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Affine Minkowski valuations

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Abstract

In geometric valuation theory, well-known examples of Minkowski valuations intertwining the special linear group are defined by the difference operator, projection operator and the moment operator. While the difference and projection operator are translation invariant the moment operator is not. The former examples can be seen as maps with values in the set of convex bodies in $\bigwedge^1\mathbb{R}^n$ and $\bigwedge^{n-1}\mathbb{R}^n$ respectively. The projection operator is homogeneous of degree $n-1$ and its support function can be expressed in terms of projections to $(n-1)$-dimensional subspaces. Similar the difference operator is homogeneous of degree 1 and its support function can be expressed in terms of projections to 1-dimensional subspaces. Therefore, one can regard the difference operator as a projection operator of degree 1. We ask whether it is possible to define projection operators homogeneous of degree k for $1 < k < n-1$. More precisely, is there a non-trivial continuous and translation invariant Minkowski valuation intertwining the special linear group which is homogeneous of degree k and has values in the set of convex bodies in $\bigwedge^k\mathbb{R}^n$? We give an answer to a more general question. We prove that for any finite-dimensional irreducible $\mathrm{SL}(n)$-representation W such a Minkowski valuation exists if and only if W equals $\mathbb{R}$, $\bigwedge^1\mathbb{R}^n$ or $\bigwedge^{n-1}\mathbb{R}^n$. In the latter cases, all such valuations are already classified by Ludwig. More precisely, such a valuation is a multiple of the difference and the projection operator respectively. In the case $W = \mathbb{R}$ such Minkowski valuations can be constructed using the Euler characteristic and the volume. Finally, we give some new examples satisfying all properties mentioned above but translation invariance. If $n \leq 3$ we show that there is a continuous and $\mathrm{SL}(n)$ equivariant Minkowski valuation defined on the set of convex bodies containing the origin in its interior with values in the set of convex bodies in W, for any finite-dimensional $\mathrm{SL}(n)$-representation W. One of these examples is a generalization of the moment operator. The existence of a Busemann-Petty type inequality for the generalized moment operator is discussed.

Zusammenfassung

Bekannte Beispiele für Minkowski-Bewertungen in der geometrischen Bewertungstheorie, die mit der speziellen linearen Gruppe kommutieren, sind der Differenzenoperator, der Projektionenoperator und der Momentenoperator. Während der Differenzenoperator und der Projektionenoperator translationsinvariante Bewertungen sind, trifft das auf den Momentenoperator nicht zu. Die erstgenannten Beispiele können als Abbildungen mit Werten in der Menge der konvexen Körper in $\bigwedge^1\mathbb{R}^n$ bzw. $\bigwedge^{n-1}\mathbb{R}^n$ betrachtet werden. Der Projektionenoperator ist homogen vom Grad $n-1$ und die zugehörige Stützfunktion kann mit Hilfe von Projektionen auf $(n-1)$-dimensionale Unterräume dargestellt werden. Ähnliches trifft auf den Differenzenoperator zu, der homogen vom Grad 1 ist und dessen Stützfunktion mit Hilfe von Projektionen auf 1-dimensionale Unterräume dargestellt werden kann. Daher kann der Differenzenoperator als Projektionenoperator von Grad 1 betrachtet werden. Wir fragen, ob es möglich ist, einen Projektionenoperator von Grad k zu definieren, wobei $1 < k < n-1$. Genauer gesagt fragen wir nach der Existenz einer nichttrivialen translationsinvarianten stetigen Minkowski-Bewertung, die mit der speziellen linearen Gruppe vertauscht und homogen von Grad k ist sowie Werte in der Menge der konvexen Körper in $\bigwedge^k\mathbb{R}^n$ annimmt. Wir beantworten diese Frage in einer allgemeineren Situation. Wir zeigen, dass für jede endlichdimensionale irreduzible $\mathrm{SL}(n)$-Darstellung W eine solche Minkowski-Bewertung genau dann existiert, wenn W mit $\mathbb{R}$, $\bigwedge^1\mathbb{R}^n$ oder $\bigwedge^{n-1}\mathbb{R}^n$ übereinstimmt. In den letztgenannten Fällen sind solche Bewertungen bereits von Ludwig charakterisiert durch ein Vielfaches des Differenzen- bzw. Projektionenoperators. Für $W = \mathbb{R}$ können solche Bewertungen mit Hilfe der Euler-Charakteristik und dem Volumen konstruiert werden. Anschließend geben wir neue Beispiele für solche Bewertungen, die nicht translationsinvariant sind. Für jede endlichdimensionale Darstellung W zeigen wir im Fall $n \leq 3$, dass es eine stetige und $\mathrm{SL}(n)$ equivariante Minkowski-Bewertung gibt, die auf der Menge der konvexen Körper, die den Ursprung in ihrem Inneren enthalten, definiert ist und Werte in der Menge der konvexen Körper in W annimmt. Eines dieser Beispiele ist eine Verallgemeinerung des Momentenoperators. Es wird diskutiert, ob die Busemann-Petty-Ungleichung für den klassischen Momentenoperator auf den verallgemeinerten Momentenoperator verallgemeinert werden kann.

Contents

1 Introduction

The isoperimetric inequality has been of interest for thousands of years: According to the legend Dido, the queen of Carthage, had to flee from her brother in 814 BC. Upon arrival on the coast of Libya, she bought a piece of land as big as she could contain within the skin of a bull. To obtain the maximal area she cut the hide into strips and laid it out in an appropriate shape [87]. The problem Dido had to deal with in this task is known as the isoperimetric inequality, stating that among all convex bodies with constant perimeter the ball is the only one of maximal volume. This was proved in the plane by Steiner in the 19th century [89] using a symmetrization tool which is known as Steiner symmetrization nowadays. There are several proofs of the classical isoperimetric inequality and also for generalizations in higher dimensions, see e.g. [25, 69]. In geometry, there is lot of research going on about other inequalities concerning the volume of convex bodies. For example the *Petty-projection inequality*

$$\mathrm{vol}_n(K)^{n-1}\,\mathrm{vol}_n(\Pi^\circ K) \leq \left(\frac{\mathrm{vol}_n(B_n)}{\mathrm{vol}_{n-1}(B^{n-1})}\right)^n,$$

with equality if and only if K is an ellipsoid, is stronger than the isoperimetric inequality [74]. Here $\Pi^\circ K$ denotes the polar of the projection body ΠK (see below) and B^n denotes the unit ball in $\mathbb{R}^n$. The projection body was also used by Petty [73] and Schneider [78] independently to give a negative answer (for $n \geq 3$) to Shephard's problem asking whether $\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)$ is true for centrally symmetric convex bodies K, L whenever all lower dimensional volumes of projections π of K and L to any subspace satisfy $\mathrm{vol}_k(\pi(K)) \leq \mathrm{vol}_k(\pi(L))$. However, the projection body and generalizations are useful to obtain information of a convex body from its projections, see e.g. [17, 22, 24, 79, 82]. Another volume inequality is the Rogers-Shephard inequality [76]

$$\mathrm{vol}_n(DK) \leq \binom{2n}{n} \cdot \mathrm{vol}_n(K),$$

where $DK := K + (-K)$ denotes the difference body of a convex body K. Here $+$ stands for the Minkowski sum of two convex bodies, which is the convex body containing all sums of two points, one lying in the first body and one lying in the second.

An important property of the difference operator D and the projection operator Π is the valuation property

$$Z(K) + Z(L) = Z(K \cup L) + Z(K \cap L) \tag{1.1}$$

for all convex bodies K, L such that $K \cup L$ is convex. The property (1.1) is a very natural assumption if we want to deal with the volume or projections since it is satisfied for Borel measures and projections. Note that the valuation property can be defined for any map $Z\colon \mathcal{K}(\mathbb{R}^n) \to (A,+)$, where $\mathcal{K}(\mathbb{R}^n)$ denotes the set of convex bodies and $(A,+)$ is any abelian semigroup. Dehn's solution [30] of Hilbert's third problem [48] can be seen as a starting point of valuation theory. The problem asks the following. Given two polyhedra of equal volume, is it always possible to cut the first one into finitely many polyhedra such that the pieces can be rearranged to yield the second one? Dehn constructed a certain map on polyhedra, nowadays known as the *Dehn invariant*, which has a nice behaviour with respect to such cuts described in Hilbert's third problem. More precisely, the Dehn invariant satisfies the valuation property (1.1).

To point out some similarities between the projection operator and the difference operator we use the support function of a convex body K given by

$$h_K\colon (\mathbb{R}^n)^* \to \mathbb{R}, \quad x \mapsto \sup_{y\in K}\langle x, y\rangle.$$

It is well-known that a convex body is uniquely determined by its support function. The projection body $\Pi K \subset (\mathbb{R}^n)^*$ is now defined by the support function

$$h_{\Pi K}(x) = \mathrm{vol}_{n-1}(K_{x^\perp}), \quad |x| = 1,$$

where $K_{x^\perp}$ is the orthogonal projection of K to $x^\perp$ and $\mathbb{R}^n$ is identified with its dual space via the euclidean structure. Similarly the support function of the difference body can be written as

$$h_{DK}(x) = 2\cdot \mathrm{vol}_1(K_{\langle x\rangle}), \quad |x| = 1.$$

The operators Π and D commute with the action of $\mathrm{SL}(n)$, where in the dual space the action is defined by $\phi\cdot v := v\circ\phi^{-1}$ [58]. The following characterization of Π and D due to Ludwig are crucial for the motivation of this work.

Theorem 1.1 ([59]). *Let $n \geq 2$. A map $Z\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ (resp. $Z\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}((\mathbb{R}^n)^*)$) is a continuous, translation invariant and* $\mathrm{SL}(n)$ *equivariant Minkowski valuation if and only if there is a non-negative constant c such that $Z = c\cdot D$ (resp. $Z = c\cdot\Pi$).*

For $n = 1$ a classification of continuous, translation invariant and $\mathrm{SL}(n)$ equivariant Minkowski valuations follows from a classical result of Blaschke (see Table 1.1 below and Section 4.2.2). Via the identification of $(\mathbb{R}^n)^*$ with $\bigwedge^{n-1}\mathbb{R}^n$ we can write $\Pi K \subset \bigwedge^{n-1}\mathbb{R}^n$ and similarly $DK \subset \bigwedge^1\mathbb{R}^n$. Therefore it makes sense to regard the difference operator as a projection operator of degree 1, while the usual projection operator is of degree $n-1$. As Table 1.1 shows, there are also rather trivial projection bodies of degree 0 and n. The notion of *representation* in the table is explained in Chapter 3. A representation of a group can be seen as a vector space equipped with a linear group action.

Representation	$\mathrm{SL}(n)$ **action**	**Valuation**	**Support function**
$\mathbb{R}\,(=\bigwedge^0 \mathbb{R}^n)$	$\phi w := w$	$K \mapsto [-1,1]$	$2 \cdot \mathrm{vol}_0(K_{\{0\}})(=2)$
$\mathbb{R}^n\,(=\bigwedge^1 \mathbb{R}^n)$	$\phi w := \phi(w)$	$K \mapsto DK$	$2 \cdot \mathrm{vol}_1(K_{\langle x\rangle})$
$(\mathbb{R}^n)^*\,(\cong \bigwedge^{n-1} \mathbb{R}^n)$	$\phi w := w \circ \phi^{-1}$	$K \mapsto \Pi K$	$\mathrm{vol}_{n-1}(K_{x^\perp})$
$\mathbb{R}\,(\cong \bigwedge^n \mathbb{R}^n)$	$\phi w := w$	$K \mapsto \mathrm{vol}_n(K) \cdot [-1,1]$	$2 \cdot \mathrm{vol}_n(K_{\mathbb{R}^n})$

Table 1.1: $\mathrm{SL}(n)$ intertwining Minkowski valuations.

In this thesis we ask whether it is possible to complete the table, i.e. is there a notion of a projection body of degree k, where $1 < k < n-1$? More precisely we ask the following question.

Question 1.2. Let W be a finite-dimensional representation of $\mathrm{SL}(n)$. Is there a non-trivial and continuous Minkowski valuation with values in the set of convex bodies in W, which commutes with the action of $\mathrm{SL}(n)$?

Since any finite-dimensional representation of $\mathrm{SL}(n)$ is a direct sum of so-called *irreducible representations* it is enough to consider irreducible representations. We answer Question 1.2 completely for translation invariant valuations in our main result as follows.

Theorem 1.3. *Let W be an irreducible finite-dimensional representation of* $\mathrm{SL}(n)$ *and* $Z\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(W)$ *a non-trivial, continuous, translation invariant and* $\mathrm{SL}(n)$ *equivariant Minkowski valuation. Then W is isomorphic (as representation) to either* $\mathbb{R}$, $\mathbb{R}^n$ *or* $(\mathbb{R}^n)^*$.

In particular, there is no projection body of degree k if $1 < k < n-1$. The first step of the proof of our main result is to show that it is enough to assume Z to be even and homogeneous degree $k \in \{0, \ldots, n\}$. In this case, Z is uniquely determined by its Klain body $Z(K)$, where K is a convex body in $\mathbb{R}^k \subset \mathbb{R}^n$ whose k-dimensional volume equals 1. This is mainly due to Hadwiger and Klain [46, 50]. Also, Schuster [86] used a similar argument for a classification of $\mathrm{SO}(n)$ equivariant valuations. Now it remains to show that the Klain body is the zero body $\{0\}$. The Klain body is invariant under transformations of type

$$\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}.$$

Using the theory of highest weights, which is a well-known tool in representation theory, one can show that the Klain body is contained in the highest weight space. If this body is not the zero body we can calculate the highest weight of the representation by acting with

$$\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{SL}(n)$$

on the Klain body and find $W = \bigwedge^k \mathbb{R}^n$. This already indicates that the exterior power is the *best guess* to find a new continuous, translation invariant and $\mathrm{SL}(n)$ equivariant Minkowski valuation. While up to this point we used methods from the theory of real valuations and representation theory, the proof becomes more analytic. Again, as for the existence of the Klain body, the real valued Klain function plays a crucial role in the following argument. It is well-known that the Klain embedding is a proper subset of the space of continuous functions on the Grassmannian $\mathrm{Gr}_k(n)$, if $1 < k < n-1$. This fact is due to Alesker and Bernstein [9] and yields to a contradiction if it is assumed that the Klain body is not the zero body.

In the case of continuous and $\mathrm{SL}(n)$ equivariant Minkowski valuations Z (not necessarily translation invariant) the answer is different from our main result. Already in the classical case $Z\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ there is not only the difference operator but also the moment operator M, which is defined via the support function

$$h_{MK}(u) = \int\limits_K |\langle u, x\rangle| dx, \quad u \in (\mathbb{R}^n)^*.$$

The operator M is a continuous Minkowski valuation which commutes with $\mathrm{SL}(n)$ (see e.g. [38], §9.1).

Also the normalized moment body $\Gamma K := \frac{1}{\mathrm{vol}(K)} MK$, known as the *centroid body* of K, is an interesting object since the name *centroid body* comes from the following fact. If $K = -K$, then the boundary points of ΓK are exactly the centroids of the intersection of K with the closed half space $H_u^+ = \{x \in \mathbb{R}^n \mid \langle x, u\rangle \geq 0\}$ for $u \in S^{n-1}$. The slightly more general L_q centroid body is a useful tool in the investigation of asymptotic properties of convex bodies, see e.g. [33, 39, 52, 70].

As in the case of D and Π we can ask for inequalities concerning the volume and the moment operator.

Theorem 1.4 (Busemann-Petty centroid inequality, [72])**.** *It holds*

$$\mathrm{vol}(MK) \geq \left(\frac{2 \,\mathrm{vol}_{n-1}(B^{n-1})}{(n+1)\,\mathrm{vol}_n(B^n)} \right)^n \mathrm{vol}(K)^{n+1}$$

for all convex bodies K with equality if and only if K is an ellipsoid centered at the origin.

We define a generalized moment body $M^p K \subset \mathrm{Sym}^p \mathbb{R}^n$, where p is a positive integer, by

$$h_{M^p K}(u) = \int\limits_K |\langle u, x\rangle| dx, \quad u \in (\mathrm{Sym}^p \mathbb{R}^n)^*.$$

As we will see, $M^p\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathrm{Sym}^p \mathbb{R}^n)$ is a continuous Minkowski valuation which commutes with $\mathrm{SL}(n)$. For $p = 1$ this valuation coincides with the classical moment operator. We try to generalize the Busemann-Petty centroid inequality to the M^p operator and find the following.

Theorem 1.5. *Let* $N := \dim(\operatorname{Sym}^p \mathbb{R}^n) = \binom{n+p-1}{p}$. *There is a positive constant* c *such that*

$$\operatorname{vol}(M^p K) \geq c \cdot \operatorname{vol}(K)^{\frac{N \cdot (n+p)}{n}}$$

for all convex bodies K.

A proof of the classical Busemann-Petty centroid inequality using Steiner symmetrization does not immediately generalize to the M^p case (see Section 6.1).

We give more examples of continuous $\mathrm{SL}(n)$ equivariant valuations $\mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*)$ for $p, q \in \mathbb{N}$. However, the examples in this case are only defined on the set of convex bodies containing the origin in its interior $\mathcal{K}_{(o)}(\mathbb{R}^n)$. These types of examples are enough to show the following theorem.

Theorem 1.6. *Let* $n \leq 3$ *and* W *be a finite-dimensional* $\mathrm{SL}(n)$*-representation. There is a continuous* $\mathrm{SL}(n)$ *equivariant valuation*

$$\mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W).$$

The structure of this thesis is as follows. Chapter 2 provides a short introduction to tools from convex geometry we need in this thesis. In Chapter 3 we give an overview on representation theory. The main goal of this Chapter is to show that any continuous homomorphism $\rho \colon \mathrm{SL}(n) \to \mathrm{GL}(W)$ is a polynomial representation if W is a real vector space of finite dimension. Also, some properties of this type of representations are discussed and the most important facts about the theory of highest weights are explained. A selection of important results concerning the theory of valuations in convex geometry is given in Chapter 4. Here, and later in this thesis, we will see that real valuations are very useful to study Minkowski valuations. In particular, the construction of the Klain function is discussed and we prove some properties of the Klain body. Our main result is proved in Chapter 5. Chapter 6 deals with new examples of continuous and $\mathrm{SL}(n)$ equivariant Minkowski valuations.

2 Convex geometry

This chapter provides an introduction to a few concepts in convex geometry. We also want to introduce some notation. To begin with, we denote by V a real (topological) vector space of finite dimension n. In some situations we would like to choose an euclidean structure on V. In this case we identify V with $\mathbb{R}^n$ equipped with the standard euclidean structure. Also if we write $\mathbb{R}^n$, then we mean the euclidean space with the standard euclidean structure and denote by e_i the i-th standard basis vector in $\mathbb{R}^n$. We sometimes identify the dual space $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. For more on convex geometry, we refer to [42, 51, 80].

2.1 Convex bodies

The notion of a convex body is crucial in this work.

Definition 2.1 (Convex body)**.** A non-empty set $K \subset V$ is called a *convex body* if K is compact and convex. The *dimension* of K is denoted by $\dim(K)$ and defined to be the dimension of the affine hull of K.

The set of convex bodies in a vector space V is denoted by $\mathcal{K}(V)$. We write $\mathcal{K}_o(V)$ for the subset of convex bodies containing the origin and $\mathcal{K}_{(o)}(V)$ for the subset of convex bodies containing the origin in its interior. If A is any subset of V we denote the set of convex bodies contained in A by $\mathcal{K}(A)$.

Example 2.2.

1. Let $v_1, \dots, v_{n+1} \subset V$ be affinely independent. The simplex $S := \operatorname{conv}\{v_i \mid i = 1, \dots, n+1\} \subset V$ is a convex body. Here conv denotes the convex hull. More general the convex hull of finitely many points is a convex body. Such a body is called a *polytope.*

2. The unit ball $B^n \subset \mathbb{R}^n$ is a convex body. We also write B instead of B^n.

3. For $x, y \in V$ the line segment denoted by $[x, y] := \{(1-\lambda)x + \lambda y \mid \lambda \in [0, 1]\}$ is a convex body.

The set of convex bodies carries an additional structure, called the *Minkowski sum.* For $K, L \in \mathcal{K}(V)$ the Minkowski sum is defined by

$$K + L := \{x + y \mid x \in K, y \in L\}.$$

It is easy to check that the Minkowski sum of two convex bodies is again a convex body. Further, this operation is clearly associative and commutative. This makes $\mathcal{K}(V)$ into an abelian semigroup. For $t \in V$ we say, that $K + \{t\}$ is a *translate* of K and also write $K + t$ for this convex body.

Also, a scalar multiplication can be defined on $\mathcal{K}(V)$ by

$$\lambda K := \{\lambda x \mid x \in K\}$$

for $\lambda \in \mathbb{R}$ and a convex body K. Note that λK is again a convex body. We write $-K$ for $(-1)K$ and if $K = -K$ we say that K is *(origin) symmetric* or *centered.*

For an affine map $L\colon V \to W$ between vector spaces we define an associated map $\widetilde{L}\colon \mathcal{K}(V) \to \mathcal{K}(W)$ by

$$\widetilde{L}(K) := \{L(x) \mid x \in K\}.$$

Note that $\widetilde{L}(K)$ is indeed a convex body. Instead of $\widetilde{L}$ we also write L.

Next, we want to define a topology on $\mathcal{K}(V)$. To do the construction we have to choose an euclidean structure on V. For $\varepsilon > 0$ we write K_ε for the body $K + \varepsilon B^n$. Now define the *Hausdorff distance* of two convex bodies K and L by

$$d_H(K, L) := \inf\{\delta > 0 \mid L \subset K_\delta, K \subset L_\delta\}.$$

One can show that the Hausdorff distance is a metric on $\mathcal{K}(V)$ ([80], §1.8) and therefore it induces a topology. Clearly, the Hausdorff distance depends on the choice of the inner product. But it is important to note that the resulting topology does not [65]. Hence we can forget the choice of the inner product on V once we have the topology.

Later we make use of several constructions of convex bodies. Two of them are mentioned below.

Definition 2.3 (Zonotope, Zonoid)**.** A *zonotope* is the Minkowski sum of finitely many line segments. A convex body is called a *zonoid* if it is the limit of a sequence of zonotopes.

Example 2.4.

1. The unit cube $C^n \subset \mathbb{R}^n$ given by

$$C^n = \sum_{i=1}^{n} [0, e_i]$$

   is a zonotope.

2. The unit ball is a zonoid ([80], use Theorem 3.5.3).

The dual space of a vector space V is denoted by V^*. Note that there is a canonical isomorphism $V \to (V^*)^*$ given by $v \mapsto \xi_v$, such that $\xi_v(\varphi) = \varphi(v)$. Now for a convex body in V we want to assign a convex body in V^*.

Definition 2.5 (Polar body)**.** For $K \in \mathcal{K}_{(o)}(V)$ the *polar body* $K^\circ \subset V^*$ is given by

$$K^\circ := \{\xi \in V^* \mid \xi(x) \leq 1, \forall x \in K\}.$$

The polar body is again a convex body (now in V^*) and contains the origin as an interior point ([80], §1.6.1).

Lemma 2.6. *Let $K, L \in \mathcal{K}_{(o)}(V)$. Then we have*

1. $(K^\circ)^\circ = K$, *where we identify* $(V^*)^*$ *with* V,
2. *If* $K \cup L$ *is convex, then* $K^\circ \cup L^\circ$ *is convex and we have*
$$(K \cup L)^\circ = K^\circ \cap L^\circ, \quad (K \cap L)^\circ = K^\circ \cup L^\circ,$$
3. $(\lambda K)^\circ = \frac{1}{\lambda} K^\circ$ *for* $\lambda \neq 0$,
4. $K \subset L \Leftrightarrow L^\circ \subset K^\circ$.

Proof. For the first statement, we refer to [80], §1.6.1. The other statements are easy to check. □

The following theorem is classical.

Theorem 2.7 (Blaschke's selection theorem, [20], §18.1)**.** *Let $B \subset \mathbb{R}^n$ be a bounded set and denote by $\mathcal{K}(B)$ the set of convex bodies contained in B. For any sequence in $\mathcal{K}(B)$ there is a subsequence converging to a convex body.*

2.2 Support function

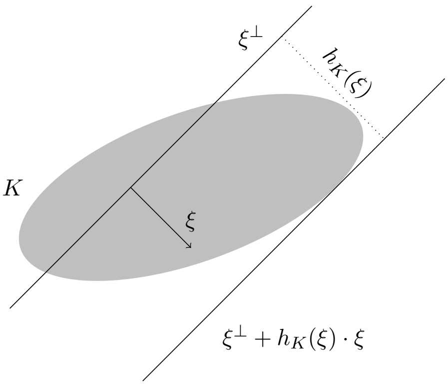
A useful tool to work with convex bodies is the support function.

Definition 2.8 (Support function)**.** Let $K \subset V$ be a convex body. The function

$$h_K \colon V^* \to \mathbb{R}, \quad \xi \mapsto \sup_{x \in K} \langle \xi, x \rangle$$

is called the *support function* of K.

