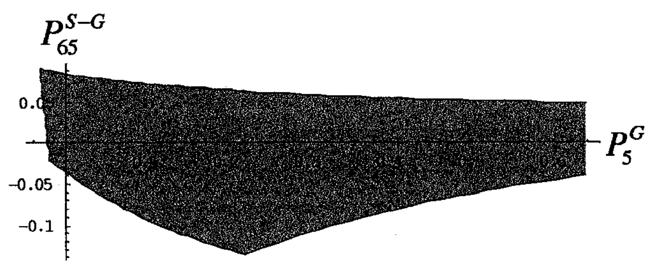
Cost is one of the two most important criteria; as it received one of the highest weights among the evaluating criteria. Cost is divided into initial costs and costs of production and use at the sub-criteria level. How changes to the weight of cost (C_5^G) and the relative importance of initial cost to total cost (C_{65}^{S-G}) will impact the decision is analyzed in the first scenario of multi-level simultaneous changes. Applying Theorem 5.1, a group of inequalities that need to be satisfied in order for the original ranking to remain unchanged are identified. Four of the inequalities bound the allowable area of perturbations induced on C_5^G and C_{65}^{S-G}:

$$
\left.\begin{array}{lll}
-0.1129P_5^G - 0.135P_{65}^{S-G} - 0.373P_5^G P_{65}^{S-G} \le 0.0048 & \mathbf{(1,2)} & (5.12a) \\
0.0408P_5^G - 0.1535P_{65}^{S-G} - 0.424P_5^G P_{65}^{S-G} \le 0.043 & \mathbf{(2,3)} & (5.12b) \\
-0.0055P_5^G + 0.0695P_{65}^{S-G} + 0.192P_5^G P_{65}^{S-G} \le 0.006 & \mathbf{(5,6)} & (5.12c) \\
-0.0914P_5^G - 0.0105P_{65}^{S-G} - 0.029P_5^G P_{65}^{S-G} \le 0.002 & \mathbf{(6,7)} & (5.12d)
\end{array}\right\} \text{Allowable Region}
$$

$$
\left.\begin{array}{ll}
-0.362 \le P_5^G \le 0.638 & (5.12e) \\
-0.6 \le P_{65}^{S-G} \le 0.4 & (5.12f)
\end{array}\right\} \text{Feasible Region}
$$

Figure 48 depicts the allowable area for P_5^G and P_{65}^{S-G} to keep the original ranking of energy sources unchanged.

Figure 48 Allowable region of P_5^G & P_{65}^{A-S}

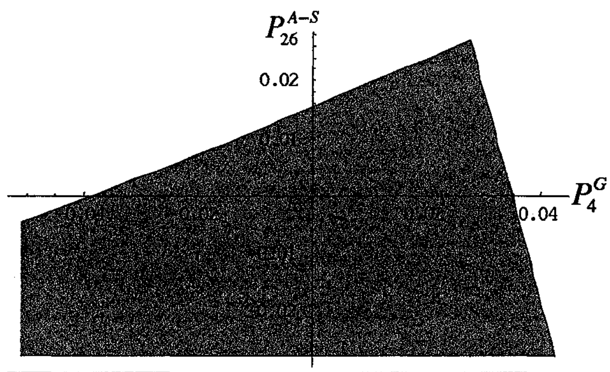


As Figure 48 shows, the original model result is relatively stable to variations of C_5^G. However, it is sensitive to increases to C_{65}^{S-G}, the relative importance of initial cost to total cost: when C_{65}^{S-G} increases by more than 0.1, no matter how C_5^G changes, the original ranking will be changed. Since all four inequalities that bound the allowable region are known, area of this region can be calculated through integration. As a result, TSC(C_5^G & C_{65}^{S-G}) = 0.097. Since area of the feasible region is one, probability of rank being changed when C_5^G and C_{65}^{S-G} vary uniformly within their feasible region is (1 - 0.097 = 90.3%).

5.2.2.3.2 Scenario two

Suppose nothing except the importance of national security was increased in the G&S model, how should the contribution of solar energy to initial cost be improved so that solar will be the most desirable energy source? To analyze this case, Theorem 5.3 is applied.

Figure 49 Allowable region of P_4^G & P_{26}^{A-S}



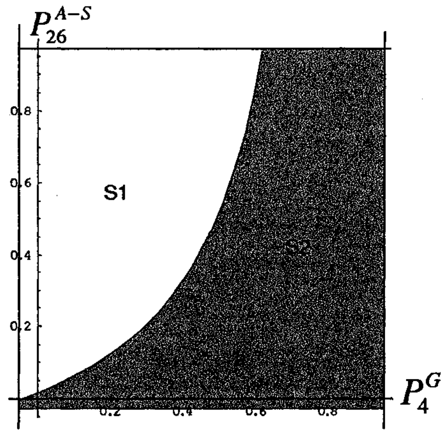
As shown in Figure 49, the inequalities defined an allowable area for perturbations on C_4^G and C_{26}^{A-S}, weight of national security and contribution of solar to initial cost. Two curves defined by the following two inequalities (5.13a) and (5.13b) bound the allowable area. If (5.13a) is violated, rank order of "coal" and "solar" which ranked as the first and the second in the original model will be reversed; If (5.13b) is violated, rank order of "oil" and "others" which ranked as the sixth and the seventh in the original model will be reversed.

$$-0.123P_4^G + 0.311P_{26}^{S-G} - 0.371P_4^G P_{26}^{S-G} \le 0.0048 \ \mathbf{(1,2)} \tag{5.13a}$$

$$0.057P_4^G + 0.016P_{26}^{S-G} - 0.019P_4^G P_{26}^{S-G} \le 0.002 \ \mathbf{(6,7)} \tag{5.13b}$$

In order for solar to be the most desirable energy source when the importance of national security increases, contribution of solar to initial cost should increase at a higher rate than the gradient of curve defined by inequality (5.13a). That means P_4^G and P_{26}^{A-S} should fall into S1 in Figure 50, so that rank order of "coal" and "solar" will be reversed to make "solar" the top-ranked energy source. If P_4^G and P_{26}^{A-S} fall into S2, the shaded area in Figure 50, where inequality (5.13a) is satisfied, rank order of coal and solar will remain unchanged.

Figure 50 Allowable region of P_4^G & P_{26}^{A-S}



5.2.2.4 Adding a new decision alternative

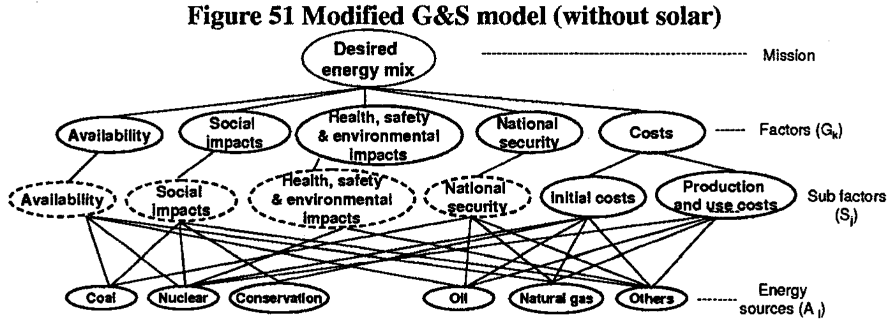
<u>5.2.2.4.1 Treating "solar" as a new decision alternative</u>

In the G&S model, energy source named "others" include a number of relatively new and rarely used energy sources. If other new energy alternatives have emerged and the same model is used to assess all the energy sources, the model can be modified by reassessing the contribution of "others" to all evaluation criteria. However, "solar," as a new energy whose commercial applications had not been seen at the time when the

G&S model was built, was evaluated as an energy source by itself in the model. As a result, inaccurate assessment of this new energy at that time lead to unrealistic proposal that 30.5% energy usage should come from solar. Some thought regarding this inaccurate result suggests that instead of involving decision alternatives that are hard to predict or rarely understood in the model and assessing their impacts together with other well established decision alternatives, treating them as "new decision alternatives" and analyzing them through HDM SA may provide more accurate analysis.

Taking G&S model as an example, instead of putting "solar" together with other energy sources under evaluation, the original model can include only coal, nuclear, conservation, oil, natural gas, and others. Contributions of solar to all the sub-criteria are brought down to zero, and contributions of all other energy sources are increased proportionately so that the sum of their contributions to a single criterion is still one. In this way, the modified model becomes the one shown in Figure 51:

Figure 51 Modified G&S model (without solar)



New $\mathbf{C}_{ij}^{A-S}$ matrix and $\mathbf{C}_i^{A}$ vector become as what are shown in Table 42 and Table 43.

Table 42 New contributions of energy sources to sub-criteria in modified G&S model

C_{ij}^{A-S}	Availability	Social impacts	Environmental impacts	National security	Initial cost	Production and use cost
Coal	0.351	0.046	0.043	0.249	0.433	0.041
Conservation	0.029	0.106	0.350	0.431	0.113	0.536
Natural gas	0.053	0.295	0.237	0.027	0.240	0.112
Others	0.106	0.089	0.071	0.066	0.045	0.032
Nuclear	0.064	0.029	0.026	0.066	0.050	0.203
Oil	0.038	0.295	0.133	0.020	0.119	0.075

Table 43 Priority of energy sources in modified G&S model

	Coal	Conservation	Natural gas	Others	Nuclear	Oil
C_i^A	0.351	0.235	0.138	0.097	0.092	0.087
Priority	**(1)**	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**

Then impact of adding a new decision alternative "solar" can be analyzed by using Corollary 4.4 in HDM SA algorithm. Now that "solar" is treated as the seventh energy source whose original contribution to all the six sub-criteria are zero (before it was identified), new contributions assigned to solar can be viewed as perturbations induced on those zero contributions. In order for solar to be more desirable than coal, the currently first-ranked energy source, inequality (5.14a) need to be violated. In order for solar to be more desirable than conservation, natural gas, others, nuclear, and oil, which ranked from the second to the sixth currently, inequalities (5.14b) through (5.14f) need to be violated respectively.

$$0.56C_{71}^{A-S} + 0.041C_{72}^{A-S} + 0.08C_{73}^{A-S} + 0.208C_{74}^{A-S} + 0.311C_{75}^{A-S} + 0.151C_{76}^{A-S} \le 0.351 \tag{5.14a}$$

$$0.38C_{71}^{A-S} + 0.044C_{72}^{A-S} + 0.107C_{73}^{A-S} + 0.242C_{74}^{A-S} + 0.242C_{75}^{A-S} + 0.223C_{76}^{A-S} \le 0.235 \tag{5.14b}$$

$$0.392C_{71}^{A-S} + 0.052C_{72}^{A-S} + 0.097C_{73}^{A-S} + 0.166C_{74}^{A-S} + 0.269C_{75}^{A-S} + 0.161C_{76}^{A-S} \le 0.138 \tag{5.14c}$$

$$0.422C_{71}^{A-S} + 0.043C_{72}^{A-S} + 0.082C_{73}^{A-S} + 0.173C_{74}^{A-S} + 0.227C_{75}^{A-S} + 0.149C_{76}^{A-S} \le 0.097 \tag{5.14d}$$

$$0.398C_{71}^{A-S} + 0.04C_{72}^{A-S} + 0.078C_{73}^{A-S} + 0.173C_{74}^{A-S} + 0.228C_{75}^{A-S} + 0.174C_{76}^{A-S} \le 0.092 \tag{5.14e}$$

$$0.383C_{71}^{A-S} + 0.052C_{72}^{A-S} + 0.088C_{73}^{A-S} + 0.165C_{74}^{A-S} + 0.243C_{75}^{A-S} + 0.156C_{76}^{A-S} \leq 0.087 \quad (5.14f)$$

The contribution values assigned to C_{7j}^{A-S} (j = 1, 2…6) in G&S model are 0.458, 0.44, 0.381, 0.141, 0.028, 0.02. These values violated inequality (5.14b) through (5.14f). For example, replacing C_{7j}^{A-S} values on the left side of inequality (5.14b) will give 0.2787, which is greater than 0.235, the right side value. As a result, ranking of this new decision alternative "solar" will be ranked as the second, the same as it was ranked in the original G&S model. Therefore, Corollary 4.4's application on analyzing new decision alternatives is verified and validated.

With the help of Corollary 4.4 in HDM SA, different values for C_{7j}^{A-S} can be tested and the rank of "solar" can be determined by evaluating which inequalities are violated.

5.2.2.4.2 Rank reversal problem

It is interesting to note that a rank reversal occurred when solar, as a new decision alternative, is removed from the original G&S model: rank order of "others" with "nuclear" and "oil" was reversed. Without solar, "others" ranked before nuclear and oil; having solar among the decision alternatives, "others" ranked behind nuclear and oil. Rank reversal problem has been reported in [12] and [13], and argued by Saaty as a result of people's inconsistency or mind changing [98]. It is not this dissertation's purpose to look at the rank reversal problem since the problem stems from the eigenvector related calculations to transform the 1-9 pair-wise comparison scale to local contribution matrix. However, HDM SA does provide a tool to evaluate situations when rank reversal will happen. Based on Corollary 4.4 in HDM SA, the conditions that need to be satisfied in order to keep rank order between any pair of decision alternatives can be identified. In the modified G&S model, when the following conditions are not satisfied, rank order of "others" and "nuclear", "others" and "oil" will reverse:

$$0.024C_{71}^{A-S} + 0.003C_{72}^{A-S} + 0.004C_{73}^{A-S} - 0.001C_{75}^{A-S} - 0.025C_{76}^{A-S} \leq 0.005 \quad (5.15a)$$

$$0.039C_{71}^{A-S} - 0.009C_{72}^{A-S} - 0.006C_{73}^{A-S} + 0.009C_{74}^{A-S} - 0.016C_{75}^{A-S} - 0.006C_{76}^{A-S} \leq 0.01 \quad (5.15b)$$

When C_{7j}^{A-S}'s take value of 0.458, 0.44, 0.381, 0.141, 0.028, and 0.02, as in the G&S model, both inequalities are violated (The left side value of (5.15a) and (5.15b) are 0.013 and 0.012). Therefore, rank reversal between "others" and "nuclear," "others" and "oil" occurred.

5.3 CONCLUSION

In this chapter, HDM SA algorithm was applied to analyze an energy portfolio model built by Gholamnezhad and Saaty in 1980 to suggest a desirable energy mix in year 2000. Theorems 1 through 5 and their corollaries in the HDM SA algorithm were again verified using data from the G&S model. With historical data about the actual consumption of different energy sources in hand, the actual ranking of the energy sources were replicated through HDM SA. Significant insights regarding new energy evaluation were gained. Whether a specific energy source contributes positively or negatively to an evaluation criterion as compared to other energy sources was revealed. The rank reversal issue, which was reported in AHP related literature

[12][13], was also evaluated. This provides a starting point to analyze the reason of rank reversal issue in a systematic way in the future.

6. CONCLUSION

In this dissertation, literature was reviewed regarding: 1) HDM using various pariwise comparison ratio scales, judgment quantification methods, and group opinion combination methods, 2) SA of general MCDM models including Linear Programming, and 3) SA for both deterministic and non-deterministic hierarchical decision models. Limitations of methods employed in previous studies to conduct SA for HDM were identified. To close the literature gap, a comprehensive HDM SA algorithm was proposed: The *direct impact* of one unit variation of local contribution to decision alternatives' overall contribution, the *allowable range/region of perturbations*, the *tolerance* of contribution values, *total sensitivity coefficient*, *operating point sensitivity coefficient*, *probability of rank changing*, and the *critical decision elements* were defined in seven groups of theorems, corollaries and propositions under different conditions regarding the rank orders of 1) specific pairs of decision alternatives, 2) all decision alternatives, and 3) the top ranked alternative. Different situations when single and multiple perturbations are induced to local contribution vector/matrices at any level(s) of the decision hierarchy were also covered.

The algorithm was applied to two previously built hierarchical decision models, one used constant sum and column-row orientation technique [52] and the other used 1-9 scales with verbal description and eigenvector based technique in deriving the local contribution matrices [43]. The two applications not only verified and validated the HDM SA algorithm with actual data, but also demonstrated significant insights gained through performing HDM SA in different fields. As a result of the first application, a strategic technology planning framework utilizing HDM SA as a critical middle step was proposed. In the second application, situations in which AHP related rank reversal problem occurs were also analyzed.

With actual data from the two models in [52] and [43], verification of the allowable range/region of perturbations proposed in the HDM SA algorithm is performed by recalculating new C_i^A values and comparing the new ranks of A_i's. As it shows in Chapter 4 and 5, whenever the perturbations' values go beyond the allowable range/region, rank order of the indicated pair of decision alternatives will be changed. The probability of rank changing was verified by performing Monte Carlo simulation on the original hierarchical decision models. In all cases, the probability of rank changing calculated from HDM SA algorithm falls in the confidence interval calculated from the point estimate of population proportion at 99.7% confidence level.

Validation of HDM SA algorithm using objective data associated with a previously built HDM model is not always appropriate: whether the HDM SA algorithm can be validated by objective data describing changes to a social phenomenon totally depends on whether that HDM model itself can be validated by objective data, whether the model is supposed to represent the actuality of such social phenomenon, and whether people actually implemented decisions suggested by the HDM model. In another words, most HDM results are only proposals or suggestions. They represent perceived value and suggested priorities of decision alternatives, not reality. A simple example would be that people keep smoking even though they are aware of the dangerous effect of smoking and they know that "smoking" is not a "suggested" option for them. In addition, as noted by Nisbett, Borgida, Crandall and Reed [83], sometimes that some kinds of information regarded as highly pertinent and compelling by scientist, the experts, are habitually ignored by people in every-day life. As a result, using historical data to validate HDM model and its sensitivity analysis is not always possible.

Since hierarchical decision models found in literature are built to suggest a desired ranking of decision alternatives, just like the two models explored in this dissertation, validating HDM SA algorithm by historical data turns out to be difficult. However, an effort is devoted to replicate the historical data through HDM SA on the G&S model. Although the G&S model did not claim to forecast energy consumption in year 2000, with actual energy consumption data in hand, the actual energy consumption ranking is replicated by performing HDM SA on the G&S model. As long as the changes are justified, the HDM SA algorithm is validated by objective data. It further demonstrates the usefulness of HDM SA algorithm for "change analysis" for hierarchical models.

Theoretical and practical contributions of this dissertation are summarized in the following section.

6.1 Contributions

6.1.1 Theoretical contribution

When a set of methods to solve a systems problem are identified, three characteristics, Computational complexity, Performance, and Generality, are compared to evaluate different methods. Computational complexity is measured by the time and space required to solve a problem. Performance of a method is the degree of its success in dealing with applicable problems. Generality is determined by the assumptions under which the method operates. [58]

Based on the definitions, when we compare the two major SA methods for deterministic HDM, mathematical deduction in symbolic form is preferred in performance and computational complexity to numerical incremental analysis, since it offers accurate and simple calculations of the sensitivity indicators such as the threshold of changes. Numerical incremental analysis uses step-by-step changes and repetitive computations for each incremental change. This makes it heavily dependent on the numerical values of the model. Further, because of the "brute force" used in the numerical incremental analysis, the more precise the results need to be, the more refined the increments must be. This increases the time and computer memory required to carry out the repetitive computations, thus increasing the computational complexity dramatically. The generality of both methods are equal since they depend on the same assumptions.

