Comment on Y.-C. Chen, E. Fried, Möbius bands, unstretchable material sheets and developable surfaces, *Proc. R. Soc. A* **472, 20160459 (2016)**

G.H.M. van der Heijden, E.L. Starostin

Department of Civil, Environmental & Geomatic Engineering
University College London
London WC1E 6BT, UK

Chen & Fried [1] claim that our solution of the shape of a material Möbius strip (taken to be inextensible) in [2] is incorrect as it involves stretching of the material. The purported proof of this claim, the calculation in Section 7, is in fact the main result of the paper. The claim is elaborated upon in subsequent publications with Fosdick [3, 4, 5]. Here we show that all these papers are based on a misunderstanding of the elastic deformation involved and what it means for such a deformation to be isometric (i.e., to preserve distances).

We all agree on the problem at issue, which is to find the equilibrium shape of a Möbius band made of a thin strip of inextensible material. This shape will be given by a developable surface that can be constructed by isometrically deforming a planar rectangular reference strip into a closed structure with half a turn of twist. An arbitrary developable strip of width $2w$ can be parametrised as

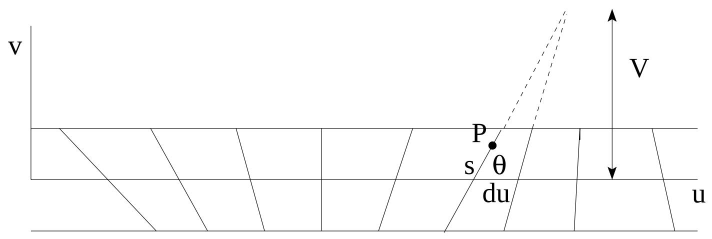
$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}(s,v) = \boldsymbol{r}(s) + v[\boldsymbol{b}(s) + \eta(s)\boldsymbol{t}(s)] \\ \tau(s) &= \eta(s)\kappa(s), \qquad s \in [0,L], \quad v \in [-w,w], \end{aligned} \tag{1}$$

where $\boldsymbol{r}$ is the centreline of the strip (of length L), $\boldsymbol{t}$ and $\boldsymbol{b}$ are the unit tangent and binormal, and κ and τ are the curvature and torsion. (There is a slight complication here that b is not defined in points where the curvature vanishes, but dealing with this is standard and not at issue here.) The surface is thus swept out by a one-parameter family of straight generators $v[\boldsymbol{b}(s) + \eta(s)\boldsymbol{t}(s)]$, $v \in [-w,w]$. With η an arbitrary function, (1) defines a ruled surface; it becomes developable if $\eta = \tau/\kappa$. Expression (1) agrees with Eq. (7.1) in [1].

The corresponding 2D stress-free reference configuration can be expressed in terms of the same coordinates (s,v) as

$$\boldsymbol{X} = u\boldsymbol{e}_3 + v\boldsymbol{e}_1 = s\boldsymbol{e}_3 + v(\boldsymbol{e}_1 + \eta(s)\boldsymbol{e}_3). \tag{2}$$

Here, $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ are the unit vectors of a Cartesian coordinate system with coordinates (u,v). The relationship between the Cartesian coordinates (u,v) and the 'skew' coordinates (s,v) of an arbitrary point P is illustrated in Fig. 1. The same coordinate transformation is used by Wunderlich



$$u = s + v\cot\theta, \qquad \eta = \tau/\kappa = \cot\theta, \qquad V = -\frac{1}{\eta'}$$

Figure 1: Reference configuration of a developable strip with generators indicated.

[6] to reduce the strain energy of the strip to its centreline. Note that the parametrisation (2) is valid provided the mapping $(s, v) \mapsto (s + v\eta(s), v)$ is locally injective, i.e., provided that $1 + v\eta'(s) > 0$ on $[0, L] \times [-w, w]$. We therefore require $w|\eta'(s)| < 1$ (i.e., generators cannot intersect in the interior of the strip).

It is well-known that a Möbius strip centreline has a so-called switching point where not only $\kappa = 0$ (i.e., it is an inflection point) but also $\eta = 0$ [7]. We may therefore parametrise the centreline of the closed strip such that $\eta(0) = \eta(L) = 0$ (we impose these boundary conditions in all our calculations in [2], as well as in our subsequent publications [8] and [9], the latter dealing with open strips). This means that the reference configuration (2) is a rectangle.