If $V = \mathbb{R}^n$ and $(\mathbb{R}^n)^*$ is identified with $\mathbb{R}^n$, we can restrict the support function to the unit sphere S^{n-1}. There is the following interpretation of this restriction. For $\xi \in S^{n-1}$ consider the hyperplane $\xi^\perp$. Then $h_K(\xi)$ is the oriented distance of the maximal shift of $\xi^\perp$ into direction ξ, such that this shift intersects K. In other words $\big(\xi^\perp + h_K(\xi) \cdot \xi\big) \cap K \neq \emptyset$ and $h_K(\xi)$ is maximal with this property.



Example 2.9.

1. For the unit ball in $\mathbb{R}^n$ we have $h_{B^n}(u) \equiv 1$ for all $u \in S^{n-1}$.
2. For a line segment it is $h_{[x,y]}(\xi) = \max\{\langle \xi, x\rangle, \langle \xi, y\rangle\}$.

It is easy to see that the support function of a convex body K is sublinear, i.e. h_K is

- subadditive: $h_K(\xi + \varphi) \leq h_K(\xi) + h_K(\varphi)$ and
- positively homogeneous: $h_K(\lambda\xi) = \lambda h_K(\xi)$ for $\lambda > 0$.

By the following statement, there is a one-to-one correspondence between convex bodies and sublinear functions.

Lemma 2.10 ([80], §1.7.1)**.** *If $f\colon V^* \to \mathbb{R}$ is a sublinear function, then there is a unique convex body $K \in \mathcal{K}(V)$ such that $f = h_K$.*

By this lemma, a convex body is uniquely described by its support function. We can say even more if we choose an euclidean structure: Using positive homogeneity a convex body is uniquely described by the restriction of its support function to the unit sphere. On the other hand, the lemma tells us that we can define a convex body by indicating a sublinear function.

One can also consider the support function as a function in K.

Lemma 2.11. *Let $K, L \in \mathcal{K}(V)$. Then*

1. $h_{K+L} = h_K + h_L$,
2. $h_{\lambda K} = \lambda h_K$ *for all positive* λ,
3. $h_K \leq h_L \Leftrightarrow K \subset L$.

Proof. For a proof of the first statement, we refer to [80], §1.7.1. The other statements are obvious. □

Last we want to mention that there is a connection between the Hausdorff metric and the support function.

Lemma 2.12 ([80], §1.8)**.** *If $V = \mathbb{R}^n$, then for two convex bodies K, L we have*

$$d_H(K, L) = \sup_{u \in S^{n-1}} |h_K(u) - h_L(u)|.$$

This tells us that a sequence $(K_i)_i$ of convex bodies converges to K if and only if the corresponding support functions restricted to the unit sphere converge with respect to the ∞-norm to the support function of K. Note that we make use of the euclidean structure here.

2.3 Radial function

In this section, we want to discuss the radial function of a convex body, which is related to the support function as we will see.

Definition 2.13 (Radial function)**.** For $K \in \mathcal{K}_{(o)}(V)$ the *radial function* is given by

$$\rho_K \colon V \setminus \{0\} \to \mathbb{R}, \quad x \mapsto \sup\{\lambda \geq 0 \mid \lambda x \in K\}.$$

The radial function can also be defined for a bigger family of sets. More precisely, we can allow K to be star-shaped with respect to the origin. However, we apply the radial function only to convex bodies containing the origin in its interior. The radial function is connected with the support function in the following way.

Lemma 2.14 ([80], §1.7.1)**.** *Let $K \in \mathcal{K}_{(o)}(V)$. Then for $x \neq 0$ we have*

$$\rho_K(x) = \frac{1}{h_{K^\circ}(x)}.$$

Here we identify $(V^*)^*$ and V via the isomorphism mentioned above.

By this lemma, a convex body containing the origin in its interior is uniquely determined by its radial function. Also, the relation implies the following corollary.

Corollary 2.15. *Let $K, L \in \mathcal{K}_{(0)}(V)$ and $\lambda > 0$. Then*

1. $\rho_K(\lambda x) = \frac{1}{\lambda}\rho_K(x)$,
2. $\rho_{\lambda K}(x) = \lambda \rho_K(x)$,
3. $\rho_K \leq \rho_L \Leftrightarrow K \subset L$.

2.4 Steiner symmetrization

In this section, we want to discuss a well-known construction called *Steiner symmetrization.* As we will see later it is a useful tool to prove inequalities in integral geometry. In this section, we assume $V = \mathbb{R}^n$ is equipped with an euclidean structure. Let K be a convex body in $\mathbb{R}^n$ and $u \in S^{n-1}$. Let H be the hyperplane through the origin and orthogonal to u. Denote by K_H the orthogonal projection of K to H and define functions

$$\overline{z}, \underline{z} \colon K_H \to \mathbb{R}, \quad \overline{z}(y) = \max\{t \in \mathbb{R} \colon y + tu \in K\}, \quad \underline{z}(y) = \min\{t \in \mathbb{R} \colon y + tu \in K\}.$$

Since K is convex the function $\overline{z}$ (resp. $\underline{z}$) is concave (resp. convex). It is easy to see that $\overline{z}$ (resp. $\underline{z}$) is upper (resp. lower) semi-continuous.

Definition 2.16 (Steiner symmetrization)**.** The Steiner symmetrization of K with respect to H is the set

$$\operatorname{st}_H K = \bigcup_{y \in K_H} [y + \underline{z}(y)u, y + \overline{z}(y)u] - \frac{\underline{z}(y) + \overline{z}(y)}{2} u.$$

It is well-known (see [80], §10.3) that $\operatorname{st}_H K$ is again a convex body and symmetric with respect to H, i.e. $\operatorname{st}_H K$ is fixed under the reflection with respect to H.

Theorem 2.17. *Let K be a convex body. There is a sequence $(u_i)_i$ in S^{n-1} such that the sequence $(K_m)_m$, where*

$$K_m := \operatorname{st}_{u_m^\perp} \cdots \operatorname{st}_{u_1^\perp} K$$

converges to a ball.

A clear and illustrated proof can be found in [13] (Theorem 9.13.6).
It is well-known and follows easily from Fubini's theorem that Steiner symmetrization preserves the volume. Further, it does not enlarge the surface area $\mathcal{H}^{n-1}(\partial K)$ of a convex body K (see [80], §10.4). Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Together with Theorem 2.17 the isoperimetric inequality

$$\left(\frac{\mathcal{H}^{n-1}(\partial K)}{\mathcal{H}^{n-1}(\partial B^n)} \right)^n \geq \left(\frac{\operatorname{vol}_n(K)}{\operatorname{vol}_n(B^n)} \right)^{n-1}$$

follows.

3 Representation theory

In this chapter, $\mathbb{F}$ stands for one of the fields $\mathbb{R}$ and $\mathbb{C}$. Our main goal in this chapter is to prove that any finite-dimensional $\mathrm{SL}(V)$-representation is already polynomial (Proposition 3.44) and discuss some properties of such representations. Also, we want to give a decomposition of certain tensor products into irreducible representations. A well-known introduction to representation theory (mostly for the complex case), where important examples are discussed is provided in [37]. We also want to refer to [53, 54, 95]. For results in the theory of polynomial representations see [40]. Depending on the reader's background this might be difficult to read, though. Therefore we recommend having a look in [55, 56].

3.1 Basics

Representation theory is a topic which is well understood in several situations. This section provides a short introduction to the general concept. We introduce important notions and give examples, which will play a central role later on. In this section, G denotes a group and W stands for a closed vector space over $\mathbb{F}$.

Definition 3.1 (Representation)**.** Let W be a closed vector space over $\mathbb{F}$. A *representation* of G is a homomorphism

$$\rho\colon G \to \mathrm{GL}(W).$$

We will also call the space W a *representation* when it is clear (or not important) how the homomorphism $\rho\colon G \to \mathrm{GL}(W)$ is defined. A representation ρ as before defines a group action of G on W by

$$g \cdot w := \rho(g)(w), \quad \forall g \in G, w \in W.$$

We also use this correspondence to define representations by group actions. If G acts on a set A, we say $A' \subset A$ is a G*-invariant* subset if $g \cdot a \in A'$ for all $g \in G$ and $a \in A'$.

Note that a representation ρ makes W into a G-module. Let us give several examples. They can be seen as a definition of representations in certain situations. Some of these examples play an important role later on.

Example 3.2.

1. The map $\rho(g) = \mathrm{id}_W$ for all $g \in G$ is a representation for any group G and any W. This is called the *trivial representation.*

2. If G is a subgroup of $\mathrm{GL}(W)$, the inclusion $\rho(g) = g$ is a representation.

3. A representation $\rho\colon \mathrm{GL}(W) \to \mathrm{GL}(\mathbb{F})$ is given by $\rho(g)(x) = \det(g) \cdot x$.

4. If W_1, W_2 are representations of G then the tensor product $W_1 \otimes W_2$ is a representation of G defined by $g(w_1 \otimes w_2) := gw_1 \otimes gw_2$ on the elementary tensors and extended by linearity to the whole tensor product. Also, the sum $W_1 \oplus W_2$ is a representation of G given by $g(w_1 + w_2) = gw_1 + gw_2$.

5. Let U be a vector space. If $\rho\colon G \to \mathrm{GL}(W)$ is a representation and $F(W, U)$ is a closed space of maps $W \to U$, then

$$\widetilde{\rho}\colon G \to \mathrm{GL}(F(W, U)), \quad \widetilde{\rho}(g)(f) = f \circ \rho(g)^{-1} \quad \forall g \in G, f \in F(W, U)$$

is a representation. In particular the *dual representation* $\rho^*\colon G \to \mathrm{GL}(W^*)$ is defined in this way and satisfies $(\rho^*(g)(u))(\rho(g)(w)) = u(w)$ for all $u \in W^*, w \in W$, or equivalently $\langle \rho^*(g)(u), \rho(g)(w)\rangle = \langle u, w\rangle$.

6. The restriction $\rho|_H$ to a subgroup $H \subset G$ of a representation $\rho\colon G \to \mathrm{GL}(W)$ is a representation of H. We sometimes write $\mathrm{Res}^G_H W$ for the H-representation W.

7. Suppose $\rho\colon G \to \mathrm{GL}(W)$ is a representation and $U \subset W$ is a closed G-invariant subspace. Then $\widetilde{\rho}\colon G \to \mathrm{GL}(U)$ given by the restriction $\widetilde{\rho}(g) = \rho(g)|_U$ is a representation called the *subrepresentation* of ρ with respect to U.

8. Denote by S_d the symmetric group. This group acts on the elementary tensors of $W^{\otimes d}$ by

$$\sigma \cdot (w_1 \otimes \cdots \otimes w_d) := w_{\sigma^{-1}(1)} \otimes \cdots \otimes w_{\sigma^{-1}(d)}$$

and by linear extension, it becomes an action on $W^{\otimes d}$. Again by linearity this action can be extended to an action on the group algebra $\mathbb{F}S_d$.

9. Define the k-th exterior power by

$$\textstyle\bigwedge^k W = \mathrm{span}\{w_1 \wedge \cdots \wedge w_k \mid w_1, \ldots, w_k \in W\} \subset W^{\otimes k},$$

where

$$w_1 \wedge \cdots \wedge w_k := \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

The exterior power is a representation of $\mathrm{GL}(W)$.

10. Define the k-th symmetric power by

$$\mathrm{Sym}^k W = \mathrm{span}\{w_1 \odot \cdots \odot w_k \mid w_1, \ldots, w_k \in W\} \subset W^{\otimes k},$$

where

$$w_1 \odot \cdots \odot w_k := \sum_{\sigma \in S_k} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

The symmetric power is a representation of $\mathrm{GL}(W)$.

Next, let us introduce some more notions. Let $\rho\colon G \to \mathrm{GL}(W)$ be a representation. We call ρ *irreducible* if there is no G-invariant proper closed subspace $U \subset W$. We say ρ is *completely reducible* if W decomposes into a direct sum

$$W = \bigoplus_{i \in I} W_i$$

of irreducible subrepresentations W_i. Another representation $\widetilde{\rho}\colon G \to \mathrm{GL}(\widetilde{W})$ is called *isomorphic* to ρ if there is an isomorphism $\alpha\colon W \to \widetilde{W}$ such that

$$\alpha \circ \rho(g) = \widetilde{\rho}(g) \circ \alpha$$

is satisfied for all $g \in G$. In this case we write $W = \tilde{W}$.

3.2 Representations of Lie groups and Lie algebras

As in the previous section, $\mathbb{F}$ stands for one of the fields $\mathbb{R}$ or $\mathbb{C}$ and W is a closed vector space over $\mathbb{F}$.

3.2.1 Lie groups

In this section we want to give a short introduction to Lie groups and Lie algebras. We need the theory to understand the behaviour of convex bodies under linear transformations.

Definition 3.3 (Lie group)**.** A *Lie group* G over $\mathbb{R}$ (respectively over $\mathbb{C}$) is a smooth (respectively complex-analytic) manifold which is also a group, such that the multiplication

$$G \times G \to G, \quad (g, h) \mapsto gh$$

and the inversion

$$G \to G, \quad g \mapsto g^{-1}$$

are smooth (respectively holomorphic) maps. The *dimension* of a Lie group is the dimension of its underlying manifold. A Lie group is called a *matrix Lie group* if it is a subgroup of some general linear group.

Example 3.4. Let W be a vector space over $\mathbb{F}$ of finite dimension.

1. The general linear group $\mathrm{GL}(W)$ is a Lie group. We also write $\mathrm{GL}(W, \mathbb{F})$ for this group to indicate that W is a vector space over $\mathbb{F}$.

2. The special linear group $\mathrm{SL}(W)$ (or $\mathrm{SL}(W, \mathbb{F})$) is a Lie group.

3. The special orthogonal group $\mathrm{SO}(n) \subset \mathrm{GL}(\mathbb{R}^n)$ is a Lie group.

4. In the case $\mathbb{F} = \mathbb{R}$ we also consider the Lie group $\mathrm{GL}^+(W)$ of orientation preserving isomorphisms T, i.e. $\det(T) > 0$.

5. Any finite group is a Lie group of dimension 0.

In the following sections, we deal with representations of Lie groups. By such a representation we mean a homomorphism $\rho\colon G \to \mathrm{GL}(W)$ as before which is also continuous. It is well-known that such a representation is necessarily smooth, if G is a matrix Lie group.

Proposition 3.5 ([47], §3.8)**.** *Let G, H be matrix Lie groups and $\varphi\colon G \to H$ be a group homomorphism. If φ is continuous, then it is smooth.*

There is a well-known theorem concerning irreducible representations, called Schur's Lemma.

Theorem 3.6 (Schur's Lemma;[37], §1.2; [75], §6.1.7)**.** *If V and W are irreducible representations over $\mathbb{F}$ of a Lie group G and $\varphi\colon V \to W$ is a homomorphism such that the group actions commute, then*

1. *φ is an isomorphism or $\varphi = 0$.*
2. *If $\mathbb{F} = \mathbb{C}$ and $V = W$ is of finite dimension, then φ is a (complex) multiple of the identity.*

A direct consequence is the following.

Corollary 3.7. *If V and W are complex irreducible representations of a Lie group G and $\varphi, \psi\colon V \to W$ are non-zero homomorphisms such that the group actions commute, then $\varphi = z \cdot \psi$ for some $z \in \mathbb{C}$.*

Proof. Since φ, ψ are non-zero, they are isomorphisms by the first assertion of Schur's Lemma. Then $\varphi \circ \psi^{-1} = z \cdot \mathrm{id}$ for some $z \in \mathbb{C}$ by the second part. This means $\varphi = z \cdot \psi$. $\square$

Schur's Lemma can also be used to show that the decomposition of completely reducible representations into irreducible subrepresentations is unique. For our purposes, it is enough to prove this fact for finite direct sums. For an integer $\lambda \geq 0$ and a vector space V over $\mathbb{F}$ we write V^λ for the direct sum $V \oplus \cdots \oplus V$ of length λ. We define $V^0 = \mathbb{F}$.

Corollary 3.8. *Let $V_1, \ldots, V_n, W_1, \ldots, W_m$ be non-zero irreducible Lie group representations, such that V_i is not isomorphic to V_j and W_i is not isomorphic to W_j for $i \neq j$. If*

$$V_1^{\lambda_1} \oplus \cdots \oplus V_n^{\lambda_n} = W_1^{\mu_1} \oplus \cdots \oplus W_m^{\mu_m},$$

then $m = n$ and up to permutation of the indices we have $V_i = W_i, \lambda_i = \mu_i$.

Proof. Let

$$\Phi\colon V_1^{\lambda_1} \oplus \cdots \oplus V_n^{\lambda_n} \to W_1^{\mu_1} \oplus \cdots \oplus W_m^{\mu_m}$$

be an isomorphism. For i, j we have a homomorphism

$$\Phi_{i,j}\colon V_i \to W_j, \quad v \mapsto (\pi_{W_j} \circ \Phi)(v),$$

where π_{W_j} denotes the projection onto W_j. For any i there is a unique index j such that $\Phi_{i,j}$ is an isomorphism. Indeed, since Φ is an isomorphism there is j such that $\Phi_{i,j}$ is not zero. Then by Schur's Lemma, it must be an isomorphism. Uniqueness follows from the fact that for two distinct indices the codomains are not isomorphic. W.l.o.g. we have $V_i = W_i$. It follows $V_i^{\lambda_i} = W_i^{\mu_i}$ which implies $\lambda_i = \mu_i$. $\square$

3.2.2 Lie algebras

A very related concept to Lie groups is the notion of Lie algebras.

Definition 3.9 (Lie algebra)**.** A *Lie algebra* $\mathfrak{g}$ is a vector space over $\mathbb{F}$ equipped with a skew symmetric bilinear map

$$[\bullet,\bullet]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$$

satisfying

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0. \tag{3.1}$$

The condition (3.1) is called *Jacobi identity*.

Let G be a Lie group. We want to assign to any Lie group a corresponding Lie algebra. For $g\in G$ consider the map

$$\Psi_g\colon G\to G,\quad h\mapsto ghg^{-1}.$$

Clearly, Ψ_g is smooth and it fixes the neutral element $e\in G$. Further one can easily check $\Psi_{g_1g_2}=\Psi_{g_1}\circ\Psi_{g_2}$ and we obtain $(\Psi_g)^{-1}=\Psi_{g^{-1}}$. In particular, Ψ_g is a diffeomorphism. The differential $(d\Psi_g)_e =: \mathrm{Ad}(g)$ at e is therefore an isomorphism $T_eG\to T_eG$. We obtain a map

$$\mathrm{Ad}\colon G\to \mathrm{GL}(T_eG)$$

which is again a homomorphism since

$$\mathrm{Ad}(gh)=(d\Psi_{gh})_e=(d(\Psi_g\circ\Psi_h))_e=(d\Psi_g)_e\circ d(\Psi_h)_e=\mathrm{Ad}(g)\circ\mathrm{Ad}(h).$$

Further it is clear that Ad is continuous and therefore smooth by Proposition 3.5. Its differential $d(\mathrm{Ad})_e =: \mathrm{ad}$ at e is a map $T_eG\to T_{\mathrm{id}}(\mathrm{GL}(T_eG))$. The space $T_{\mathrm{id}}(\mathrm{GL}(T_eG))$ can be identified with $\mathrm{End}(T_eG)$. The bilinear map

$$[\bullet,\bullet]\colon T_eG\times T_eG\to T_eG,\quad (X,Y)\mapsto \mathrm{ad}(X,Y) \tag{3.2}$$

is called the *Lie bracket.* It is a fact that T_eG equipped with $[\bullet,\bullet]$ is a Lie algebra, i.e. we have the following lemma.

Lemma 3.10 ([37], §8.1)**.** *The bilinear map defined in (3.2) is skew symmetric and satisfies the Jacobi identity.*

Definition 3.11 (Lie algebra of a Lie group)**.** Let G be a Lie group. The corresponding Lie algebra is given by $\mathfrak{g}:=T_eG$ equipped with the Lie bracket. The Lie algebra of the Lie group $\mathrm{GL}(W)$ is denoted by $\mathfrak{gl}(W)$.

For the Lie groups in Example 3.4 mentioned above we have the following corresponding Lie algebras, where we identify W with $\mathbb{F}^n$.

Lie group	**Lie algebra**	**Lie bracket**
$\mathrm{GL}(\mathbb{F}^n)$	$\mathfrak{gl}(\mathbb{F}^n) := \mathrm{End}(\mathbb{F}^n)$	$[X,Y] = X \cdot Y - Y \cdot X$
$\mathrm{SL}(\mathbb{F}^n)$	$\mathfrak{sl}(\mathbb{F}^n) := \{X \in \mathrm{End}(\mathbb{F}^n) \mid \mathrm{tr}(X) = 0\}$	$[X,Y] = X \cdot Y - Y \cdot X$
$\mathrm{SO}(\mathbb{R}^n)$	$\mathfrak{so}(\mathbb{F}^n) := \{X \in \mathrm{End}(\mathbb{F}^n) \mid X^t = -X\}$	$[X,Y] = X \cdot Y - Y \cdot X$
$\mathrm{GL}^+(\mathbb{R}^n)$	$\mathfrak{gl}(\mathbb{R}^n)$	$[X,Y] = X \cdot Y - Y \cdot X$

Table 3.1: Some Lie groups and their Lie algebras

We saw that for any Lie group there is an associated Lie algebra. As we will see, this is not the end of the story. There is also a notion of *representation* for Lie algebras. It turns out that this notion is highly connected with the notion of Lie group representations.

Definition 3.12 (Lie algebra representation)**.** Let $\mathfrak{g}$ be a Lie algebra and W be a vector space over $\mathbb{R}$ (resp. $\mathbb{C}$). A *(Lie algebra) representation* is a smooth (resp. holomorphic) homomorphism

$$\rho\colon \mathfrak{g} \to \mathfrak{gl}(W)$$

such that $\rho([X,Y]) = [\rho(X), \rho(Y)]$.

In order to investigate Lie group representations, it is (in certain situations) helpful to study representations of its corresponding Lie algebra. To specify the term *in certain situations* we need some more preparation. First of all let us mention that any representation of a Lie group admits a Lie algebra representation by its differential, provided it exists.

Proposition 3.13 ([75], §4.3.2)**.** *Let G be a Lie group. If $\rho\colon G \to \mathrm{GL}(W)$ is a representation, then the differential $d\rho\colon \mathfrak{g} \to \mathfrak{gl}(W)$ is also a representation.*

If our Lie algebra has the property that any Lie algebra representation comes from a Lie group representation, then these concepts are equivalent and instead of studying Lie group representations one can also study Lie algebra representations. But we have to be careful: In general it, is not true that every Lie algebra representation comes from a Lie group representation. This property is what we meant by the term *in certain situations* above.

Definition 3.14 (Simple and semisimple Lie algebras)**.** A Lie algebra is called *simple* if it is not abelian, i.e. $[\cdot,\cdot] \neq 0$, and contains no non-zero proper ideal. It is called *semisimple* if it is a direct sum of simple Lie algebras.

It is well-known that the Lie algebra $\mathfrak{sl}(\mathbb{F}^n)$ is simple (and therefore semisimple), but $\mathfrak{gl}(\mathbb{F}^n)$ is neither simple nor semisimple. The notions of complete reducibility and irreducible representations also transfer for Lie algebra representations.

Definition 3.15 (Reducibility)**.** A non-zero $\mathfrak{g}$-representation W is called *irreducible* if W and $\{0\}$ are the only $\mathfrak{g}$ invariant subspaces. W is called *completely reducible* if M decomposes into a direct sum of irreducible subrepresentations.

For semisimple Lie algebras, we have the following fact. In particular the proposition applies to the case of $\mathfrak{sl}(W)$.

Proposition 3.16 ([49], Chapter III.7)**.** *If $\mathfrak{g}$ is a semisimple Lie algebra then every finite-dimensional $\mathfrak{g}$-representation is completely reducible.*

To finish our discussion about the correspondence between Lie group representations and Lie algebra representations we want to give a condition, such that the representation theory of Lie groups and Lie algebras is equivalent.

Proposition 3.17 ([75], §3.4)**.** *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. If $f\colon \mathfrak{g} \to \mathfrak{gl}(W)$ is a representation, then there is a representation $\rho\colon G \to \mathrm{GL}(W)$ such that $f = d\rho$.*

Since the group $\mathrm{SL}(\mathbb{C}^n)$ is simply connected the following corollary is a direct consequence from the previous proposition.

Corollary 3.18. *The classification of irreducible $\mathrm{SL}(\mathbb{C}^n)$-representations is exactly the same as for $\mathfrak{sl}(\mathbb{C}^n)$-representations.*

The previous corollary tells us, that a classification of the irreducible representations of $\mathrm{SL}(\mathbb{C}^n)$ is equivalent to a classification of the corresponding Lie algebra representations, which seems to be an easier task. Unfortunately, the Lie group $\mathrm{SL}(\mathbb{R}^n)$ is not simply connected. Therefore we have to do more work to obtain such a classification for $\mathrm{SL}(\mathbb{R}^n)$. The plan for the next sections is to give a classification of irreducible representations of $\mathfrak{sl}(\mathbb{C}^n)$ and to show that this classification is the same as for $\mathfrak{sl}(\mathbb{R}^n)$. Finally, we show that Corollary 3.18 is still true when $\mathbb{C}^n$ is replaced by $\mathbb{R}^n$.