Among the methods using mathematical deduction in symbolic form to study the sensitivity of hierarchical models, the HDM SA algorithm proposed in this dissertation is by far the most accurate and comprehensive one. While other studies in this category either proposed a sensitivity coefficient as a likelihood of rank change or only studied a single perturbation in the first level contribution vector without normalizing the threshold value, the proposed HDM SA addresses all the limitations and inaccuracies in previous literature. A comparison of HDM SA to other major methods/studies has been summarized in Table 1 in chapter 2.

Compared to numerical incremental analysis, the computational complexity of HDM SA algorithm developed in this dissertation is much lower and the performance is higher: people only need to solve a number of inequalities, as listed in Table 2, to get a 100% accurate threshold of a change.

One assumption that Theorems 1 through 5 and their corollaries in the HDM SA algorithm are based on is that the local contribution vector and matrices are aggregated using an additive relationship. The same logic could be used to deduce HDM SA for multiplicative HDM, since the multiplicative relationship is not widely used as the additive one, and has been pointed out to be invalid [108], the present Theorems and Corollaries apply only to HDM based on additive relationship. Although numerical incremental analysis is free from this assumption, which made it relatively more general than HDM SA, considering the joint preference of computational complexity, performance and generality of the two methods, the proposed HDM SA is preferred overall.

As noted by Phillips, a model can be considered as requisite only when no new intuitions emerge about the problem [86]. Significant insights gained through performing HDM SA on previous decision problems further demonstrate the importance and necessity of HDM SA in completing problem solutions and making hierarchical decision models requisite.

6.1.2 Empirical contribution

HDM have been widely applied to assist decision makings for different problems in a variety of fields [37]. For example, several researchers have applied HDM to address problems in R&D portfolio management (e.g., [64][106][91]), technology assessment (e.g., [52]), and technology roadmap (e.g., [42]). However, those problems have a common characteristic, which is the high uncertainty involved throughout the technology/product development life cycle [101] especially when the technology is new or rapidly changing [81]. HDM SA is a helpful toolbox that assists managers in visualizing the impact of changes at the policy and strategy levels on decisions at the operational level, and figuring out scenarios of possible situations and corresponding solutions. Therefore, when the two major sources of uncertainty, technological uncertainty and market uncertainty [23], are being eliminated as more information becomes available, managers will be able to respond more quickly and make better decisions.

Uncertainty does not only exist in problems mentioned above. In fact, decisions in any field are mostly made in the absence of perfect information [10]. Although HDM offers an effective way to organize decision elements in complex problems and analyze contributions of decision alternatives to the overall objective, the impact of highly uncertain and rapidly changing input data must be dealt with. Insights not easily seen using the hierarchical model itself are revealed by performing HDM SA.

To fully demonstrate the application and empirical contribution of HDM SA, the whole set of algorithm is applied to two hierarchical decision models reported in literature, one dealing with decision making in technology assessment [52] and another that provides an energy portfolio forecast [43]. In general, the detailed analysis and interpretation of the results help to: 1) improve the understanding of how changes at policy and strategy levels affect decisions at the operational level, 2) test the robustness of the recommended decision, 3) show scenarios of different situations and the corresponding decisions, 4) call special attention to the critical elements in the decision making process, 5) answer "what if" questions, and 6) replicate historical facts to help understand changes.

By applying HDM SA to technology assessment, a strategic technology planning framework was proposed in this dissertation. The model links *synoptic planning mode* and *adaptive planning* mode, the two types of models pervading the strategic planning literature, by improving people's understanding of cause-effect relationships among the decision elements and providing "what if" scenarios for technology managers. This not only improves the comprehensiveness of initial planning, but also provides a base against which business and technology environmental changes are compared periodically, and thus enables the company to respond more quickly in an unstable environment. With the proposed framework, companies can start their technology plan synoptically and follow up with frequent incremental adaptations in order to influence changes in the environment and thereby shape favorable contingencies, reposition competitive goals, shift company strategies, and evaluate newly emerged technologies.

6.2 Limitations and future work

Being independent of different judgment quantification methods is strength of HDM SA algorithm. However, the fact that the SA is performed on intermediate inputs, instead of the pair-wise comparison judgment, may be perceived as a limitation in certain situations [21]. Future extensions will link HMD SA algorithm to the two major pair-wise comparison scales, 1-9 scale with verbal description and constant sum scales, to enable people to visualize the sensitivity of changing pair-wise comparison judgments.

Although an effort is devoted to clearly represent contributions/perturbations among different decision elements at different levels of the MOGSA hierarchy, the cognitive burden on the users of HDM SA algorithm is quite heavy. To ease the difficulties in applying HDM SA algorithm, a computer software program will be developed. Just as the software "Experts Choice®" increased the use of AHP, software that automatically helps people perform HDM SA will inevitably improve the acceptance and popularity of the underlying algorithm.

Limitations of the technology planning framework proposed in this dissertation include the fact that it only deals with changes to decision elements included in the hierarchical model. Unforeseeable changes in the decision environment cannot be predicted and analyzed by the framework. The framework is, however, able to look at the impact of emerging new technologies to the original decision. However, this evaluation is limited to the new technologies' contributions to existing technology strategies. It has been noted while presenting the strategic technology planning framework that when disruptive changes occur, and when the existing hierarchical model is not able to accommodate new decision elements, the hierarchical model must be rebuilt.

REFERENCES

[1] J. Aguaron, and J. M. Moreno-Jimenez, 2000, "Local stability intervals in the analytic hierarchy process," *European Journal of Operational Research*, 125(2000), 113-132

[2] E. R. Alexander, 1989, "Sensitivity analysis in complex decision models," *APA Journal*, 1989(4), 323-333

[3] K. R. Andrews, *The Concept of Corporate Strategy*, Homewood, III.: Dow Jones-Irwin, 1971.

[4] Ansoff, H. I., *Corporate Strategy*, New York: McGraw-Hill, 1965

[5] A. Arbel, and L.G. Vargas, 1990, "The analytic hierarchy process with interval judgments," *Proceedings of the MCDM Conference*, Washington DC, August 1990

[6] R. Armacost, and J. Hosseini, 1994, "Identification of determinant attribute using the analytic hierarchy process," *Journal of the Academy of Marketing Science*, 22(4), 383-392

[7] A. H. Ashton, and R. H. Ashton, 1985, "Aggregating subjective forecasts: Some empirical results," *Management Science*, 31, 1499-1508

[8] R. U. Ayres, 1999, "What have we learned?" *Technological Forecasting and Social Change*, 62 (1999), 9-12

[9] J. Barzilai, and F. A. Lootsma, 1997, "Power Relations and Group Aggregation in the Multiplicative AHP and SMART". *Journal of Multi-Criteria Decision Analysis*, Vol.6, 155 - 165

[10] B. F. Baird, 1989, *Managerial Decisions Under Uncertainty*, John Wiley & Sons, New York

[11] H. Barron, and C. P. Schmidt, 1988, "Sensitivity analysis of additive multiattribute value models, *Operations Research*," 36(1), 122-127

[12] V. Belton, and T. Gear, 1983, "On a short-coming of Saaty's method of analytic hierarchies," Omega, Vol. 11, No. 3, 228-230

[13] V. Belton, and T. Gear, 1985, "The legitimacy of rank reversal – A comment," Omega, Vol. 13, No. 3, 143-144

[14] F. Betz, *Managing Technological Innovation: Competitive Advantage from Change*, 2nd Ed, John Wiley & Sons, Inc, 2003

[15] A. F. Borthick, and J. H. Scheiner, 1998, "Selection of small business computer system: structuring a multi-criteria approach," *Journal of Information System*, Fall 1999, 10-29

[16] D. Braybrooke, C. E. Lindblom, *A Strategy of Decision: Policy Evaluation as a Social Process*, New York: Free Press, 1970

[17] A. H. Briggs, A. E. Ades, and M. J. Price, 2003, "Probabilistic sensitivity analysis for decision trees with multiple branches: use of the dirichlet distribution in a Bayesian framework," *Medical Decision Making*, Jul-Aug 2003, 341-350

[18] N. Bryson, and A. Mobolurin, 1994, "An approach to using the Analytic Hierarchy Process for solving multiple criteria decision making problems," *European Journal of Operational Research*, Vol. 76 Issue 3, 440-455

[19] J. Butler, J. Jia, and J. Dyer, 1997, "Simulation techniques for the sensitivity analysis of multi-criteria decision models," *European Journal of Operational Research*, 103 (1997), 531-546

[20] A. D. Chandler, "Strategy and structure: chapters in the history of the industrial enterprise," MIT press research monographs, Cambridge: MIT Press, 1962, xiv, 463.

[21] H. Chen, D. F. Kocaoglu, 2007,"A sensitivity analysis algorithm for Hierarchical Decision Models," *European Journal of Operational Research*, xx-xx

[22] H. Chen, J. Ho, D. F. Kocaoglu, 2007, "A strategic technology planning framework: A case of Taiwan's semiconductor foundry industry," *IEEE Transactions on Engineering Management*, under revision since Apr 2007.

[23] J. Chen, R. R. Reilly, and G. S. Lynn, 2005, "Speed: Too much of a good thing?" *Technology Management: A Unifying Discipline for Melting the Boundaries*, Proceedings of PICMET '05, Portland, US, 520-532

[24] A. T. W. Chu, R. E. Kalaba, and K. Spingarn, 1979, "A comparison of two methods for determining the weights of belonging to fuzzy sets," *Journal of Optimization Theory and Applications*, Vol. 27, No. 4, 531-538

[25] D. I. Cleland, and D. F. Kocaoglu, 1981, *Engineering Management*, McGraw-Hill, Inc.

[26] R. T. Clemen, *Making hard decisions: An introduction to decision analysis.* Boston, MA: PWS-Kent, 1996.

[27] D. W. Collier, "Linking Business and Technology Strategy," *Planning Review*, Chicago, 1985, 13(5), pp. 28 -35.

[28] A. L. Comrey, 1950, "A proposed method for absolute ratio scaling," *Psychometrika*, Vol. 15, 317-325

[29] R. Cook, and T. Stewart, 1979, "A comparison of seven methods for obtaining objective descriptions of judgmental policy," *Organizational Behavior and Human Performance*, 13, 31-45

[30] G. B. Dantzig, 1963, *Linear Programming and Extensions*, Princeton, NJ: Princeton University Press

[31] J. R. Evans, 1984, "Sensitivity analysis in decision theory," *Decision Sciences*, 15(1), 239-247

[32] *Expert Choice* (Computer Software), 1990, Pittsburgh, PA: Expert Choice Inc.

[33] A. Farkas, A. Gyorgy, and P. Rozsa, 2004, "On the spectrum of pairwise comparison matrices," *Linear Algebra and its Applications*, 385 (2004), 443-462

[34] C. Farrukh, R. Phaal and D. Probert, "Technology roadmapping: linking technology resources into business planning," *International Journal of Technology Management*, 2003, 26(1), pp. 2–19.

[35] J. Fenn, A. Linden, and S. Fairchok, "Strategic technology planning: Picking the winners," *Gartner Research*, R-20-3354, Jul. 2003
[36] W. R. Ferrell, 1985, "Combining individual judgments," *Behavioral Decision Making*, edited by G. Wright, Plenum Press, London, England, 111-145
[37] E. H. Forman, and S. I. Gass, 2001, "The Analytic Hierarchy Process – an exposition," *Operations Research*, 49(4), 469-485
[38] J. W. Fredrickson, T. R. Mitchell, "Strategic decision process: comprehensiveness and performance in an industry with an unstable environment," *Academy of Management Journal*, 27(2), pp. 399-423, 1984.
[39] S. French, and D. Rios Insua, 1989, "Partial information and sensitivity analysis in multi-objective decision making," *Improving Decision Making in Organizations*, LNEMS, Springer-Verlag. 23-41
[40] A. Fusfeld, "How to put Technologies into Corporate Planning," *Technology Review*, May 1978.
[41] W. Gabrielli, and D. Von Winterfeldt, 1978, "Are weights sensitive to the range of alternatives in multiattribute utility measurement?" SSRI Research Report 78-6, Social Science Research Institute, University of Southern California, Los Angeles, CA
[42] N. Gerdsri, 2005, *An Analytical Approach to Building a Technology Development Envelope (TDE) for Roadmapping of Emerging Technologies*, Ph.D. Dissertation, Portland State University
[43] A. H. Gholamnezhad and T. L. Saaty, "A desired energy mix for the United States in the year 2000: An analytic hierarchy process," *International Journal of Policy Analysis and Information Systems*, 6(1), pp. 47-64, 1982
[44] A. H. Gholamnezhad, "1995: The turning point in oil prices," In: *The Global Economy: Today, Tomorrow, and the Transition*, H. G. Didsbury (ed.), World Future Society, Bethesda, MD, pp. 296-314
[45] J. H. Grant, W. R. King, *The Logic of Strategic Planning*, Boston: Little, Brown and Company, pp. 104-122, 1979
[46] J. P. Guilford, 1954, *Psychometric Methods*, McGraw Hill Book Company, INC.
[47] Harrell, Ghosh, and Bowden, 2000, *Simulation Using ProModel*, McGraw-Hill
[48] R. Hastie and T. Kameda, 2005, "The robust beauty of majority rules in group decisions," *Psychological Review*, Vol. 112, No, 2, 494-508
[49] D. Hauser, and P. Tadikamalla, 1996, "The Analytic Hierarchy Process in an uncertain environment: simulation approach," *European Journal of Operational Research*, 91(1996), 27-37
[50] J. M. Hihn, 1980, *A Hierarchical Approach to Decision Making: Economic Theory and Methodology*, Ph. D. Dissertation, University of Maryland, College Park
[51] R. A. Howard, 1968, "The foundations of decision analysis," *IEEE Transactions on Systems Science and Cybernetics*, SSC-4(3)

[52] C. Ho, 2004, *Strategic Evaluation of Emerging Technologies in the Semiconductor Foundry Industry (Special Case: Taiwan Semiconductor Foundry Industry)*, Ph.D. Dissertation, Portland State University

[53] Y. Huang, 2002, "Enhancement on sensitivity analysis of priority in analytical hierarchy process," *International Journal of General System*, 31(5), 531-542

[54] P. Jaramillo, 2005, "Multi-decision-makers equalizer: A multi-objective decision support system for multiple decision makers," Annals of Operations Research, 138, 97-111

[55] R. E. Jensen, 1984, "An alternative scaling method of priorities in hierarchical structures," *Journal of Mathematical Psychology*, Vol. 28, 317-332

[56] C. R. Johnson, W. B. Beine, and T. J. Wang, 1980, "Right-left asymmetry in an eigenvector ranking procedure," *Journal of Mathematical Psychology*, Vol. 19, 61-64

[57] W. D. Kelton, R. P. Sadowski, and D. A. Sadowski, 1998, *Simulation with Arena*, McGraw-Hill, 2nd Edition

[58] G. J. Klir, 2001, *Facets of Systems Science*, 2nd Edition, Kluwer Academic/Plenum Publishers

[59] D. F. Kocaoglu, 1976, *A Systems Approach to the Resource Allocation Process in Police Patrol*, Ph.D. dissertation, University of Pittsburgh

[60] D. F. Kocaoglu, 1983, "A participative approach to program evaluation", *IEEE Transactions on Engineering Management*, Vol. EM-30, No. 3, August 1983

[61] D. F. Kocaoglu, 1983, "Hierarchical Decision Modeling," unpublished paper, University of Pittsburgh

[62] D. F. Kocaoglu, 2002, Decision Making course handout, Portland State University

[63] K. Krajewski, "Five steps to effective strategic technology initiatives," *International IT*, pp. 46 – 47, Sep 2003

[64] M. J. Liberatore, 1989, "A decision support for R&D project selection" in B. L. Golden, E. A. Wasil, and P. T. Harker (eds.) *The Analytic Hierarchy Process—Applications and Studies*, Springer-Verlag Berlin, Heidelberg, 82-100

[65] C. E. Lindblom, 1959, "The science of muddling through," *Public Administration Review*, 19, pp. 79-88

[66] C. E. Lindblom, 1979, "Still muddling, not yet through," *Public Administration Review*, 39, pp. 517-526

[67] F. A. Lootsma, 1999, *Multi-Criteria Decision Analysis via Ratio and Difference Judgment*, Applied Optimization Series, Vol. 29, Kluwer Academic Publishers, Dordrecht, the Netherlands.

[68] S. Makridakis, and R. L. Winkler, 1983, "Averages of forecasts: Some empirical results," Management Science, 29, 987-966

[69] T. Masuda, 1990, "Hierarchical sensitivity analysis of the priority used in analytic hierarchy process," *International Journal of System Science*, 21(2), 415-427

[70] N. W. McGuaghey, "Corporate technology planning," *Industrial Management*, pp. 1 – 2, Mar/Apr 1990

[71] R. G. McGrath, "A real options logic for initiating technology positioning investments," *Academy of Management Review*, 22(4), pp. 974-996, 1997

[72] R. E. Miles and C. C. Snow, 1978, *Organizational Strategy, Structure, and Process*. McGraw-Hill series in management. New York, NY: McGraw-Hill, xiii, 274.

[73] R. E. Miles and C. C. Snow, 1994, *Fit, Failure, and The Hall of Fame: How Companies Succeed or Fail*. New York, NY: Free Press, iv, 215.

[74] A. Miller, "A taxonomy of technological settings, with related strategies and performance levels," *Strategic Management Journal*, 9(3) pp. 239 - 254, 1988.

[75] G. A. Miller, 1956, "The Magical Number Seven Plus or Minus Two: Some Limits on our Capacity for Processing Information," *Psychological Review* 63, pp. 81-97

[76] H. Mintzberg, 1973, "Strategy-making in three modes," *California Management Review*, 16(2), pp. 44-53.

[77] H. Mintzberg, 1978, "Patterns in strategy formation," *Management Science*, 24(9), pp. 934-948.

[78] G. R. Mitchell, 1990, "Alternative frameworks for technology strategy," *European Journal of Operational Research*, 10(5), pp. 153 – 161.

[79] V. L. Mitchell, R. W. Zmud, "Endogenous adaptation: the effects of technology position and planning mode on IT-enabled change," *Decision Sciences*, 37(3), pp. 325-355, 2006

[80] J. M. Moreno-Jimenez, and L.G. Vargas, 1993, "A probabilistic study of preference structures in the Analytic Hierarchy Process with interval judgments," *Mathematical and Computer Modeling*, 17, 73-81

[81] R. T. Moriarty, and T. J. Kosnik, 1990, "High-tech marketing: Concepts, continuity, and change," *IEE Engineering Management Review*, 25-35

[82] K. G. Murty, 1976, *Linear and Combinatorial Programming*, John Wiley & Sons, Inc.

[83] R. E. Nisbett, E. Borgida, R. Crandall, and H. Reed, "Popular induction: Information is not necessarily informative," in J. S. Carroll & J. W. Payne (Eds.), Cognition and Social Behavior, 2, 227-236, 1976.

[84] C. K. Park, W.J. Kim, and S. Park, 2005, "On the properties of e-sensitivity analysis for linear programming," *Asia-Pacific Journal of Operational Research*, Vol. 22, No. 2(2005), 135-151

[85] F. Y. Partovi, 1994, "Determining what to benchmark: an Analytic Hierarchy Process approach", *International Journal of Operations & Production Management*, Vol. 14, Issue 6, 25-40

[86] D. T. Phillips, A. Ravindran, and J. J. Solberg, 1976, *Operations Research Principles and Practice*, John Wiley & Sons, Inc.