The deformation $\boldsymbol{X} \mapsto \mathbf{x} = \boldsymbol{f}(\boldsymbol{X})$ can be imagined as lifting up the planar strip and bending it everywhere about the (fixed) local generator to form the Möbius strip (the pattern of generators thus completely determines the shape of the strip in space). No stretching is involved. If we were to vary the width $2w$ of the strip, or to apply end loads, then this would cause a rearrangement of the generators over the surface of the strip. The generators are not attached to the material under these actions. Chen & Fried seem to think that the generators are material lines and confuse the 'lifting-up' deformation described above with a rearrangement of generators (which, if they were material, would indeed induce stretching). The lifting-up action is the real deformation at issue here, not the shearing action of generators under parameter change.

Chen & Fried make a big point about the fact that a deformation from a planar region to a developable surface need not be isometric. This is a simple fact: a rectangle cannot be mapped into a parallelogram without stretching. We do not consider such non-isometric deformations. Our boundary conditions prevent them. It may be worth pointing out though that for *closed* strips the boundary conditions $\eta(0) = \eta(L) = 0$ above can be replaced by $\eta(0) = \eta(L)$: it is possible to isometrically deform a parallelogram into a cylinder and a trapezoid into a Möbius strip. The short sides of the strip (at $s = 0$ and $s = L$) need *not* be generators in such cases. By the way, these examples show that the discussion of the so-called 'covering requirement' in [5] completely misses the point: it is pointless to discuss these isometric deformations without considering boundary conditions that would put kinematic constraints on the allowable deformations.

To confirm that the deformation does not involve stretching of the material we compute the relevant strain tensor to show that the deformation $\boldsymbol{X} \mapsto \mathbf{x}$ is indeed isometric. This calculation is also carried out in [1] to support the authors' statements. We follow the corrected version in [10] but explain in more detail precisely where Chen & Fried go wrong. Let

$$\boldsymbol{G}_i = \frac{\partial \boldsymbol{X}}{\partial \xi^i}, \quad \boldsymbol{g}_i = \frac{\partial \boldsymbol{x}}{\partial \xi^i} \quad (i = 1, 2), \qquad (\xi^1, \xi^2) = (s, v).$$

Then the deformation gradient can be written as

$$\boldsymbol{F} = \nabla \boldsymbol{f} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} = \frac{\partial \boldsymbol{x}}{\partial \xi^i} \otimes \boldsymbol{G}^i = \boldsymbol{g}_i \otimes \boldsymbol{G}^i.$$

The right Cauchy-Green strain tensor is

$$\boldsymbol{C} = \boldsymbol{F}^T \boldsymbol{F} = (\boldsymbol{G}^i \otimes \boldsymbol{g}_i) \cdot (\boldsymbol{g}_j \otimes \boldsymbol{G}^j) = (\boldsymbol{g}_i \cdot \boldsymbol{g}_j)(\boldsymbol{G}^i \otimes \boldsymbol{G}^j) = g_{ij} \boldsymbol{G}^i \otimes \boldsymbol{G}^j.$$

Here $g_{ij} := \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ are the components of the spatial metric tensor. For an isometric deformation, $\boldsymbol{C} = \boldsymbol{I}$.

On using the developability condition $\tau = \eta\kappa$ and the Frenet-Serret equations $\boldsymbol{t}' = \kappa\boldsymbol{n}$, $\boldsymbol{n}' = -\kappa\boldsymbol{t} + \tau\boldsymbol{b}$, $\boldsymbol{b}' = -\tau\boldsymbol{n}$ ($\boldsymbol{n}$ being the unit principal normal), we have

$$\frac{\partial \boldsymbol{x}}{\partial \xi^1} = \boldsymbol{r}' + v(\boldsymbol{b}' + \eta'\boldsymbol{t} + \eta\boldsymbol{t}') = (1 + v\eta')\boldsymbol{t}, \qquad \frac{\partial \boldsymbol{x}}{\partial \xi^2} = \boldsymbol{b} + \eta\boldsymbol{t}.$$

Thus

$$g_{11} = (1 + v\eta')^2, \qquad g_{12} = g_{21} = \eta(1 + v\eta'), \qquad g_{22} = 1 + \eta^2$$

and hence

$$\boldsymbol{C} = g_{ij}\boldsymbol{G}^i \otimes \boldsymbol{G}^j = (1 + v\eta')^2\boldsymbol{G}^1 \otimes \boldsymbol{G}^1 + \eta(1 + v\eta')(\boldsymbol{G}^1 \otimes \boldsymbol{G}^2 + \boldsymbol{G}^2 \otimes \boldsymbol{G}^1) + (1 + \eta^2)\boldsymbol{G}^2 \otimes \boldsymbol{G}^2.$$

This agrees with Eq. (7.10) in [1]. From this result the authors conclude that "$\boldsymbol{x}$ elongates material filaments ... unless $\tau \equiv 0$" and hence that the deformation $\boldsymbol{f}$ is isometric only if η is identically zero, i.e., if the generator is everywhere perpendicular to the centreline. It would of course follow from this that the only non-planar developable surfaces are cylinders. Chen & Fried stop short of drawing this conclusion, although the claim "our analysis shows that the mapping from a prescribed planar region to a rectifying developable surface is isometric only if that surface is cylindrical" in their Abstract comes very close.