3.2.3 The theory of highest weights

In this section let $\mathfrak{g}$ be a complex semisimple Lie algebra coming from a Lie group. We need the theory of representations of semisimple Lie algebras only for $\mathfrak{g} = \mathfrak{sl}(\mathbb{C}^n)$. However, we explain the general case and as an application, we will deal with the case $\mathfrak{sl}(\mathbb{C}^n)$ explicitly.

Definition 3.19 (Cartan subalgebra)**.** An element $X \in \mathfrak{g}$ is called *semisimple* if it is diagonalizable. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *Cartan subalgebra* if it consists of semisimple elements and is abelian. The latter means $[h_1, h_2] = 0$ for all $h_1, h_2 \in \mathfrak{h}$.

Now fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For $\alpha \in \mathfrak{h}^* \setminus \{0\}$ let

$$\mathfrak{g}_\alpha := \{g \in \mathfrak{g} \mid [h, g] = \alpha(h)g, \forall h \in \mathfrak{h}\}.$$

Denote by Φ the set containing all $\alpha \in \mathfrak{h}^* \setminus \{0\}$ such that $\mathfrak{g}_\alpha$ is not zero. Then $\alpha \in \Phi$ is called a *root* of $\mathfrak{g}$ (with respect to $\mathfrak{h}$) and the set $\mathfrak{g}_\alpha$ is called the corresponding *root* *space.* It is well-known that $\mathfrak{g}$ decomposes into a direct sum of the Cartan subalgebra and its root spaces ([31], §10.3), i.e.

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha. \tag{3.3}$$

Further, the set Φ is called the *root system* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. It has the following properties.

Lemma 3.20 ([31], §11.1)**.**
- *Φ is finite and $0 \notin \Phi$.*
- *If $\alpha \in \Phi$, then the only real scalar multiples of α contained in Φ are $\pm\alpha$.*

We can choose a subset $\Phi^+ \subset \Phi$ such that

- For each root $\alpha \in \Phi$ exactly one of $\alpha, -\alpha$ is contained in Φ^+.
- For any two distinct roots $\alpha, \beta \in \Phi^+$ such that $\alpha+\beta$ is a root, we have $\alpha+\beta \in \Phi^+$.

The roots contained in Φ^+ are called *positive roots.* The set of *negative roots* is defined by $\Phi^- := \Phi \setminus \Phi^+$. We have a similar decomposition for an irreducible representation W of $\mathfrak{g}$ of finite dimension. For $\alpha \in \mathfrak{h}^*$ consider the space

$$W_\alpha := \{w \in W \mid h \cdot w = \alpha(h)w, \forall h \in \mathfrak{h}\}$$

Let Ψ be the set of $\alpha \in \mathfrak{h}^*$ such that W_α is not zero. An element in Ψ is called *weight* and the corresponding space W_λ is called the *weight space.* It is well-known that W decomposes into a direct sum of its weight spaces ([31], §15.1), i.e. we have a decomposition

$$W = \bigoplus_{\alpha \in \Psi} W_\alpha. \tag{3.4}$$

Let $w \in W_\lambda, g \in \mathfrak{g}_\alpha$ and $h \in \mathfrak{h}$. Note that it holds

$$h(g \cdot w) = ([h, g] + gh) \cdot w = \alpha(h)g \cdot w + \lambda(h)g \cdot w = (\alpha + \lambda)(h)g \cdot w.$$

Hence $g \cdot w \in W_{\lambda+\alpha}$. Since W has finite dimension, the set Ψ is finite. Hence there is $\lambda \in \Psi$ such that $\lambda + \alpha \notin \Psi$ for all positive roots α. Such λ is called a *highest weight* and W_λ is the corresponding *highest weight space.* In this case, a non-zero element $w \in W_\lambda$ is called *highest weight vector.* If W is irreducible it is well-known that there is a unique highest weight and the highest weight space is of dimension 1 ([31], §15.1.). The fact that two finite-dimensional irreducible representations of a semisimple Lie algebra are isomorphic if and only if they have the same highest weight becomes important later on ([75], §5.2).

Example 3.21. Recall that the Lie algebra $\mathfrak{sl}(\mathbb{C}^n)$ is given by the endomorphisms on V whose trace vanishes. As a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(\mathbb{C}^n)$ we choose

$$\mathfrak{h} = \left\{ \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathbb{C}^{n \times n} \colon \sum_{i=1}^{n} a_i = 0 \right\}.$$

Let $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ given by $\alpha_i(h) = a_i$. Obviously $\alpha_1 + \dots + \alpha_n = 0$. Then the roots are given by $\alpha_{ij} = \alpha_i - \alpha_j$ for $i \neq j$ and the corresponding root spaces $\mathfrak{g}_{ij}$ are spanned by X_{ij}, the matrix with 1 on the entry (i, j) and zero otherwise. A set of positive roots is given by $\{\alpha_{ij} \mid j > i\}$. Now it is easy to see that the highest weight of a finite-dimensional irreducible $\mathfrak{sl}(\mathbb{C}^n)$-representation can be expressed as a linear combination of $\alpha_1, \dots, \alpha_n$ whose coefficients are non-negative integers.

Consider the representation $W = \bigwedge^k \mathbb{C}^n$ and $w = e_1 \wedge \dots \wedge e_k \in W$. As we will see later it is an irreducible representation. For $j > i$ let $\gamma_{ij}(t) = \mathrm{id} + tX_{ij} \in \mathrm{SL}(\mathbb{C}^n)$. Then $\gamma(t)w = w$ and therefore

$$0 = \left.\frac{d}{dt}\right|_{t=0} \gamma(t)w = X_{ij}w.$$

Further $h \in \mathfrak{h}$ can be written as $h = \frac{d}{dt}\big|_{t=0} D(t \cdot (a_1, \dots, a_n))$, where $a_1 + \dots + a_n = 0$ and

$$D(t \cdot (a_1, \dots, a_n)) = \begin{pmatrix} e^{ta_1} & & \\ & \ddots & \\ & & e^{ta_n} \end{pmatrix}.$$

Clearly $D(t \cdot (a_1, \dots, a_n))w = e^{t(a_1 + \dots + a_k)}w$ and therefore $hw = (a_1 + \dots + a_k)w$. By the previous discussion w must be the highest weight vector and a corresponding highest weight is given by $\alpha_1 + \dots + \alpha_k$.

3.2.4 A classification of irreducible $\mathfrak{sl}(\mathbb{C}^n)$-representations

Young tableaux and the Littlewood-Richardson coefficient

Let V be a real or complex vector space of finite dimension n. In the complex case a classification of finite-dimensional irreducible $\mathfrak{sl}(V)$-representations is given by Weyl modules. As we will see later, the classification in the real case is exactly the same. In the construction of Weyl modules, the notion of a Young tableau plays an important role. As we will see, Weyl modules are certain subrepresentations of $V^{\otimes d}$. The action of the group algebra $\mathbb{F}S_d$ on $V^{\otimes d}$, where S_d is the symmetric group (see again Example 3.2) will be crucial in this construction. First, we construct certain elements in $\mathbb{F}S_d$.

Definition 3.22 (Partition, Young tableau)**.** A *partition* of size d is a m-tuple $\nu = (\nu_1, \dots, \nu_m)$ of non-negative integers such that $\nu_1 \geq \dots \geq \nu_m$ and $d = \nu_1 + \dots + \nu_m$. A *Young tableau* of ν is a scheme of d boxes arranged in m rows, such that in the i-th row there are exactly ν_i boxes.

Since there is clearly a one-to-one correspondence between partitions and Young tableaux we also write ν for the Young tableau corresponding to the partition ν. A *label* of ν is a numbering of the boxes of the Young tableau, where each box is labeled with a number between 1 and d such that any number occurs exactly once.

Example 3.23. A label of the Young tableau $\nu = (4, 4, 2, 1)$ is given by

1	2	3	4
5	6	7	8
9	10		
11			

For a given label of a Young tableau ν let R_ν (resp. C_ν) be the subgroup of S_d containing all permutations fixing the rows (resp. columns) of the label of ν. Now define elements in the group algebra $\mathbb{R}S_d$ of the symmetric group S_d by

$$a_\nu := \sum_{\sigma \in R_\nu} \sigma, \quad b_\nu := \sum_{\sigma \in C_\nu} \operatorname{sgn}(\sigma) \cdot \sigma.$$

The product

$$c_\nu := a_\nu \cdot b_\nu \in \mathbb{F}S_d \tag{3.5}$$

is called the *Young symmetrizer* of ν. In Example 3.2 we defined an action of $\mathbb{F}S_d$ on $V^{\otimes d}$. In this way, c_ν becomes an endomorphism on $V^{\otimes d}$.

Next, we want to rearrange boxes of two given Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_s), \mu = (\mu_t, \ldots, \mu_t)$ of size d_λ, d_μ respectively, to another Young tableau $\nu = (\nu_1, \ldots, \nu_r)$ of size $d_\nu = d_\lambda + d_\mu$ by a certain rule, called the *Littlewood-Richardson rule.* We start with the Young tableau λ and add the boxes of μ step by step to the tableau λ. In the first step, we add all the boxes in the first row of μ to λ but not two boxes in the same column. Further, we want the obtained scheme again to be a Young tableau, i.e. the i-th row contains at least as many boxes as the $(i+1)$-th row. In the second step, we continue with the second row of μ in the same way and so on. The boxes we added in the i-th step will be labeled with i. Now the extension of λ by μ is valid according to the Littlewood-Richardson rule if the following holds. List the integers of the added boxes from right to left, starting with the top row and working down. Then in the first j entries in this list (for any $j = 1, \ldots, d_\mu$) each integer p between 1 and $t-1$ occurs at least as many times as the next integer $p+1$. The number of ways to extend λ by μ to ν according to the Littlewood-Richardson rule is called the Littlewood-Richardson coefficient and is denoted by $N^\nu_{\lambda,\mu}$.

Example 3.24. Let $\lambda = (4, 2, 1), \mu = (3, 1, 1)$ and $\nu = (6, 3, 2, 1)$. There are three ways to extend λ by μ according to the Littlewood-Richardson rule such that ν is obtained and therefore $N^{\nu}_{\lambda,\mu} = 3$. The extensions are given by

$$\begin{array}{cccccc} \, & \, & \, & \, & 1 & 1 \\ \, & \, & 1 & & & \\ \, & 2 & & & & \\ 3 & & & & & \end{array} \qquad \begin{array}{cccccc} \, & \, & \, & \, & 1 & 1 \\ \, & \, & 2 & & & \\ \, & 1 & & & & \\ 3 & & & & & \end{array} \qquad \begin{array}{cccccc} \, & \, & \, & \, & 1 & 1 \\ \, & \, & 2 & & & \\ \, & 3 & & & & \\ 1 & & & & & \end{array} .$$

A Young tableau $\nu = (\nu_1, \dots, \nu_m)$ is called *rectangular* if $\nu_1 = \dots = \nu_m$. For rectangular Young tableaux the Littlewood-Richardson coefficient is bounded from above by 1.

Lemma 3.25 ([68], Theorem 2.4; [88], Lemma 3.3)**.** *Let λ, μ be rectangular Young tableaux and ν another Young tableau. Then the Littlewood-Richardson coefficient $N^{\nu}_{\lambda,\mu}$ is either* 0 *or* 1.

Weyl modules and a classification of $\mathfrak{sl}(\mathbb{C}^n)$-representations

As before let V be a vector space over $\mathbb{F}$ of dimension n. The representation $W = V^{\otimes d}$ from Example 3.2 is our main example we want to deal with in this section. Almost all of the considered representations in this section occur as subrepresentations and restrictions of this representation. Therefore, by a representation U for a subspace $U \subset W$ we mean the corresponding subrepresentation. Recall that in the last section we introduced the Young symmetrizer $c_\nu \in \mathbb{R}S_d$, where ν is a Young tableau equipped with a label.

Definition 3.26 (Weyl module)**.** The *Weyl module* with respect to a label of a Young tableau ν of size d is the subspace

$$\mathbb{S}_\nu V := \operatorname{im}(c_\nu \colon V^{\otimes d} \to V^{\otimes d}) \subset V^{\otimes d}.$$

Remark 3.27. The notation $\mathbb{S}_\nu V$ comes from the so-called *Schur functor* given by $V \mapsto \mathbb{S}_\nu V$.

It is clear from the construction that we have

$$\operatorname{Sym}^k V = \mathbb{S}_{(k,0,\dots,0)} V, \quad \bigwedge^k V = \mathbb{S}_{(1,\dots,1,0,\dots,0)} V,$$

where in the latter Young tableau the $1'$s occur k times. It is known that any Weyl module is an irreducible representation of $\mathrm{GL}(V)$ ([40], §4.7, §4.8). Further, as the notation already indicates, this representation does not depend on the label of ν, i.e. if we choose another label of ν the corresponding Weyl modules are isomorphic as $\mathrm{GL}(V)$-representations. As we will see, Weyl modules are also useful for a classification of finite-dimensional irreducible representations of $\mathrm{SL}(V)$ and $\mathfrak{sl}(V)$. Therefore, for this work, it is worth it to study Weyl modules.

First of all, we want to mention that there are various formulas for the dimension of a Weyl module (see for example [37], §4.1, §6.1, §15.3). One of them is given by

$$\dim\left(\mathbb{S}_\nu V\right) = \begin{cases} 0 & ,\nu_{n+1} > 0 \\ \prod\limits_{1\le i<j\le n} \frac{\nu_i - \nu_j + j - i}{j-i} & ,\nu_{n+1} = 0. \end{cases} \tag{3.6}$$

In particular, if $\nu_{n+1} > 0$ then $\mathbb{S}_\nu V$ and $\mathbb{S}_\mu V$ are isomorphic if and only if $\mu_{n+1} > 0$. The following theorem gives a classification of finite-dimensional irreducible representations of $\mathfrak{sl}(V)$ in terms of Weyl modules and highest weights in the complex case.

Theorem 3.28 ([37], §15.3)**.** *Let $\mathbb{F} = \mathbb{C}$ and $\nu = (\nu_1, \ldots, \nu_n)$ be a Young tableau. The Weyl module $\mathbb{S}_\nu V$ is the irreducible representation of $\mathfrak{sl}(V)$ with highest weight $\nu_1\alpha_1 + \cdots + \nu_n\alpha_n$.*

In Equation 3.6 we saw that $\mathbb{S}_\nu V = \{0\}$ if $\nu_{n+1} > 0$. By Corollary 3.18 we conclude that in the case $\mathbb{F} = \mathbb{C}$ any Weyl module is irreducible as $\mathrm{SL}(V)$-representation and two Weyl modules $\mathbb{S}_\nu V, \mathbb{S}_\mu V$ are isomorphic if and only if $\nu_{n+1}, \mu_{n+1} > 0$ or $\nu_1\alpha_1 + \cdots + \nu_n\alpha_n = \mu_1\alpha_1 + \cdots + \mu_n\alpha_n$. Since $\alpha_1 + \cdots + \alpha_n = 0$ the latter is equivalent to $\nu_i - \mu_i = const.$ We want to formulate this assertion as a corollary.

Corollary 3.29. *The finite-dimensional irreducible representations of $\mathfrak{sl}(\mathbb{C}^n)$ are exactly the Weyl modules. Two Weyl modules $\mathbb{S}_\nu V$, $\mathbb{S}_\mu V$ are isomorphic if and only if $\nu_{n+1}, \mu_{n+1} > 0$ or $\nu_i - \mu_i$ is constant.*

3.2.5 Complexification of Lie algebras

In this section, we want to discuss the interplay between real representations of real Lie algebras and complex representations of complex Lie algebras. The main goal is to prove Theorem 3.32 below. In this section, V denotes a real vector space of finite dimension n. Further, we write $\mathfrak{g}$ for a semisimple real Lie algebra. All the considered representations are supposed to be of finite dimension.

Definition 3.30 (Complexification)**.** We write $V^{\mathbb{C}}$ for the vector space $V \otimes_{\mathbb{R}} \mathbb{C}$ called the *complexification* of V. By definition $V^{\mathbb{C}}$ is a real vector space of twice the dimension of V.

Note that we can decompose $V^{\mathbb{C}}$ as real vector space

$$V^{\mathbb{C}} = V \oplus iV = \{v_1 + iv_2 \mid v_1, v_2 \in V\},$$

where i is the imaginary unit. We can define a complex scalar multiplication by $z \cdot (v \otimes x) := v \otimes (z \cdot x)$. In this way, $V^{\mathbb{C}}$ becomes a complex vector space. The following lemma is a direct consequence of the construction of Weyl modules.

Lemma 3.31. *Let ν be a Young tableaux. As complex representations of $\mathfrak{sl}(V^{\mathbb{C}})$ we have*

$$\mathbb{S}_\nu V^{\mathbb{C}} = \left(\mathbb{S}_\nu V\right)^{\mathbb{C}}.$$

For the complexification $\mathfrak{g}^{\mathbb{C}}$ there is a unique extension of the Lie bracket that makes $\mathfrak{g}^{\mathbb{C}}$ into a complex Lie algebra ([47], §3.6). This extension is given by the linear extension of the Lie bracket of $\mathfrak{g}$. For example the complexification of $\mathfrak{sl}(V)$ is given by $\mathfrak{sl}(V^{\mathbb{C}})$ ([47], §3.6). Now we want to give several constructions for real and complex representations of real and complex Lie algebras.

- If W is a $\mathfrak{g}$ representation, then $W^{\mathbb{C}}$ is also a $\mathfrak{g}$ representation via

$$g \cdot (w \otimes z) := gw \otimes z.$$

- If W is a $\mathfrak{g}$ representation, then $W^{\mathbb{C}}$ is a $\mathfrak{g}^{\mathbb{C}}$ representation via

$$(g \otimes y) \cdot (w \otimes z) := gw \otimes yz.$$

- If W is a $\mathfrak{g}^{\mathbb{C}}$ representation, then W is also a representation of $\mathfrak{g} \otimes \{1\} = \mathfrak{g}$.

It is well-known that $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g}^{\mathbb{C}}$ is semisimple (this is a consequence of Cartan's criterion for semisimplicity, see [37], §C.1). This implies that any representation of $\mathfrak{g}^{\mathbb{C}}$ is completely reducible if $\mathfrak{g}$ is semisimple. Some of the results in this section only hold for Lie algebras with an additional property. For our purposes, it is enough to assume this property to be true for $\mathfrak{g}$. We assume that any irreducible complex representation $\widetilde{W}$ of $\mathfrak{g}^{\mathbb{C}}$ is of *real type*, i.e. it is isomorphic to $W \otimes \mathbb{C}$ where W is a real representation of $\mathfrak{g}$. Note that this is true for the Lie algebra $\mathfrak{g} = \mathfrak{sl}(V)$. The goal of this section is to classify all irreducible real representations of $\mathfrak{g}$ in terms of the classification of irreducible complex representations of $\mathfrak{g}^{\mathbb{C}}$. We want to show the following theorem.

Theorem 3.32. *Let $\mathfrak{g}$ be semisimple such that any complex representation of $\mathfrak{g}^{\mathbb{C}}$ is of real type. Then W is a real irreducible representation of $\mathfrak{g}$ if and only if $W^{\mathbb{C}}$ is a complex irreducible representation of $\mathfrak{g}^{\mathbb{C}}$. Two irreducible representations $W, \widetilde{W}$ of $\mathfrak{g}$ are isomorphic if and only if $W \otimes \mathbb{C}$ and $\widetilde{W} \otimes \mathbb{C}$ are isomorphic as complex representations of $\mathfrak{g}^{\mathbb{C}}$.*

Since the representation theory of $\mathfrak{sl}(\mathbb{C}^n)$ is well understood the application of this statement to $\mathfrak{g} = \mathfrak{sl}(V)$ gives a complete classification of irreducible real representations of $\mathfrak{sl}(V)$. Before we can prove Theorem 3.32 we need some preparation.

Proposition 3.33. *Let W_1, W_2 be finite-dimensional representations of $\mathfrak{g}$. If $W_1^{\mathbb{C}}$ is isomorphic to $W_2^{\mathbb{C}}$ as $\mathfrak{g}^{\mathbb{C}}$-representations and both representations are irreducible, then W_1 is isomorphic to W_2 as $\mathfrak{g}$-representations and W_1, W_2 are irreducible.*

Proof. Let

$$F\colon W_1 \otimes \mathbb{C} \to W_2 \otimes \mathbb{C}$$

be an isomorphism and consider the conjugations

$$\sigma_1\colon W_1 \otimes \mathbb{C} \to W_1 \otimes \mathbb{C}, \quad \sigma_1(w \otimes z) = w \otimes \overline{z}$$
$$\sigma_2\colon W_2 \otimes \mathbb{C} \to W_2 \otimes \mathbb{C}, \quad \sigma_2(w \otimes z) = w \otimes \overline{z}$$

Note that σ_1, σ_2 are antilinear and F is linear. Hence the composition

$$\sigma_2 \circ F \circ \sigma_1 \colon W_1 \otimes \mathbb{C} \to W_2 \otimes \mathbb{C}$$

is again linear and an isomorphism (one can easily check that this composition commutes with the action of the Lie algebra.) By Corollary 3.7 we have $\sigma_2 \circ F \circ \sigma_1 = c \cdot F$ for some $c \in \mathbb{C}$. Since σ_1, σ_2 are involutions we obtain

$$F = \sigma_2 \circ c \cdot F \circ \sigma_1 = c \cdot (\sigma_2 \circ F \circ \sigma_1) = c^2 F.$$

Thus $c = \pm 1$.

Case 1: c=1.
Then it is $F = \sigma_2 \circ F \circ \sigma_1$ and we obtain

$$F(w \otimes 1) = (\sigma_2 \circ F \circ \sigma_1)(w \otimes 1) = \sigma_2 \left(F(w \otimes 1) \right).$$

This implies $F(w \otimes 1) \in W_2 \otimes \mathbb{R} \cong W_2$. Hence the map $W_1 \to W_2$ given by $w \mapsto F(w \otimes 1)$ is an isomorphism.

Case 2: c=-1.
Then it is $F = -(\sigma_2 \circ F \circ \sigma_1)$ and we obtain

$$F(w \otimes 1) = -(\sigma_2 \circ F \circ \sigma_1)(v \otimes 1) = -\sigma_2 \left(F(w \otimes 1) \right).$$

This implies $F(w \otimes 1) \in W_2 \otimes i\mathbb{R} \cong W_2$. Hence the map $W_1 \to W_2$ given by $w \mapsto F(w \otimes 1)$ is an isomorphism. $\square$

It is clear from the definition that a representation W of $\mathfrak{g}$ is irreducible if $W^{\mathbb{C}}$ is irreducible as a representation of $\mathfrak{g}^{\mathbb{C}}$. By the next two lemmas, the converse is also true.

Lemma 3.34. *Let W be a real finite-dimensional irreducible representation of $\mathfrak{g}$. Then $W^{\mathbb{C}}$ as a representation of $\mathfrak{g}^{\mathbb{C}}$ decomposes into a sum of not more than two irreducible $\mathfrak{g}^{\mathbb{C}}$-representations.*

Proof. The decomposition of $W^{\mathbb{C}}$ into irreducible representations of $\mathfrak{g}^{\mathbb{C}}$ is given by

$$W^{\mathbb{C}} = \bigoplus_{i=1}^{m} W_i \otimes \mathbb{C},$$

where W_i is a representation of $\mathfrak{g}$. On one hand, the restriction to $\mathfrak{g}$ is given by

$$\operatorname{Res}_{\mathfrak{g}}^{\mathfrak{g}^{\mathbb{C}}} W^{\mathbb{C}} = W \oplus W.$$

On the other hand, the restriction

$$\operatorname{Res}_{\mathfrak{g}}^{\mathfrak{g}^{\mathbb{C}}} \bigoplus_{i \in I} W_i = \bigoplus_{i \in I} \operatorname{Res}_{\mathfrak{g}}^{\mathfrak{g}^{\mathbb{C}}} (W_i \otimes \mathbb{C})$$

decomposes into a sum of at least $|I|$ summands. Corollary 3.8 now implies $|I| \leq 2$. $\square$

Lemma 3.35. *Let W be a real finite-dimensional irreducible representation of $\mathfrak{g}$. Then $W^{\mathbb{C}}$ is irreducible as a representation of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. We prove this by contradiction. Assume that $W^{\mathbb{C}}$ is not irreducible. By Lemma 3.34 we have a decomposition

$$W^{\mathbb{C}} = W_1 \oplus W_2$$

into exactly two irreducible $\mathfrak{g}^{\mathbb{C}}$-representations. By assumption there are real representations $\widetilde{W_1}, \widetilde{W_2}$ of $\mathfrak{g}$ such that $W_i = \widetilde{W_i} \otimes \mathbb{C}$. But as a representation of $\mathfrak{g}$ the space $\widetilde{W_i} \otimes \mathbb{C}$ is not irreducible since $W_i \otimes \{1\}$ is a proper subspace invariant under $\mathfrak{g}$. Hence the restriction

$$\operatorname{Res}_{\mathfrak{g}}^{\mathfrak{g}^{\mathbb{C}}}(W_1 \oplus W_2)$$

decomposes into a sum of at least 4 irreducible representations. As in the proof of the previous lemma the restriction

$$\operatorname{Res}_{\mathfrak{g}}^{\mathfrak{g}^{\mathbb{C}}} W^{\mathbb{C}} = W \oplus W$$

decomposes into a sum of 2 irreducible representations. This contradicts with Corollary 3.8. □

We are now able to prove Theorem 3.32.