[87] L. D. Phillips, "Requisite decision modeling," *Journal of the Operational Research Society*, 33, pp. 303-312, 1982

[88] M. E. Porter, 1980, *Competitive Strategy: Techniques for Analyzing Industries and Competitors*. 1st ed. New York, NY: Free Press. xxviii, 396.

[89] M. E. Porter, 1983, "The technological dimension of competitive strategy," *Management and Policy*, 1, pp. 1 – 33.

[90] M. E. Porter, 1996, "What is strategy?" *Harvard Business Review*, 74(6): p. 61-78.

[91] T. R. Prabhu, and K. Vizayakumar, "Technology choice using FHDM: A case of iron-making technology," *IEEE transactions on Engineering Management*, 48(2), pp. 209-222, 2001

[92] J. B. Quinn, "Strategic change: Logical incrementalism," *Sloan Management Review*, 20(1), pp. 7-21, 1978

[93] J.W. Ra, 1988, *Analysis of Expert Judgments in Hierarchical Decision Process*, Ph.D. dissertation, University of Pittsburgh

[94] V. Ramanujam and T. L. Saaty, "Technology choice in the less developed countries: An analytic hierarchy process," *Technological Forecasting & Social Changes*, 1994, vol. 19, pp. 81-98.

[95] C. T. Ragsdale, 2001, *Spreadsheet Modeling and Decision Analysis*, South-Western: Thomson Publishers, 3rd Edition

[96] T. Reilly, 2000, "Sensitivity analysis for dependant variables," *Decision Sciences*, Vol. 31 Issue 3, p551, 22p

[97] T. L. Saaty, 1980, *The Analytic Hierarchy Process*, New York: McGraw Hill

[98] T. L. Saaty, 2000, *Fundamentals of Decision Making and Priority Theory with the Analytic Hierarchy Process*, Vol. 6 of the AHP Series, RWS Publications

[99] T. L. Saaty and A. H. Gholamnezhad, 1981, "Oil prices: 1985 and 1990," *Energy Systems and Policy*, 5(4), pp. 303-318

[100] A. Saltelli, 2004, "Global sensitivity analysis: an introduction," Tutorial, International Conference on Sensitivity Analysis, Santa Fe, New Mexico, March 8-11, 2004

[101] L. P. Santiago, and T. G. Bifano, 2005, "Management of R&D projects under uncertainty: A multidimensional approach to managerial flexibility," *IEEE Transactions on Engineering Management*, Vol. 52, No. 2, 269-280

[102] A. Segars, V. Grover, "Profiles of strategic information system planning," *Information Systems Research*, 10(3), pp. 199-233, 1999

[103] Y. E. Spanos and S. Lioukas, "An examination into the causal logic of rent generation: Contrasting Porter's competitive strategy framework and the resource-based perspective," *Strategic Management Journal*, 22(10), pp. 907-934, 2001.

[104] R. U. Subba, N. Rangaraj and A. W. Date, "The influence of development perspectives on the choice of technology," *Technological Forecasting & Social Changes*, vol. 48, pp. 27-43, 1995.

[105] K. Sugihara, and H. Tanaka, 2001, "Interval evaluations in the Analytic Hierarchy Process by possibility analysis", *Computational Intelligence*, Vol. 17, No. 3, 567-579

[106] C. Suh, E. Suh, and K. Baek, 2004, "Prioritizing Telecommunications technologies for long-range R&D planning to the year 2006," *IEEE Transactions on Engineering Management*, Vol. 41, No. 3, 264-275

[107] E. Triantaphyllou, and A. Sanchez, 1997, "A sensitivity analysis approach for some deterministic multi-criteria decision-making methods," *Decision Science*, 28(1), 151-194

[108] L. G. Vargas, 1998, "Comments on Barzilai and Lootsma: Why the multiplicative AHP is invalid: a practical counterexample," *Journal of Multi-criteria Decision Analysis*, Vol. 6, Issue 3, 169-170

[109] K. Wang, C. K. Wang and C. Hu, 2005, "Analytic Hierarchy Process with fuzzy scoring in evaluating multidisciplinary R&D projects in China," *IEEE Transactions on Engineering Management*, Vol. 52, No. 1, 119-129

[110] Y. Wang, J. Yang, and D. Xu, 2005, "Interval weight generation approaches based on consistency test and interval comparison matrices", *Applied Mathematics & Computation*, Vol. 167, Issue 1, 252-273

[111] S. Watson, and D. Buede, 1987, *Decision Synthesis*, Cambridge: Cambridge University Press.

[112] R. E. Wendel, 1992, "Sensitivity analysis revisited and extended", *Decision Sciences*, 23(5), 1127-1142

[113] J. J. Winebrake, and B.P. Creswick, 2003, "The future of hydrogen fueling systems for transportation: An application of perspective-based scenario analysis using the analytic hierarchy process," *Technological Forecasting and Social Change*, 70 (2003), 359-384

[114] J. Yeh, B. Kreng, and C. Lin, 2001, "A consensus approach for synthesizing the elements of comparison matrix in the Analytic Hierarchy Process," *International Journal of System Science*, 32(11), 1353-1363

[115] "Bush state of the union: Work with me," news retrieved on Feb 28, 2007, from http://www.cnn.com/2007/POLITICS/01/23/bush.sotu/index.html

[116] Energy information administration, at http://www.eia.doe.gov/

APPENDIX

A.1 Mathematical Deduction for Theorem 1.1

When a perturbation $P_{\ell^*}^O$ $(-C_{\ell^*}^O < P_{\ell^*}^O < 1 - C_{\ell^*}^O)$ is induced on one of the C_ℓ^O's, which is $C_{\ell^*}^O$, the new value of $C_{\ell^*}^O$ is:

$$C_{\ell^*}^O(new) = C_{\ell^*}^O + P_{\ell^*}^O$$

The new values of other C_ℓ^O's are:

$$C_\ell^O(new) = C_\ell^O + P_\ell^O, \text{ with } P_\ell^O = -\frac{P_{\ell^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}$$

Therefore, the new value of C_i^A can be represented as:

$$C_i^A(new) = (C_{\ell^*}^O + P_{\ell^*}^O) \times C_{i\ell^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell^*}^{L} (C_\ell^O + P_\ell^O) \times C_{i\ell}^{A-O}$$

$$= (C_{\ell^*}^O \times C_{i\ell^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O \times C_{i\ell}^{A-O}) + P_{\ell^*}^O \times C_{i\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{i\ell}^{A-O} \times \frac{P_{\ell^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}$$

Since $C_{\ell^*}^O \times C_{i\ell^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O \times C_{i\ell}^{A-O} = C_i^A$

Then
$$C_i^A(new) = C_i^A + P_{\ell^*}^O \times (C_{i\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{i\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}) \qquad \text{(B1.1)}$$

Therefore, the perturbation $P_{\ell^*}^O$ induced on any C_ℓ^O value will result in a change to the values of C_i^A equaling to $P_{\ell^*}^O \times (C_{i\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{i\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O})$.

A.2 Mathematical Deduction for Theorem 1.2

When a perturbation $P_{k^*\ell^*}^{G-O}$ $(-C_{k^*\ell^*}^{G-O} < P_{k^*\ell^*}^{G-O} < 1 - C_{k^*\ell^*}^{G-O})$ is induced on one of the $C_{k\ell}^{G-O}$, the new value of $C_{k^*\ell^*}^{G-O}$ is: $C_{k^*\ell^*}^{G-O}(new) = C_{k^*\ell^*}^{G-O} + P_{k^*\ell^*}^{G-O}$

And the new value of other $C_{k\ell^*}^{G-O}$ will be:

$$C_{k\ell^*}^{G-O}(new) = C_{k\ell^*}^{G-O} + P_{k\ell^*}^{G-O}, \text{ with } P_{k\ell^*}^{G-O} = -P_{k^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}$$

Therefore, the new values of C_i^A can be represented as:

$$C_i^A(new) = \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{k=1}^{K} C_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G}$$

$$= \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{k=1,k\neq k^*}^{K} C_{\ell^*}^O (C_{k\ell^*}^{G-O} - P_{k^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}) C_{ik}^{A-G} + C_{\ell^*}^O (C_{k^*\ell^*}^{G-O} + P_{k^*\ell^*}^{G-O}) C_{ik^*}^{A-G}$$

$$= \sum_{\substack{\ell=1\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G} + C_{\ell^*}^O C_{k^*\ell^*}^{G-O} C_{ik^*}^{A-G} - \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{\ell^*}^O C_{ik}^{A-G} P_{k^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}$$

$$+ C_{\ell^*}^O C_{ik^*}^{A-G} P_{k^*\ell^*}^{G-O}$$

Since $C_i^A = \sum_{\substack{\ell=1\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G} + C_{\ell^*}^O C_{k^*\ell^*}^{G-O} C_{ik^*}^{A-G}$

Then $C_i^A(new) = C_i^A - \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{\ell^*}^O C_{ik}^{A-G} P_{k^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}} + C_{\ell^*}^O C_{ik^*}^{A-G} P_{k^*\ell^*}^{G-O}$

$$= C_i^A + P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{ik^*}^{A-G} - \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{ik}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}) \quad \text{(B1.2)}$$

Therefore, the perturbation $P_{k^*\ell^*}^{G-O}$ induced on any $C_{k\ell}^{G-O}$ value will result in a change to the values of C_i^A equaling to: $P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{ik^*}^{A-G} - \sum_{\substack{k=1\\ k\neq k^*}}^{K} C_{ik}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}})$

A.3 Mathematical Deduction for Theorem 1.3

When a perturbation $P_{i^*j^*}^{A-S}$ ($-C_{i^*j^*}^{A-S} < P_{i^*j^*}^{A-S} < 1 - C_{i^*j^*}^{A-S}$) is induced on one of the C_{ij}^{A-S}, denoted as $C_{i^*j^*}^{A-S}$, the new value of $C_{i^*j^*}^{A-S}$ is:

$$C_{i^*j^*}^{A-S}(new) = C_{i^*j^*}^{A-S} + P_{i^*j^*}^{A-S}$$

And the new value of $C_{ij^*}^{A-S}$ will be:

$$C_{ij^*}^{A-S}(new) = C_{ij^*}^{A-S} + P_{ij^*}^{A-S}, \text{ with } P_{ij^*}^{A-S} = -P_{i^*j^*}^{A-S} \times \frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij^*}^{A-S}}$$

Therefore, the new values of C_i^A can be represented as:

$$C_{i^*}^{A}(new)=\sum_{j=1,j\neq j^*}^{J}C_j^S C_{i^*j}^{A-S}+C_{j^*}^{S}\times(C_{i^*j^*}^{A-S}+P_{i^*j^*}^{A-S})=C_{i^*}^{A}+C_{j^*}^{S}\times P_{i^*j^*}^{A-S} \qquad \text{(B1.3a)}$$

and

$$C_i^A(new)=\sum_{j=1,j\neq j^*}^{J}C_j^S C_{ij}^{A-S}+C_{j^*}^{S}\times(C_{ij^*}^{A-S}-P_{i^*j^*}^{A-S}\times\frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij^*}^{A-S}})=C_i^A-C_{j^*}^{S}\times P_{i^*j^*}^{A-S}\times\frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij^*}^{A-S}}$$

(B1.3b)

Therefore, the perturbation $P_{i^*j^*}^{A-S}$ induced on any C_{ij}^{A-S} value will result in a change to the values of C_i^A equaling to: $C_{j^*}^{S}\times P_{i^*j^*}^{A-S}$ (when $P_{i^*j^*}^{A-S}$ is induced on C_{ij}^{A-S}, so $i=i^*$) or $-C_{j^*}^{S}\times P_{i^*j^*}^{A-S}\times\frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij^*}^{A-S}}$ (when $P_{i^*j^*}^{A-S}$ is not induced on C_{ij}^{A-S}, so $i\neq i^*$).

A.4 Mathematical Deduction for Theorem 2

When M perturbations $P_{\ell_m^*}^{O}$ ($-C_{\ell_m^*}^{O}<P_{\ell_m^*}^{O}<1-C_{\ell_m^*}^{O}$, $\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_\ell^O-1<\sum_{m=1}^{M}P_{\ell_m^*}^{O}<\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_\ell^O$) are induced on M of the C_ℓ^O's, which are $C_{\ell_m^*}^{O}$, the new values of $C_{\ell_m^*}^{O}$ are:

$$C_{\ell_m^*}^{O}(new)=C_{\ell_m^*}^{O}+P_{\ell_m^*}^{O}$$

Based on the assumption, the other C_ℓ^O's will be changed according to their original ratio scales. Therefore, new values of other C_ℓ^O's are:

$$C_\ell^O(new)=C_\ell^O+P_\ell^O\text{, with }P_\ell^O=-\sum_{m=1}^{M}P_{\ell_m^*}^{O}\times\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_\ell^O}$$

Therefore, the new values of C_i^A can be represented as:

$$C_i^A(new)=\sum_{m=1}^{M}(C_{\ell_m^*}^{O}+P_{\ell_m^*}^{O})\times C_{i\ell_m^*}^{A-O}+\sum_{\ell=1,\ell\neq\ell_m^*}^{L}(C_\ell^O+P_\ell^O)\times C_{i\ell}^{A-O}$$

$$=\sum_{m=1}^{M}C_{\ell_m^*}^{O}\times C_{i\ell_m^*}^{A-O}+\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_\ell^O\times C_{i\ell}^{A-O}+\sum_{m=1}^{M}P_{\ell_m^*}^{O}\times C_{i\ell_m^*}^{A-O}-\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_{i\ell}^{A-O}\times\frac{\sum_{m=1}^{M}P_{\ell_m^*}^{O}\times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L}C_\ell^O}$$

Since $\sum_{m=1}^{M}C_{\ell_m^*}^{O}\times C_{i\ell_m^*}^{A-O}+\sum_{\ell=1,\ell\neq\ell_1^*\ldots\ell_M^*}^{L}C_\ell^O\times C_{i\ell}^{A-O}=C_i^A$

then $C_i^A(new) = C_i^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{i\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{i\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$ (B2.1)

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B2.1 in the inequality, we get:

$$C_r^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} \geq C_{r+n}^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r+n,\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r+n,\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} - \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r\ell_m^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} [P_{\ell_m^*}^O \times (C_{r+n,\ell_m^*}^{A-O} - C_{r\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O})] \quad \text{(B2.2)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new)$, $C_1^A(new) \geq C_3^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq \ldots \geq C_r^A(new) \geq \ldots \geq C_I^A(new)$.

A.5 Mathematical Deduction for Corollary 2.1

When a perturbation $P_{\ell^*}^O$ $(-C_{\ell^*}^O < P_{\ell^*}^O < 1 - C_{\ell^*}^O)$ is induced on one of the C_ℓ^O's, which is $C_{\ell^*}^O$, the new C_i^A's are: $C_i^A(new) = C_i^A + P_{\ell^*}^O \times (C_{i\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{i\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O})$,

according to (B1.1) in section 8.1.

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B1.1 in this inequality, we get:

$$C_r^A + P_{\ell^*}^O \times (C_{r\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}) \geq C_{r+n}^A + P_{\ell^*}^O \times (C_{r+n,\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O})$$

$$C_r^A - C_{r+n}^A \geq P_{\ell^*}^O \times (C_{r+n,\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}) - P_{\ell^*}^O \times (C_{r\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O})$$

$$C_r^A - C_{r+n}^A \geq P_{\ell^*}^O (C_{r+n,\ell^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} - C_{r\ell^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O}) \qquad \text{(B2.3)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new)$, $C_1^A(new) \geq C_3^A(new)$,…, $C_1^A(new) \geq C_r^A(new)$,…, $C_1^A(new) \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq ... \geq C_r^A(new) \geq ... \geq C_I^A(new)$.

A.6 Mathematical Deduction for Corollary 2.2

When M perturbations $P_{\ell_m^*}^O$ ($-C_{\ell_m^*}^O < P_{\ell_m^*}^O < 1 - C_{\ell_m^*}^O$, $\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O - 1 < \sum_{m=1}^{M} P_{\ell_m^*}^O < \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O$) are induced on M of the C_ℓ^O's, which are $C_{\ell_m^*}^O$, the new values of $C_{\ell_m^*}^O$ are:

$$C_{\ell_m^*}^O(new) = C_{\ell_m^*}^O + P_{\ell_m^*}^O$$

And the new values of other C_ℓ^O's are:

$$C_\ell^O(new) = C_\ell^O + P_\ell^O \text{, with } P_\ell^O = -\sum_{m=1}^{M} P_{\ell_m^*}^O \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

Therefore, the new values of C_i^A can be represented as:

$$C_i^A(new) = \sum_{m=1}^{M} (C_{\ell_m^*}^O + P_{\ell_m^*}^O) \times C_{i\ell_m^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} (C_\ell^O + P_\ell^O) \times C_{i\ell}^{A-O}$$

$$= \sum_{m=1}^{M} C_{\ell_m^*}^O \times C_{i\ell_m^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O \times C_{i\ell}^{A-O} + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{i\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{i\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

Since $\sum_{m=1}^{M} C_{\ell_m^*}^O \times C_{i\ell_m^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell_1^*...\ell_M^*}^{L} C_\ell^O \times C_{i\ell}^{A-O} = C_i^A$

Then $C_i^A(new) = C_i^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{i\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{i\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$ (B2.4)

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B2.4 in the inequality, we get:

$$C_r^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} \geq C_{r+n}^A + \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r+n,\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r+n,\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} - \sum_{m=1}^{M} P_{\ell_m^*}^O \times C_{r\ell_m^*}^{A-O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{\sum_{m=1}^{M} P_{\ell_m^*}^O \times C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} [P_{\ell_m^*}^O \times (C_{r+n,\ell_m^*}^{A-O} - C_{r\ell_m^*}^{A-O} - \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r+n,\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O} + \sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_{r\ell}^{A-O} \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell_m^*}^{L} C_\ell^O})] \quad \text{(B2.5)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq \ldots \geq C_r^A(new) \geq \ldots \geq C_I^A(new)$.