However, contrary to what Chen & Fried write, (s, v) are *not* Cartesian coordinates and $(\boldsymbol{G}_1, \boldsymbol{G}_2)$ is not an orthogonal basis. To complete the calculation, we note that

$$\boldsymbol{G}_i = g_{ij}\boldsymbol{G}^j \qquad \text{and} \qquad \det \boldsymbol{g} = (1 + v\eta')^2$$

and hence

$$\boldsymbol{G}^1 = \frac{1}{(1 + v\eta')^2}\left[(1 + \eta^2)\boldsymbol{G}_1 - \eta(1 + v\eta')\boldsymbol{G}_2\right],$$
$$\boldsymbol{G}^2 = \frac{1}{(1 + v\eta')^2}\left[-\eta(1 + v\eta')\boldsymbol{G}_1 + (1 + v\eta')^2\boldsymbol{G}_2\right],$$

where

$$\boldsymbol{G}_1 = \frac{\partial \boldsymbol{X}}{\partial \xi^1} = \boldsymbol{e}_3 + v\eta'\boldsymbol{e}_3 = (1 + v\eta')\boldsymbol{e}_3, \qquad \boldsymbol{G}_2 = \frac{\partial \boldsymbol{X}}{\partial \xi^2} = \boldsymbol{e}_1 + \eta\boldsymbol{e}_3.$$

Inserting this into the expression for $\boldsymbol{C}$ above we finally obtain

$$\boldsymbol{C} = g_{ij}\boldsymbol{G}^i \otimes \boldsymbol{G}^j = \boldsymbol{G}^i \otimes \boldsymbol{G}_i = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

(In geometrical terms, this calculation shows that the spatial metric pulled back to the reference configuration (source manifold) is equal to the Cartesian identity.)

This confirms that the deformation $\boldsymbol{f}$ is isometric, *provided* the boundary conditions on η are compatible. There is no stretching. Because of this we do not need to impose unstretchability constraints (as suggested in the final section of [1]) in our subsequent variational analysis minimising the Wunderlich functional subject to appropriate boundary conditions. This analysis gives us Euler-Lagrange equations that we can solve for $\boldsymbol{r}$, κ and η, after which we can draw the equilibrium shape through (1).

References

[1] Y.-C. Chen, E. Fried, Möbius bands, unstretchable material sheets and developable surfaces, *Proc. R. Soc. A* **472**, 20160459 (2016)

[2] E.L. Starostin, G.H.M. van der Heijden, The shape of a Möbius strip, *Nature Materials* **6**, 563–567 (2007)

[3] Y.-C. Chen, R. Fosdick, E. Fried, Isometric deformations of unstretchable material surfaces, a spatial variational treatment, *Journal of the Mechanics and Physics of Solids* **116**, 290–322 (2018)

[4] Y.-C. Chen, R. Fosdick, E. Fried, Representation of a smooth isometric deformation of a planar material region into a curved surface, *Journal of Elasticity* **130**, 145–195 (2018)

[5] Y.-C. Chen, R. Fosdick, E. Fried, Issues concerning isometric deformations of planar regions to curved surfaces, *Journal of Elasticity* **132**, 1–42 (2018)

[6] W. Wunderlich, Über ein abwickelbares Möbiusband, *Monatshefte für Mathematik* **66**, 276–289 (1962)

[7] T. Randrup, P. Røgen, Sides of the Möbius strip, *Archiv der Mathematik* **66**, 511–521, 1996)

[8] E.L. Starostin, G.H.M. van der Heijden, Equilibrium shapes with stress localisation for inextensible elastic Möbius and other strips, *Journal of Elasticity* **119**, 67–112 (2015)

[9] A.P. Korte, E.L. Starostin, G.H.M. van der Heijden, Triangular buckling patterns of twisted inextensible strips, *Proc. R. Soc. A* **467**, 285–303 (2011)

[10] A. Moore, T. Healey, Computation of elastic equilibria of complete Möbius bands and their stability, *Mathematics and Mechanics of Solids* **24**, 939–967 (2019)