Proof of Theorem 3.32. Let W be an irreducible representation of $\mathfrak{g}$. Then $W^{\mathbb{C}}$ is a finite-dimensional irreducible representation of $\mathfrak{g}^{\mathbb{C}}$ by Lemma 3.35. If W is not irreducible then it decomposes into a finite sum of irreducible representations

$$W = \bigoplus_{i \in I} W_i.$$

The complexification $W^{\mathbb{C}}$ is also not irreducible as $\mathfrak{g}^{\mathbb{C}}$-representation since

$$W^{\mathbb{C}} = \bigoplus_{i \in I} W_i^{\mathbb{C}}.$$

If W and $\widetilde{W}$ are irreducible and isomorphic as $\mathfrak{g}$-representations then clearly $W^{\mathbb{C}}$ and $\widetilde{W}^{\mathbb{C}}$ are isomorphic as $\mathfrak{g}^{\mathbb{C}}$-representations. On the other hand, if $W^{\mathbb{C}}$ and $\widetilde{W}^{\mathbb{C}}$ are irreducible and isomorphic as $\mathfrak{g}^{\mathbb{C}}$ representations then W and $\widetilde{W}$ are isomorphic as $\mathfrak{g}$-representations by Proposition 3.33 □

Applying Theorem 3.32 to $\mathfrak{g} = \mathfrak{sl}(V)$ together with the classification of finite-dimensional irreducible representations of $\mathfrak{sl}(\mathbb{C}^n)$ (see Corollary 3.29) we end up with a classification of finite-dimensional irreducible representations of $\mathfrak{sl}(V)$.

Lemma 3.36. *The finite-dimensional irreducible representations of $\mathfrak{sl}(V)$ are exactly the Weyl modules. Two Weyl modules $\mathbb{S}_\lambda V$, $\mathbb{S}_\mu V$ are isomorphic if and only if $\lambda_i - \mu_i$ is constant or $\lambda_{n+1}, \mu_{n+1} > 0$.*

3.3 Rational representations

3.3.1 Complete reducibility of rational representations

In this section $\mathbb{F}$ again stands for one of the fields $\mathbb{R}$ or $\mathbb{C}$. All the results are needed only for $\mathbb{F} = \mathbb{R}$, though. Let V, W be vector spaces over $\mathbb{F}$ of finite dimension and let $n = \dim(V)$. We write $\mathrm{End}(W)$ for the space of endomorphisms on W.

Definition 3.37 (Polynomial and rational representations)**.** A function $f\colon \mathrm{GL}(V) \to \mathbb{F}$ is called *polynomial* if there is a polynomial $\tilde{f} \in \mathbb{F}[\mathrm{End}(V)]$ such that $f = \tilde{f}|_{\mathrm{GL}(V)}$. It is called *regular* if $\det^r \cdot f$ is polynomial for some $r \in \mathbb{N}$.
A representation $\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(W)$ is called *polynomial* (resp. *rational*) if for any basis of W the matrix entries $\rho(g)_{ij}$ are polynomial (resp. regular) functions on $\mathrm{GL}(V)$.

It is clear that a representation is polynomial (resp. rational) if for one particular basis the matrix entries $\rho(g)_{ij}$ are polynomial (resp. regular). This follows from the fact that for another basis of W the matrix entries are given by linear combinations of the entries of the initial matrix entries. For a basis $b_1, \ldots, b_m$ of W we have an associated basis of the vector space $\mathrm{End}(W)$ consisting of E_{ij}, the matrix with 1 on the entry (i, j) and 0 otherwise. Denote by X_{ij} the corresponding dual basis. Then we can identify $\mathbb{F}[\mathrm{End}(W)]$ with $\mathbb{F}[\{X_{ij} \colon i, j = 1, \ldots, m\}]$.

Example 3.38. The standard representation $\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(V)$ is clearly polynomial. Indeed, for a basis $b_1, \ldots, b_n$ of V the entry $\rho(g)_{ij}$ is given by the polynomial X_{ij}.

Example 3.39. If W_1, W_2 are polynomial (resp. rational) representations of $\mathrm{GL}(V)$, then $W_1 \otimes W_2$ and $W_1 \oplus W_2$ are polynomial (resp. rational) representations of $\mathrm{GL}(V)$. Indeed, it is not hard to see that for a given basis the coordinate functions of $g \in \mathrm{GL}(V)$ for the tensor product and the direct sum are products and sums of the coordinate functions of the representations W_1, W_2.

Example 3.40. If W is a polynomial (resp. rational) representation of $\mathrm{GL}(V)$, then clearly any subrepresentation U is a polynomial (resp. rational) representation of $\mathrm{GL}(V)$.

These examples show that finite-dimensional subrepresentations of $\bigoplus_{m\geq 0} V^{\otimes m}$ are polynomial $\mathrm{GL}(V)$-representations. As we will see later the converse is also true. More precisely we will see that any finite-dimensional irreducible polynomial representation is contained in some tensor power of V.

3.3.2 Weight spaces

As before V denotes a vector space over $\mathbb{F}$ and $n = \dim(V)$. In this section we fix a basis $b_1, \ldots, b_n$ of V. After this choice we can identify V with $\mathbb{F}^n$. Denote by

$$T^n := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mid t_i \in \mathbb{F}^\times \right\} \subset \mathrm{GL}(n, \mathbb{F})$$

the *n-dimensional torus.* Further, let $\varepsilon_i \in (T^n)^*$ be the element defined by

$$\varepsilon_i\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right) = t_i.$$

The 1-dimensional rational representations of T^n are given by $r_1\varepsilon_1 + \cdots + r_n\varepsilon_n$ defined by

$$(r_1\varepsilon_1 + \cdots + r_n\varepsilon_n)\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right) = t_1^{r_1} \cdot \ldots \cdot t_n^{r_n}$$

for $r_i \in \mathbb{Z}$ ([56], §5.6). Clearly, these types of representations form a group via multiplication which we denote by $\chi(T^n)$.

It is well-known that for rational representations W we have a certain decomposition, called the *weight space decomposition* ([40], §3.2; [56], §5.6, also compare with Section 3.2.3). This is given by

$$W = \bigoplus_{\lambda \in \chi(T^n)} W_\lambda,$$

where

$$W_\lambda := \{w \in W \mid tw = \lambda(t)w, \forall t \in T^n\}.$$

If $W_\lambda \neq 0$ we call λ a *weight* and W_λ is the corresponding *weight space.* If $w \in W_\lambda$ is non-zero we say w is a *weight vector* for λ.

Denote by U_n the subgroup of the matrices

$$\begin{pmatrix} 1 & * & * \\ & \ddots & * \\ & & 1 \end{pmatrix}.$$

It is well-known that U_n is spanned by the matrices $u_{ij}(s)$ for $j > i$, where

$$u_{ij}(s) = \mathrm{id} + sE_{ij}$$

and E_{ij} is the matrix with 1 on the entry (i, j) and zero otherwise. The following lemma describes the action of certain elements of U_n on weight vectors.

Lemma 3.41 ([40], §4.8;[56], §5.7)**.** *Let λ be a weight of W and $w \in W_\lambda$ a weight vector. There are elements $w_k \in W_{\lambda+k(\varepsilon_i-\varepsilon_j)}$ for $k \in \mathbb{N}$, where $w_0 = w$, such that*

$$u_{ij}(s)w = \sum_{k \geq 0} s^k w_k.$$

As described in [56] (§5.7) for irreducible representations W the space

$$W^{U_n} := \{w \in W \mid \phi(w) = w, \forall \phi \in U_n\}$$

has dimension 1 and coincides with a weight space W_λ. This weight λ is called the *highest weight* of the representation W.

Lemma 3.42 ([56], §5.9)**.** *Two irreducible rational* $\mathrm{GL}(V)$*-representations are isomorphic if and only if they have the same highest weight.*

Example 3.43. For the irreducible $\mathrm{GL}(n,\mathbb{R})$-representation

$$W = (\textstyle\bigwedge^n \mathbb{R}^n)^{\otimes m} \otimes \textstyle\bigwedge^k \mathbb{R}^n$$

the vector

$$w = (e_1 \wedge \cdots \wedge e_n)^{\otimes m} \otimes (e_1 \wedge \cdots \wedge e_k)$$

is a highest weight vector. The highest weight is given by

$$(m+1)(\varepsilon_1 + \cdots + \varepsilon_k) + m(\varepsilon_{k+1} + \cdots + \varepsilon_n).$$

3.4 Representations of $\mathrm{SL}(V)$

In this section, V is a real vector space of finite dimension n. The main goal is to classify all finite-dimensional $\mathrm{SL}(V)$-representations.

3.4.1 Polynomial representations of $\mathrm{SL}(V)$

In this section, W denotes an irreducible real $\mathrm{SL}(V)$-representation of finite dimension. The goal of this section is to show the following proposition.

Proposition 3.44. *Let W be a finite-dimensional vector space over $\mathbb{R}$. If $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(W)$ is a representation then it is isomorphic to the restriction of a polynomial* $\mathrm{GL}(V)$*-representation.*

Recall that we assume a representation of a Lie group to be continuous. Hence ρ in the previous proposition is continuous. We prove Proposition 3.44 in several steps. First, let us introduce the notion of rational and polynomial representations of a subgroup $G \subset \mathrm{GL}(V)$.

Definition 3.45. Let $G \subset \mathrm{GL}(V)$ be a subgroup. A function $f\colon G \to \mathbb{R}$ is called *regular* (resp. *polynomial*) if it is the restriction of a regular (resp. polynomial) function on $\mathrm{GL}(V)$. A representation $\rho\colon G \to \mathrm{GL}(W)$ is called *rational* (resp. *polynomial*) if the matrix entries ρ_{ij} with respect to any basis of W are regular (resp. polynomial) functions on G.

As in the case of $\mathrm{GL}(V)$, tensor products and subrepresentations of rational (resp. polynomial) G-representations are again rational (resp. polynomial). Let us first make sure that an investigation of irreducible polynomial representations of $\mathrm{GL}(V)$ is useful to describe all polynomial representations of $\mathrm{GL}(V)$. The first step is the following lemma.

Lemma 3.46 ([56], §5.3; [40], Corollary 2.6e)**.** *Every polynomial representation of* $\mathrm{GL}(V)$ *is completely reducible.*

Lemma 3.47 ([56], §5.3, [55], Exercise 5.3b)**.** *Every irreducible polynomial G-representation W of* $\mathrm{GL}(V)$ *occurs in a unique $V^{\otimes m}$.*

The next lemma shows that the classification of irreducible polynomial $\mathrm{SL}(V)$-representations is similar to the classification of $\mathrm{GL}(V)$-representations. It turns out that this is a crucial fact later on.

Lemma 3.48 ([56], §5.4)**.** *Every rational* $\mathrm{SL}(V)$*-representation is completely reducible. Further, it is the restriction of a rational* $\mathrm{GL}(V)$*-representation ρ and irreducible if and only if ρ is irreducible.*

Another ingredient for the proof of our main goal in this section is needed.

Lemma 3.49. *If $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(W)$ is a finite-dimensional irreducible representation, then W is a subset of some tensor power of V.*

Proof. If $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(W)$ is a representation then $\mathrm{d}\rho\colon \mathfrak{sl}(V) \to \mathrm{End}(W)$ is a representation of the Lie algebra $\mathfrak{sl}(V)$. In Section 3.2.5 we classified all such representations. In particular, we showed that W is contained in a (finite) sum of tensor powers of V. Since W is irreducible as $\mathrm{SL}(V)$-representation it is contained in exactly one tensor power of V. □

Example 3.38 and Example 3.39 imply the following corollary.

Corollary 3.50. *If $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(W)$ is a finite-dimensional irreducible representation, then it is a polynomial representation of* $\mathrm{SL}(V)$*.*

In fact one can prove that any such representation W is isomorphic to a Weyl module (see next section). We are now able to give a proof of Proposition 3.44.

Proof of Proposition 3.44. W is a polynomial representation by Corollary 3.50. Lemma 3.48 tells us that W is also an irreducible rational $\mathrm{GL}(V)$-representation. For $r \in \mathbb{N}$ sufficiently large the representation $(\bigwedge^n V)^{\otimes r} \otimes W$ is polynomial (and also irreducible). Since $(\bigwedge^n V)^{\otimes r}$ corresponds to the trivial $\mathrm{SL}(V)$-representation $\phi \mapsto \det(\phi)^r$ its restriction to $\mathrm{SL}(V)$ is isomorphic to W. □

3.4.2 A classification of irreducible $\mathrm{SL}(V)$-representations

In this section let V be a real vector space of dimension n. The goal of this section is to show the following theorem.

Theorem 3.51. *The finite-dimensional irreducible representations of* $\mathrm{SL}(V)$ *are exactly the Weyl modules. Two Weyl modules $\mathbb{S}_\lambda V$, $\mathbb{S}_\mu V$ are isomorphic if and only if $\mu_{n+1}, \lambda_{n+1} > 0$ or $\lambda_i - \mu_i$ is constant.*

We prove this theorem step by step. First, we show that any Weyl module is irreducible as $\mathrm{SL}(V)$-representation.

Lemma 3.52. *Let W be a Weyl module. Then W is irreducible as* $\mathrm{SL}(V)$*-representation.*

Proof. In Section 3.2.4 we mentioned, that W is irreducible as $\mathrm{GL}(V)$-representation. Note that W is a subrepresentation of some tensor power of V. Hence it is polynomial. By Lemma 3.48 it is also irreducible as $\mathrm{SL}(V)$-representation. □

To complete the proof of the first part of Theorem 3.51 we want to show that any finite-dimensional irreducible $\mathrm{SL}(V)$-representation is isomorphic to a Weyl module.

Lemma 3.53. *Let W be a finite-dimensional irreducible $\mathrm{SL}(V)$-representation. Then W is isomorphic to a Weyl module.*

Proof. Let $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(W)$ be a finite-dimensional representation. Then $d\rho\colon \mathfrak{sl}(V) \to \mathfrak{gl}(W)$ is a Lie algebra representation. By the classification of such representations, we obtain that W is a sum of Weyl modules $W_1, \ldots, W_m$. Since W is of finite dimension it is indeed a finite sum. Since W_i is also a representation of $\mathrm{SL}(V)$ it follows that $m = 1$ by irreducibility of W. □

It remains to find a condition for two Weyl modules to be isomorphic (as $\mathrm{SL}(V)$-representations). By the following lemma, we can reduce the problem to the case of $\mathfrak{sl}(V)$-representations.

Lemma 3.54 ([37], §8.1)**.** *Let G be a connected Lie group. Then a representation $\rho\colon G \to \mathrm{GL}(W)$ is uniquely determined by its differential.*

We are now able to finish the proof of Theorem 3.51.

Lemma 3.55. *The Weyl modules $\mathbb{S}_\lambda V$, $\mathbb{S}_\mu V$ are isomorphic as $\mathrm{SL}(V)$-representations if and only if $\lambda_{n+1}, \mu_{n+1} > 0$ or $\lambda_i - \mu_i$ is constant.*

Proof. If $\lambda_{n+1}, \mu_{n+1} > 0$ then both Weyl modules are $\{0\}$ by (3.6). Since $\mathrm{SL}(V)$ is connected, the corresponding homomorphisms ρ_λ, ρ_μ are uniquely determined by their differentials. Hence both Weyl modules are isomorphic as $\mathrm{SL}(V)$-representations if and only if they are isomorphic as $\mathfrak{sl}(V)$-representations. Now apply Lemma 3.36. □

Note that this proves Theorem 3.51.

3.4.3 A decomposition of tensor products

Again we assume V to be a real vector space of finite dimension n. In general, a tensor product of irreducible representations is not irreducible. Since any polynomial representation is completely reducible as $\mathrm{SL}(V)$-representation it decomposes into a sum of irreducible representations. In this section, we want to find the decomposition of a tensor product of Weyl modules. We want to prove the following decomposition rule.

Theorem 3.56. *Let λ, μ be Young tableaux. The decomposition of the tensor product $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ into irreducible representations of $\mathrm{SL}(V)$ is given by*

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V = \bigoplus_\nu (\mathbb{S}_\nu V)^{N^\nu_{\lambda,\mu}}.$$

The theorem is a corollary of the analogous decomposition for $\mathrm{GL}(V)$. A proof of the following theorem can be found in [40] (D.10, together with (3.4c) and Theorem 3.5a).

Theorem 3.57. *Consider the Weyl modules* $\mathbb{S}_\lambda V$, $\mathbb{S}_\mu V$ *as* $\mathrm{GL}(V)$*-representations. Then the decomposition of the tensor product* $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V$ *into irreducible representations is given by*

$$\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V = \bigoplus_\nu (\mathbb{S}_\nu V)^{N^\nu_{\lambda,\mu}}.$$

Now Theorem 3.56 follows by the previous theorem and Lemma 3.48. Note that this also implies symmetry of the Littlewood-Richardson coefficient $N^\nu_{\lambda,\mu}$ in the variables λ, μ, since $\mathbb{S}_\lambda V \otimes \mathbb{S}_\mu V = \mathbb{S}_\mu V \otimes \mathbb{S}_\lambda V$.

4 Valuations in convex geometry

Now we want to introduce the most important concept for this work. As before V denotes a real vector space of dimension n and W is another real vector space, also of finite dimension.

Definition 4.1 (Valuation)**.** Let $(A,+)$ be an abelian semigroup. A map $\Phi\colon \mathcal{K}(V) \to A$ is called *valuation* if

$$\Phi(K \cup L) + \Phi(K \cap L) = \Phi(K) + \Phi(L) \tag{4.1}$$

is satisfied for all convex bodies K, L such that $K \cup L$ is convex. If $A = \mathbb{R}$ we say that Φ is a *real valuation.* In the case $A = \mathcal{K}(W)$ we call Φ a *Minkowski valuation.*

Besides real and Minkowski valuations, other types of valuations such as *vector-valued valuations and tensor valuations* [44, 45, 57, 94] and modifications of the given definition to *valuations on functions* [26, 27] or *valuations on manifolds* [5, 6, 7, 12, 15, 16, 32] with connections to Alexandrov spaces [8] are also of interest in current research. In this thesis, we are mostly interested in Minkowski valuations. As we will see several times, we can learn a lot about Minkowski valuations from the theory of real valuations. Therefore real valuations are also discussed in this thesis.

A valuation Φ is called *translation invariant* if $\Phi(K+t) = \Phi(K)$ for all $K \in \mathcal{K}(V), t \in V$. We say that Φ is *even* (resp. *odd*) if $\Phi(-K) = \Phi(K)$ (resp. $\Phi(-K) = -\Phi(K)$) for all convex bodies K. Further if Φ is a real valuation or a Minkowski valuation we call Φ *homogeneous* of degree k if $\Phi(\lambda K) = \lambda^k \Phi(K)$ for all convex bodies K and $\lambda > 0$.

Example 4.2.

1. The Euler characteristic $\chi\colon \mathcal{K}(V) \to \mathbb{R}$ on the space of convex bodies given by $\chi \equiv 1$ is a real, translation invariant and even valuation.

2. The restriction of any Borel measure to $\mathcal{K}(V)$ is a real valuation. We write $\mathrm{vol}\colon \mathcal{K}(V) \to \mathbb{R}$ (or vol_n) for the n-dimensional Lebesgue measure. If $V = \mathbb{R}^n$ we denote by ω_n the volume of the unit ball.

3. The identity on $\mathcal{K}(V)$ is a Minkowski valuation homogeneous of degree 1 ([80], Lemma 3.1.1).

The last example together with Lemma 2.11 implies that the support function considered as a functional $\mathcal{K}(V) \to C(V^*), K \mapsto h_K$ is a valuation homogeneous of degree 1. Here $C(V^*)$ denotes the space of continuous functions on V^*.

4.1 Real valuations

This section provides a short introduction to real valuations. The set $\mathrm{Val}(V)$ of translation invariant, continuous, real valuations on V is of particular interest in this section. Note that $\mathrm{Val}(V)$ is a real vector space. As explained in [11] (§1.1.1), $\mathrm{Val}(V)$ becomes a Fréchet space, if it is equipped with the topology of uniform convergence on compact subsets of $\mathcal{K}(V)$. If an euclidean structure on V is fixed, then $\mathrm{Val}(V)$ is a Banach space with norm given by

$$|\phi| := \sup_{K \subset B} |\phi(K)|,$$

where B denotes the unit ball. In this case a dense subspace $\mathrm{Val}^\infty(V)$ of *smooth valuations* can be defined, where ϕ is *smooth* if

$$\mathrm{GL}(V) \to \mathrm{Val}(V), \quad g \mapsto \phi \circ g^{-1}$$

is smooth ([11], §1.1.6).

4.1.1 Mixed volumes

The volume serves as an easy but, as we will see, important example for a real valuation. In 1840 Steiner proved a formula (called the Steiner formula) for the volume of parallel sets $K + tB^n$, where K is a fixed body in $\mathbb{R}^n$ and B^n is the unit ball. More precisely he found that this quantity is a polynomial in $t \geq 0$ of degree at most n with coefficients depending on K and n [90]. The Steiner formula is a special case of the following fact. Denote by $K_1, \ldots, K_m$ convex bodies in V. It is well-known ([66, 67], see also [21], §7) that there is a function $V_n \colon (\mathcal{K}(V))^n \to \mathbb{R}$ such that

$$\mathrm{vol}(t_1 K_1 + \cdots + t_m K_m) = \sum_{i_1, \ldots, i_n = 1}^{m} t_{i_1} \cdots t_{i_n} V_n(K_{i_1}, \ldots, K_{i_n}) \tag{4.2}$$

holds for all $t_i \geq 0$. The function V_n is called the *mixed volume*. It is easy to see that V_n is symmetric and translation invariant in each component and continuous. Further, it is non-negative and multilinear. The latter means

$$V_n(\lambda K + \mu L, K_2, \ldots, K_n) = \lambda V_n(K, K_2, \ldots, K_n) + \mu V_n(L, K_2, \ldots, K_n)$$

for all $\lambda, \mu \geq 0$. There is a lot more to say about mixed volumes (see [80], §5.1). For this work the last important property we want to state here is the valuation property. More precisely we have the following fact. The map

$$K \mapsto V_n(K[l], K_{l+1}, \ldots K_n)$$

is a valuation ([80], §5.1), where $V_n(K[l], K_1, \ldots, K_{n-l})$ stands for $V_n(K, \ldots, K, K_1, \ldots, K_{n-l})$, where K occurs in the first l entries. If we fix an euclidean structure on V and identify V with $\mathbb{R}^n$ we have

- $V_n(K[n])$ is proportional to the volume of K,
- $V_n(K[n-1], B^n)$ is proportional to the surface area of K,
- $V_n(K, B^n[n-1])$ is proportional to the mean width of K,
- if $v \in S^{n-1}$ then $V_n(K[n-1], [-v, v])$ is proportional $(n-1)$-dimensional volume of the orthogonal projection of K onto $v^\perp$.

It is well-known (see [38], §A.3) that for convex bodies $K_1, \ldots, K_n$ we have

$$V_n(K_1, \ldots, K_n) = \frac{1}{n!} \sum_{j=1}^{n} (-1)^{n+j} \sum_{i_1 < \cdots < i_j} \mathrm{vol}(K_{i_1} + \cdots + K_{i_j}). \tag{4.3}$$

4.1.2 McMullen's decomposition

In this section, we want to talk about decompositions of $\mathrm{Val}(V)$. Denote by $\mathrm{Val}_k(V)$ the subspace of valuations homogeneous of degree k. The following theorem tells us, that any $\varphi \in \mathrm{Val}(V)$ can be written uniquely as a sum of homogeneous, translation invariant and continuous valuations, where the degree of homogeneity ranges from 0 to n.

Theorem 4.3 (McMullen's Decomposition, [63])**.** *The space of translation invariant, continuous and real valuations* $\mathrm{Val}(V)$ *decomposes into a direct sum*

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{n} \mathrm{Val}_i(V). \tag{4.4}$$

In particular for $k \notin \{0, \ldots, n\}$ the space $\mathrm{Val}_k(V)$ is zero. It is easy to verify that $\mathrm{Val}_0(V)$ is spanned by the Euler characteristic. By a result of Hadwiger $\mathrm{Val}_n(V)$ is also of dimension 1.

Theorem 4.4 ([46])**.**

$$\mathrm{Val}_n(V) = \mathrm{span}\{\mathrm{vol}_n\}.$$

Obviously, we have another decomposition given by

$$\mathrm{Val}(V) = \mathrm{Val}^+(V) \oplus \mathrm{Val}^-(V),$$

where $\mathrm{Val}^+(V)$ (resp. $\mathrm{Val}^-(V)$) denotes the subspace of even (resp. odd) valuations. Using a similar notation we can also write

$$\mathrm{Val}_k(V) = \mathrm{Val}_k^+(V) \oplus \mathrm{Val}_k^-(V).$$

Hence the decomposition in 4.4 can be written as

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{n} \mathrm{Val}_i^+(V) \oplus \mathrm{Val}_i^-(V).$$

Note that $\mathrm{Val}_0^-(V) = \mathrm{Val}_n^-(V) = \{0\}$.