A.7 Mathematical Deduction for Theorem 3

When M perturbations $P_{k_m^*\ell_\alpha^*}^{G-O}$ (m=1, 2…M) are induced in M of the $C_{k\ell_\alpha}^{G-O}$'s, denoted as $C_{k_m^*\ell_\alpha^*}^{G-O}$, T perturbations $P_{k_t^*\ell_\beta^*}^{G-O}$ (t = 1, 2…T) are induced in T of the $C_{k\ell_\beta}^{G-O}$'s, denoted as $C_{k_t^*\ell_\beta^*}^{G-O}$, Q perturbations $P_{k_q^*\ell_\gamma^*}^{G-O}$ (q = 1, 2…Q) are induced in Q of the $C_{k\ell_\gamma}^{G-O}$'s, denoted as $C_{k_q^*\ell_\gamma^*}^{G-O}$, based on the assumptions, the new values of $C_{k_m^*\ell_\alpha^*}^{G-O}$'s and other $C_{k\ell_\alpha^*}^{G-O}$'s will be:

$$C_{k_m^*\ell_\alpha^*}^{G-O}(new) = C_{k_m^*\ell_\alpha^*}^{G-O} + P_{k_m^*\ell_\alpha^*}^{G-O}$$

$$C_{k\ell_\alpha^*}^{G-O}(new) = C_{k\ell_\alpha^*}^{G-O} + P_{k\ell_\alpha^*}^{G-O}, \text{ with } P_{k\ell_m^*}^{G-O} = -\sum_{m=1}^{M} P_{k_m^*\ell_\alpha^*}^{G-O} \times \frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{k=1, k\neq k_m^*}^{K} C_{k\ell_\alpha^*}^{G-O}}$$

The new values of $C_{k_t^*\ell_\beta^*}^{G-O}$'s and of other $C_{k\ell_\beta^*}^{G-O}$'s will be:

$$C_{k_t^*\ell_\beta^*}^{G-O}(new) = C_{k_t^*\ell_\beta^*}^{G-O} + P_{k_t^*\ell_\beta^*}^{G-O}$$

$$C_{k\ell_\beta^*}^{G-O}(new) = C_{k\ell_\beta^*}^{G-O} + P_{k\ell_\beta^*}^{G-O}, \text{ with } P_{k\ell_\beta^*}^{G-O} = -\sum_{t=1}^{T} P_{k_t^*\ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K} C_{k\ell_\beta^*}^{G-O}}$$

The new values of $C_{k_q^*\ell_\gamma^*}^{G-O}$'s and of other $C_{k\ell_\gamma^*}^{G-O}$'s will be:

$$C_{k_q^*\ell_\gamma^*}^{G-O}(new) = C_{k_q^*\ell_\gamma^*}^{G-O} + P_{k_q^*\ell_\gamma^*}^{G-O}$$

$$C_{k\ell_\gamma^*}^{G-O}(new) = C_{k\ell_\gamma^*}^{G-O} + P_{k\ell_\gamma^*}^{G-O}, \text{ with } P_{k\ell_\gamma^*}^{G-O} = -\sum_{q=1}^{Q} P_{k_q^*\ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K} C_{k\ell_\gamma^*}^{G-O}}$$

Therefore, the new values of C_i^A's can be represented as:

$$C_i^A(new) = \sum_{\substack{\ell=1\\ \ell\neq\ell_\alpha^*\\ \ell\neq\ell_\beta^*\\ \ell\neq\ell_\gamma^*}}^{L} \sum_{\substack{k=1\\ k\neq k_m^*\\ k\neq k_t^*\\ k\neq k_q^*}}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{\ell_\alpha^*}^O \left(C_{k\ell_\alpha^*}^{G-O} - \sum_{m=1}^{M} P_{k_m^*\ell_\alpha^*}^{G-O} \times \frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{k\ell_\alpha^*}^{G-O}}\right) C_{ik}^{A-G}$$

$$+ \sum_{m=1}^{M} C_{\ell_\alpha^*}^O \left(C_{k_m^*\ell_\alpha^*}^{G-O} + P_{k_m^*\ell_\alpha^*}^{G-O}\right) C_{ik_m^*}^{A-G} + \sum_{\substack{k=1\\k\neq k_t^*}}^{K} C_{\ell_\beta^*}^O \left(C_{k\ell_\beta^*}^{G-O} - \sum_{t=1}^{T} P_{k_t^*\ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K} C_{k\ell_\beta^*}^{G-O}}\right) C_{ik}^{A-G}$$

$$+ \sum_{t=1}^{T} C_{\ell_\beta^*}^O \left(C_{k_t^*\ell_\beta^*}^{G-O} + P_{k_t^*\ell_\beta^*}^{G-O}\right) C_{ik_t^*}^{A-G} + \sum_{\substack{k=1\\k\neq k_q^*}}^{K} C_{\ell_\gamma^*}^O \left(C_{k\ell_\gamma^*}^{G-O} - \sum_{q=1}^{Q} P_{k_q^*\ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K} C_{k\ell_\gamma^*}^{G-O}}\right) C_{ik}^{A-G}$$

$$+ \sum_{q=1}^{Q} C_{\ell_\gamma^*}^O \left(C_{k_q^*\ell_\gamma^*}^{G-O} + P_{k_q^*\ell_\gamma^*}^{G-O}\right) C_{ik_q^*}^{A-G}$$

$$C_i^A(new)=\sum_{\ell=1}^{L}\sum_{k=1}^{K}C_\ell^O C_{k\ell}^{G-O}C_{ik}^{A-G}-\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{\ell_\alpha^*}^O(\sum_{m=1}^{M}P_{k_m^*\ell_\alpha^*}^{G-O}\times\frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{k\ell_\alpha^*}^{G-O}})C_{ik}^{A-G}+\sum_{m=1}^{M}C_{\ell_\alpha^*}^O P_{k_m^*\ell_\alpha^*}^{G-O}C_{ik_m^*}^{A-G}$$

$$-\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{\ell_\beta^*}^O(\sum_{t=1}^{T}P_{k_t^*\ell_\beta^*}^{G-O}\times\frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{k\ell_\beta^*}^{G-O}})C_{ik}^{A-G}+\sum_{t=1}^{T}C_{\ell_\beta^*}^O P_{k_t^*\ell_\beta^*}^{G-O}C_{ik_t^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{\ell_\gamma^*}^O(\sum_{q=1}^{Q}P_{k_q^*\ell_\gamma^*}^{G-O}\times\frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{k\ell_\gamma^*}^{G-O}})C_{ik}^{A-G}$$

$$+\sum_{q=1}^{Q}C_{\ell_\gamma^*}^O P_{k_q^*\ell_\gamma^*}^{G-O}C_{ik_q^*}^{A-G}$$

Since $C_i^A=\sum_{\ell=1}^{L}\sum_{k=1}^{K}C_\ell^O C_{k\ell}^{G-O}C_{ik}^{A-G}$, then

$$C_i^A(new)=C_i^A-\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{\ell_\alpha^*}^O(\sum_{m=1}^{M}P_{k_m^*\ell_\alpha^*}^{G-O}\times\frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{k\ell_\alpha^*}^{G-O}})C_{ik}^{A-G}+\sum_{m=1}^{M}C_{\ell_\alpha^*}^O P_{k_m^*\ell_\alpha^*}^{G-O}C_{ik_m^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{\ell_\beta^*}^O C_{ik}^{A-G}(\sum_{t=1}^{T}P_{k_t^*\ell_\beta^*}^{G-O}$$

$$\times\frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{k\ell_\beta^*}^{G-O}})+\sum_{t=1}^{T}C_{\ell_\beta^*}^O P_{k_t^*\ell_\beta^*}^{G-O}C_{ik_t^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{\ell_\gamma^*}^O(\sum_{q=1}^{Q}P_{k_q^*\ell_\gamma^*}^{G-O}\times\frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{k\ell_\gamma^*}^{G-O}})C_{ik}^{A-G}+\sum_{q=1}^{Q}C_{\ell_\gamma^*}^O P_{k_q^*\ell_\gamma^*}^{G-O}C_{ik_q^*}^{A-G}$$

(B3.1)

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new)\geq C_{r+n}^A(new)$. By substituting equation B3.1 in the inequality, we get:

$$C_r^A-\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{\ell_\alpha^*}^O C_{rk}^{A-G}(\sum_{m=1}^{M}P_{k_m^*\ell_\alpha^*}^{G-O}\times\frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{k\ell_\alpha^*}^{G-O}})+\sum_{m=1}^{M}C_{\ell_\alpha^*}^O P_{k_m^*\ell_\alpha^*}^{G-O}C_{rk_m^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{\ell_\beta^*}^O C_{rk}^{A-G}(\sum_{t=1}^{T}P_{k_t^*\ell_\beta^*}^{G-O}\times\frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{k\ell_\beta^*}^{G-O}})$$

$$+\sum_{t=1}^{T}C_{\ell_\beta^*}^O P_{k_t^*\ell_\beta^*}^{G-O}C_{rk_t^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{\ell_\gamma^*}^O C_{rk}^{A-G}(\sum_{q=1}^{Q}P_{k_q^*\ell_\gamma^*}^{G-O}\times\frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{k\ell_\gamma^*}^{G-O}})+\sum_{q=1}^{Q}C_{\ell_\gamma^*}^O P_{k_q^*\ell_\gamma^*}^{G-O}C_{rk_q^*}^{A-G}\geq$$

$$C_{r+n}^A-\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{\ell_\alpha^*}^O C_{r+n,k}^{A-G}(\sum_{m=1}^{M}P_{k_m^*\ell_\alpha^*}^{G-O}\times\frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K}C_{k\ell_\alpha^*}^{G-O}})+\sum_{m=1}^{M}C_{\ell_\alpha^*}^O P_{k_m^*\ell_\alpha^*}^{G-O}C_{r+n,k_m^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{\ell_\beta^*}^O C_{r+n,k}^{A-G}(\sum_{t=1}^{T}P_{k_t^*\ell_\beta^*}^{G-O}\times\frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_t^*}}^{K}C_{k\ell_\beta^*}^{G-O}})$$

$$+\sum_{t=1}^{T}C_{\ell_\beta^*}^O P_{k_t^*\ell_\beta^*}^{G-O}C_{r+n,k_t^*}^{A-G}-\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{\ell_\gamma^*}^O C_{r+n,k}^{A-G}(\sum_{q=1}^{Q}P_{k_q^*\ell_\gamma^*}^{G-O}\times\frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_q^*}}^{K}C_{k\ell_\gamma^*}^{G-O}})+\sum_{q=1}^{Q}C_{\ell_\gamma^*}^O P_{k_q^*\ell_\gamma^*}^{G-O}C_{r+n,k_q^*}^{A-G}$$

....

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} C_{\ell_\alpha^*}^{O} P_{k_m^*\ell_\alpha^*}^{G-O} (C_{r+n,k_m^*}^{A-G} - C_{rk_m^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_m^*}}^{K} C_{\ell_\alpha^*}^{O} (\sum_{m=1}^{M} P_{k_m^*\ell_\alpha^*}^{G-O} \times \frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_m^*}}^{K} C_{k\ell_\alpha^*}^{G-O}})(C_{rk}^{A-G} - C_{r+n,k}^{A-G})$$

$$+ \sum_{t=1}^{T} C_{\ell_\beta^*}^{O} P_{k_t^*\ell_\beta^*}^{G-O} (C_{r+n,k_t^*}^{A-G} - C_{rk_t^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_t^*}}^{K} C_{\ell_\beta^*}^{O} (\sum_{t=1}^{T} P_{k_t^*\ell_\beta^*}^{G-O} \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_t^*}}^{K} C_{k\ell_\beta^*}^{G-O}})(C_{rk}^{A-G} - C_{r+n,k}^{A-G})$$

$$+ \sum_{q=1}^{Q} C_{\ell_\gamma^*}^{O} P_{k_q^*\ell_\gamma^*}^{G-O} (C_{r+n,k_q^*}^{A-G} - C_{rk_q^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C_{\ell_\gamma^*}^{O} (\sum_{q=1}^{Q} P_{k_q^*\ell_\gamma^*}^{G-O} \times \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C_{k\ell_\gamma^*}^{G-O}})(C_{rk}^{A-G} - C_{r+n,k}^{A-G})$$

....

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} P_{k_m^*\ell_\alpha^*}^{G-O} \times [C_{\ell_\alpha^*}^{O} (C_{r+n,k_m^*}^{A-G} - C_{rk_m^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_m^*}}^{K} C_{\ell_\alpha^*}^{O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \frac{C_{k\ell_\alpha^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_m^*}}^{K} C_{k\ell_\alpha^*}^{G-O}}] +$$

$$\sum_{t=1}^{T} P_{k_t^*\ell_\beta^*}^{G-O} \times [C_{\ell_\beta^*}^{O} (C_{r+n,k_t^*}^{A-G} - C_{rk_t^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_t^*}}^{K} C_{\ell_\beta^*}^{O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \times \frac{C_{k\ell_\beta^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_t^*}}^{K} C_{k\ell_\beta^*}^{G-O}}] +$$

$$\sum_{q=1}^{Q} P_{k_q^*\ell_\gamma^*}^{G-O} [C_{\ell_\gamma^*}^{O} (C_{r+n,k_q^*}^{A-G} - C_{rk_q^*}^{A-G}) + \sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C_{\ell_\gamma^*}^{O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \frac{C_{k\ell_\gamma^*}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k_q^*}}^{K} C_{k\ell_\lambda^*}^{G-O}}] \quad \text{(B3.2)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new)$, $C_1^A(new) \geq C_3^A(new)$, …, $C_1^A(new) \geq C_I^A(new)$. The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq ... \geq C_r^A(new) \geq ... \geq C_I^A(new)$.

A.8 Mathematical Deduction for Corollary 3.1

When a perturbation $P_{k^*\ell^*}^{G-O}$ $(-C_{k^*\ell^*}^{G-O} < P_{k^*\ell^*}^{G-O} < 1 - C_{k^*\ell^*}^{G-O})$ is induced on one of the $C_{k\ell}^{G-O}$, based on (B1.2), $C_i^A(new) = C_i^A + P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^{O} \times (C_{ik^*}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{ik}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}})$.

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B1.2 in the inequality, we get:

$$C_r^A + P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{rk^*}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{rk}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}) \geq C_{r+n}^A + P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{r+n,k^*}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{r+n,k}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}})$$

$$C_r^A - C_{r+n}^A \geq P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{r+n,k^*}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{r+n,k}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}}) -$$

$$P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times (C_{rk^*}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{rk}^{A-G} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}})$$

$$C_r^A - C_{r+n}^A \geq P_{k^*\ell^*}^{G-O} \times C_{\ell^*}^O \times \left[C_{r+n,k^*}^{A-G} - C_{rk^*}^{A-G} - (\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{r+n,k}^{A-G} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{rk}^{A-G}) \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*}^{G-O}} \right] \quad \text{(B3.3)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq \ldots \geq C_r^A(new) \geq \ldots \geq C_I^A(new)$.

A.9 Mathematical Deduction for Corollary 3.2

When M perturbations $P_{k_m^*\ell^*}^{G-O}$ (m=1, 2…M) are induced in matrix $\mathbf{C}_{k\ell}^{G-O}$ on contributions of M goals $G_{k_m^*}$'s to a specific objective O_{ℓ^*}, the new values of $C_{k_m^*\ell^*}^{G-O}$'s will be: $C_{k_m^*\ell^*}^{G-O}(new) = C_{k_m^*\ell^*}^{G-O} + P_{k_m^*\ell^*}^{G-O}$

and the new values of the other $C_{k\ell^*}^{G-O}$'s will be:

$$C_{k\ell^*}^{G-O}(new) = C_{k\ell^*}^{G-O} + P_{k\ell^*}^{G-O}, \text{ with } P_{k\ell^*}^{G-O} = -\sum_{m=1}^{M} P_{k_m^*\ell^*}^{G-O} \times \frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell^*}^{G-O}}$$

Therefore, the new values of C_i^A can be represented as:

$$C_i^A(\text{new}) = \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{k=1}^{K} C_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G}$$

$$=\sum_{\substack{\ell=1\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}C_\ell^O C_{k\ell}^{G-O}C_{ik}^{A-G}+\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O\Big(C_{k\ell^*}^{G-O}-\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)C_{ik}^{A-G}+\sum_{m=1}^{M}C_{\ell^*}^O\big(C_{k_m^*\ell^*}^{G-O}+P_{k_m^*\ell^*}^{G-O}\big)C_{ik_m^*}^{A-G}$$

$$=\sum_{\substack{\ell=1\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}C_\ell^O C_{k\ell}^{G-O}C_{ik}^{A-G}+\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{k\ell^*}^{G-O}C_{ik}^{A-G}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{ik}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)$$

$$+\sum_{m=1}^{M}C_{\ell^*}^O C_{k_m^*\ell^*}^{G-O}C_{ik_m^*}^{A-G}+\sum_{m=1}^{M}C_{\ell^*}^O C_{ik_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}$$

Since $C_i^A=\sum_{\substack{\ell=1\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}C_\ell^O C_{k\ell}^{G-O}C_{ik}^{A-G}+\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{k\ell^*}^{G-O}C_{ik}^{A-G}+\sum_{m=1}^{M}C_{\ell^*}^O C_{k_m^*\ell^*}^{G-O}C_{ik_m^*}^{A-G}$

Then $C_i^A(new)=C_i^A+\sum_{m=1}^{M}C_{\ell^*}^O C_{ik_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{ik}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)$ (B3.4)

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new)\geq C_{r+n}^A(new)$. By substituting equation B3.4 in the inequality, we get:

$$C_r^A+\sum_{m=1}^{M}C_{\ell^*}^O C_{rk_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{rk}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)\geq$$

$$C_{r+n}^A+\sum_{m=1}^{M}C_{\ell^*}^O C_{r+n,k_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{r+n,k}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)$$

$$C_r^A-C_{r+n}^A\geq\sum_{m=1}^{M}C_{\ell^*}^O C_{r+n,k_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{r+n,k}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)$$

$$-\sum_{m=1}^{M}C_{\ell^*}^O C_{rk_m^*}^{A-G}P_{k_m^*\ell^*}^{G-O}+\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{rk}^{A-G}\times\Big(\sum_{m=1}^{M}P_{k_m^*\ell^*}^{G-O}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)$$

$$C_r^A-C_{r+n}^A\geq$$

$$\sum_{m=1}^{M}\left[P_{k_m^*\ell^*}^{G-O}\times\Big(C_{\ell^*}^O C_{r+n,k_m^*}^{A-G}-C_{\ell^*}^O C_{rk_m^*}^{A-G}-\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{r+n,k}^{A-G}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}+\sum_{\substack{k=1\\ k\neq k_m^*}}^{K}C_{\ell^*}^O C_{rk}^{A-G}\times\frac{C_{k\ell^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K}C_{k\ell^*}^{G-O}}\Big)\right]$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new)\geq C_2^A(new),\ldots,C_1^A(new)\geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new)\geq C_2^A(new)\geq\ldots\geq C_r^A(new)\geq\ldots\geq C_I^A(new)$.

A.10 Mathematical Deduction for Corollary 3.3

When M perturbations $P^{G-O}_{k^*_m \ell^*_m}$ (m=1, 2…M) are induced in matrix $\mathbf{C}^{G-O}_{k\ell}$ for contributions of M goals $G_{k^*_m}$'s to M objectives $O_{\ell^*_m}$'s, the new value of $C^{G-O}_{k^*_m \ell^*_m}$'s will be: $C^{G-O}_{k^*_m \ell^*_m}(new) = C^{G-O}_{k^*_m \ell^*_m} + P^{G-O}_{k^*_m \ell^*_m}$

The new values of other $C^{G-O}_{k\ell^*_m}$'s will be:

$$C^{G-O}_{k\ell^*_m}(new) = C^{G-O}_{k\ell^*_m} + P^{G-O}_{k\ell^*_m}, \text{ with } P^{G-O}_{k\ell^*_m} = -P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^{G-O}_{k\ell^*_m}}$$

Therefore, the new values of C^A_i's can be represented as:

$$C^A_i(new) = \sum_{\substack{\ell=1\\ \ell\neq\ell^*_m}}^{L} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_\ell C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{m=1}^{M} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} (C^{G-O}_{k\ell^*_m} - P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^{G-O}_{k\ell^*_m}}) C^{A-G}_{ik}$$

$$+ \sum_{m=1}^{M} C^O_{\ell^*_m} (C^{G-O}_{k^*_m \ell^*_m} + P^{G-O}_{k^*_m \ell^*_m}) C^{A-G}_{ik^*_m}$$

$$= \sum_{\substack{\ell=1\\ \ell\neq\ell^*_m}}^{L} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_\ell C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{m=1}^{M} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} C^{G-O}_{k\ell^*_m} C^{A-G}_{ik} - \sum_{m=1}^{M} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} (P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^{G-O}_{k\ell^*_m}}) C^{A-G}_{ik}$$

$$+ \sum_{m=1}^{M} C^O_{\ell^*_m} C^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{ik^*_m} + \sum_{m=1}^{M} C^O_{\ell^*_m} P^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{ik^*_m}$$

Since $C^A_i = \sum\limits_{\substack{\ell=1\\ \ell\neq\ell^*_m}}^{L} \sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_\ell C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum\limits_{m=1}^{M} \sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} C^{G-O}_{k\ell^*_m} C^{A-G}_{ik} + \sum\limits_{m=1}^{M} C^O_{\ell^*_m} C^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{ik^*_m}$

Then $C^A_i(new) = C^A_i - \sum\limits_{m=1}^{M} \sum\limits_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} (P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{k=1,k\neq k^*_m}^{K} C^{G-O}_{k\ell^*_m}}) C^{A-G}_{ik} + \sum\limits_{m=1}^{M} C^O_{\ell^*_m} P^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{ik^*_m}$ (B3.5)

The ranking of A_r and A_{r+n} will not be reversed if $C^A_r(new) \geq C^A_{r+n}(new)$. By substituting equation B3.5 in the inequality, we get:

$$C^A_r - \sum_{m=1}^{M} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} (P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{k=1,k\neq k^*_m}^{K} C^{G-O}_{k\ell^*_m}}) C^{A-G}_{rk} + \sum_{m=1}^{M} C^O_{\ell^*_m} P^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{rk^*_m} \geq$$

$$C^A_{r+n} - \sum_{m=1}^{M} \sum_{\substack{k=1\\k\neq k^*_m}}^{K} C^O_{\ell^*_m} (P^{G-O}_{k^*_m \ell^*_m} \times \frac{C^{G-O}_{k\ell^*_m}}{\sum\limits_{k=1,k\neq k^*_m}^{K} C^{G-O}_{k\ell^*_m}}) C^{A-G}_{r+n,k} + \sum_{m=1}^{M} C^O_{\ell^*_m} P^{G-O}_{k^*_m \ell^*_m} C^{A-G}_{r+n,k^*_m}$$

$$C_r^A - C_{r+n}^A \geq -\sum_{m=1}^{M}\sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{\ell_m^*}^O (P_{k_m^*\ell_m^*}^{G-O} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell_m^*}^{G-O}}) C_{r+n,k}^{A-G} + \sum_{m=1}^{M} C_{\ell_m^*}^O P_{k_m^*\ell_m^*}^{G-O} C_{r+n,k_m^*}^{A-G}$$

$$+\sum_{m=1}^{M}\sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{\ell_m^*}^O (P_{k_m^*\ell_m^*}^{G-O} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell_m^*}^{G-O}}) C_{rk}^{A-G} - \sum_{m=1}^{M} C_{\ell_m^*}^O P_{k_m^*\ell_m^*}^{G-O} C_{rk_m^*}^{A-G}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} P_{k_m^*\ell_m^*}^{G-O} \times C_{\ell_m^*}^O \left[C_{r+n,k_m^*}^{A-G} - C_{rk_m^*}^{A-G} + \sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{rk}^{A-G} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell_m^*}^{G-O}} - \sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{r+n,k}^{A-G} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{k=1,k\neq k_m^*}^{K} C_{k\ell_m^*}^{G-O}} \right] \quad \text{(B3.6)}$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2…I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq \ldots \geq C_I^A(new)$.

A.11 Mathematical Deduction for Corollary 3.4

When M perturbations $P_{k^*\ell_m^*}^{G-O}$ (m=1, 2…M) are induced in matrix $\mathbf{C}_{k\ell}^{G-O}$ for contributions of a specific goal G_{k^*} to M specific objectives $O_{\ell_m^*}$'s, the new value of $C_{k^*\ell_m^*}^{G-O}$'s will be:

$$C_{k^*\ell_m^*}^{G-O}(new) = C_{k^*\ell_m^*}^{G-O} + P_{k^*\ell_m^*}^{G-O}$$

The new values of other $C_{k\ell_m^*}^{G-O}$'s will be:

$$C_{k\ell_m^*}^{G-O}(new) = C_{k\ell_m^*}^{G-O} + P_{k\ell_m^*}^{G-O}, \text{ with } P_{k\ell_m^*}^{G-O} = -P_{k^*\ell_m^*}^{G-O} \times \frac{C_{k\ell_m^*}^{G-O}}{\sum_{\substack{k=1\\k\neq k_m^*}}^{K} C_{k\ell_m^*}^{G-O}}$$

Therefore, the new values of C_i^A's can be represented as:

$$C_i^A(new) = \sum_{\substack{\ell=1 \\ \ell\neq\ell^*_m}}^{L}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(C_{k\ell^*_m}^{G-O} - P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{ik}^{A-G} +$$

$$\sum_{m=1}^{M} C_{\ell^*_m}^O \left(C_{k^*\ell^*_m}^{G-O} + P_{k^*\ell^*_m}^{G-O}\right) C_{ik^*}^{A-G}$$

$$= \sum_{\substack{\ell=1 \\ \ell\neq\ell^*_m}}^{L}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O C_{k\ell^*_m}^{G-O} C_{ik}^{A-G} - \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{ik}^{A-G}$$

$$+ \sum_{m=1}^{M} C_{\ell^*_m}^O C_{k^*\ell^*_m}^{G-O} C_{ik^*}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{ik^*}^{A-G}$$

Since $C_i^A = \sum_{\substack{\ell=1 \\ \ell\neq\ell^*_m}}^{L}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O C_{k\ell^*_m}^{G-O} C_{ik}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O C_{k^*\ell^*_m}^{G-O} C_{ik^*}^{A-G}$

Then $C_i^A(new) = C_i^A - \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{ik}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{ik^*}^{A-G}$ (B3.7)

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B3.7 in the inequality, we get:

$$C_r^A - \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{rk}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{rk^*}^{A-G} \geq$$

$$C_{r+n}^A - \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{r+n,k}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{r+n,k^*}^{A-G}$$

$$C_r^A - C_{r+n}^A \geq -\sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{r+n,k}^{A-G} + \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{r+n,k^*}^{A-G}$$

$$+ \sum_{m=1}^{M}\sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{\ell^*_m}^O \left(P_{k^*\ell^*_m}^{G-O} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}}\right) C_{rk}^{A-G} - \sum_{m=1}^{M} C_{\ell^*_m}^O P_{k^*\ell^*_m}^{G-O} C_{rk^*}^{A-G}$$

$$C_r^A - C_{r+n}^A \geq \sum_{m=1}^{M} P_{k^*\ell^*_m}^{G-O} \times C_{\ell^*_m}^O \left[C_{r+n,k^*}^{A-G} - C_{rk^*}^{A-G} + \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{rk}^{A-G} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}} - \sum_{\substack{k=1 \\ k\neq k^*}}^{K} C_{r+n,k}^{A-G} \times \frac{C_{k\ell^*_m}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell^*_m}^{G-O}} \right]$$

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

The rank order for all A_i's will remain the same if the above condition is satisfied for all r=1, 2...I-1, and n=1, which means $C_1^A(new) \geq C_2^A(new) \geq ... \geq C_r^A(new) \geq ... \geq C_I^A(new)$.

A.12 Mathematical Deduction for Theorem 4

When M perturbations $P_{i_m^* j_\alpha^*}^{A-S}$ (m=1, 2...M) are induced in M of the $C_{ij_\alpha}^{A-S}$'s, denoted as $C_{i_m^* j_\alpha^*}^{A-S}$ (contributions of M actions A_{i*} to the α^{th} changing strategy $S_{j_\alpha^*}$), T perturbations $P_{i_t^* j_\beta^*}^{A-S}$ (t = 1, 2...T) are induced in T of the $C_{ij_\beta}^{A-S}$'s, denoted as $C_{i_t^* j_\beta^*}^{A-S}$ (contributions of T actions G_{k*} to the β^{th} changing strategy $S_{j_\beta^*}$), Q perturbations $P_{i_q^* j_\gamma^*}^{A-S}$ (q = 1, 2...Q) are induced in Q of the $C_{k\ell_\gamma}^{G-O}$'s, denoted as $C_{k_q^* \ell_\gamma^*}^{G-O}$ (contributions of Q actions G_{k*} to the γ^{th} changing strategy $S_{j_\gamma^*}$), based on the assumption, the new values of $C_{i^*}^A$'s and of other C_i^A's will be:

$$C_{i^*}^A(new) = C_{i^*}^A + C_{j_\alpha^*}^S P_{i_m^* j_\alpha^*}^{A-S} - C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{ij_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{ij_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \tag{B4.1}$$

$$C_i^A(new) = C_i^A - C_{j_\alpha^*}^S \times \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{ij_\alpha^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} - C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{ij_\beta^*}^{A-S}}{\sum_{i=1, i \neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}$$

$$- C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{ij_\gamma^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \tag{B4.2}$$

In (B4.1), α, β, and γ can be replaced with each other according to situation.

Suppose a pair of A_i's (A_r and A_{r+n}, $C_r^A \geq C_{r+n}^A$) are being compared. There will be four situations: (1) r=i* and r+n$\neq$i*; (2) r$\neq$i* and r+n=i*; (3) r$\neq$i* and r+n$\neq$i*; or (4) r=i* and r+n=i*. In these four situations, the ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$; therefore, we have the following inequalities regarding the four situations:

(1) r=i* and r+n ≠ i*

$$C^A_{r^*} + C^S_{j^*_\alpha} P^{A-S}_{r^*_m j^*_\alpha} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{rj^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{r^*_q j^*_\gamma}\times\frac{C^{A-S}_{rj^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}}$$

$$\geq C^A_{r+n} - C^S_{j^*_\alpha}\times\sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{r+n,j^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{r+n,j^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{i^*_q j^*_\gamma}\times\frac{C^{A-S}_{r+n,j^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}}$$

$$C^A_{r^*} - C^A_{r+n} \geq -C^S_{j^*_\alpha}\left(P^{A-S}_{r^*_m j^*_\alpha} + \sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{r+n,j^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}}\right) + C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{rj^*_\beta} - C^{A-S}_{r+n,j^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}}$$

$$+ C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{r^*_q j^*_\gamma}\times\frac{C^{A-S}_{rj^*_\gamma} - C^{A-S}_{r+n,j^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}} \qquad \text{(B4.3)}$$

(2) r ≠ i* and r+n=i*

$$C^A_{r} - C^S_{j^*_\alpha}\times\sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{rj^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{rj^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{i^*_q j^*_\gamma}\times\frac{C^{A-S}_{rj^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}}$$

$$\geq C^A_{r+n^*} + C^S_{j^*_\alpha}\times P^{A-S}_{r+n^* j^*_\alpha} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{r+n,j^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{r^*_q j^*_\gamma}\times\frac{C^{A-S}_{r+n,j^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}}$$

$$C^A_{r} - C^A_{r+n^*} \geq C^S_{j^*_\alpha}\left(\sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{rj^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}} + P^{A-S}_{r+n^* j^*_\alpha}\right) + C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{rj^*_\beta} - C^{A-S}_{r+n,j^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}}$$

$$+ C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{i^*_q j^*_\gamma}\times\frac{C^{A-S}_{rj^*_\gamma} - C^{A-S}_{r+n,j^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}} \qquad \text{(B4.4)}$$

(3) r ≠ i* and r+n ≠ i*

$$C^A_{r} - C^S_{j^*_\alpha}\times\sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{rj^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{rj^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{i^*_q j^*_\gamma}\times\frac{C^{A-S}_{rj^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}} \geq$$

$$C^A_{r+n} - C^S_{j^*_\alpha}\times\sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_\alpha}\times\frac{C^{A-S}_{r+n,j^*_\alpha}}{\sum_{i=1,i\neq i^*_m}^{I} C^{A-S}_{ij^*_\alpha}} - C^S_{j^*_\beta}\times\sum_{t=1}^{T} P^{A-S}_{i^*_t j^*_\beta}\times\frac{C^{A-S}_{r+n,j^*_\beta}}{\sum_{i=1,i\neq i^*_t}^{I} C^{A-S}_{ij^*_\beta}} - C^S_{j^*_\gamma}\times\sum_{q=1}^{Q} P^{A-S}_{i^*_q j^*_\gamma}\times\frac{C^{A-S}_{r+n,j^*_\gamma}}{\sum_{i=1,i\neq i^*_q}^{I} C^{A-S}_{ij^*_\gamma}}$$

$$C_r^A - C_{r+n}^A \geq C_{j_\alpha^*}^S \times \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{rj_\alpha^*}^{A-S} - C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \quad \text{(B4.5)}$$

(4.1) r=i* and r+n=i* (when the perturbations of r* and r+n* are induced to the same S_j, assumed to be $C_{j_\alpha^*}^S$ here.)

$$C_{r^*}^A + C_{j_\alpha^*}^S P_{r_m^* j_\alpha^*}^{A-S} - C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \geq$$

$$C_{r+n^*}^A + C_{j_\alpha^*}^S P_{r+n_m^* j_\alpha^*}^{A-S} - C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}}$$

$$C_{r^*}^A - C_{r+n^*}^A \geq C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S} - C_{r+n,j_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} + C_{j_\alpha^*}^S (P_{r+n_m^* j_\alpha^*}^{A-S} - P_{r_m^* j_\alpha^*}^{A-S}) \quad \text{(B4.6)}$$

(4.2) r=i* and r+n=i* (when the perturbations of r* and r+n* are induced to different S_j, assumed to be $C_{j_\alpha^*}^S$ and $C_{j_\beta^*}^S$ here.)

$$C_{r^*}^A + C_{j_\alpha^*}^S P_{r_m^* j_\alpha^*}^{A-S} - C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \geq$$

$$C_{r+n^*}^A + C_{j_\beta^*}^S P_{r+n_m^* j_\beta^*}^{A-S} - C_{j_\alpha^*}^S \times \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} - C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}}$$

$$C_{r^*}^A - C_{r+n^*}^A \geq C_{j_\beta^*}^S P_{r+n_m^* j_\beta^*}^{A-S} - C_{j_\alpha^*}^S P_{r_m^* j_\alpha^*}^{A-S} - C_{j_\alpha^*}^S \times \sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S} \times \frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^S \times \sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^S \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}}$$

Since $\sum_{m=1}^{M} P_{i_m^* j_\alpha^*}^{A-S}$ includes $P_{r_m^* j_\alpha^*}^{A-S}$, and $\sum_{t=1}^{T} P_{i_t^* j_\beta^*}^{A-S}$ includes $P_{r+n_m^* j_\beta^*}^{A-S}$, then

$$C_{r^*}^{A} - C_{r+n^*}^{A} \geq P_{r+n_m^* j_\beta^*}^{A-S} C_{j_\beta^*}^{S} \left(1+\frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}}\right) - P_{r_m^* j_\alpha^*}^{A-S} C_{j_\alpha^*}^{S} \left(1+\frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}}\right) - C_{j_\alpha^*}^{S} \sum_{\substack{i_m^*=i_1^* \\ i_m^*\neq r_m^*}}^{M} P_{i_m^* j_\alpha^*}^{A-S}$$

$$\times \frac{C_{r+n,j_\alpha^*}^{A-S}}{\sum_{i=1,i\neq i_m^*}^{I} C_{ij_\alpha^*}^{A-S}} + C_{j_\beta^*}^{S} \times \sum_{\substack{i_t^*=i_1^*, \\ i_t^*\neq r+n_m^*}}^{T} P_{i_t^* j_\beta^*}^{A-S} \times \frac{C_{rj_\beta^*}^{A-S}}{\sum_{i=1,i\neq i_t^*}^{I} C_{ij_\beta^*}^{A-S}} + C_{j_\gamma^*}^{S} \times \sum_{q=1}^{Q} P_{i_q^* j_\gamma^*}^{A-S} \times \frac{C_{rj_\gamma^*}^{A-S} - C_{r+n,j_\gamma^*}^{A-S}}{\sum_{i=1,i\neq i_q^*}^{I} C_{ij_\gamma^*}^{A-S}} \quad \text{(B4.7)}$$

For the ranking of all A_i's to remain the same, the condition $C_r^A(new) \geq C_{r+n}^A(new)$ needs to be satisfied for all r=1, 2…I-1 and n=1, which means $C_1^A(new) \geq C_2^A(new)$, $C_1^A(new) \geq C_3^A(new)$, …, $C_1^A(new) \geq C_I^A(new)$. This includes all the situations being discussed above.

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \geq C_2^A(new)$,…, $C_1^A(new) \geq C_I^A(new)$. This includes all the situations being discussed above.

A.13 Mathematical Deduction for Corollary 4.1

When a perturbation $P_{i^*j^*}^{A-S}$ $(-C_{i^*j^*}^{A-S} < P_{i^*j^*}^{A-S} < 1-C_{i^*j^*}^{A-S})$ is induced on one of the C_{ij}^{A-S}'s, denoted as $C_{i^*j^*}^{A-S}$, based on (B1.3a) and (B1.3b) in section 8.3, the new values of C_i^A are: $C_{i^*}^{A}(new) = \sum_{j=1,j\neq j^*}^{J} C_j^S C_{i^*j}^{A-S} + C_{j^*}^{S} \times (C_{i^*j^*}^{A-S} + P_{i^*j^*}^{A-S}) = C_{i^*}^{A} + C_{j^*}^{S} \times P_{i^*j^*}^{A-S}$ (B1.3a), and

$$C_i^A(new) = \sum_{j=1,j\neq j^*}^{J} C_j^S C_{ij}^{A-S} + C_{j^*}^{S} \times \left(C_{ij^*}^{A-S} - P_{i^*j^*}^{A-S} \times \frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij^*}^{A-S}}\right) = C_i^A - C_{j^*}^{S} \times P_{i^*j^*}^{A-S} \times \frac{C_{ij^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij^*}^{A-S}}$$

(B1.3b)

Suppose a pair of A_i's (A_r and A_t, $C_r^A \geq C_t^A$) are selected to be compared, there will be three kinds of situations: (1) r=i*; (2) t=i*; or (3) r≠i* and t≠i*. In these three situations, the ranking of A_r and A_t will not be reversed if $C_r^A(new) \geq C_t^A(new)$; therefore, we have the following inequalities regarding the three situations:

(1) $$C_{r^*}^{A} + C_{j^*}^{S} \times P_{r^*j^*}^{A-S} \geq C_t^A - C_{j^*}^{S} \times P_{r^*j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij^*}^{A-S}}$$

$$C_{r^*}^A - C_t^A \geq -P_{r^*j^*}^{A-S} \times C_{j^*}^S \times (1 + \frac{C_{ij^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}})$$

(B4.8)

(2) $$C_r^A - C_{j^*}^S \times P_{t^*j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}} \geq C_{t^*}^A + C_{j^*}^S \times P_{t^*j^*}^{A-S}$$

$$C_r^A - C_{t^*}^A \geq P_{t^*j^*}^{A-S} \times C_{j^*}^S \times (1 + \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}}) \quad \text{(B4.9)}$$

(3) $$C_r^A - C_{j^*}^S \times P_{i^*j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}} \geq C_t^A - C_{j^*}^S \times P_{i^*j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}}$$

$$C_r^A - C_t^A \geq P_{i^*j^*}^{A-S} \times C_{j^*}^S \times \frac{C_{rj^*}^{A-S} - C_{tj^*}^{A-S}}{\sum_{i=1, i \neq i^*}^{I} C_{ij^*}^{A-S}}$$

(B4.10)

For the ranking of all A_i's to remain the same, the condition $A_r^{new} \geq A_{r+n}^{new}$ needs to be satisfied for all r=1, 2...I-1, and n=1. This includes all the situations being discussed above. Therefore, by replacing "t" with "r+n" in inequalities B4.8 to B4.10, we get the condition for A_i's to remain the original ranking.