4.1.3 Irreducibility Theorem

In this section, we want to view the space $\mathrm{Val}(V)$ as a representation of $\mathrm{GL}(V)$. Recall that the representation is given by

$$g \cdot \varphi := \varphi \circ g^{-1}, \quad \forall g \in \mathrm{GL}(V), \varphi \in \mathrm{Val}(V).$$

It is easy to see that $\mathrm{Val}_k^+(V)$ and $\mathrm{Val}_k^-(V)$ are subrepresentations of $\mathrm{Val}(V)$. It was conjectured by McMullen that the span of valuations of type

$$K \mapsto V_n(K[l], K_{l+1}, \ldots, K_n) \tag{4.5}$$

form a dense subset of $\mathrm{Val}(V)$ [64]. We call a valuation as in (4.5) *mixed volume valuation.* Alesker's Irreducibility Theorem below implies McMullen's conjecture.

Theorem 4.5 (Irreducibility Theorem, [3, 4])**.** *As* $\mathrm{GL}(V)$*-representations the spaces* $\mathrm{Val}_k^+(V)$ *and* $\mathrm{Val}_k^-(V)$ *are irreducible for all* k*.*

Let us discuss that the Irreducibility Theorem indeed implies McMullen's conjecture. Clearly, the space of mixed volume valuations contained in $\mathrm{Val}_k^\varepsilon(V)$ is invariant under the action of $\mathrm{GL}(V)$, where $\varepsilon \in \{\pm 1\}$. The Irreducibility Theorem implies that this space is either dense in $\mathrm{Val}_k^\varepsilon(V)$ or zero. Obviously, it is not zero. Now by McMullen's decomposition (Theorem 4.3) the space of mixed volume valuations is dense in $\mathrm{Val}_k^\varepsilon(V)$.

4.1.4 Klain function

Now we want to construct a map on the Grassmannian $\mathrm{Gr}_k(V)$, i.e. the set of all k-dimensional subspaces of V, associated with a valuation $\varphi \in \mathrm{Val}_k(V)$ which plays a central role later in this work.

For a subspace $E \subset V$ denote by $\mathrm{Dens}(E)$ the space of densities on E. Let L be the line bundle of densities over $\mathrm{Gr}_k(V)$. Note that L can be written as $\bigsqcup_{E \in \mathrm{Gr}_k(V)} \mathrm{Dens}(E)$ and its global trivialization is given by $\mathrm{Gr}_k(V) \times \mathbb{R}$. Now let $\varphi \in \mathrm{Val}_k(V)$ and $E \in \mathrm{Gr}_k(V)$. By Theorem 4.4 the restriction $\varphi|_{\mathcal{K}(E)} \colon \mathcal{K}(E) \to \mathbb{R}$ is a multiple of the volume. In other words, $\varphi|_{\mathcal{K}(E)}$ is contained in $\mathrm{Dens}(E)$. This defines a map

$$\mathrm{Kl}_\varphi \colon \mathrm{Gr}_k(V) \to L, \quad E \mapsto \varphi|_{\mathcal{K}(E)} \in \mathrm{Dens}(E)$$

and finally

$$\mathrm{Kl} \colon \mathrm{Val}_k(V) \to \Gamma(L), \quad \varphi \mapsto \mathrm{Kl}_\varphi,$$

where $\Gamma(L)$ denotes the space of global continuous sections of L.

Definition 4.6 (Klain function, Klain map)**.** The map Kl_φ is called the *Klain function* for $\varphi \in \mathrm{Val}_k(V)$ and Kl is the *Klain map.*

It is easy to see that Kl_φ is indeed continuous since φ is continuous. Further, it is clear from the construction that the Klain map is linear.

On the space $\Gamma(L)$ we have a natural action of $\mathrm{GL}(V)$ given by

$$(g \cdot s)(E)(\bullet) := s(g^{-1}E)(g^{-1}\bullet),$$

where $s \in \Gamma(L)$ and $g \in \mathrm{GL}(V)$.

Lemma 4.7. *The action of* $\mathrm{GL}(V)$ *on* $\mathrm{Val}_k(V)$ *commutes with the Klain map.*

Proof. Let $g \in \mathrm{GL}(V), E \in \mathrm{Gr}_k(V)$ and $K \in \mathcal{K}(E)$. Then

$$\mathrm{Kl}_{g\varphi}(E)(K) = (g \cdot \varphi)_{\mathcal{K}(E)}(K) = \varphi_{\mathcal{K}(g^{-1}E)}(g^{-1}K) = (g \cdot \mathrm{Kl}_{\varphi})(E)(K).$$

□

One important fact about the Klain map is the following theorem due to Klain which says, that if $\varphi \in \mathrm{Val}_k(V)$ is even, then it is uniquely described by its Klain function.

Theorem 4.8 ([50])**.** *The restriction* $\mathrm{Kl}\,|_{\mathrm{Val}_k^+(V)}$ *is injective.*

Remark 4.9. If we choose an euclidean structure on V, we have a natural choice of a k-dimensional volume in $E \in \mathrm{Gr}_k(V)$. This allows us to write $\varphi_{\mathcal{K}(E)} = c_E \cdot \mathrm{vol}_E$. In this case, we can define the Klain function by $\mathrm{Kl}_{\varphi}(E) = c_E \in \mathbb{R}$ and the Klain map becomes a map with values in $C(\mathrm{Gr}_k(V))$, the space of continuous functions on the Grassmannian.

4.2 Minkowski valuations

This section deals with Minkowski valuations $Z\colon \mathcal{K}(V) \to \mathcal{K}(W)$ (see Definition 4.1). It also serves as a motivation for our main question. The notion of *Minkowski valuations* goes back to Ludwig in 2005 [59]. However, this type of valuations was already part of Ludwig's research earlier [58]. Initially they were defined as valuations whose domain and codomain coincide, i.e. as valuations $\mathcal{K}(V) \to \mathcal{K}(V)$. We are interested in Minkowksi valuations with certain properties. The most important one is $\mathrm{SL}(V)$ equivariance, that is

$$Z(\phi K) = \phi Z(K),$$

for all $K \in \mathcal{K}(V)$ and $\phi \in \mathrm{SL}(V)$ if W is a $\mathrm{SL}(V)$-representation. Similar one can also define equivariance with respect to other groups as $\mathrm{GL}(V), \mathrm{O}(n)$ or $\mathrm{SO}(n)$, which is, as well as the case $\mathrm{SL}(V)$, of interest in the theory of real valuations and Minkowski valuations [10, 14, 43, 58, 60, 84, 85, 86, 93].

As for real valuations we denote by $\mathrm{MVal}(V, W)$ the space of continuous and translation invariant Minkowski valuations $\mathcal{K}(V) \to \mathcal{K}(W)$. The space $\mathrm{MVal}_k(V, W)$ stands for the subspace of k-homogeneous valuations and $\mathrm{MVal}^+(V, W)$ (resp. $\mathrm{MVal}^-(V, W)$) denotes the subspace of even (resp. odd) valuations. Finally, it is clear what we mean by the spaces $\mathrm{MVal}_k^{\varepsilon}(V, W)$ for $\varepsilon = \pm 1$.

As we will see in this section, from the theory of real valuations we can learn a lot about Minkowski valuations. For a Minkowski valuation as before and fixed $u \in W^*$ certain properties of Z are given to the support function $h_{Z(\bullet)}(u)$ evaluated in u. More precisely, if Z is a continuous (resp. translation invariant, resp. even, resp. $\mathrm{SL}(V)$ equivariant) valuation, then $h_{Z(\bullet)}(u)$ is also a continuous (resp. translation invariant, resp. even, resp. $\mathrm{SL}(V)$ equivariant) valuation. As a first assertion, it follows immediately from McMullen's decomposition (Theorem 4.3) that $\mathrm{MVal}_k(V, W) = \{0\}$ if $k \notin \{0, \dots, n\}$ (see also Lemma 5.6). Therefore we always assume $k \in \{0, \dots, n\}$. Schneider and Schuster asked for a stronger result, namely whether there is a McMullen decomposition for Minkowski valuations (in certain situations [81]). Parapatits and Wannerer showed that the answer to this question is negative for general continuous and translation invariant Minkowski valuations [71]. As we will see in this section, the method described above can also be used to define a Klain map for Minkowksi valuations (similarly to a result by Schuster in [86]) which in turn we can use to classify $\mathrm{SL}(V)$ equivariant Minkowski valuations in Chapter 5.

4.2.1 Klain body

In this section, we want to discuss the existence of a Klain map for Minkowski valuations. It is convenient in this section to choose an euclidean structure on V and identify V with $\mathbb{R}^n$ equipped with the standard euclidean product. Instead of $\mathrm{Gr}_k(\mathbb{R}^n)$ we write $\mathrm{Gr}_k(n)$. Let W be a real vector space of finite dimension. The goal of this section is to show the following theorem.

Theorem 4.10. *Let W be a finite-dimensional $\mathrm{SL}(n)$-representation and $Z \in \mathrm{MVal}_k^+(\mathbb{R}^n, W)$. Then Z is uniquely determined by $Z(K)$, where $K \in \mathbb{R}^k \subset \mathbb{R}^n$ has k-dimensional volume equal to 1. If $L \subset \mathbb{R}^k$ is another convex body whose k-dimensional volume equals r, then $Z(K) = r \cdot Z(L)$.*

We prove this theorem in two steps. As a by-product, we obtain a Klain function for Minkowski valuations.

Proposition 4.11. *Let $Z \in \mathrm{MVal}_k(\mathbb{R}^n, W)$. There is a continuous map $\mathrm{Kl}_Z\colon \mathrm{Gr}_k(n) \to \mathcal{K}(W)$ such that*

$$Z|_{\mathcal{K}(E)} = \mathrm{vol}_k \cdot \mathrm{Kl}_Z(E), \quad \forall E \in \mathrm{Gr}_k(\mathbb{R}^n). \tag{4.6}$$

If Z is even, then it is uniquely determined by Kl_Z.

The map Kl_Z plays the role of the Klain function for real valuations. Since in the case of Minkowski valuations it takes values in the set of convex bodies, it is a slightly different object. However, we call Kl_Z the *Klain function of Z*. If we speak about the Klain function, it will be clear from the context (and from the subscript in Kl_Z) whether we mean the Klain function for real valuations or Minkowski valuations. We call the body $\mathrm{Kl}_Z(E)$ the *Klain body* of Z in E.

Proof of Proposition 4.11. For $u \in W^*$ define

$$F_u\colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}, \quad K \mapsto h_{Z(K)}(u).$$

This function is a translation invariant, continuous valuation homogeneous of degree k. By the construction in Section 4.1.4, for $E \in \mathrm{Gr}_k(n)$ we have $F_u|_{\mathcal{K}(E)} = \mathrm{Kl}_{F_u}(E) \cdot \mathrm{vol}_k$. If K is a convex body in E it follows

$$h_{Z(K)}(u) = \mathrm{vol}_k(K) \cdot \mathrm{Kl}_{F_u}(E).$$

By ranging over all $u \in W^*$ the last equation shows that the convex body corresponding to the support function $h_{Z(K)}(u)$ does only depend on E and the volume of K. More precisely there is a convex body $\mathrm{Kl}_Z(E) \subset W$ such that

$$Z(K) = \mathrm{vol}_k(K) \cdot \mathrm{Kl}_Z(E), \quad \forall K \in \mathcal{K}(E). \tag{4.7}$$

Now suppose that Z is even and $Y \colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(W)$ is also an even, translation invariant, continuous valuation homogeneous of degree k. Assume $Z \neq Y$. We have to show $\mathrm{Kl}_Z \neq \mathrm{Kl}_Y$. For

$$G_u \colon \mathcal{K}(\mathbb{R}^n) \to \mathbb{R}, \quad K \mapsto h_{Y(K)}(u)$$

we have $F_u, G_u \in \mathrm{Val}_k^+(\mathbb{R}^n)$. Since a convex body is uniquely determined by its support function we can find $u \in W^*$ and $K \in \mathcal{K}(\mathbb{R}^n)$ such that $F_u(K) \neq G_u(K)$. Theorem 4.8 implies $\mathrm{Kl}_{F_u} \neq \mathrm{Kl}_{G_u}$, i.e. we can find $E \in \mathrm{Gr}_k(n)$ such that $\mathrm{Kl}_{F_u}(E) \neq \mathrm{Kl}_{G_u}(E)$. For $K \in \mathcal{K}(E)$ with $\mathrm{vol}_k(K) > 0$ we have

$$h_{Z(K)}(u) = \mathrm{vol}_k(K) \cdot \mathrm{Kl}_{F_u}(E) \neq \mathrm{vol}_k(K) \cdot \mathrm{Kl}_{G_u}(E) = h_{Y(K)}(u).$$

In particular $Z(K) \neq Y(K)$ and finally we obtain $\mathrm{Kl}_Z(E) \neq \mathrm{Kl}_Y(E)$ by (4.7). □

Proof of Theorem 4.10. Let K, L be as in the theorem. Then $Z(K) = \mathrm{Kl}_Z(\mathbb{R}^k) = r \cdot Z(L)$ by Proposition 4.11. Let $E \in \mathrm{Gr}_k(n)$. The group $\mathrm{SL}(n)$ acts faithfully on $\mathrm{Gr}_k\, n$, i.e. we can find $g \in \mathrm{SL}(n)$ such that $g\mathbb{R}^k = E$. Further, we can choose g such that $\det(g|_{\mathbb{R}^k}) = 1$. Now

$$\mathrm{Kl}_Z(E) = Z(gK) = gZ(K) = g\,\mathrm{Kl}_Z(\mathbb{R}^k).$$

This shows that the Klain function is uniquely described by the Klain body in $\mathbb{R}^k$. By Proposition 4.10 the Minkowski valuation Z is uniquely described by $Z(K) = \mathrm{Kl}_Z(\mathbb{R}^k)$. □

4.2.2 $\mathrm{SL}(V)$ invariant Minkowski valuations

Let W be the uniquely determined (up to isomorphism) irreducible $\mathrm{SL}(V)$-representation with the trivial $\mathrm{SL}(V)$ action $\phi w = w$. Then W is isomorphic to $\mathbb{R}$ and the term $\mathrm{SL}(V)$ *equivariant* Minkowski valuation $Z \colon \mathcal{K}(V) \to \mathcal{K}(\mathbb{R})$ can be replaced by $\mathrm{SL}(V)$ *invariant* Minkowski valuation. That means, in this case we have $Z(\phi K) = Z(K)$ for all $\phi \in \mathrm{SL}(V)$. Our goal in this section is to classify all continuous, translation invariant and $\mathrm{SL}(V)$ invariant Minkowski valuations $\mathcal{K}(V) \to \mathcal{K}(\mathbb{R})$. Note that $\mathcal{K}(\mathbb{R})$ is just the set of closed intervals. The classification follows from a classical result due to Blaschke.

Theorem 4.12 ([19]). *$Z \colon \mathcal{K}(V) \to \mathbb{R}$ is a continuous, translation invariant and $\mathrm{SL}(V)$ invariant valuation if and only if there are real numbers c_1, c_2 such that*

$$Z(K) = c_1 + \mathrm{vol}(K) \cdot c_2.$$

Other classifications of $\mathrm{SL}(V)$ invariant real valuations without assuming translation invariance and with weaker continuity assumptions can be found in [61].

Corollary 4.13. *$Z\colon \mathcal{K}(V) \to \mathcal{K}(\mathbb{R})$ is a continuous, translation invariant and* $\mathrm{SL}(V)$ *invariant Minkowski valuation if and only if there are closed intervals $I_1, I_2 \subset \mathbb{R}$ such that*

$$Z(K) = I_1 + \mathrm{vol}(K) \cdot I_2.$$

Proof. It is clear that $K \mapsto I_1 + \mathrm{vol}(K) \cdot I_2$ is a continuous, translation invariant and $\mathrm{SL}(V)$ invariant Minkowski valuation.

Conversely, assume Z to be a continuous, translation invariant and $\mathrm{SL}(V)$ invariant Minkowski valuations. Then for fixed $u \in \mathbb{R}^*$ the map

$$H_u\colon \mathcal{K}(V) \to \mathbb{R}, \quad K \mapsto h_{Z(K)}(u)$$

is contained in $\mathrm{Val}(V)$ and $\mathrm{SL}(V)$ invariant. By Theorem 4.12 there are $c_{1,u}, c_{2,u}$ such that $H_u(K) = c_{1,u} + \mathrm{vol}(K) \cdot c_{2,u}$. The body $Z(K)$ is uniquely determined by the functions H_1, H_{-1}, where $\mathbb{R}^*$ is identified with $\mathbb{R}$ in the usual way. Now using the support function it is easy to verify

$$Z(K) = [-c_{1,-1}, c_{1,1}] + \mathrm{vol}(K) \cdot [-c_{2,-1}, c_{2,1}].$$

□

4.2.3 Difference body

As an example for a Minkowski valuation in the classical sense (by this we mean a valuation $\mathcal{K}(V) \to \mathcal{K}(V)$) we have the *difference operator*

$$D\colon \mathcal{K}(V) \to \mathcal{K}(V), \quad K \mapsto K + (-K).$$

For $K \in \mathcal{K}(V)$ the body DK is called the *difference body.* It follows from Example 4.2 that D is indeed a valuation. Note that D is continuous, even, translation invariant and homogeneous of degree 1. Further, we have $\mathrm{SL}(V)$ equivariance:

$$D(\phi K) = \phi D(K), \quad \forall \phi \in \mathrm{SL}(V).$$

The following classification theorem is due to Ludwig.

Theorem 4.14 ([59])**.** *Let $n \geq 2$. A map $Z\colon \mathcal{K}(V) \to \mathcal{K}(V)$ is contained in* $\mathrm{MVal}(V, V)$ *and* $\mathrm{SL}(V)$ *equivariant if and only if $Z = c \cdot D$ for some $c \geq 0$.*

The difference operator occurs in the following inequality which is a special case of the so-called *Blaschke-Santaló inequality* [18, 77]. It says

$$\mathrm{vol}(DK) \cdot \mathrm{vol}((DK)^\circ) \leq \omega_n^2 \tag{4.8}$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$ with non-empty interior. Equality holds if and only if K is an ellipsoid [74].

4.2.4 Projection body

The next example of a Minkowski valuation we want to discuss is given by the *projection operator* $\Pi \colon \mathcal{K}(V) \to \mathcal{K}(V^*)$, which is defined via the support function

$$h_{\Pi K} \colon V \to \mathbb{R}, \quad u \mapsto \tfrac{n}{2} \cdot V_n(K[n-1], [-u, u]). \tag{4.9}$$

Here we identify $(V^*)^*$ with V. Clearly, this function is sublinear and therefore a support function of a convex body ΠK, the projection body of K. Further, from (4.9) we deduce that Π is even, translation invariant, continuous and homogeneous of degree $n-1$. As well as for the difference operator we have $\mathrm{SL}(V)$ equivariance of the projection operator ([62]), i.e.

$$\Pi(\phi K) = \phi \Pi(K), \quad \forall \phi \in \mathrm{SL}(V).$$

As for the difference operator, there is a classification result for the projection operator due to Ludwig.

Theorem 4.15 ([59])**.** *Let $n \geq 2$. A map $Z \colon \mathcal{K}(V) \to \mathcal{K}(V^*)$ is contained in* $\mathrm{MVal}(V, V^*)$ *and* $\mathrm{SL}(V)$ *equivariant if and only if $Z = c \cdot \Pi$ for some $c \geq 0$.*

The following inequality is called the *Petty projection inequality* [74]. It says

$$\mathrm{vol}(K)^{n-1} \cdot \mathrm{vol}((\Pi K)^\circ) \leq \left(\frac{\omega_n}{\omega_{n-1}} \right)^n$$

for all $K \in \mathcal{K}(\mathbb{R}^n)$ with non-empty interior. The equality cases are also characterized and given by the ellipsoids. A lower bound of the left-hand side is given by the *Zhang projection inequality* [100] stating that

$$\frac{1}{n^n} \binom{2n}{n} \leq \mathrm{vol}(K)^{n-1} \cdot \mathrm{vol}((\Pi K)^\circ) \tag{4.10}$$

with equality if and only if K is a simplex.

4.2.5 Moment body

The last example of a Minkowski valuation we want to mention is the *moment operator* $M \colon \mathcal{K}(V) \to \mathcal{K}(V)$. As the projection operator, it is defined via the support function

$$h_{MK} \colon V^* \to \mathbb{R}, \quad u \mapsto \int_K |\langle u, x \rangle| dx.$$

Again, it is easy to see that this function is sublinear and therefore it defines a convex body, more precisely the moment body MK. Using $\mathbb{I}_K + \mathbb{I}_L = \mathbb{I}_{K \cap L} + \mathbb{I}_{K \cup L}$, where

$$\mathbb{I}_K(x) := \begin{cases} 1, & x \in K \\ 0, & x \notin K \end{cases}$$

one can see that the moment operator satisfies the valuation property. It is continuous, but in contrast to the difference operator and the projection operator, it is not translation invariant. It is not difficult to prove that M is $\mathrm{SL}(V)$ equivariant ([62], see also Section 6.1). As in the previous examples we want to state a geometric inequality for the moment body, the *Busemann-Petty centroid inequality.* It says

$$\mathrm{vol}(MK) \geq \left(\frac{2\omega_{n-1}}{(n+1)\omega_n}\right)^n \mathrm{vol}(K)^{n+1} \tag{4.11}$$

with equality if and only if K is a centered ellipsoid [72]. As described in [38] the Busemann-Petty centroid inequality implies the Petty-projection inequality. For more information on the moment operator, we refer to Section 6.1.

4.2.6 Projection bodies of general degree

Now we want to explain the notion of a projection operator of degree k. To do so let us point out some similarities between the difference operator D and the projection operator Π. As mentioned before they are both continuous, translation invariant and $\mathrm{SL}(V)$ equivariant Minkowski valuations. Further, they are (up to normalization) classified by these properties as maps $\mathcal{K}(V) \to \mathcal{K}(V)$ and $\mathcal{K}(V) \to \mathcal{K}(V^*)$ respectively.

Let us now state some facts about the projection operator. As $\mathrm{SL}(V)$-representation the space V^* is isomorphic to $\bigwedge^{n-1} V$. Hence we can write $\Pi\colon \mathcal{K}(V) \to \mathcal{K}(\bigwedge^{n-1})$. Further Π is homogeneous of degree $n-1$. By choosing an euclidean structure and identifying V^* with V the support function of the projection body can be written as

$$h_{\Pi K}(u) = \mathrm{vol}_{n-1}(\pi_{u^\perp}(K)), \quad u \in S^{n-1},$$

where $\pi_{u^\perp}$ denotes the orthogonal projection onto $u^\perp$.

Now let us do the same for the difference operator. Since $V = \bigwedge^1 V$ we can write $D\colon \mathcal{K}(V) \to \mathcal{K}(\bigwedge^1 V)$. Further D is homogeneous of degree 1. By choosing an euclidean structure and identifying V^* with V we can write

$$h_{DK}(u) = 2 \cdot \mathrm{vol}_1(\pi_{\langle u \rangle}(K)), \quad u \in S^{n-1},$$

where $\pi_{\langle u \rangle}$ denotes the orthogonal projection onto $\langle u \rangle$.

By these similarities, we can speak about the difference operator to be a projection operator of degree 1, while the usual projection operator is of degree $n-1$. We ask, whether there is a notion of a projection operator of degree k, where $1 < k < n-1$. More precisely we ask the following question.

Question 4.16. Is there a valuation $Z \in \mathrm{MVal}(V, \bigwedge^k V)$, which is $\mathrm{SL}(V)$ equivariant?

Generalizations of the difference and the projection operator were already studied by Abardia and Bernig in [1, 2], who also used Ludwig's characterizations we mentioned earlier, to define complex versions of the difference and the projection operator.

We generalize Question 4.16 to the following one.

Question 4.17. Let W be a finite-dimensional irreducible $\mathrm{SL}(V)$-representation. Is there a non-trivial, continuous and $\mathrm{SL}(V)$ equivariant valuation $Z\colon \mathcal{K}(V) \to \mathcal{K}(W)$?

As we see in the next section, we classify all finite-dimensional irreducible $\mathrm{SL}(V)$-representations W such that a non-trivial and $\mathrm{SL}(V)$ equivariant map $Z \in \mathrm{MVal}(V, W)$ exists. In particular, we show that there are no new maps of this type. This also implies that there is no projection body of degree k if $1 < k < n-1$.

5 A classification of $\mathrm{SL}(V)$ equivariant Minkowski valuations

In this chapter, we want to prove our main result. Throughout this chapter, V denotes a real vector space of finite dimension n and W a real $\mathrm{SL}(V)$-representation of finite dimension. We answer Question 4.16 under the additional assumption on $Z\colon \mathcal{K}(V) \to \mathcal{K}(W)$ being translation invariant. More precisely, we want to prove the following Theorem.