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

A.14 Mathematical Deduction for Corollary 4.2

If M perturbations $P_{i_m^*j^*}^{A-S}$ are induced on M of the C_{ij}^{A-S} 's, denoted as $C_{i_m^*j^*}^{A-S}$ (contributions of M actions A_{i^*}'s to a specific strategy S_{j^*}), the new values of C_i^A's are: $C_{i^*}^A(new) = C_{i^*}^A + C_{j^*}^S P_{i_m^*j^*}^{A-S}$ and $C_i^A(new) = C_i^A - C_{j^*}^S \times \sum_{m=1}^{M} P_{i_m^*j^*}^{A-S} \times \frac{C_{ij^*}^{A-S}}{\sum_{i=1, i \neq i_m^*}^{I} C_{ij^*}^{A-S}}$

Suppose a pair of A_i's (A_r and A_t, $C_r^A \geq C_t^A$) are being compared. There will be four situations: (1) r=i* and t≠i*; (2) r≠i* and t=i*; (3) r≠i* and t≠i*; or (4) r=i* and t=i*. In these four situations, the ranking of A_r and A_t will not be reversed if $C_r^A(new) \geq C_t^A(new)$; therefore, we have the following inequalities regarding the four situations:

$$(1)\ C_{r}^{A} + C_{j^*}^{S} \times P_{r_m^* j^*}^{A-S} \ge C_{t}^{A} - C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}}$$

$$C_{r}^{A} - C_{t}^{A} \ge -C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} - C_{j^*}^{S} \times P_{r_m^* j^*}^{A-S}$$

$$C_{r}^{A} - C_{t}^{A} \ge -C_{j^*}^{S} \times \sum_{\substack{m=1 \\ m \ne r}}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} - C_{j^*}^{S} \times P_{r_m^* j^*}^{A-S} \times \left(1 + \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}}\right) \qquad \text{(B4.11)}$$

$$(2)\ C_{r}^{A} - C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} \ge C_{t}^{A} + C_{j^*}^{S} \times P_{t_m^* j^*}^{A-S}$$

$$C_{r}^{A} - C_{t}^{A} \ge C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} + C_{j^*}^{S} \times P_{t_m^* j^*}^{A-S}$$

$$C_{r}^{A} - C_{t}^{A} \ge C_{j^*}^{S} \times \sum_{\substack{m=1 \\ m \ne t}}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} + C_{j^*}^{S} \times P_{t_m^* j^*}^{A-S} \times \left(1 + \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}}\right) \qquad \text{(B4.12)}$$

$$(3)\ C_{r}^{A} - C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} \ge C_{t}^{A} - C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}}$$

$$C_{r}^{A} - C_{t}^{A} \ge C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{rj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} - C_{j^*}^{S} \times \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times \frac{C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}}$$

$$C_{r}^{A} - C_{t}^{A} \ge \sum_{m=1}^{M} P_{i_m^* j^*}^{A-S} \times C_{j^*}^{S} \times \frac{C_{rj^*}^{A-S} - C_{tj^*}^{A-S}}{\sum_{i=1, i \ne i_m^*}^{I} C_{ij^*}^{A-S}} \qquad \text{(B4.13)}$$

$$(4)\ C_{r}^{A} + C_{j^*}^{S} \times P_{r_m^* j^*}^{A-S} \ge C_{t}^{A} + C_{j^*}^{S} \times P_{t_m^* j^*}^{A-S} \longrightarrow C_{r}^{A} - C_{t}^{A} \ge C_{j^*}^{S} \times (P_{t_m^* j^*}^{A-S} - P_{r_m^* j^*}^{A-S}) \qquad \text{(B4.14)}$$

For the ranking of all A_i's to remain the same, the condition $C_r^A(new) \ge C_{r+n}^A(new)$ needs to be satisfied for all r=1, 2…I-1, and n=1. This includes all the situations being discussed above. Therefore, by replacing "t" with "r+n" in inequalities B4.11 to B4.14, we get the condition for all A_i's to remain the original ranking.

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2…I-1, which means $C_1^A(new) \ge C_2^A(new), \ldots, C_1^A(new) \ge C_I^A(new)$.

A.15 Mathematical Deduction for Corollary 4.3

When M perturbations $P_{i^* j_m^*}^{A-S}$ (m=1, 2...M) are induced in M of the C_{ij}^{A-S}'s, denoted as $C_{i^* j_m^*}^{A-S}$ (contributions of a specific action A_{i^*} to specific strategies S_{j^*}'s), the new values of C_i^A's are: $C_{i^*}^A(\text{new}) = C_{i^*}^A + \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S}$ (B4.15)

and $$C_i^A(\text{new}) = C_i^A - \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S} \times \frac{C_{ij_m^*}^{A-S}}{\sum_{i=1}^{I} C_{ij_m^*}^{A-S}} \quad \text{(B4.16)}$$

Suppose a pair of A_i's (A_r and A_t, $C_r^A \geq C_t^A$) are being compared, there are three kinds of situations: (1) r=i^* and t$\neq i^*$; (2) r$\neq i^*$and t=i^*; (3) r$\neq i^*$ and t$\neq i^*$. In these three situations, the ranking of A_r and A_t will not be reversed if $C_r^A(new) \geq C_t^A(new)$, therefore, we have the following inequalities regarding the three situations:

(1) $$C_{r^*}^A + \sum_{m=1}^{M} C_{j_m^*}^S \times P_{r^* j_m^*}^{A-S} \geq C_t^A - \sum_{m=1}^{M} C_{j_m^*}^S \times P_{r^* j_m^*}^{A-S} \times \frac{C_{tj_m^*}^{A-S}}{\sum_{i=1, i\neq r}^{I} C_{ij_m^*}^{A-S}}$$

$$C_{r^*}^A - C_t^A \geq -\sum_{m=1}^{M} C_{j_m^*}^S \times P_{r^* j_m^*}^{A-S} (1+\frac{C_{tj_m^*}^{A-S}}{\sum_{i=1, i\neq r}^{I} C_{ij_m^*}^{A-S}}) \quad \text{(B4.17)}$$

(2) $$C_r^A - \sum_{m=1}^{M} C_{j_m^*}^S \times P_{t^* j_m^*}^{A-S} \times \frac{C_{rj_m^*}^{A-S}}{\sum_{i=1, i\neq t}^{I} C_{ij_m^*}^{A-S}} \geq C_{t^*}^A + \sum_{m=1}^{M} C_{j_m^*}^S \times P_{t^* j_m^*}^{A-S}$$

$$C_r^A - C_{t^*}^A \geq \sum_{m=1}^{M} C_{j_m^*}^S \times P_{t^* j_m^*}^{A-S} \times (1+\frac{C_{rj_m^*}^{A-S}}{\sum_{i=1, i\neq t}^{I} C_{ij_m^*}^{A-S}}) \quad \text{(B4.18)}$$

(3) $$C_r^A - \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S} \times \frac{C_{rj_m^*}^{A-S}}{\sum_{i=1}^{I} C_{ij_m^*}^{A-S}} \geq C_t^A - \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S} \times \frac{C_{tj_m^*}^{A-S}}{\sum_{i=1}^{I} C_{ij_m^*}^{A-S}}$$

$$C_r^A - C_t^A \geq \sum_{m=1}^{M} C_{j_m^*}^S \times P_{i^* j_m^*}^{A-S} \times \frac{C_{rj_m^*}^{A-S} - C_{tj_m^*}^{A-S}}{\sum_{i=1}^{I} C_{ij_m^*}^{A-S}} \quad \text{(B4.19)}$$

For the ranking of all A_i's to remain the same, the condition $C_r^A(new) \geq C_{r+n}^A(new)$ needs to be satisfied for all r=1, 2...I-1 and n=1. This includes all the situations being discussed above. Therefore, replacing "t" with "r+n" in inequalities B4.17 to B4.19, we get the condition for A_i's to remain the original ranking.

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1, which means $C_1^A(new) \geq C_2^A(new), \ldots, C_1^A(new) \geq C_I^A(new)$.

A.16 Mathematical Deduction for Corollary 4.4

When M perturbations $P_{i_m^* j_m^*}^{A-S}$ (m=1, 2...M) are induced in M of the C_{ij}^{A-S}'s, denoted as $C_{i_m^* j_m^*}^{A-S}$ (contributions of M actions A_{i^*}'s to M strategies S_{j^*}'s), the new values of the C_i^A are:

$$C_{i_m^*}^A(\text{new}) = C_{i_m^*}^A + C_{j_m^*}^S \times P_{i_m^* j_m^*}^{A-S} - \sum_{\substack{n=1 \\ n \neq m}}^{M} C_{j_n^*}^S \times P_{i_n^* j_n^*}^{A-S} \times \frac{C_{ij_n^*}^{A-S}}{\sum_{i=1, i \neq i_n^*}^{I} C_{ij_n^*}^{A-S}} \quad \text{(B4.20)}$$

and

$$C_i^A(\text{new}) = C_i^A - \sum_{n=1}^{M} C_{j_n^*}^S \times P_{i_n^* j_n^*}^{A-S} \times \frac{C_{ij_n^*}^{A-S}}{\sum_{i=1, i \neq i_n^*}^{I} C_{ij_n^*}^{A-S}} \quad \text{(B4.21)}$$

Suppose a pair of A_i's (A_r and A_t, $C_r^A \geq C_t^A$) are being compared. There are four situations: (1) r=i_m^* and t≠ i_m^* ; (2) r≠ i_m^* and t=i_m^*; (3) r=i_m^* and t=i_m^*; or (4) r≠ i_m^* and t ≠ i_m^* . In these four situations, the ranking of A_r and A_t will not be reversed if $C_r^A(new) \geq C_t^A(new)$, therefore, we have the following inequalities regarding the four situations:

(1)

$$C_{r_m^*}^A + C_{j_m^*}^S \times P_{r_m^* j_m^*}^{A-S} - \sum_{\substack{q=1 \\ q \neq m}}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{r_m^* j_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}} \geq C_t^A - \sum_{q=1}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{tj_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}}$$

$$C_{r_m^*}^A - C_t^A \geq -\sum_{q=1}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{tj_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}} - C_{j_m^*}^S \times P_{r_m^* j_m^*}^{A-S} + \sum_{\substack{q=1 \\ q \neq m}}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{r_m^* j_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}}$$

$$C_{r_m^*}^A - C_t^A \geq \sum_{\substack{q=1 \\ q \neq m}}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{r_m^* j_q^*}^{A-S} - C_{tj_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}} - C_{j_m^*}^S \times P_{r_m^* j_m^*}^{A-S} \times \left(1 + \frac{C_{tj_m^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_m^*}^{A-S}}\right) \quad \text{(B4.22)}$$

(2)

$$C_r^A - \sum_{q=1}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{rj_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}} \geq C_{t_m^*}^A + C_{j_m^*}^S \times P_{t_m^* j_m^*}^{A-S} - \sum_{\substack{q=1 \\ q \neq m}}^{M} C_{j_q^*}^S \times P_{i_q^* j_q^*}^{A-S} \times \frac{C_{t_m^* j_q^*}^{A-S}}{\sum_{i=1, i \neq i_q^*}^{I} C_{ij_q^*}^{A-S}}$$

$$C^A_r - C^A_{t^*_m} \ge C^S_{j^*_m}\times P^{A-S}_{t^*_m j^*_m} - \sum_{\substack{q=1\\ q\ne m}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{t^*_m j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} + \sum_{q=1}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{rj^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}$$

$$C^A_r - C^A_{t^*_m} \ge \sum_{\substack{q=1\\ q\ne m}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{rj^*_q} - C^{A-S}_{t^*_m j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} + P^{A-S}_{t^*_m j^*_m}\times C^S_{j^*_m}\times\left(1+\frac{C^{A-S}_{rj^*_m}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_m}}\right) \qquad \text{(B4.23)}$$

(3)

$$C^A_{r^*_m} + C^S_{j^*_m}\times P^{A-S}_{r^*_m j^*_m} - \sum_{\substack{q=1\\ q\ne m}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{r^*_m j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} \ge C^A_{t^*_p} + C^S_{j^*_p}\times P^{A-S}_{t^*_p j^*_p} - \sum_{\substack{q=1\\ q\ne p}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}$$

$$C^A_{r^*_m} - C^A_{t^*_p} \ge C^S_{j^*_p}\times P^{A-S}_{t^*_p j^*_p} - \sum_{\substack{q=1\\ q\ne p}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}} - C^S_{j^*_m}\times P^{A-S}_{r^*_m j^*_m} + \sum_{\substack{q=1\\ q\ne m}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{r^*_m j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}$$

$$C^A_{r^*_m} - C^A_{t^*_p} \ge C^S_{j^*_p}\times P^{A-S}_{t^*_p j^*_p}\times\left(1+\frac{C^{A-S}_{r^*_m j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}\right) - C^S_{j^*_m}\times P^{A-S}_{r^*_m j^*_m}\times\left(1+\frac{C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}\right) + \sum_{\substack{q=1\\ q\ne m\\ q\ne p}}^{M} C^S_{j^*_q}\times P^{A-S}_{i^*_q j^*_q}\times \frac{C^{A-S}_{r^*_m j^*_q} - C^{A-S}_{t^*_p j^*_q}}{\sum_{i=1,i\ne i^*_q}^{I} C^{A-S}_{ij^*_q}}$$

(B4.24)

(4)
$$C^A_r - \sum_{m=1}^{M} C^S_{j^*_m}\times P^{A-S}_{i^*_m j^*_m}\times \frac{C^{A-S}_{rj^*_m}}{\sum_{i=1,i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}} \ge C^A_t - \sum_{m=1}^{M} C^S_{j^*_m}\times P^{A-S}_{i^*_m j^*_m}\times \frac{C^{A-S}_{tj^*_m}}{\sum_{i=1,i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}}$$

$$C^A_r - C^A_t \ge \sum_{m=1}^{M} C^S_{j^*_m}\times P^{A-S}_{i^*_m j^*_m}\times \frac{C^{A-S}_{rj^*_m}}{\sum_{i=1,i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}} - \sum_{m=1}^{M} C^S_{j^*_m}\times P^{A-S}_{i^*_m j^*_m}\times \frac{C^{A-S}_{tj^*_m}}{\sum_{i=1,i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}}$$

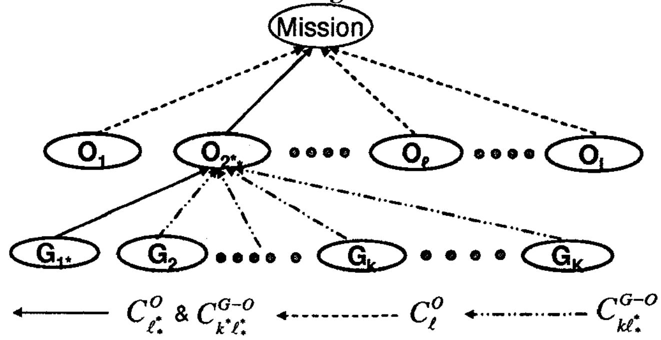
$$C^A_r - C^A_t \ge \sum_{m=1}^{M} P^{A-S}_{i^*_m j^*_m}\times C^S_{j^*_m}\times \frac{C^{A-S}_{rj^*_m} - C^{A-S}_{tj^*_m}}{\sum_{i=1,i\ne i^*_m}^{I} C^{A-S}_{ij^*_m}} \qquad \text{(B4.25)}$$

For the ranking of all A_i's to remain the same, the condition $C^A_r(new) \ge C^A_{r+n}(new)$ needs to be satisfied for all r=1, 2...I-1 and n=1. This includes all the situations being discussed above. Therefore, replacing "t" with "r+n" in inequalities B4.22 to B4.25, we get the condition for A_i's to remain the original ranking.

The top choice will remain at the top rank if the above condition is satisfied for all r=1 and n=1, 2...I-1, which means $C^A_1(new) \ge C^A_2(new), \ldots, C^A_1(new) \ge C^A_I(new)$.

A.17 Mathematical deduction for Theorem 5.1

Fig 2.



As shown in Fig 2, when a perturbation $P^O_{\ell^*}$ $(-C^O_{\ell_*} < P^O_{\ell^*} < 1 - C^O_{\ell_*})$ is induced on one of the C^O_ℓ's, which is $C^O_{\ell_*}$, and a perturbation $P^{G-O}_{k^*\ell_*}$ $(-C^{G-O}_{k^*\ell_*} < P^{G-O}_{k^*\ell_*} < 1 - C^{G-O}_{k^*\ell_*})$ is induced on one of the $C^{G-O}_{k\ell_*}$'s, which is $C^{G-O}_{k^*\ell_*}$; The new value of $C^O_{\ell_*}$ is:

$$C^O_{\ell_*}(new) = C^O_{\ell_*} + P^O_{\ell^*}$$

The new values of other C^O_ℓ's are:

$$C^O_\ell(new) = C^O_\ell + P^O_\ell \text{, with } P^O_\ell = -\frac{P^O_{\ell^*} \times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}$$

The new value of $C^{G-O}_{k^*\ell_*}$ is:

$$C^{G-O}_{k^*\ell_*}(new) = C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*} \tag{B5.1}$$

The new value of other $C^{G-O}_{k\ell_*}$ will be:

$$C^{G-O}_{k\ell_*}(new) = C^{G-O}_{k\ell_*} + P^{G-O}_{k\ell_*} \text{, with } P^{G-O}_{k\ell_*} = -P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}} \tag{B5.2}$$

Therefore, the new values of C^A_i can be represented as:

$$C^A_i(new) = \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{k=1}^{K} C^O_\ell(new) C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{k=1,k\neq k^*}^{K} C^O_{\ell_*}(new) C^{G-O}_{k\ell_*}(new) C^{A-G}_{ik}$$
$$+ C^O_{\ell_*}(new) C^{G-O}_{k^*\ell_*}(new) C^{A-G}_{ik^*}$$
$$= \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{k=1}^{K} \left(C^O_\ell - \frac{P^O_{\ell^*} \times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\right) C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{k=1,\, k\neq k^*}^{K} (C^O_{\ell_*} + P^O_{\ell^*})\left(C^{G-O}_{k\ell_*} - P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}}\right) C^{A-G}_{ik}$$
$$+ (C^O_{\ell_*} + P^O_{\ell^*})(C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*}) C^{A-G}_{ik^*}$$

$$= \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}\Big(C^O_\ell - \frac{P^O_{\ell^*}\times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C^O_\ell}\Big)C^{G-O}_{k\ell}C^{A-G}_{ik} + \sum_{k=1,k\neq k^*}^{K}P^O_{\ell^*}\Big(C^{G-O}_{k\ell^*} - P^{G-O}_{k^*\ell^*}\times\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}\Big)C^{A-G}_{ik}$$

$$+\sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}\Big(C^{G-O}_{k\ell^*} - P^{G-O}_{k^*\ell^*}\times\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}\Big)C^{A-G}_{ik} + (C^O_{\ell^*}+P^O_{\ell^*})(C^{G-O}_{k^*\ell^*}+P^{G-O}_{k^*\ell^*})C^{A-G}_{ik^*}$$

$$= \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}C^O_\ell C^{G-O}_{k\ell}C^{A-G}_{ik} - \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}\frac{P^O_{\ell^*}\times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C^O_\ell}C^{G-O}_{k\ell}C^{A-G}_{ik} + \sum_{k=1,k\neq k^*}^{K}P^O_{\ell^*}C^{G-O}_{k\ell^*}C^{A-G}_{ik}$$

$$-\sum_{k=1,k\neq k^*}^{K}P^O_{\ell^*}P^{G-O}_{k^*\ell^*}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}C^{A-G}_{ik} + \sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}C^{G-O}_{k\ell^*}C^{A-G}_{ik} - \sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}P^{G-O}_{k^*\ell^*}\times\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}C^{A-G}_{ik}$$

$$+C^O_{\ell^*}C^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*} + C^O_{\ell^*}P^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*} + P^O_{\ell^*}C^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*} + P^O_{\ell^*}P^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*}$$

$$= C^A_i - \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}\frac{C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C^O_\ell}C^{G-O}_{k\ell}C^{A-G}_{ik}P^O_{\ell^*} + \sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}C^{A-G}_{ik}P^O_{\ell^*} + P^O_{\ell^*}C^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*} + C^O_{\ell^*}C^{A-G}_{ik^*}P^{G-O}_{k^*\ell^*}$$

$$-\sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}C^{A-G}_{ik}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}P^{G-O}_{k^*\ell^*} + C^{A-G}_{ik^*}P^O_{\ell^*}P^{G-O}_{k^*\ell^*} - \sum_{k=1,k\neq k^*}^{K}C^{A-G}_{ik}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}P^O_{\ell^*}P^{G-O}_{k^*\ell^*}$$

Then $$C^A_i(new) = C^A_i - \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}\frac{C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C^O_\ell}C^{G-O}_{k\ell}C^{A-G}_{ik}P^O_{\ell^*} + \sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}C^{A-G}_{ik}P^O_{\ell^*} + P^O_{\ell^*}C^{G-O}_{k^*\ell^*}C^{A-G}_{ik^*}$$

$$+C^O_{\ell^*}C^{A-G}_{ik^*}P^{G-O}_{k^*\ell^*} - \sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}C^{A-G}_{ik}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}P^{G-O}_{k^*\ell^*} + C^{A-G}_{ik^*}P^O_{\ell^*}P^{G-O}_{k^*\ell^*} - \sum_{k=1,k\neq k^*}^{K}C^{A-G}_{ik}\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}}P^O_{\ell^*}P^{G-O}_{k^*\ell^*}$$

(B5.3)

The ranking of A_r and A_{r+n} will not be reversed if $C^A_r(new) \geq C^A_{r+n}(new)$. By substituting equation B 5.3 in this inequality, we get:

$$C^A_r - \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}C^{G-O}_{k\ell}C^{A-G}_{rk}P^O_{\ell^*}\times\frac{C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C^O_\ell} - \sum_{k=1,k\neq k^*}^{K}C^O_{\ell^*}C^{A-G}_{rk}P^{G-O}_{k^*\ell^*}\times\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}} + C^O_{\ell^*}P^{G-O}_{k^*\ell^*}C^{A-G}_{rk^*}$$

$$+P^O_{\ell^*}C^{G-O}_{k^*\ell^*}C^{A-G}_{rk^*} - \sum_{k=1,k\neq k^*}^{K}P^O_{\ell^*}P^{G-O}_{k^*\ell^*}C^{A-G}_{rk}\times\frac{C^{G-O}_{k\ell^*}}{\sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}} + P^O_{\ell^*}P^{G-O}_{k^*\ell^*}C^{A-G}_{rk^*} + \sum_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell^*}C^{A-G}_{rk}P^O_{\ell^*} \geq$$

$$C^{A}_{r+n}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}C^{G-O}_{k\ell}C^{A-G}_{r+n,k}P^{O}_{\ell^*}\times\frac{C^{O}_{\ell}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L}C^{O}_{\ell}}+P^{O}_{\ell^*}P^{G-O}_{k^*\ell_*}C^{A-G}_{r+n,k^*}-\sum_{\substack{k=1,\\ k\neq k^*}}^{K}C^{O}_{\ell_*}C^{A-G}_{r+n,k}P^{G-O}_{k^*\ell_*}\times\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}$$

$$+C^{O}_{\ell_*}C^{A-G}_{r+n,k^*}P^{G-O}_{k^*\ell_*}+C^{G-O}_{k^*\ell_*}C^{A-G}_{r+n,k^*}P^{O}_{\ell^*}-\sum_{\substack{k=1,\\ k\neq k^*}}^{K}P^{O}_{\ell^*}P^{G-O}_{k^*\ell_*}C^{A-G}_{r+n,k}\times\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}+\sum_{\substack{k=1,\\ k\neq k^*}}^{K}C^{G-O}_{k\ell_*}C^{A-G}_{r+n,k}P^{O}_{\ell^*}$$

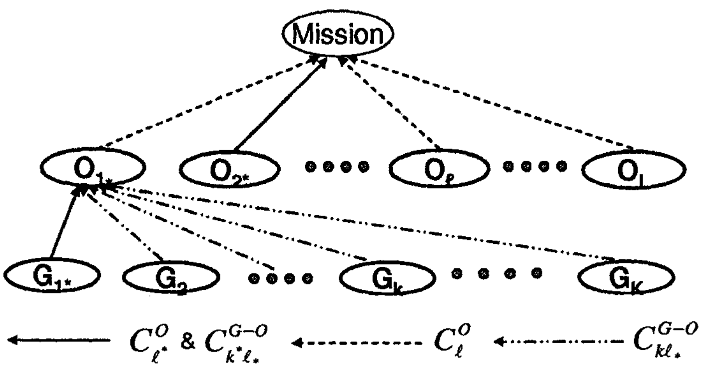
$$C^{A}_{r}-C^{A}_{r+n}\geq\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{k=1}^{K}C^{G-O}_{k\ell}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})P^{O}_{\ell^*}\times\frac{C^{O}_{\ell}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L}C^{O}_{\ell}}+\sum_{\substack{k=1,\\ k\neq k^*}}^{K}P^{O}_{\ell^*}P^{G-O}_{k^*\ell_*}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})$$

$$\times\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}+\sum_{\substack{k=1,\\ k\neq k^*}}^{K}C^{O}_{\ell_*}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})P^{G-O}_{k^*\ell_*}\times\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}-C^{O}_{\ell_*}P^{G-O}_{k^*\ell_*}(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})$$

$$-P^{O}_{\ell^*}C^{G-O}_{k^*\ell_*}(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})-P^{O}_{\ell^*}P^{G-O}_{k^*\ell_*}(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})-\sum_{\substack{k=1,\\ k\neq k^*}}^{K}C^{G-O}_{k\ell_*}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})P^{O}_{\ell^*}$$

$$\begin{aligned}C^{A}_{r}-C^{A}_{r+n}\geq{}&[\sum_{\substack{k=1,\\ k\neq k^*}}^{K}\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})-(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})]C^{O}_{\ell_*}P^{G-O}_{k^*\ell_*}\\&+[\sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{k=1}^{K}C^{G-O}_{k\ell}\frac{C^{O}_{\ell}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L}C^{O}_{\ell}}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})-\sum_{\substack{k=1,\\ k\neq k^*}}^{K}C^{G-O}_{k\ell_*}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})-C^{G-O}_{k^*\ell_*}(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})]P^{O}_{\ell^*}\\&+[\sum_{\substack{k=1,\\ k\neq k^*}}^{K}\frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K}C^{G-O}_{k\ell_*}}(C^{A-G}_{rk}-C^{A-G}_{r+n,k})-(C^{A-G}_{rk^*}-C^{A-G}_{r+n,k^*})]P^{O}_{\ell^*}P^{G-O}_{k^*\ell_*}\end{aligned}\qquad\text{(B5.4)}$$

A.18 Mathematical Deduction for Theorem 5.2



When a perturbation $P^{G-O}_{k^*\ell_*}$ $(-C^{G-O}_{k^*\ell_*}<P^{G-O}_{k^*\ell_*}<1-C^{G-O}_{k^*\ell_*})$ is induced on one of the $C^{G-O}_{k\ell}$, and a perturbation $P^{O}_{\ell^*}$ $(-C^{O}_{\ell^*}<P^{O}_{\ell^*}<1-C^{O}_{\ell^*})$ is induced on one of the C^{O}_{ℓ}'s, which is $C^{O}_{\ell^*}$. The new value of $C^{O}_{\ell^*}$ is:

$$C^O_{\ell^*}(new) = C^O_{\ell^*} + P^O_{\ell^*}$$

The new values of other C^O_ℓ's are:

$$C^O_\ell(new) = C^O_\ell + P^O_\ell \text{, with } P^O_\ell = -\frac{P^O_{\ell^*} \times C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}$$

The new value of $C^{G-O}_{k^*\ell_*}$ is:

$$C^{G-O}_{k^*\ell_*}(new) = C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*} \qquad \text{(B5.5)}$$

The new value of other $C^{G-O}_{k\ell_*}$ will be:

$$C^{G-O}_{k\ell_*}(new) = C^{G-O}_{k\ell_*} + P^{G-O}_{k\ell_*} \text{, with } P^{G-O}_{k\ell_*} = -P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}} \qquad \text{(B5.6)}$$

Therefore, the new values of C^A_i can be represented as:

$$C^A_i(new) = \sum_{\substack{\ell=1,\ell\neq\ell^*\\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} C^O_\ell(new) C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{k=1}^{K} C^O_{\ell^*}(new) C^{G-O}_{k\ell^*} C^{A-G}_{ik}$$

$$+ \sum_{k=1,k\neq k^*}^{K} C^O_{\ell_*}(new) C^{G-O}_{k\ell_*}(new) C^{A-G}_{ik} + C^O_{\ell_*}(new) C^{G-O}_{k^*\ell_*} C^{A-G}_{ik^*}$$

$$C^A_i(new) = \sum_{\substack{\ell=1,\ell\neq\ell^*\\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} \left(C^O_\ell - P^O_{\ell^*} \times \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\right) C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{k=1}^{K} (C^O_{\ell^*} + P^O_{\ell^*}) C^{G-O}_{k\ell^*} C^{A-G}_{ik}$$

$$+ \sum_{k=1,k\neq k^*}^{K} \left(C^O_{\ell_*} - P^O_{\ell^*} \times \frac{C^O_{\ell_*}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\right)\left(C^{G-O}_{k\ell_*} - P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}}\right) C^{A-G}_{ik}$$

$$+ \left(C^O_{\ell_*} - P^O_{\ell^*} \times \frac{C^O_{\ell_*}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\right)\left(C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*}\right) C^{A-G}_{ik^*}$$

$$C^A_i(new) = \sum_{\substack{\ell=1,\ell\neq\ell^*\\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} C^O_\ell C^{G-O}_{k\ell} C^{A-G}_{ik} - \sum_{\substack{\ell=1,\ell\neq\ell^*\\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} P^O_{\ell^*} \times \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{G-O}_{k\ell} C^{A-G}_{ik} + \sum_{k=1}^{K} C^O_{\ell^*} C^{G-O}_{k\ell^*} C^{A-G}_{ik}$$

$$+ \sum_{k=1}^{K} P^O_{\ell^*} C^{G-O}_{k\ell^*} C^{A-G}_{ik} + \sum_{k=1,k\neq k^*}^{K} C^O_{\ell_*}\left(C^{G-O}_{k\ell_*} - P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}}\right) C^{A-G}_{ik} + C^O_{\ell_*}\left(C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*}\right) C^{A-G}_{ik^*}$$

$$- \sum_{k=1,k\neq k^*}^{K} P^O_{\ell^*} \times \frac{C^O_{\ell_*}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\left(C^{G-O}_{k\ell_*} - P^{G-O}_{k^*\ell_*} \times \frac{C^{G-O}_{k\ell_*}}{\sum\limits_{k=1,k\neq k^*}^{K} C^{G-O}_{k\ell_*}}\right) C^{A-G}_{ik} - P^O_{\ell^*} \times \frac{C^O_{\ell_*}}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\left(C^{G-O}_{k^*\ell_*} + P^{G-O}_{k^*\ell_*}\right) C^{A-G}_{ik^*}$$

$$C_i^A(new) = \sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} C_\ell^O C_{k\ell}^{G-O} C_{ik}^{A-G} - \sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} P_{\ell^*}^O \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{k=1}^{K} C_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G}$$

$$+ \sum_{k=1}^{K} P_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G} + \sum_{k=1,k\neq k^*}^{K} C_{\ell_*}^O C_{k\ell_*}^{G-O} C_{ik}^{A-G} - \sum_{k=1,k\neq k^*}^{K} C_{\ell_*}^O P_{k^*\ell_*}^{G-O} \times \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G} + C_{\ell_*}^O C_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G}$$

$$+ C_{\ell_*}^O P_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G} - \sum_{k=1,k\neq k^*}^{K} P_{\ell^*}^O \times \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_*}^{G-O} C_{ik}^{A-G} + \sum_{k=1,k\neq k^*}^{K} P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} P_{k^*\ell_*}^{G-O} \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G}$$

$$- P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G} - P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} P_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G}$$

$$C_i^A(new) = C_i^A - \sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} P_{\ell^*}^O \times \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} C_{ik}^{A-G} + \sum_{k=1}^{K} P_{\ell^*}^O C_{k\ell^*}^{G-O} C_{ik}^{A-G} - P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G}$$

$$- \sum_{\substack{k=1, \\ k\neq k^*}}^{K} P_{\ell^*}^O \times \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_*}^{G-O} C_{ik}^{A-G} + C_{\ell_*}^O P_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G} - \sum_{\substack{k=1, \\ k\neq k^*}}^{K} C_{\ell_*}^O P_{k^*\ell_*}^{G-O} \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G}$$

$$+ \sum_{\substack{k=1, \\ k\neq k^*}}^{K} P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} P_{k^*\ell_*}^{G-O} \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G} - P_{\ell^*}^O \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} P_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G}$$

$$C_i^A(new) = C_i^A - \Bigg(\sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_*}}^{L} \sum_{k=1}^{K} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} C_{ik}^{A-G} - \sum_{k=1}^{K} C_{k\ell^*}^{G-O} C_{ik}^{A-G} + \sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_*}^{G-O} C_{ik}^{A-G}$$

$$+ \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_*}^{G-O} C_{ik^*}^{A-G} \Bigg) P_{\ell^*}^O + \Bigg(C_{\ell_*}^O C_{ik^*}^{A-G} - \sum_{\substack{k=1, \\ k\neq k^*}}^{K} C_{\ell_*}^O \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G} \Bigg) P_{k^*\ell_*}^{G-O}$$

$$+ \Bigg(\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{k\ell_*}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_*}^{G-O}} C_{ik}^{A-G} - \frac{C_{\ell_*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{ik^*}^{A-G} \Bigg) P_{\ell^*}^O P_{k^*\ell_*}^{G-O} \qquad \text{(B 5.7)}$$

The ranking of A_r and A_{r+n} will not be reversed if $C_r^A(new) \geq C_{r+n}^A(new)$. By substituting equation B 5.7 in this inequality, we get:

$$C_r^A - \Big(\sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_\bullet}}^{L} \sum_{k=1}^{K} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} C_{rk}^{A-G} - \sum_{k=1}^{K} C_{k\ell^*}^{G-O} C_{rk}^{A-G} + \sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_\bullet}^{G-O} C_{rk}^{A-G}$$

$$+ \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_\bullet}^{G-O} C_{rk^*}^{A-G} \Big) P_{\ell^*}^O + \Big(C_{\ell_\bullet}^O C_{rk^*}^{A-G} - \sum_{\substack{k=1, \\ k\neq k^*}}^{K} C_{\ell_\bullet}^O \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} C_{rk}^{A-G} \Big) P_{k^*\ell_\bullet}^{G-O}$$

$$+ \Big(\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} C_{rk}^{A-G} - \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{rk^*}^{A-G} \Big) P_{\ell^*}^O P_{k^*\ell_\bullet}^{G-O} \geq$$

$$C_{r+n}^A - \Big(\sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_\bullet}}^{L} \sum_{k=1}^{K} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} C_{r+n,k}^{A-G} - \sum_{k=1}^{K} C_{k\ell^*}^{G-O} C_{r+n,k}^{A-G} + \sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_\bullet}^{G-O} C_{r+n,k}^{A-G}$$

$$+ \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_\bullet}^{G-O} C_{r+n,k^*}^{A-G} \Big) P_{\ell^*}^O + \Big(C_{\ell_\bullet}^O C_{r+n,k^*}^{A-G} - \sum_{\substack{k=1, \\ k\neq k^*}}^{K} C_{\ell_\bullet}^O \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} C_{r+n,k}^{A-G} \Big) P_{k^*\ell_\bullet}^{G-O}$$

$$+ \Big(\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} C_{r+n,k}^{A-G} - \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{r+n,k^*}^{A-G} \Big) P_{\ell^*}^O P_{k^*\ell_\bullet}^{G-O}$$

$$C_r^A - C_{r+n}^A \geq$$

$$\Big[\sum_{\substack{\ell=1,\ell\neq\ell^* \\ \ell\neq\ell_\bullet}}^{L} \sum_{k=1}^{K} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) - \sum_{k=1}^{K} C_{k\ell^*}^{G-O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) + \sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell_\bullet}^{G-O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G})$$

$$+ \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k^*\ell_\bullet}^{G-O} (C_{rk^*}^{A-G} - C_{r+n,k^*}^{A-G}) \Big] P_{\ell^*}^O - \Big[C_{\ell_\bullet}^O (C_{rk^*}^{A-G} - C_{r+n,k^*}^{A-G}) - \sum_{\substack{k=1, \\ k\neq k^*}}^{K} C_{\ell_\bullet}^O \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \Big] P_{k^*\ell_\bullet}^{G-O}$$