Theorem 5.1. *Let W be a real finite-dimensional irreducible $\mathrm{SL}(V)$-representation. There exists a non-trivial and $\mathrm{SL}(V)$ equivariant valuation $Z \in \mathrm{MVal}(V, W)$ if and only if W is isomorphic to V, V^* or $\mathbb{R}$.*

5.1 Vector-valued valuations

In this section we consider $Z \in \mathrm{MVal}_k(V, W)$ to be $\mathrm{SL}(V)$ equivariant and vector-valued. The latter means $Z(K)$ contains only one point v_K (depending on K). Instead of $Z(K) = \{v_k\}$ we also write $Z(K) = v_K$. We use the Irreducibility Theorem to prove a statement about vector-valued valuations. The goal of this section is to show that Z has to be the trivial valuation in most situations.

Lemma 5.2. *Let W be a finite-dimensional, irreducible $\mathrm{SL}(V)$-representation. Let $k \in \{0, \dots, n\}$ and $Z \in \mathrm{MVal}_k(V, W)$ be $\mathrm{SL}(V)$ equivariant. Further let Z be vector-valued and odd (resp. even). If $k \neq 0, n$ then $Z(K) = 0$ for all $K \in \mathcal{K}(V)$. If $k = 0, n$ and $W \neq \mathbb{R}$ then it is $Z(K) = 0$.*

Proof. Fix $g_0 \in \mathrm{GL}(V)$ with $\det(g_0) = -1$ and define the space

$$U_Z := \{\langle \xi, Z(\bullet)\rangle \mid \xi \in W^*\} + \{\langle \xi, (Z \circ g_0^{-1})(\bullet)\rangle \mid \xi \in W^*\}.$$

Note that U_Z is a subspace of $\mathrm{Val}_k^-(V)$ (resp. $\mathrm{Val}_k^+(V)$) since Z is odd (resp. even). We show that U_Z is invariant under $\mathrm{GL}(V)$. Let $g \in \mathrm{GL}(V)$. Then $g\xi := \xi \circ g^{-1}$ is again contained in W^*. First let $\det(g) = 1$. Then we can compute

$$g\langle \xi, Z(\bullet)\rangle = \langle \xi, Z(g^{-1}\bullet)\rangle = \langle \xi, g^{-1}Z(\bullet)\rangle = \langle g\xi, Z(\bullet)\rangle \in U_Z.$$

Similarly we obtain

$$g\langle \xi, (Z \circ g_0^{-1})(\bullet)\rangle = \langle \xi, Z(g_0^{-1}g^{-1}\bullet)\rangle = \langle \xi, Z(g_0^{-1}g^{-1}g_0g_0^{-1}\bullet)\rangle = \langle \xi, Z(hg_0^{-1}\bullet)\rangle$$

for $h = g_0^{-1}g^{-1}g_0$. Since $h \in \mathrm{SL}(V)$ it is

$$\langle \xi, Z(hg_0^{-1}\bullet)\rangle = \langle \xi, hZ(g_0^{-1}\bullet)\rangle = \langle h^{-1}\xi, Z(g_0^{-1}\bullet)\rangle \in U_Z.$$

For $c > 0$ we have

$$c\langle \xi, Z(\bullet)\rangle = \langle \xi, Z(c^{-1}\bullet)\rangle = \langle \xi, c^{-l}Z(\bullet)\rangle = \langle c^l\xi, Z(\bullet)\rangle \in U_Z.$$

Similarly one shows

$$c\langle \xi, (Z \circ g_0^{-1})(\bullet)\langle = \langle c^l\xi, (Z \circ g_0^{-1})(\bullet)\rangle \in U_Z.$$

We have shown that U_Z is invariant under $\mathrm{GL}^+(V)$.

Now assume $\det(g) = -1$. We have $\det(g^{-1}g_0) = 1$ and therefore

$$g\langle \xi, Z(\bullet)\rangle = \langle \xi, Z(g^{-1}g_0g_0^{-1}\bullet)\rangle = \langle \xi, g^{-1}g_0Z(g_0^{-1}\bullet)\rangle = \langle g_0^{-1}g\xi, Z(g_0^{-1}\bullet)\rangle \in U_Z.$$

Also we have

$$g\langle \xi, (Z \circ g_0^{-1})(\bullet)\rangle = \langle \xi, Z(g_0^{-1}g\bullet)\rangle = \langle \xi, g_0^{-1}gZ(\bullet)\rangle = \langle g^{-1}g_0\xi, Z(\bullet)\rangle \in U_Z.$$

Now let $g \in \mathrm{GL}(V)$ be arbitrarily. Then for $c_g = |\det(g)|^{-n}$ we have $\det(c_g \cdot g) = \pm 1$ and therefore

$$g\langle \xi, Z(\bullet)\rangle = c_g^{-1}(c_gg)\langle \xi, Z(\bullet)\rangle \in U_Z.$$

Further, we have

$$g\langle \xi, Z(g_0^{-1}\bullet)\rangle = c_g^{-1}(c_gg)\langle \xi, Z(g_0^{-1}\bullet)\rangle \in U_Z.$$

Finally, U_Z is invariant under $\mathrm{GL}(V)$. Clearly, U_Z has finite dimension. Hence it is closed. By the Irreducibility Theorem U_Z is equal to $\mathrm{Val}_k^-(V)$ (resp. $\mathrm{Val}_k^+(V)$) or $\{0\}$. But $U_Z = \mathrm{Val}_k^-(V)$ (resp. $U_Z = \mathrm{Val}_k^+(V)$) is impossible if $k \neq 0, n$ since $\mathrm{Val}_k^-(V)$ (resp. $\mathrm{Val}_k^+(V)$) has infinite dimension if $k \neq 0, n$. Therefore $U_Z = \{0\}$, which implies $Z \equiv 0$.

In the case $k = n$ we obtain $U_Z = \langle \mathrm{vol}_n\rangle$ if Z is even. In particular for $\phi \in \mathrm{SL}(V)$ we have

$$\langle \xi, Z(K)\rangle = \langle \xi, Z(\phi K)\rangle = \langle \xi, \phi Z(K)\rangle.$$

But this is only possible if ϕ acts trivial (which means $W = \mathbb{R}$) or $Z(K) = 0$. If Z is odd we obtain $U_Z = \{0\}$, thus $Z \equiv 0$.

To prove the lemma for the case $k = 0$ note that for all $K \in \mathcal{K}(V)$ we have

$$Z(\{0\}) = Z(\lim_{\lambda\to 0} \lambda K) = \lim_{\lambda\to 0} Z(\lambda K) = \lim_{\lambda\to 0} \lambda^0 Z(K) = Z(K).$$

This means Z is constant. In particular

$$Z(K) = Z(\phi K) = \phi Z(K)$$

for all $\phi \in \mathrm{SL}(V)$. This is only possible if $W = \mathbb{R}$ or $Z(K) = 0$. □

Corollary 5.3. *Let $k \in \{0, \dots, n\}$ and $Z \in \mathrm{MVal}_k(V, W)$ be vector-valued and $\mathrm{SL}(V)$ equivariant. If $k \neq 0, n$ then $Z \equiv 0$. If $W \neq \mathbb{R}$ and $k = 0, n$ then $Z \equiv 0$.*

Proof. Define $Z^+(K) = \frac{1}{2}(Z(K) + Z(-K))$ and $Z^-(K) = \frac{1}{2}(Z(K) - Z(-K))$. Then Z^+ (resp. Z^-) is an even (resp. odd) vector-valued valuation. If $k \neq 0, n$ or $W \neq \mathbb{R}$ Lemma 5.2 implies $Z^- \equiv 0 \equiv Z^+$ and therefore $Z = Z^+ + Z^- \equiv 0$. □

5.2 Even and homogeneous Minkowski valuations

Now we turn into the case of Minkowski valuations not necessarily vector-valued. In this section we assume $Z \in \mathrm{MVal}(V, W)$ to be $\mathrm{SL}(V)$ equivariant. The purpose of this section is to show that it is enough to assume $Z \in \mathrm{MVal}_k^+(V, W)$ for some $k \in \{0, \dots, n\}$ and satisfying $Z = -Z$. More precisely we want to show the following proposition.

Proposition 5.4. *Let W be a finite-dimensional, irreducible $\mathrm{SL}(V)$-representation. The following statements are equivalent:*

1. *There exists $k \in \{0, \dots, n\}$ and a non-trivial $\mathrm{SL}(V)$ equivariant valuation $Z \in \mathrm{MVal}_k^+(V, W)$ such that $Z = -Z$.*
2. *There exists a non-trivial $\mathrm{SL}(V)$ equivariant valuation $Z \in \mathrm{MVal}(V, W)$.*

First we will show that we can reduce the problem to even valuations Z such that $Z = -Z$.

Lemma 5.5. *Let W be a finite-dimensional, irreducible $\mathrm{SL}(V)$-representation. The following statements are equivalent:*

1. *There exists a non-trivial $\mathrm{SL}(V)$ equivariant valuation $Z \in \mathrm{MVal}_k^+(V, W)$ such that $Z = -Z$.*
2. *There exists a non-trivial $\mathrm{SL}(V)$ equivariant valuation $Z \in \mathrm{MVal}_k(V, W)$.*

Proof. Obviously (1) $\Rightarrow$ (2) holds. To prove (2) $\Rightarrow$ (1) let $Z \in \mathrm{MVal}_k(V, W)$ be non-trivial. First, consider the case $k \neq 0, n$ or $W \neq \mathbb{R}$. Then Z is not vector-valued by Corollary 5.3. Hence the map

$$\tilde{Z} \colon \mathcal{K}(V) \to \mathcal{K}(W), \quad K \mapsto Z(K) + Z(-K) - (Z(K) + Z(-K))$$

is non-trivial. Clearly, we have $\tilde{Z} \in \mathrm{MVal}_k^+(V, W)$. Further $\tilde{Z}$ is $\mathrm{SL}(V)$ equivariant and we have $\tilde{Z}(K) = -\tilde{Z}(K)$.

In the case $k = 0$ and $W = \mathbb{R}$ both assertions are true since the valuation $Z \equiv [-1, 1]$ satisfies all the conditions in (1). In the remaining case, where $k = n$ and $W = \mathbb{R}$, we argue similarly. Again, both assertions are true since $Z(K) = \mathrm{vol}(K)[-1, 1]$ satisfies all the conditions in (1). $\square$

To continue with the reduction to the case of homogeneous valuations we want to discuss the possible degrees of homogeneity. The following lemma follows easily from McMullens decomposition.

Lemma 5.6. *Let $Z \in \mathrm{MVal}_k(V, W) \colon \mathcal{K}(V) \to \mathcal{K}(W)$. If $k \notin \{0, \dots, n\}$ then $Z = 0$.*

Proof. Fix $\xi \in W^*$ and consider

$$F \colon \mathcal{K}(V) \to \mathbb{R}, \quad K \mapsto h_{Z(K)}(\xi).$$

It is easy to see that F is a translation invariant continuous real valuation. By McMullen's decomposition (Theorem 4.3) there are continuous, translation invariant, real valuations Φ_i homogeneous of degree $i \in \{0, \dots, n\}$ such that

$$F = \sum_{i=0}^{n} \Phi_i.$$

But obviously, F is homogeneous of degree k. Hence $F \equiv 0$ if $k \notin \{0, \dots, n\}$. Since the choice of ξ was arbitrary we obtain $h_{Z(K)} = 0$ for all $K \in \mathcal{K}(V)$. $\square$

Next, we want to use McMullen's decomposition again to prove that it is enough to consider homogeneous valuations.

Lemma 5.7. *Let W be a finite-dimensional, irreducible* $\mathrm{SL}(V)$*-representation. The following statements are equivalent:*

1. *There exists $k \in \{0, \dots, n\}$ and a non-trivial* $\mathrm{SL}(V)$ *equivariant valuation* $Z \in \mathrm{MVal}_k(V, W)$.
2. *There exists a non-trivial* $\mathrm{SL}(V)$ *equivariant valuation* $Z \in \mathrm{MVal}(V, W)$.

Proof. Obviously (1) $\Rightarrow$ (2) holds. To prove (2) $\Rightarrow$ (1) assume that $Z \in \mathrm{MVal}(V, W)$ is non-trivial and $\mathrm{SL}(V)$ equivariant. Consider the function

$$H \colon \mathcal{K}(V) \times W^* \to \mathbb{R}, \quad (K, u) \mapsto h_{Z(K)}(u).$$

McMullen's decomposition tells us that there are translation invariant, continuous valuations $\varphi^i(\bullet, u) \colon \mathcal{K}(V) \to \mathbb{R}$ homogeneous of degree i such that

$$H(\bullet, u) = \sum_{i=0}^{n} \varphi^i(\bullet, u).$$

Let k be the maximum index i such that $\varphi^i(\bullet, u)$ does not vanish for some u. We repeat a standard argument which is used for example in [59] to show that $\varphi^k(\bullet, u)$ is a support function for a non-trivial valuation $Y \in \mathrm{MVal}_k(V, W)$. For $K \in \mathcal{K}(V)$ and $\lambda > 0$ we have

$$H(\lambda K, u) = \sum_{i=0}^{k} \lambda^i \varphi^i(K, u).$$

It follows

$$\lim_{\lambda \to \infty} \frac{H(\lambda K, u)}{\lambda^k} = \varphi^k(K, u).$$

Since $H(\lambda K, u)$ is sublinear in the second argument we derive sublinearity of $\varphi^k(K, u)$ in the second argument. Then $\varphi^k(K, u)$ must be a support function $h_{Y(K)}(u)$ for a convex body $Y(K) \in \mathcal{K}(W)$. Clearly, $Y \colon \mathcal{K}(V) \to \mathcal{K}(W)$ must be translation invariant, continuous and homogeneous of degree k. Also, it is easy to see that it is a valuation and further, it is $\mathrm{SL}(V)$ equivariant, since

$$\begin{aligned} h_{Y(\psi K)}(u) = \lim_{\lambda\to\infty} \frac{h_{Z(\psi K)}(u)}{\lambda^k} &= \lim_{\lambda\to\infty} \frac{h_{\psi Z(K)}(u)}{\lambda^k} \\ &= \lim_{\lambda\to\infty} \frac{h_{Z(K)}(\psi u)}{\lambda^k} = h_{Y(K)}(\psi u) = h_{\psi Y(K)}(u). \end{aligned}$$

□

Proof of Proposition 5.4. Clearly, (1) ⇒ (2). The other direction follows from Lemma 5.5 and Lemma 5.7. □

5.3 The Klain body

As before V denotes a real vector space of dimension n and let W be an irreducible $\mathrm{SL}(V)$-representation of finite dimension. As we saw in Section 3.4 the representation W is isomorphic to the restriction of an irreducible polynomial $\mathrm{GL}(V)$-representation. We also discussed that W is a subrepresentation of some tensor power of V, say $W \subset V^{\otimes d}$. In this section, we assume $Z \in \mathrm{MVal}_k^+(V, W)$ to be non-trivial, $\mathrm{SL}(V)$ equivariant and satisfying $Z = -Z$. The degree of homogeneity k is supposed to be an integer between 0 and n. We want to discuss the behaviour of the Klain function of Z under certain transformations. The main goal of this section is to reduce our problem to the case $W = \bigwedge^k V$.

Lemma 5.8. *For all $T \in \mathrm{GL}^+(V)$ we have*

$$Z(TK) = \det(T)^{-m} TZ(K),$$

where

$$m := \frac{d-k}{n}.$$

Proof. Let $T \in \mathrm{GL}^+(V)$. Then $(\det T)^{-\frac{1}{n}} T \in \mathrm{SL}(V)$ and by homogeneity

$$Z(TK) = (\det T)^{\frac{k}{n}} \cdot Z\left((\det T)^{-\frac{1}{n}} TK\right).$$

Now use $\mathrm{SL}(V)$ equivariance to obtain that the right-hand side is equal to

$$(\det T)^{\frac{k}{n}} \cdot \left((\det T)^{-\frac{1}{n}} T\right) Z(K) = \det(T)^{-m} TZ(K).$$

For the last equality, we used the fact that $Z(K)$ is a subset of $V^{\otimes d}$. □

Now it is convenient to fix an euclidean structure on V to comfortably speak about the Klain function of Z. Identify V with $\mathbb{R}^n$ via the euclidean structure, where $\mathbb{R}^n$ carries the standard euclidean structure. Recall from Section 4.2.1 that Z is uniquely determined by the Klain body $\mathrm{Kl}_Z(\mathbb{R}^k)$. Since in this section we suppose Z to be non-trivial, the Klain body is also not the zero body $\{0\}$.

Lemma 5.9. *Let $E \in \mathrm{Gr}_k(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}^+(n)$ such that $\phi(E) = E$. Then we have*

$$|\det(\phi|_E)|\,\mathrm{Kl}_Z(E) = \det(\phi)^{-m}\phi\,\mathrm{Kl}_Z(E).$$

Proof. Let $K \subset E$ be a convex body with $\mathrm{vol}_k(K) = 1$. By (4.6) we have

$$Z(\phi K) = \mathrm{vol}_k(\phi K)\cdot \mathrm{Kl}_Z(E) = |\det(\phi|_E)|\cdot \mathrm{Kl}_Z(E).$$

On the other hand, using Lemma 5.8 we obtain

$$Z(\phi K) = \det(\phi)^{-m}\phi Z(K) = \det(\phi)^{-m}\phi\,\mathrm{Kl}_Z(E)$$

and hence the claim. □

Recall that $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$. The previous lemma says that $\mathrm{Kl}_Z(\mathbb{R}^k)$ is invariant under U_n (see Section 3.3.2 for the notation). The next goal is to show that $\mathrm{Kl}_Z(\mathbb{R}^k)$ is contained in the highest weight space with respect to the standard basis.

Lemma 5.10. *Let $w \in \mathrm{Kl}_Z(\mathbb{R}^k)$. Then U_n acts trivial on w.*

Proof. Since W is a $\mathrm{GL}(n)$-representation we have a decomposition into weight spaces

$$W = \bigoplus_{\lambda\in\Lambda} W_\lambda,$$

where Λ denotes the set of weights of W. Now let

$$w = \sum_{\lambda\in\Lambda} w_\lambda$$

be a point in the Klain body and $w_\lambda \in W_\lambda$. The group U_n is spanned by the elements $u_{ij}(s) = \mathrm{id} + sE_{ij}$ for $j > i$. Hence it suffices to show that $u_{ij}(s)$ acts trivially on w. By Lemma 3.41 there are $w_{\lambda,l} \in W_{\lambda + l(\varepsilon_i - \varepsilon_j)}$ for $l \geq 0$ such that $w_{\lambda,0} = w_\lambda$ and

$$u_{ij}(s)w = \sum_{l\geq 0} s^l w_{\lambda,l}.$$

Note that this is a finite sum since W has finite dimension. Denote by $l_{\max}$ the maximal value such that $w_{\lambda,l_{\max}}$ is non-zero for some λ. Now we can calculate

$$u_{ij}(s)w = \sum_{\lambda\in\Lambda}\sum_{l\geq 0} s^k w_{\lambda,l} = \sum_{\lambda\in\Lambda} w_{\lambda,0} + s\sum_{\lambda\in\Lambda} w_{\lambda,1} + \dots + s^{l_{\max}}\sum_{\lambda\in\Lambda} w_{\lambda,l_{\max}}.$$

Since the Klain body is invariant under U_n we have $u_{ij}(s)w \in \mathrm{Kl}_Z(\mathbb{R}^k)$ for all $s \in \mathbb{R}$. For $s \to \infty$ we see that $\sum_{\lambda\in\Lambda} w_{\lambda,i}$ has to be zero for $i > 0$ by compactness of the Klain body. Hence

$$u_{ij}(s)w = \sum_{\lambda\in\Lambda} w_{\lambda,0} = \sum_{\lambda\in\Lambda} w_\lambda = w.$$

□

Corollary 5.11. *There is a highest weight vector $w \in W$ such that* $\mathrm{Kl}_Z(\mathbb{R}^n) = [-w, w]$.

Proof. The comment after Lemma 3.41 says $\mathrm{Kl}_Z(\mathbb{R}^k) \subset W^{U_n}$. But we already know that W^{U_n} equals the highest weight space which is of dimension 1. Since $Z = -Z$ we have $\mathrm{Kl}_Z(\mathbb{R}^k) = [-w, w]$ for a highest weight vector w. $\square$

The next step is to compute the highest weight of the representation W. This gives a condition on W, as the following proposition says.

Proposition 5.12. *The highest weight of W is*

$$\lambda = (m+1)(\varepsilon_1 + \cdots + \varepsilon_k) + m(\varepsilon_{k+1} + \cdots + \varepsilon_n).$$

In particular, as $\mathrm{SL}(n)$*-representation W is isomorphic to* $\bigwedge^k V$.

Proof. We compute the action of the n-dimensional torus T^n on the highest weight vector w to find the highest weight of W. Let

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in T^n.$$

By Lemma 5.9 we have

$$|t_1 \cdots t_l|[-w, w] = (t_1 \cdots t_n)^{-m} T[-w, w],$$

provided $\det(T) > 0$. Since the action of T is linear it follows

$$Tw = \varepsilon(T)|t_1 \cdots t_l|(t_1 \cdot \ldots \cdot t_n)^m w,$$

where ε is a function on T^n with values in $\{+1, -1\}$. Since the action of T^n is supposed to be continuous we obtain that the resriction $\varepsilon|_{\{T \in T^n \,:\, t_1 \cdots t_l > 0\}}$ is constant. Hence

$$Tw = \pm (t_1 \cdots t_k)^{m+1} \cdot (t_{k+1} \cdots t_n)^m w \tag{5.1}$$

if $t_1 \cdots t_k > 0$. Since the representation is also rational, (5.1) must hold for all $T \in T^n$. For both cases ($+$ or $-$), this action is isomorphic to

$$Tw = (t_1 \cdot \ldots \cdot t_k)^{m+1} \cdot (t_{k+1} \cdot \ldots \cdot t_n)^m w.$$

It follows $m \in \mathbb{Z}$ and the the highest weight is given by

$$\lambda = (m+1)(\varepsilon_1 + \cdots + \varepsilon_k) + m(\varepsilon_{k+1} + \cdots + \varepsilon_n).$$

Example 3.43 implies

$$W = (\textstyle\bigwedge^n \mathbb{R}^n)^{\otimes m} \otimes \bigwedge^k \mathbb{R}^n.$$

Since the factor $(\bigwedge^n \mathbb{R}^n)^{\otimes m}$ is just multiplication with the determinant, W is isomorphic to $\bigwedge^k \mathbb{R}^n$ as representation of $\mathrm{SL}(n)$. $\square$

5.4 The case $\wedge^k V$

To finish the proof of our main result, by Proposition 5.12 it is enough to consider $Z \in \mathrm{MVal}_k^+(V, \bigwedge^k V)$. As before we assume Z to be $\mathrm{SL}(V)$ equivariant and satisfying $Z = -Z$. Recall from Section 4.2 that there is a complete classification of such Z in the cases $k \in \{0, 1, n-1, n\}$. Therefore it suffices to consider the case $1 < k < n-1$. In particular, we suppose $n \geq 4$ in this whole section. As in the previous section, we equip V with an euclidean structure and identify it with $\mathbb{R}^n$ equipped with the standard euclidean structure. The goal of this section is to show that in this case Z is necessarily the trivial valuation, or equivalently $\mathrm{Kl}_Z(\mathbb{R}^k) = \{0\}$. Some preparation is needed.

5.4.1 Spherical harmonics

The goal of this section is to study $C(S^{n-1})$, the space of continuous functions on the unit sphere S^{n-1}, by introducing the reader to spherical harmonics. As for valuations we write $C^{\pm}(S^{n-1})$ for the subspace of even/odd functions. For more information on the topic we refer to [28, 41]. Let $p\colon \mathbb{R}^n \to \mathbb{R}$ be a polynomial. We say that p is *homogeneous of degree k* for some non-negative integer k, if $p(\lambda x) = \lambda^k p(x)$ for all $\lambda \in \mathbb{R}$. Further we say that p is *harmonic* if

$$\Delta p := \sum_{i=1}^{n} \frac{\partial^2 p}{\partial x_i^2} = 0.$$

Definition 5.13 (Spherical harmonic)**.** A function f on the unit sphere is called *spherical harmonic (of degree k)* if it is the restriction of a harmonic polynomial homogeneous of degree k. We write $\mathcal{H}_k$ for the space of spherical harmonics of degree k and define

$$\mathcal{H} := \bigoplus_{k \geq 0} \mathcal{H}_k. \tag{5.2}$$

The space $\mathcal{H}$ can be equipped with the inner product given by

$$\langle f, g \rangle_{L_2} := \int_{S^{n-1}} f(u) g(u) d\sigma(u),$$

where σ denotes the Lebesgue measure on the sphere. It is well-known ([41], §3.2) that for $k \neq l$ the functions $f \in \mathcal{H}_k$ and $g \in \mathcal{H}_l$ are orthogonal, i.e.

$$\langle f, g \rangle = 0.$$

Hence the sum in 5.2 is indeed a direct sum. Further, $\mathcal{H}_k$ is of finite dimension ([41], §3.2). Note that

$$\mathcal{H}_k \subset C^+(S^{n-1}) \text{ if } k \text{ is even}, \quad \mathcal{H}_k \subset C^-(S^{n-1}) \text{ if } k \text{ is odd}. \tag{5.3}$$

Theorem 5.14 ([41], §3.3)**.** *The space $\mathcal{H}_k$ is an irreducible representation of* $\mathrm{SO}(n)$.