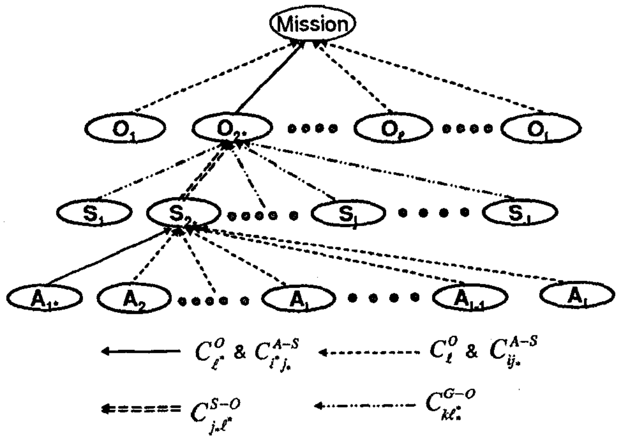
$$- \Big[\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) - \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} (C_{rk^*}^{A-G} - C_{r+n,k^*}^{A-G}) \Big] P_{\ell^*}^O P_{k^*\ell_\bullet}^{G-O}$$

$$C_r^A - C_{r+n}^A \geq \Big[\sum_{\substack{\ell=1, \\ \ell\neq\ell^*}}^{L} \sum_{k=1}^{K} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{k\ell}^{G-O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) - \sum_{k=1}^{K} C_{k\ell^*}^{G-O} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) \Big] P_{\ell^*}^O$$

$$+ \Big[\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) - (C_{rk^*}^{A-G} - C_{r+n,k^*}^{A-G}) \Big] C_{\ell_\bullet}^O P_{k^*\ell_\bullet}^{G-O}$$

$$- \Big[\sum_{\substack{k=1, \\ k\neq k^*}}^{K} \frac{C_{k\ell_\bullet}^{G-O}}{\sum_{k=1,k\neq k^*}^{K} C_{k\ell_\bullet}^{G-O}} (C_{rk}^{A-G} - C_{r+n,k}^{A-G}) - (C_{rk^*}^{A-G} - C_{r+n,k^*}^{A-G}) \Big] \frac{C_{\ell_\bullet}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} P_{\ell^*}^O P_{k^*\ell_\bullet}^{G-O}$$

A.19 Mathematical Deduction for Theorem 5.3



When a perturbation $P^{A-S}_{i^*j.}$ ($-C^{A-S}_{i^*j.} < P^{A-S}_{i^*j.} < 1-C^{A-S}_{i^*j.}$) is induced on one of the C^{A-S}_{ij}, and a perturbation $P^O_{\ell^*}$ ($-C^O_{\ell^*} < P^O_{\ell^*} < 1-C^O_{\ell^*}$) is induced on one of the C^O_ℓ's, which is $C^O_{\ell^*}$. The new value of $C^O_{\ell^*}$ is: $C^O_{\ell^*}(new) = C^O_{\ell^*} + P^O_{\ell^*}$

The new values of other C^O_ℓ's are: $C^O_\ell(new) = C^O_\ell + P^O_\ell$, with $P^O_\ell = -\dfrac{P^O_{\ell^*} \times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}$

The new value of $C^{A-S}_{i^*j.}$ is: $C^{A-S}_{i^*j.}(new) = C^{A-S}_{i^*j.} + P^{A-S}_{i^*j.}$

The new value of other $C^{A-S}_{ij.}$ will be:

$$C^{A-S}_{ij.}(new) = C^{A-S}_{ij.} + P^{A-S}_{ij.}, \text{ with } P^{A-S}_{ij.} = -P^{A-S}_{i^*j.} \times \frac{C^{A-S}_{ij.}}{\sum_{i=1,i\neq i^*}^{I} C^{A-S}_{ij.}}$$

Therefore, the new values of C^A_i can be represented as:

$$C^A_i(new) = \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{j=1}^{J} C^O_\ell(new) C^{S-O}_{j\ell} C^{A-O}_{ij}(new) + \sum_{j=1}^{J} C^O_{\ell^*}(new) C^{S-O}_{j\ell^*} C^{A-O}_{ij}(new)$$

$$C^A_i(new) = \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \left(C^O_\ell - \frac{P^O_{\ell^*} \times C^O_\ell}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell}\right) C^{S-O}_{j\ell} C^{A-O}_{ij}(new) + \sum_{j=1}^{J} (C^O_{\ell^*} + P^O_{\ell^*}) C^{S-O}_{j\ell^*} C^{A-O}_{ij}(new)$$

$$= \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{j=1}^{J} C^O_\ell C^{S-O}_{j\ell} C^{A-O}_{ij}(new) - \sum_{\ell=1,\ell\neq\ell^*}^{L} \sum_{j=1}^{J} \frac{P^O_{\ell^*}}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^O_\ell C^{S-O}_{j\ell} C^{A-O}_{ij}(new)$$

$$+ \sum_{j=1}^{J} C^O_{\ell^*} C^{S-O}_{j\ell^*} C^{A-O}_{ij}(new) + \sum_{j=1}^{J} P^O_{\ell^*} C^{S-O}_{j\ell^*} C^{A-O}_{ij}(new)$$

$$= \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_\ell^O C_{j\ell}^{S-O} C_{i^*j}^{A-O} + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O}(C_{i^*j_*}^{A-O} + P_{i^*j_*}^{A-O}) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} \frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j\ell}^{S-O} C_{i^*j}^{A-O}$$

$$- \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j_*\ell}^{S-O}(C_{i^*j_*}^{A-O} + P_{i^*j_*}^{A-O}) + \sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_{\ell^*}^O C_{j\ell^*}^{S-O} C_{i^*j}^{A-O} + C_{\ell^*}^O C_{j_*\ell^*}^{S-O}(C_{i^*j_*}^{A-O} + P_{i^*j_*}^{A-O})$$

$$+ \sum_{j=1, j\neq j_*}^{J} P_{\ell^*}^O C_{j\ell^*}^{S-O} C_{i^*j}^{A-O} + P_{\ell^*}^O C_{j_*\ell^*}^{S-O}(C_{i^*j_*}^{A-O} + P_{i^*j_*}^{A-O})$$

$$= \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_\ell^O C_{j\ell}^{S-O} C_{i^*j}^{A-O} + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O} C_{i^*j_*}^{A-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} \frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j\ell}^{S-O} C_{i^*j}^{A-O}$$

$$+ \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O} P_{i^*j_*}^{A-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j_*\ell}^{S-O} C_{i^*j_*}^{A-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j_*\ell}^{S-O} P_{i^*j_*}^{A-O}$$

$$+ C_{\ell^*}^O C_{j_*\ell^*}^{S-O} P_{i^*j_*}^{A-O} + \sum_{j=1, j\neq j_*}^{J} P_{\ell^*}^O C_{j\ell^*}^{S-O} C_{i^*j}^{A-O} + P_{\ell^*}^O C_{j_*\ell^*}^{S-O} C_{i^*j_*}^{A-O} + P_{\ell^*}^O C_{j_*\ell^*}^{S-O} P_{i^*j_*}^{A-O}$$

$$+ \sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_{\ell^*}^O C_{j\ell^*}^{S-O} C_{i^*j}^{A-O} + C_{\ell^*}^O C_{j_*\ell^*}^{S-O} C_{i^*j_*}^{A-O}$$

$$= C_{i^*}^A + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O} P_{i^*j_*}^{A-S} + C_{\ell^*}^O C_{j_*\ell^*}^{S-O} P_{i^*j_*}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{i^*j}^{A-S} P_{\ell^*}^O$$

$$- \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O} C_{i^*j_*}^{A-S} P_{\ell^*}^O - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O} P_{i^*j_*}^{A-S} P_{\ell^*}^O$$

$$+ \sum_{j=1, j\neq j_*}^{J} C_{j\ell^*}^{S-O} C_{i^*j}^{A-S} P_{\ell^*}^O + C_{j_*\ell^*}^{S-O} C_{i^*j_*}^{A-S} P_{\ell^*}^O + C_{j_*\ell^*}^{S-O} P_{i^*j_*}^{A-S} P_{\ell^*}^O$$

$$= C_{i^*}^A + (C_{j_*\ell^*}^{S-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O}) P_{\ell^*}^O P_{i^*j_*}^{A-S} + (\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O}) P_{i^*j_*}^{A-S}$$

$$+ (\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{i^*j}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{i^*j}^{A-S}) P_{\ell^*}^O \qquad \text{(B5.8)}$$

or

$$C_i^A(new) = \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{j=1}^{J} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O}(new) - \sum_{\ell=1,\ell\neq\ell^*}^{L}\sum_{j=1}^{J}\frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O}(new)$$

$$+\sum_{j=1}^{J} C_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O}(new) + \sum_{j=1}^{J} P_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O}(new)$$

$$= \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O}\left(C_{ij_*}^{A-O} - P_{i^*j_*}^{A-S}\times\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J}\frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O}$$

$$-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j_*\ell}^{S-O}\left(C_{ij_*}^{A-O} - P_{i^*j_*}^{A-S}\times\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) + \sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O} +$$

$$C_{\ell^*}^O C_{j_*\ell^*}^{S-O}\left(C_{ij_*}^{A-O} - P_{i^*j_*}^{A-S}\times\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) + \sum_{j=1,j\neq j_*}^{J} P_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O} + P_{\ell^*}^O C_{j_*\ell^*}^{S-O}\Big(C_{ij_*}^{A-O}$$

$$- P_{i^*j_*}^{A-S}\times\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\Big) + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O}$$

$$= \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O} + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O} C_{ij_*}^{A-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} C_\ell^O C_{j_*\ell}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{i^*j_*}^{A-S}$$

$$-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{\substack{j=1,\\ j\neq j_*}}^{J}\frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j\ell}^{S-O} C_{ij}^{A-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{P_{\ell^*}^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_\ell^O C_{j_*\ell}^{S-O} C_{ij_*}^{A-O} + \sum_{\substack{j=1,\\ j\neq j_*}}^{J} C_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O}$$

$$+\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{\ell^*}^O P_{i^*j_*}^{A-S} + C_{\ell^*}^O C_{j_*\ell^*}^{S-O} C_{ij_*}^{A-O} - C_{\ell^*}^O C_{j_*\ell^*}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{i^*j_*}^{A-S}$$

$$+\sum_{j=1,j\neq j_*}^{J} P_{\ell^*}^O C_{j\ell^*}^{S-O} C_{ij}^{A-O} + P_{\ell^*}^O C_{j_*\ell^*}^{S-O} C_{ij_*}^{A-O} - C_{j_*\ell^*}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{\ell^*}^O P_{i^*j_*}^{A-S}$$

$$=\sum_{\ell=1}^{L}\sum_{j=1}^{J}C_\ell^O C_{j\ell}^{S-O}C_{ij}^{A-O}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j\ell}^{S-O}C_{ij}^{A-O}P_{\ell^*}^O+\sum_{j=1}^{J}P_{\ell^*}^O C_{j\ell^*}^{S-O}C_{ij}^{A-O}$$

$$-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}C_\ell^O C_{j_*\ell}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}P_{i^*j_*}^{A-S}-C_{\ell^*}^O C_{j_*\ell^*}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}P_{i^*j_*}^{A-S}$$

$$+\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}C_{j_*\ell}^{S-O}P_{\ell^*}^O P_{i^*j_*}^{A-S}-\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}C_{j_*\ell^*}^{S-O}P_{\ell^*}^O P_{i^*j_*}^{A-S}$$

$$=C_i^A-\sum_{\ell=1}^{L}C_\ell^O C_{j_*\ell}^{S-O}\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}P_{i^*j_*}^{A-S}+[\sum_{j=1}^{J}C_{j\ell^*}^{S-O}C_{ij}^{A-O}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j\ell}^{S-O}C_{ij}^{A-O}]P_{\ell^*}^O$$

$$+(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j_*\ell}^{S-O}-C_{j_*\ell^*}^{S-O})\frac{C_{ij_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}P_{\ell^*}^O P_{i^*j_*}^{A-S} \tag{B5.9}$$

Suppose a pair of A_i's (A_r and A_t, $C_r^A \geq C_t^A$) are selected to be compared, there will be three kinds of situations: (1) r=i*; (2) t=i*; or (3) r≠i* and t≠i*. In these three situations, the ranking of A_r and A_t will not be reversed if $C_r^A(new)\geq C_t^A(new)$; therefore, we have the following inequalities regarding the three situations:

(1)

$$C_{r^*}^A+(C_{j_*\ell^*}^{S-O}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j_*\ell}^{S-O})P_{\ell^*}^O P_{r^*j_*}^{A-S}+(\sum_{\ell=1}^{L}C_\ell^O C_{j_*\ell}^{S-O})P_{r^*j_*}^{A-S}$$

$$+(\sum_{j=1}^{J}C_{j\ell^*}^{S-O}C_{r^*j}^{A-S}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j\ell}^{S-O}C_{r^*j}^{A-S})P_{\ell^*}^O\geq C_t^A-(\sum_{\ell=1}^{L}C_\ell^O C_{j_*\ell}^{S-O}\frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}})P_{r^*j_*}^{A-S}$$

$$+(\sum_{j=1}^{J}C_{j\ell^*}^{S-O}C_{tj}^{A-S}-\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\sum_{j=1}^{J}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j\ell}^{S-O}C_{tj}^{A-S})P_{\ell^*}^O$$

$$+(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L}\frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L}C_\ell^O}C_{j_*\ell}^{S-O}-C_{j_*\ell^*}^{S-O})\frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I}C_{ij_*}^{A-S}}P_{\ell^*}^O P_{r^*j_*}^{A-S}$$

$$C^A_{r^*} - C^A_t \ge -\Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell} \frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}}\Big) P^{A-S}_{r^*j_*} - \Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell}\Big) P^{A-S}_{r^*j_*} +$$

$$\Big(\sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{tj} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{tj}\Big) P^O_{\ell^*} - \Big(\sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{r^*j} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{r^*j}\Big) P^O_{\ell^*}$$

$$-\Big(C^{S-O}_{j_*\ell^*} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell}\Big) P^O_{\ell^*} P^{A-S}_{r^*j_*} + \Big(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*}\Big) \frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} P^O_{\ell^*} P^{A-S}_{r^*j_*}$$

$$C^A_{r^*} - C^A_t \ge -\Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell}\Big)\Big(1 + \frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}}\Big) P^{A-S}_{r^*j_*}$$

$$\Big(\sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{tj} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{tj} - \sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{r^*j} + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{r^*j}\Big) P^O_{\ell^*}$$

$$+\Big[\Big(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*}\Big) \frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*}\Big] P^O_{\ell^*} P^{A-S}_{r^*j_*}$$

$$C^A_{r^*} - C^A_t \ge -\Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell}\Big)\Big(1 + \frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}}\Big) P^{A-S}_{r^*j_*} + \Big[\sum_{j=1}^{J} C^{S-O}_{j\ell^*}\big(C^{A-S}_{tj} - C^{A-S}_{r^*j}\big) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell}$$

$$\big(C^{A-S}_{tj} - C^{A-S}_{r^*j}\big)\Big] P^O_{\ell^*} + \Big(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*}\Big)\Big(\frac{C^{A-S}_{tj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} + 1\Big) P^O_{\ell^*} P^{A-S}_{r^*j_*} \qquad \text{(B5.10)}$$

(2)

$$C^A_r - \Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell} \frac{C^{A-S}_{rj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}}\Big) P^{A-S}_{t^*j_*} + \Big[\sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{rj} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{rj}\Big] P^O_{\ell^*}$$

$$+\Big(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} \frac{C^{A-S}_{rj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}} C^{S-O}_{j_*\ell} - C^{S-O}_{j_*\ell^*} \frac{C^{A-S}_{rj_*}}{\sum\limits_{i=1,i\neq i^*}^{I} C^{A-S}_{ij_*}}\Big) P^O_{\ell^*} P^{A-S}_{t^*j_*} \ge C^A_{t^*} + \Big(\sum_{\ell=1}^{L} C^O_\ell C^{S-O}_{j_*\ell}\Big) P^{A-S}_{t^*j_*} +$$

$$\Big(C^{S-O}_{j_*\ell^*} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j_*\ell}\Big) P^O_{\ell^*} P^{A-S}_{t^*j_*} + \Big(\sum_{j=1}^{J} C^{S-O}_{j\ell^*} C^{A-S}_{t^*j} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C^O_\ell}{\sum\limits_{\ell=1,\ell\neq\ell^*}^{L} C^O_\ell} C^{S-O}_{j\ell} C^{A-S}_{t^*j}\Big) P^O_{\ell^*}$$

$$C_r^A - C_t^A \geq (\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O}) P_{t^* j_*}^{A-S} + (C_{j_*\ell^*}^{S-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O}) P_{\ell^*}^O P_{t^* j_*}^{A-S} + (\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{t^* j_*}^{A-S}$$

$$+ (\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{t^* j}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{t^* j}^{A-S}) P_{\ell^*}^O - [\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{rj}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{rj}^{A-S}] P_{\ell^*}^O$$

$$+ (C_{j_*\ell^*}^{S-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O}) \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{\ell^*}^O P_{t^* j_*}^{A-S}$$

$$C_r^A - C_t^A \geq (\frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} + 1)(\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O}) P_{t^* j_*}^{A-S} + (C_{j_*\ell^*}^{S-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O})$$

$$(1 + \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{\ell^*}^O P_{t^* j_*}^{A-S} + [\sum_{j=1}^{J} C_{j\ell^*}^{S-O} (C_{t^* j}^{A-S} - C_{rj}^{A-S}) - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} (C_{t^* j}^{A-S} - C_{rj}^{A-S})] P_{\ell^*}^O$$

(5.11)

(3)

$$C_r^A - (\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{i^* j_*}^{A-S} + [\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{rj}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{rj}^{A-S}] P_{\ell^*}^O$$

$$+ (\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} C_{j_*\ell}^{S-O} - C_{j_*\ell^*}^{S-O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{\ell^*}^O P_{i^* j_*}^{A-S} \geq$$

$$C_t^A - (\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{i^* j_*}^{A-S} + [\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{tj}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{tj}^{A-S}] P_{\ell^*}^O$$

$$+ (\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} C_{j_*\ell}^{S-O} - C_{j_*\ell^*}^{S-O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}) P_{\ell^*}^O P_{i^* j_*}^{A-S}$$

$$C_r^A - C_t^A \ge \left(\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) P_{i^*j_*}^{A-S} - \left(\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) P_{i^*j_*}^{A-S}$$

$$+\left[\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{tj}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{tj}^{A-S}\right] P_{\ell^*}^O - \left[\sum_{j=1}^{J} C_{j\ell^*}^{S-O} C_{rj}^{A-S} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O} C_{rj}^{A-S}\right] P_{\ell^*}^O$$

$$+\left(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} C_{j_*\ell}^{S-O} - C_{j_*\ell^*}^{S-O} \frac{C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) P_{\ell^*}^O P_{i^*j_*}^{A-S}$$

$$-\left(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} C_{j_*\ell}^{S-O} - C_{j_*\ell^*}^{S-O} \frac{C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) P_{\ell^*}^O P_{i^*j_*}^{A-S}$$

$$C_r^A - C_t^A \ge \left(\sum_{\ell=1}^{L} C_\ell^O C_{j_*\ell}^{S-O} \frac{C_{rj_*}^{A-S} - C_{tj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}}\right) P_{i^*j_*}^{A-S} + \left(\sum_{j=1}^{J} C_{j\ell^*}^{S-O} - \sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \sum_{j=1}^{J} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j\ell}^{S-O}\right)$$

$$(C_{tj}^{A-S} - C_{rj}^{A-S}) P_{\ell^*}^O + \left(\sum_{\substack{\ell=1,\\ \ell\neq\ell^*}}^{L} \frac{C_\ell^O}{\sum_{\ell=1,\ell\neq\ell^*}^{L} C_\ell^O} C_{j_*\ell}^{S-O} - C_{j_*\ell^*}^{S-O}\right) \frac{C_{tj_*}^{A-S} - C_{rj_*}^{A-S}}{\sum_{i=1,i\neq i^*}^{I} C_{ij_*}^{A-S}} P_{\ell^*}^O P_{i^*j_*}^{A-S} \qquad (5.12)$$