Theorem 5.15 ([41], §3.2). *For any function $f \in C(S^{n-1})$ and all $\varepsilon > 0$ there exist spherical harmonics $f_1, \dots, f_k$ such that $f_i \in \mathcal{H}_i$ and*

$$\left| f(u) - \sum_{i=0}^{k} f_i(u) \right| < \varepsilon, \quad \forall u \in S^{n-1}.$$

In particular, $\mathcal{H}$ is dense in $C(S^{n-1})$ with respect to the supremum norm.

The last two statements tell us that, as $\mathrm{SO}(n)$-representation, $C(S^{n-1})$ decomposes into the irreducible subspaces $\mathcal{H}_k$ for $k \geq 0$. In other words, we have

$$C(S^{n-1}) = \overline{\bigoplus_{k\geq 0} \mathcal{H}_k}. \tag{5.4}$$

Together with 5.3, we obtain

$$C^+(S^{n-1}) = \overline{\bigoplus_{k\geq 0} \mathcal{H}_{2k}}. \tag{5.5}$$

5.4.2 Cosine transform

In this section, we want to define the cosine transform of a continuous function on the Grassmannian and also the cosine transform of a continuous function on the sphere. By $C(\mathrm{Gr}_k(V))$ we denote the space of continuous functions on $\mathrm{Gr}_k(V)$. We write $\mathrm{Gr}_k^+(V)$ for the set of oriented k-dimensional subspaces of V and $f \in C^+(\mathrm{Gr}_k^+(V))$ if $f(E) = f(F)$, if E equals F as a space (but possibly have different orientation). Note that there is a natural identification $C^+(\mathrm{Gr}_k^+(V)) = C(\mathrm{Gr}_k(V))$.

As before we fix an euclidean structure on V and identify V with $\mathbb{R}^n$. For $F \in \mathrm{Gr}_k(n)$ let us denote by $\pi_F \colon \mathbb{R}^n \to F$ the orthogonal projection. We define the *cosine of the angle* between E and F both contained in $\mathrm{Gr}_k(n)$ by

$$|\cos(E, F)| := \frac{\mathrm{vol}_k\,(\pi_F(A))}{\mathrm{vol}_k(A)},$$

where $A \in \mathcal{K}(E)$ with $\mathrm{vol}_k(A) > 0$. Since π_F is linear this definition does not depend on A.

Via the identification

$$\mathrm{Gr}_k(n) = \mathrm{O}(n)/(\mathrm{O}(k) \times \mathrm{O}(n-k))$$

we have a Haar measure on $\mathrm{Gr}_k(n)$ (for more on the Haar measure see [53], §IV.2).

Definition 5.16 (Cosine transform for the Grassmannian). Let $f \colon \mathrm{Gr}_k(n) \to \mathbb{R}$ be continuous. The *cosine transform* is defined by

$$C_k(f)\colon\ \mathrm{Gr}_k(n) \to \mathbb{R}, \quad E \mapsto \int_{\mathrm{Gr}_k(V)} |\cos(E, F)| f(F) dF,$$

where integration is with respect to the Haar measure on $\mathrm{Gr}_k(V)$.

We want to study the range of the cosine transform as a map $C_k\colon C(\mathrm{Gr}_k(n)) \to C(\mathrm{Gr}_k(n))$. The goal is to discuss that the image of C_k is a proper subset of $C(\mathrm{Gr}_k(n))$ if $1 < k < n-1$. The proof uses representations of $\mathrm{SO}(n)$. Recall that $C(\mathrm{Gr}_k(n))$ is a representation of $\mathrm{SO}(n)$ by the action

$$\phi \cdot f := f \circ \phi^{-1}.$$

This representation decomposes into a sum of irreducible $\mathrm{SO}(n)$-representations given by

$$C(\mathrm{Gr}_k(V)) = \bigoplus_{\lambda} W_\lambda, \tag{5.6}$$

where W_λ is the $\mathrm{SO}(n)$-representation with highest weight λ ranging over all integer partitions of the form

$$(m_1, \ldots, m_{\lfloor \frac{n}{2} \rfloor}), \quad m_i \text{ even}, m_j = 0, \forall j > k.$$

For a proof see [91, 92].

For a range characterization of the cosine transform, we use a statement by Alesker and Bernstein which implies that the image of C_k is a proper subset of $C(\mathrm{Gr}_k(V))$ if $1 < k < n-1$.

Theorem 5.17 ([9])**.** $\mathrm{Im}(C_k) \subset C(\mathrm{Gr}_k(V))$ *decomposes into a sum of irreducible* $\mathrm{SO}(n)$*-representations with highest weights*

$$\lambda = (m_1, \ldots, m_{\lfloor \frac{n}{2} \rfloor}), \quad m_i \textit{ even}, \ \forall j > \min\{k, n-k\}\colon m_j = 0, \ |m_2| \geq 2.$$

Each weight space occurs with multiplicity 1.

Note that by comparing the decomposition in Theorem 5.17 with the decomposition in (5.6) it follows

$$\mathrm{Im}(C_k) \subsetneq C(\mathrm{Gr}_k(V)), \quad \text{if } 1 < k < n-1.$$

As a by-product of the proof of Theorem 5.17, the following assertion comes out.

Theorem 5.18 ([9])**.** *Let* $\phi \in \mathrm{Val}_k^+(\mathbb{R}^n)$ *be smooth. Then* $\mathrm{Kl}_\phi\colon \ \mathrm{Gr}_k(n) \to \mathbb{R}$ *is contained in the image of the cosine transform.*

In particular, we have the following Corollary.

Corollary 5.19. *If* $1 < k < n-1$ *it is*

$$\overline{\mathrm{span}\{\mathrm{Kl}_\phi \mid \phi \in \mathrm{Val}_k^+(\mathbb{R}^n)\}} \subsetneq C(\mathrm{Gr}_k(n)).$$

Proof. Let $\phi \in \mathrm{Val}_k^+(\mathbb{R}^n)$. Then it can be approximated by smooth valuations $\phi_i \in \mathrm{Val}_k^{+,\infty}(\mathbb{R}^n)$. By Theorem 5.18 we have $\mathrm{Kl}_{\phi_i} \in \mathrm{Im}(C_k)$ and by continuity of the Klain embedding we obtain $\mathrm{Kl}_\phi \in \overline{\mathrm{Im}(C_k)}$. Since $\overline{\mathrm{Im}(C_k)}$ is a linear subspace it follows

$$\overline{\mathrm{span}\{\mathrm{Kl}_\phi \mid \phi \in \mathrm{Val}_k^+(\mathbb{R}^n)\}} \subset \overline{\mathrm{Im}(C_k)}.$$

It remains to show $\overline{\mathrm{Im}(C_k)} \subsetneq C(\mathrm{Gr}_k(n))$. By (5.6) and Theorem 5.17 there is a decomposition

$$\overline{\mathrm{Im}(C_k)} = \overline{\bigoplus_\lambda W_\lambda} \subsetneq \overline{\bigoplus_\mu W_\mu} \oplus \overline{\bigoplus_\lambda W_\lambda} = C(\mathrm{Gr}_k(n)).$$

Fix μ such that W_μ occurs in the decomposition in (5.6) but not in the decomposition in Theorem 5.17. The projection $\pi_\mu \colon C(\mathrm{Gr}_k(n)) \to W_\mu$ is then non-trivial. Now $\overline{\mathrm{Im}(C_k)} \subset \ker(\pi_\mu) \subsetneq C(\mathrm{Gr}_k(n))$. $\square$

There is another notion of the cosine transform not for functions on the Grassmannian but for functions on the sphere.

Definition 5.20 (Cosine transform for functions on S^{n-1})**.** Let f be a function on S^{n-1} and σ the Lebesgue measure on S^{n-1}. The *(spherical) cosine transform* is defined by

$$C(f) \colon S^{n-1} \to \mathbb{R}, \quad u \mapsto \int_{S^{n-1}} |\langle u, v\rangle| f(v) d\sigma(v).$$

In general, it is not true that the (spherical) cosine transform is injective. However, restricted to even function it turns out that this becomes true.

Theorem 5.21 ([41], §3.4)**.** *The restriction of the spherical cosine transform to even functions of the sphere is injective.*

5.4.3 The proof

Recall that in this section we assume $Z \in \mathrm{MVal}_k^+(\mathbb{R}^n, \bigwedge^k \mathbb{R}^n)$ is non-trivial and $\mathrm{SL}(n)$ equivariant and satisfies $Z = -Z$, where $1 < k < n-1$. The plan is to cause a contradiction to show that such Z does not exist.

By Corollary 5.11 and $\mathrm{SL}(n)$ equivariance of Z it is

$$\mathrm{Kl}_Z(E) = [-w_E, w_E] \tag{5.7}$$

for $w_E \in W$. Note that $w_E \neq 0$ if Z is non-trivial by Theorem 4.10 and $\mathrm{SL}(n)$ equivariance.

Lemma 5.22. *Let $Z \in \mathrm{MVal}_k^+(\mathbb{R}^n, \bigwedge^k \mathbb{R}^n)$ be non-trivial with $k \in \{2, \dots, n-2\}$ and $\mathrm{SL}(n)$ equivariant such that $Z = -Z$. With the notation from above we have*

$$\overline{\mathrm{span}\{E \mapsto |\langle u, w_E\rangle| \colon u \in \textstyle\bigwedge^k \mathbb{R}^n\}} \subsetneq C(\mathrm{Gr}_k(n)).$$

Proof. Fix $u \in \bigwedge^k \mathbb{R}^n$. Note that

$$Z(K) = \mathrm{vol}_k(K)\,\mathrm{Kl}_Z(E) = \mathrm{vol}_k(K)[-w_E, w_E],$$

for all convex bodies $K \in \mathcal{K}(E)$. Hence

$$h_{Z(K)}(u) = \mathrm{vol}_k(K)|\langle u, w_E\rangle|$$

for all $K \in \mathcal{K}(E)$. It follows immediately

$$\mathrm{Kl}_{h_{Z(K)(u)}}(E) = |\langle u, w_E \rangle|.$$

Now apply Corollary 5.19.

$\square$

Example 5.23. Recall that we write $\mathbb{R}^k$ for the subspace $\mathbb{R}^k \times \{0\}^{n-k} \subset \mathbb{R}^n$. A highest weight vector for $\bigwedge^k \mathbb{R}^n$ with respect to the standard basis $e_1, \dots, e_n$ is given by

$$w = c \cdot e_1 \wedge \cdots \wedge e_k,$$

where c is a real non-zero real number (see also Example 3.43). It follows

$$\mathrm{Kl}_Z(\mathbb{R}^k) = c[-e_1 \wedge \cdots \wedge e_k, e_1 \wedge \cdots \wedge e_k].$$

Lemma 5.24. *Let* $F := \overline{\mathrm{span}\{|\langle \bullet, v\rangle| \colon v \in S^{n-1}\}}$. *Then we have*

$$F = C^+(S^{n-1}).$$

Proof. It is clear that $F \subset C^+(S^{n-1})$. We prove $F = C^+(S^{n-1})$ by contradiction. Assume

$$F \subsetneq C^+(S^{n-1}) \left(= \overline{\bigoplus_{i \geq 0} \mathcal{H}_{2i}} \right).$$

Denote by $\pi_i \colon C^+(S^{n-1}) \to \mathcal{H}_i$ the projection. Note that $\pi_i(F)$ is a $\mathrm{SO}(n)$ invariant subspace of $\mathcal{H}_i$, and $\mathcal{H}_i$ is irreducible as $\mathrm{SO}(n)$-representation by Theorem 5.14. Then $\pi_i(F)$ is either $\{0\}$ or dense in $\mathcal{H}_i$. If it is dense in $\mathcal{H}_i$, it must be equal to $\mathcal{H}_i$ since the space is of finite dimension. Recall that we suppose $F \subsetneq C^+(S^{n-1})$. Therefore there is a non-negative integer m such that $\pi_{2m}(F) = \{0\}$. It follows

$$F \subset \overline{\bigoplus_{i \geq 0, i \neq m} \mathcal{H}_{2i}}.$$

For $f \in F$ there is a sequence f_j in $\bigoplus_{i \geq 0, i \neq m} \mathcal{H}_{2i}$ converging to f. For all $h \in \mathcal{H}_{2m}$ it follows

$$\langle f, h \rangle = \lim_{j \to \infty} \langle f_i, h \rangle = 0.$$

This shows $\mathcal{H}_{2m} \subset F^\perp$. In particular $F^\perp$ is not trivial. Let $h \in F^\perp$ be non-zero. Then

$$C(h)(v) = \int_{S^{n-1}} |\langle u, v \rangle| \cdot h(u) du = \langle |\langle \bullet, v \rangle|, h \rangle = 0.$$

Since the cosine transform is injective on $C^+(S^{n-1})$ by Theorem 5.21, it follows $h = 0$. A contradiction. $\square$

There are several ways to define an euclidean structure on the exterior power $\bigwedge^k \mathbb{R}^n \subset (\mathbb{R}^n)^{\otimes k}$. The most natural way is to define it first on $(\mathbb{R}^n)^{\otimes k}$ by

$$\langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k \rangle := \langle v_1, w_1 \rangle \cdots \langle v_k, w_k \rangle,$$

and restrict this to $\bigwedge^k \mathbb{R}^n$. In the same way, one can define an euclidean structure on other subspaces of $(\mathbb{R}^n)^{\otimes k}$ as $\operatorname{Sym}^k \mathbb{R}^n$. Now we can speak about balls, cubes, spheres and lower dimensional volumes in $\bigwedge^k \mathbb{R}^n$ and $\operatorname{Sym}^k \mathbb{R}^n$. We denote the sphere in a subspace $U \subset (\mathbb{R}^n)^{\otimes k}$ by $S(U)$.

It is well-known that the Grassmannian $\operatorname{Gr}_k(n)$ can be embedded into $\mathbb{P}(\bigwedge^k \mathbb{R}^n)$, the projective space of $\bigwedge^k \mathbb{R}^n$, via so-called *Plücker coordinates*. More precisely the embedding is given by

$$E := \langle b_1, \dots, b_k \rangle \mapsto \langle b_1 \wedge \cdots \wedge b_k \rangle.$$

Via the identification

$$\mathbb{P}(\textstyle\bigwedge^k \mathbb{R}^n) = S(\textstyle\bigwedge^k \mathbb{R}^n)/_\sim,$$

where $\sim$ denotes the equivalence relation $x \sim y \Leftrightarrow x = \pm y$, we can already see, how to generalize Plücker coordinates to $\operatorname{Gr}_k^+(\mathbb{R}^n)$. Namely, we can identify $\operatorname{Gr}_k^+(\mathbb{R}^n)$ with $S(\bigwedge^k \mathbb{R}^n)$ by

$$E \mapsto \frac{1}{|b_1 \wedge \cdots \wedge b_k|} b_1 \wedge \cdots \wedge b_k,$$

where $b_1, \dots, b_k$ is a positively oriented basis of E.

Lemma 5.25. *Let $Z \in \operatorname{MVal}_k^+(\mathbb{R}^n, \bigwedge^k \mathbb{R}^n)$ be non-trivial and $\operatorname{SL}(n)$ equivariant. We have*

$$\overline{\operatorname{span}\{E \mapsto |\langle u, \omega_E \rangle| \colon u \in \textstyle\bigwedge^k \mathbb{R}^n\}} = C(\operatorname{Gr}_k(n)).$$

Proof. Recall that $\operatorname{Gr}_k^+(n)$ denotes the space of oriented k-dimensional subspaces in $\mathbb{R}^n$. Denote by $\psi \colon \operatorname{Gr}_k^+(n) \to S(\bigwedge^k \mathbb{R}^n)$ the embedding via Plücker coordinates described above, i.e.

$$\psi(E) = \frac{1}{|b_1 \wedge \cdots \wedge b_k|} b_1 \wedge \cdots \wedge b_k,$$

where $b_1, \dots, b_k$ is a positively oriented basis of E. Then we have

$$\overline{\operatorname{span}\{|\langle \psi(\bullet), u \rangle| \colon u \in S(\textstyle\bigwedge^k \mathbb{R}^n)\}} = C^+(\operatorname{Gr}_k^+(n))$$

by the Lemma 5.24. Now use the identification $C^+(\operatorname{Gr}_k^+(n)) = C(\operatorname{Gr}_k(n))$. □

Proof of Theorem 5.1. Suppose $Z \in \operatorname{MVal}(V, W)$ is non-trivial and $\operatorname{SL}(V)$ equivariant. By Proposition 5.4 we can assume $Z \in \operatorname{MVal}_k^+(V, W)$ for some $k \in \{0, \dots, n\}$ and $Z = -Z$. In this case Proposition 5.12 implies $W = \bigwedge^k V$. By Lemma 5.25 and Lemma 5.22 we have $k \in \{0, 1, n-1, n\}$. Finally, W is isomorphic to one of the spaces

$$\textstyle\bigwedge^0 \mathbb{R}^n \,(= \mathbb{R}),\ \bigwedge^1 \mathbb{R}^n \,(= \mathbb{R}^n),\ \bigwedge^{n-1} \mathbb{R}^n \,(= (\mathbb{R}^n)^*),\ \bigwedge^n \mathbb{R}^n \,(= \mathbb{R}).$$

□

6 New $\mathrm{SL}(V)$ equivariant Minkowski valuations

In this section, we want to define new continuous and $\mathrm{SL}(V)$ equivariant Minkowski valuations.

6.1 The Moment body in $\mathrm{Sym}^p V$

6.1.1 Definition and properties

In this section, we want to define a continuous $\mathrm{SL}(V)$ equivariant Minkowski valuation $\mathcal{K}(V) \to \mathcal{K}(\mathrm{Sym}^p V)$ for a positive integer p. Recall that for $u_1, \dots, u_p \in V$ we write $u_1 \odot \cdots \odot u_p$ for the symmetric tensor, i.e.

$$u_1 \odot \cdots \odot u_p := \frac{1}{p!} \sum_{\sigma \in S_p} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(p)} \in \mathrm{Sym}^p V.$$

The proofs in this section are generalizations of proofs in [38] (§9.1, §9.2).

Definition 6.1 (Generalized moment operator)**.** We define a map $M^p \colon \mathcal{K}(V) \to \mathcal{K}(\mathrm{Sym}^p V)$ via the support function

$$h_{M^p K} \colon (\mathrm{Sym}^p V)^* \to \mathbb{R}, \quad u \mapsto \int\limits_K |\langle u, x^{\odot p} \rangle| dx \tag{6.1}$$

Clearly, $h_{M^p K}$ is sublinear and hence indeed a support function for a convex body $M^p K$. Further, M^p is a valuation. Also, it is easy to see that M^p is continuous but not translation invariant. Note that for $p = 1$ the defined valuation equals the moment body (see Section 4.2.5). In this section, we want to state some first properties of the generalized moment body.

For the first lemma we need to fix an euclidean structure on V. We identify V with $\mathbb{R}^n$ via this structure.

Lemma 6.2. *If K is a convex body in $\mathbb{R}^n$ containing the origin in its interior, then*

$$h_{M^p K}(u) = \frac{1}{p+n} \int\limits_{S^{n-1}} \rho_K(v)^{p+n} |\langle u, v^{\odot p} \rangle| dv, \quad u \in (\mathrm{Sym}^p \mathbb{R}^n)^*,$$

where the integration is with respect to the spherical Lebesgue measure.

Proof. This follows by a calculation using polar coordinates. We have

$$h_{M^pK}(u) = \int\limits_K |\langle u, x^{\odot p}\rangle| dx = \int\limits_{S^{n-1}} \int\limits_0^{\rho_K(v)} |\langle u, (tv)^{\odot p}\rangle| t^{n-1} dt dv$$

$$= \int\limits_{S^{n-1}} \int\limits_0^{\rho_K(v)} |\langle u, v^{\odot p}\rangle| t^{p+n-1} dt dx = \frac{1}{p+n} \int\limits_{S^{n-1}} |\langle u, v^{\odot p}\rangle| \rho_K(v)^{p+n} dv.$$

□

For convex bodies containing the origin in its interior, it is an immediate consequence that M^pK is homogeneous of degree $p+n$. It is not hard to see that this is true in general, as the following lemma shows.

Lemma 6.3. *M^p is homogeneous of degree $n+p$.*

Proof. Let $\lambda > 0$. Then

$$h_{M^p\lambda K}(u) = \int\limits_{\lambda K} |\langle u, x^{\odot p}\rangle| dx = \int\limits_K |\langle u, (\lambda x)^{\odot p}\rangle| \lambda^n dx$$

$$= \lambda^{n+p} \int\limits_K |\langle u, x^{\odot p}\rangle| dx = \lambda^{n+p} \cdot h_{M^pK}(u) = h_{\lambda^{n+p}M^pK}(u).$$

□

Similarly one can show that M^p commutes with transformations of the special linear group.

Lemma 6.4. *M^p is* $\mathrm{SL}(n)$ *equivariant.*

Proof. Let $T \in \mathrm{SL}(n)$. Then

$$h_{M^pTK}(u) = \int\limits_{TK} |\langle u, x^{\odot p}\rangle| dx = \int\limits_K |\langle u, (Tx)^{\odot p}\rangle| dx = \int\limits_K |\langle u, Tx^{\odot p}\rangle| dx$$

$$= \int\limits_K |\langle Tu, x^{\odot p}\rangle| dx = h_{M^pK}(Tu) = h_{TM^pK}(u).$$

□

6.1.2 The volume of M^pK

In this section, we want to deduce a formula for the volume of M^pK. We will also see that M^pK is a zonoid. Again, all the proofs of this section are generalizations of the proofs in [38] (§9.1), except for the proofs of Lemma 6.5, Proposition 6.6 and Lemma 6.11.

Recall that the euclidean structure of $\mathbb{R}^n$ induces an euclidean structure in $\mathrm{Sym}^p(\mathbb{R}^n)$ as follows. On the p-th tensor power $V^{\otimes p}$ an induced euclidean structure is given by

$$\langle x_1 \otimes \cdots \otimes x_p, y_1 \otimes \cdots \otimes y_p \rangle = \langle x_1, y_1 \rangle \cdots \langle x_p, y_p \rangle.$$

and the euclidean structure on $\mathrm{Sym}^p V$ is just the restriction of the euclidean structure in $V^{\otimes k}$. One important property of this definition is the following. For $x, y \in V$ we have

$$\langle x^{\odot p}, y^{\odot p} \rangle = \langle x, y \rangle^p. \tag{6.2}$$

This follows easily from the fact $x^{\odot p} = x^{\otimes p}$. The downside of this definition is that for an orthonormal basis $b_1, \dots, b_n \in V$ the symmetric tensors

$$b_1^{i_1} \odot \cdots \odot b_n^{i_n}, \quad i_1 + \cdots + i_n = p \tag{6.3}$$

do not form an orthonormal basis. At least this basis is orthogonal.

Lemma 6.5. *Let $b_1 \dots, b_n$ be an orthonormal basis of V. Then*

$$\langle b_1^{i_1} \odot \cdots \odot b_n^{i_n}, b_1^{j_1} \odot \cdots \odot b_n^{j_n} \rangle = \begin{cases} \frac{i_1! \cdots i_n!}{p!}, & i_1 = j_1, \dots, i_n = i_n \\ 0, & \textit{otherwise.} \end{cases}$$

In particular, the basis in 6.3 is orthogonal.

Proof. It is easy to see that

$$\langle b_1^{i_1} \odot \cdots \odot b_n^{i_n}, b_1^{j_1} \odot \cdots \odot b_n^{j_n} \rangle = 0$$

if $i_k \neq j_k$ for some k. Further, we have

$$\begin{aligned} \langle b_1^{i_1} \odot \cdots \odot b_n^{i_n}, b_1^{i_1} \odot \cdots \odot b_n^{i_n} \rangle &= \tfrac{1}{p!^2} p! \sum_{\sigma \in S_p} \langle \sigma \cdot (b_1^{i_1} \otimes \cdots \otimes b_n^{i_n}), b_1^{i_1} \otimes \cdots \otimes b_n^{i_n} \rangle \\ &= \frac{i_1! \cdots i_n!}{p!}. \end{aligned}$$

□

The choice of an inner product induces a volume on convex bodies in $\mathrm{Sym}^p \mathbb{R}^n$ in the usual way. We denote by N the dimension of $\mathrm{Sym}^p V$. It is well-known (but not important for this thesis) that $N = \binom{n+p-1}{p}$.

Proposition 6.6. *If* $\dim(K) = n$, *then* $\dim(M^p K) = N$.

Proof. Consider the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. Then

$$e_1^{k_1} \odot \cdots \odot e_n^{k_n}, \quad k_1 + \cdots + k_n = p$$

is a basis for $\operatorname{Sym}^p \mathbb{R}^n$. Let $x = \sum\limits_{i=1}^n x_i e_i \in K$ be an interior point and $\varepsilon > 0$ such that $B_{2\varepsilon}(x) \subset K$. Then

$$\begin{aligned} y(t) &:= (x+t)^{\odot p} \\ &= \sum_{k_1+\cdots+k_n=p} \binom{p}{k_1,\ldots,k_n} (x_1+t_1)^{k_1}\cdots(x_n+t_n)^{k_n} e_1^{\odot k_1} \odot \cdots \odot e_n^{k_n} \in M^p K \end{aligned}$$

for all $t = \sum\limits_{i=1}^n t_i e_i \in \varepsilon \cdot S^{n-1}$. In particular, for any basis element $e_{i_1} \odot \cdots \odot e_{i_p}$ we can find $t \in \varepsilon \cdot S^{n-1}$ such that

$$\langle e_1^{k_1} \odot \cdots \odot e_n^{k_n}, y(t)\rangle = \frac{k_1! \cdots k_n!}{p!} \binom{p}{k_1,\ldots,k_n} (x_1+t_1)^{k_1}\cdots(x_n+t_n)^{k_n} > 0,$$

where we used Lemma 6.5. It follows

$$h_{M^pK}(e_1^{k_1} \odot \cdots \odot e_n^{k_n}) > 0$$

for all $k_1, \ldots, k_n$. In particular, M^pK contains a small cross polytope

$$\delta \operatorname{conv}\{e_1^{k_1} \odot \cdots \odot e_n^{k_n} \mid k_1 + \cdots + k_n = p\}$$

for some positive δ. Clearly, the cross polytope has full dimension N. This shows $\dim(M^pK) = N$. $\square$

Our next task is to show that M^pK is a zonoid.

Lemma 6.7. *Let K be a convex body in $\mathbb{R}^n$ and $0 < \varepsilon < 1$. Then there is $\delta > 0$ with the following property. There is a partition $E_1, \ldots, E_m$ of K into Borel sets of diameter less than δ such that for $u \in S(\operatorname{Sym}^p \mathbb{R}^n), x_j \in E_j$ we have*

$$\left| h_{M^pK}(u) - \sum_{j=1}^m \operatorname{vol}(E_j) |\langle u, x_j^{\odot p}\rangle| \right| < \varepsilon \cdot \operatorname{vol}(K).$$

Proof. Note that the map $f(y) = y^{\odot p}$ is continuous. Hence for all $x \in K$ there is $\delta_x > 0$ such that

$$|x-y| < \delta_x \Rightarrow |x^{\odot p} - y^{\odot p}| < \varepsilon.$$

Denote by $U_r(x)$ the open ball around x with radius r. By compactness the open cover

$$K \subset \bigcup_{x\in K} U_{\delta_x}(x)$$

has a finite subcover, i.e. there are $x_1, \ldots, x_m \in K$ such that

$$K \subset \bigcup_{i=1}^m F_i,$$

where $F_i = U_{\delta_{x_i}}(x_i)$. Now define $E_1 := F_1$ and $E_{i+1} := F_{i+1} \setminus E_i$ inductively. Then $E_1, \dots, E_m$ is a partition of K into Borel sets of diameter less than $\delta := 3 \cdot \max\{\delta_{x_1}, \dots, \delta_{x_m}\}$. It is

$$\begin{aligned}
\left| h_{M^pK}(u) - \sum_{j=1}^m \mathrm{vol}(E_j) |\langle u, x_j^{\odot p}\rangle| \right| &= \left| \int_K |\langle u, x^{\odot p}\rangle| dx - \sum_{j=1}^m \mathrm{vol}(E_j)|\langle u, x_j^{\odot p}\rangle| \right| \\
&= \left| \sum_{j=1}^m \int_{E_j} |\langle u, x^{\odot p}\rangle| - |\langle u, x_j^{\odot p}\rangle| dx \right| \\
&\leq \sum_{j=1}^m \int_{E_j} \big| |\langle u, x^{\odot p}\rangle| - |\langle u, x_j^{\odot p}\rangle| \big| dx \\
&\leq \sum_{j=1}^m \int_{E_j} |\langle u, x^{\odot p} - x_j^{\odot p}\rangle| dx \\
&\leq \sum_{j=1}^m \int_{E_j} |x^{\odot p} - x_j^{\odot p}| dx < \varepsilon \cdot \mathrm{vol}(K).
\end{aligned}$$

□

For a partition $E_1, \dots, E_m$ of K as in the previous lemma we write

$$Z_\varepsilon(x_1, \dots x_m) := \sum_{i=1}^m \mathrm{vol}(E_i)[-x_i^{\odot p}, x_i^{\odot p}] \subset \mathrm{Sym}^p \mathbb{R}^n, \tag{6.4}$$

where we assume $x_i \in E_i$. To simplify the notation we sometimes write only Z_ε.

Lemma 6.8. *Let K be a convex body in $\mathbb{R}^n$. Then M^pK is a centered zonoid in* $\mathrm{Sym}^p \mathbb{R}^n$.

Proof. Note that Z_ε is a centered zonotope. Its support function is given by

$$h_{Z_\varepsilon}(u) = \sum_{i=1}^m \mathrm{vol}(E_i)|\langle u, x_i^{\odot p}\rangle|.$$

Lemma 6.7 and Lemma 2.12 imply $Z_\varepsilon \to M^pK$ as $\varepsilon \to 0$. Hence M^pK is a centered zonoid. □

Recall that N denotes the dimension of $\mathrm{Sym}^p \mathbb{R}^n$. For points $y_1, \dots y_N \in \mathbb{R}^n$ we define

$$[y_1, \dots, y_N]_p := \mathrm{vol}(S(0, y_1^{\odot p}, \dots, y_N^{\odot p})),$$

where

$$S(0, y_1^{\odot p}, \dots, y_N^{\odot p}) = \mathrm{conv}\{0, y_1^{\odot p}, \dots, y_N^{\odot p}\}.$$

If the points $y_1^{\odot p}, \dots, y_N^{\odot p}$ are linearly independent, then $S(0, y_1^{\odot p}, \dots, y_N^{\odot p})$ is a simplex. In particular $[y_1, \dots, y_N]_p > 0$. Otherwise we have $[y_1, \dots, y_N]_p = 0$. Our next goal is to prove a formula for the volume of M^pK.

Theorem 6.9. *It is*

$$\mathrm{vol}(M^pK) = 2^N \int\limits_K \cdots \int\limits_K [x_1, \ldots, x_N]_p dx_1 \ldots dx_N.$$

Proof. Let $E_1, \ldots, E_m$ and $x_1, \ldots, x_m$ as in (6.4). By (4.2) we have

$$\mathrm{vol}(Z_\varepsilon) = \sum_{i_1,\ldots,i_N=1}^{m} \mathrm{vol}(E_{i_1}) \cdot \ldots \cdot \mathrm{vol}(E_{i_N}) \cdot V_N([-x_{i_1}^{\odot p}, x_{i_1}^{\odot p}], \ldots, [-x_{i_N}^{\odot p}, x_{i_N}^{\odot p}]).$$

The polarization formula 4.3 implies

$$V_N([-x_{i_1}^{\odot p}, x_{i_1}^{\odot p}], \ldots, [-x_{i_N}^{\odot p}, x_{i_N}^{\odot p}]) = \frac{1}{N!} \mathrm{vol}([-x_1^{\odot p}, x_1^{\odot p}] + \cdots + [-x_N^{\odot p}, x_N^{\odot p}]).$$

Now use well-known formulas for the volume of a parallelepiped and a simplex to obtain

$$\begin{aligned}\frac{1}{N!} \mathrm{vol}([-x_1^{\odot p}, x_1^{\odot p}] + \cdots + [-x_N^{\odot p}, x_N^{\odot p}]) &= \frac{2^N}{N!} \mathrm{vol}([0, x_1^{\odot p}] + \cdots + [0, x_N^{\odot p}]) \\ &= 2^N [x_1, \ldots, x_N]_p.\end{aligned}$$

Finally we have

$$\mathrm{vol}(Z_\varepsilon) = 2^N \sum_{i_1,\ldots,i_N=1}^{m} \mathrm{vol}(E_{i_1}) \cdot \ldots \cdot \mathrm{vol}(E_{i_N}) \cdot [x_1, \ldots, x_N]_p$$

and for $\varepsilon \to 0$ the claim follows. □

For specific calculations, it could be helpful to express the volume of M^pK in terms of integrals of the radial function of K. Of course, this is only possible if the radial function exists.

Corollary 6.10. *Let K be a convex body containing the origin in its interior. Then*

$$\mathrm{vol}(M^pK) = \frac{2^N}{(n+p)^N} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [y_1, \ldots, y_N]_p \rho_K(y_1)^{n+p} \ldots \rho_K(y_N)^{n+p} dy_1 \ldots dy_N.$$

Proof. Using polar coordinates we compute

$$\begin{aligned}\mathrm{vol}(M^pK) &= 2^N \int\limits_K \cdots \int\limits_K [x_1, \ldots, x_N]_p dx_1 \ldots dx_N \\ &= 2^N \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \int\limits_0^{\rho_K(y_1)} \cdots \int\limits_0^{\rho_K(y_N)} [t_1y_1, \ldots, t_Ny_N]_p t_1^{n-1} \ldots t_N^{n-1} dt_1 \ldots dt_N dy_1 \ldots dy_N \\ &= 2^N \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} \int\limits_0^{\rho_K(y_1)} \cdots \int\limits_0^{\rho_K(y_N)} [y_1, \ldots, y_N]_p t_1^{n-1+p} \ldots t_N^{n-1+p} dt_1 \ldots dt_N dy_1 \ldots dy_N\end{aligned}$$

$$= \frac{2^N}{(n+p)^N} \int\limits_{S^{n-1}} \cdots \int\limits_{S^{n-1}} [y_1, \dots, y_N]_p \rho_K(y_1)^{n+p} \dots \rho_K(y_N)^{n+p} dy_1 \dots dy_N.$$

□

If $p = 1$ it is obvious that the volume of M^pK and $\phi M^pK = M^p(\phi K)$ is the same if $\phi \in \mathrm{SL}(n)$. We show that this is also true for general p. This comes down to the fact that the image of the representation $\mathrm{SL}(n) \to \mathrm{GL}(\mathrm{Sym}^p \mathbb{R}^n)$ is contained in $\mathrm{SL}(\mathrm{Sym}^p \mathbb{R}^n)$.

Lemma 6.11. *For all $\phi \in \mathrm{SL}(n)$ we have*

$$\mathrm{vol}(\phi M^pK) = \mathrm{vol}(M^pK).$$

Proof. We show that the image of the representation $\rho\colon \mathrm{SL}(V) \to \mathrm{GL}(\mathrm{Sym}^p V)$ is contained in $\mathrm{SL}(\mathrm{Sym}^p V)$. In doing so we show the equivalent condition for the corresponding Lie algebras, i.e. $d\rho$ is a map on $\mathfrak{sl}(V)$ whose image is contained in $\mathfrak{sl}(\mathrm{Sym}^p V)$. Let us fix the basis

$$e_{i_1} \odot \cdots \odot e_{i_p}, \quad 1 \le i_1 \le \cdots \le i_p \le n$$

of $\mathrm{Sym}^p V$. We have to show that $\mathrm{tr}(d\rho(\phi)) = 0$ for all $\phi \in \mathfrak{sl}(V)$. Consider the decomposition

$$\mathfrak{sl}(V) = \mathfrak{h} \oplus \bigoplus_{i,j=1, i\neq j}^{n} \mathfrak{g}_{ij},$$

where $\mathfrak{h}$ is the space of all diagonal matrices with vanishing trace and $\mathfrak{g}_{ij}$ is spanned by X_{ij}, the matrix with 1 on the entry (i,j) and 0 otherwise. To compute the trace of $d\rho(\phi)$ we study the action on the basis given before. We write

$$d\rho(X_{ij})(e_{k_1} \odot \cdots \odot e_{k_p}) = \sum_{1 \le l_1 \le \cdots \le l_p \le n} c_{l_1,\dots,l_p} e_{l_1} \odot \cdots \odot e_{l_p}.$$

To compute the trace of $d\rho(X_{ij})$ we are only interested in the coefficient $c_{k_1,\dots,k_p}$. Since $i \neq j$ it is easy to see that this coefficient vanishes. It is

$$d\rho(X_{ij})(e_{k_1} \odot \cdots \odot e_{k_p}) = \sum_{m=1}^{p} e_{k_1} \odot \cdots \odot e_{k_{m-1}} \odot X_{ij} e_{k_m} \odot e_{k_{m+1}} \odot \cdots \odot e_{k_p}$$

Since $X_{ij} e_k = \delta_j^k e_i$ it follows

$$d\rho(X_{ij})(e_{k_1} \odot \cdots \odot e_{k_p}) = \sum_{m=1}^{p} \delta_j^{k_m} e_{k_1} \odot \cdots \odot e_{k_{m-1}} \odot e_i \odot e_{k_{m+1}} \odot \cdots \odot e_{k_p}.$$

Finally we obtain $c_{k_1,\dots,k_p} = \sum\limits_{m=1}^{p} \delta_j^{k_m} \delta_i k_m = p\delta_i^j$. But $\delta_i^j = 0$ since we assumed $i \neq j$. Hence

$$\mathrm{tr}(d\rho(X_{ij})) = \sum_{1 \le k_1 \le \cdots \le k_p \le n} c_{k_1,\dots,k_p} = 0.$$

If $\phi = \mathrm{diag}(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}$ then a similar calculation shows

$$d\rho(\phi)(e_{i_1} \odot \cdots \odot e_{i_p}) = (\lambda_{i_1} + \cdots + \lambda_{i_p}) e_{i_1} \odot \cdots \odot e_{i_p}.$$

It follows

$$\mathrm{tr}(\rho(\phi)) = \sum_{1 \leq i_1 \leq \cdots \leq i_p \leq n} \lambda_{i_1} + \cdots + \lambda_{i_p}.$$

It is an easy exercise to see that each λ_i occurs exactly

$$m = \sum_{j=0}^{p} \binom{n+j-2}{j} (p-j)$$

times. In particular m does not depend on i. Hence $\mathrm{tr}(\rho(\phi)) = m \cdot \mathrm{tr}(\phi) = 0$. □

6.1.3 On a Busemann-Petty type inequality for M^p

In this section, we want to discuss if it is possible to generalize the Busemann-Petty centroid inequality for the generalized moment operator for higher symmetric powers, i.e. we ask the following question:

Question 6.12. Is there a positive constant c only depending on n and p such that

$$\mathrm{vol}(M^p K) \geq c \cdot \mathrm{vol}(K)^{\frac{(n+p)\cdot N}{n}} \tag{6.5}$$

for all convex bodies $K \in \mathcal{K}(\mathbb{R}^n)$? Further, if this is true, are there equality cases for optimal c and how can they be classified?

Note that the choice of the exponent in (6.5) ensures compatibility with the degrees of homogeneity. It is well-known that these types of inequalities hold if the functionals that occur are $\mathrm{SL}(V)$ invariant and translation invariant (see e.g. [83]). But unfortunately, the left-hand side is not translation invariant. For $p = 1$ the question is answered by the usual Busemann-Petty centroid inequality. We show, that such a constant c exists for all p. We explain why the proof for the equality cases for $p = 1$ does not immediately generalize for general p. To do so we show that in the case $n = p = 2$ a convexity condition breaks.

Existence of a Minimizer

Now, we want to show that the answer to the first part of Question 6.12 is positive. The first ingredient we need is an estimate for the volume of $M^p B^n$.

Lemma 6.13. *Let $B \in \mathcal{K}(\mathbb{R}^n)$ be the centered ball with radius 1 and $t \in S^{n-1}$. There is a constant C not depending on t such that*

$$\mathrm{vol}(M^p B) \leq \mathrm{vol}(M^p(B + \lambda t)), \quad \forall \lambda > C. \tag{6.6}$$

Proof. By Lemma 6.11 and $\mathrm{SL}(n)$ equivariance of M^p we can assume $t = e_1$. By Theorem 6.9 it is

$$\begin{aligned}
\mathrm{vol}(M^p(B+\lambda e_1)) &= \int\limits_{B+\lambda e_1} \cdots \int\limits_{B+\lambda e_1} [x_1, \ldots, x_N]_p dx_1 \ldots dx_N \\
&= \int\limits_B \cdots \int\limits_B [x_1 + \lambda e_1, \ldots, x_N + \lambda e_1]_p dx_1 \ldots dx_N.
\end{aligned}$$

Note that

$$\begin{aligned}
[x_1 + \lambda e_1, \ldots, x_N + \lambda e_1]_p &= \frac{1}{N!} \mathrm{vol}([0, (x_1 + \lambda e_1)^{\odot p}] + \cdots + [0, (x_N + \lambda e_1)^{\odot p}]) \\
&= \frac{1}{N!} \sqrt{\det(\langle (x_i + \lambda e_1)^{\odot p}, (x_j + \lambda e_1)^{\odot p} \rangle)} \\
&= \frac{1}{N!} \sqrt{\det(\langle x_i + \lambda e_1, x_j + \lambda e_1 \rangle^{\odot p})} =: \frac{1}{N!} \sqrt{p_{x_1, \ldots, x_N}(\lambda)}.
\end{aligned}$$

Since $p_{x_1,\ldots,x_N}(\lambda)$ is the determinant of a matrix, whose entries are polynomials in λ, it is a polynomial in λ itself and the coefficients depend continuously on $x_1, \ldots, x_N$. Further, the polynomial is non-negative, since it is the determinant of a symmetric matrix. By construction, the degree is not bigger than $N \cdot p$. Let

$$m := \max\{\deg(p_{x_1,\ldots,x_N}) \mid x_1, \ldots, x_N \in B\}.$$

If $m = 0$ then we clearly have equality in 6.6 for all λ. If $m > 0$ let $y_1, \ldots, y_N \in B$ such that $\deg(p_{y_1,\ldots,y_N}) = m$. Since all the coefficients in the polynomial depend continuously on the points $y_1, \ldots, y_N$ we can find neighborhoods U_i of y_i such that $\deg(p_{x_1,\ldots,x_N}) = m$ for all $x_i \in U_i$. Then there are compact sets $C_i \subset B \cap U_i$ with non-empty interior and $y_i \in C_i$. By continuity and compactness of B the coefficients of the polynomials $p_{x_1,\ldots,x_N}(\lambda)$ are uniformly bounded for $x_i \in C_i$. Since $p_{x_1,\ldots,x_N}(\lambda)$ is non-negative for all λ, the leading coefficient must be positive and by the previous argument bigger than $\varepsilon > 0$. Finally, given a constant $K > 0$ we can find $k > 0$ not depending on $x_1 \in C_1, \ldots, x_N \in C_N$ such that

$$p_{x_1,\ldots,x_N}(\lambda) > K, \quad \forall \lambda > k.$$

Now

$$\begin{aligned}
\mathrm{vol}(M^p(B+\lambda e_1)) &= \int\limits_B \cdots \int\limits_B [x_1 + \lambda e_1, \ldots, x_N + \lambda e_1]_p dx_1 \ldots dx_N \\
&\geq \int\limits_{C_N} \cdots \int\limits_{C_1} [x_1 + \lambda e_1, \ldots, x_N + \lambda e_1]_p dx_1 \ldots dx_N \\
&= \frac{1}{N!} \int\limits_{C_N} \cdots \int\limits_{C_1} \sqrt{p_{x_1,\ldots,x_N}(\lambda)} dx_1 \ldots dx_N.
\end{aligned}$$

The right-hand side of this inequality tends to infinity as λ increases by the previous discussion. This implies

$$\mathrm{vol}(M^p(B + \lambda e_1)) \to \infty, \quad \text{for } \lambda \to \infty$$

and hence the claim. $\square$

Now for a convex body with non-empty interior we define

$$\Phi(K) = \frac{\mathrm{vol}(M^p K)}{\mathrm{vol}(K)^{\frac{(p+n)N}{n}}}.$$

We want to show that Φ is bounded below by a positive constant. To answer the first part of Question 6.12 we need one more ingredient which is the notion of John ellipsoid. Let K be a convex body with non-empty interior. Among all the ellipsoids contained in K there is a unique one of maximal volume called the *John ellipsoid* [29, 97].

Theorem 6.14 ([80], §10.12)**.** *Let K be a convex body with non-empty interior and $\underline{J}(K)$ its John ellipsoid with center c. Let $\overline{J}(K) = n(\underline{J}(K) - c) + c$. Then*

$$\underline{J}(K) \subset K \subset \overline{J}(K).$$

In the next theorem, we prove that the function Φ is bounded below by a positive constant. This answers the first part of Question 6.12. However, this does not imply that we can choose c such that equality is attained for some convex body. In the following $B(r)$ denotes the ball in $\mathbb{R}^n$ with radius r centered at the origin.

Theorem 6.15. *There is a positive constant c such that*

$$\Phi(K) \geq c$$

for all convex bodies K with non-empty interior.

Proof. Let $K \subset \mathbb{R}^n$ be a convex body with non-empty interior. Let $T \in \mathrm{SL}(n)$, $t \in \mathbb{R}^n$ and $r > 0$ such that $\underline{J}(K) = TB(r) + t$ is the John ellipsoid. For $\overline{J}(K) = nTB(r) + t$ we have

$$\mathrm{vol}(\underline{J}(K)) \leq \mathrm{vol}(K) \leq \mathrm{vol}(\overline{J}(K))$$

and also

$$\mathrm{vol}(M^p \underline{J}(K)) \leq \mathrm{vol}(M^p K) \leq \mathrm{vol}(M^p \overline{J}(K))$$

by monotonicity of M^p. It follows

$$\begin{aligned}
\Phi(K) &\geq \frac{\mathrm{vol}(M^p \underline{J}(K))}{\mathrm{vol}(\overline{J}(K))^{\frac{(n+p)N}{n}}} = \frac{\mathrm{vol}(M^p(TB(r) + t))}{\mathrm{vol}(n(TB(r) + t))^{\frac{(n+p)N}{n}}} \\
&= \left(\frac{1}{n}\right)^{(n+p)N} \cdot \Phi(TB(r) + t) = \left(\frac{1}{n}\right)^{(n+p)N} \cdot \Phi(B(1) + \tfrac{1}{r}T^{-1}t),
\end{aligned}$$

where for the last equality we used Lemma 6.11. It remains to show that

$$c' := \inf\{\Phi(B(1)+x) \mid x \in \mathbb{R}^n\}$$

is positive. By Lemma 6.13 it is

$$c' = \inf\{\Phi(B(1)+x) \mid |x| \leq C\}$$

for C sufficiently large. Since Φ is continuous the infimum is taken over a compact set. Hence there is $x_0 \in \mathbb{R}^n$ such that

$$c' = \Phi(B(1)+x_0).$$

Note that $M^p(B_1+x_0)$ has full dimension by Lemma 6.6. It follows $\mathrm{vol}(M^p(B(1)+x_0)) > 0$ and finally $c' > 0$. Now put $c = \left(\frac{1}{n}\right)^{(n+p)N} \cdot c'$. $\square$

This answers the first part of Question 6.12. Next, we want to explain, why a proof of the characterization of the equality cases for $p = 1$ not immediately generalizes.

On the equality cases

A proof of the classical Busemann-Petty centroid inequality

$$\mathrm{vol}(MK) \geq \left(\frac{2\omega_{n-1}}{(n+1)\omega_n}\right)^n \mathrm{vol}(K)^{n+1}$$

given in [38] (§9.2) is based on Steiner symmetrization. The proof also gives a classification of the equality cases. Let us now discuss that this proof does not immediately generalize to a proof of a Busemann-Petty type inequality for $M^p, p > 1$. In particular, we do not obtain a classification of the equality cases in the case $p > 1$ by this method. The classical Busemann-Petty centroid inequality is a direct consequence of the following theorem.

Theorem 6.16 (Busemann random simplex inequality; [23]; see also [38], §9.2)**.** *Let K be a convex body. Then*

$$\int_K \cdots \int_K [y_1, \ldots, y_n]_1 dy_1 \ldots dy_n \geq \left(\frac{\omega_{n-1}}{(n+1)\omega_n}\right)^n \mathrm{vol}(K)^{n+1},$$

with equality if and only if K is a centered ellipsoid.

We define

$$\Psi(K) := \int_K \cdots \int_K [y_1, \ldots, y_n]_1 dy_1 \ldots dy_n.$$

The proof of the Busemann random simplex inequality is based on Steiner symmetrization. Besides Theorem 2.17 the main ingredient is the following assertion.

Lemma 6.17 ([38], §9.2)**.** *Let K be a convex body containing the origin in its interior. Further, let $u \in S^{n-1}$ and H be the hyperplane orthogonal to u through the origin. Then*

$$\Psi(K) \geq \Psi(\operatorname{st}_H K).$$

Equality holds if and only if the midpoints of chords of K parallel to u lie in a hyperplane containing the origin.

Following [38] (§9.2) the proof of this lemma uses convexity of a certain function, explained in the next Lemma.

Lemma 6.18. *Let $u \in S^{n-1}$ and H be the hyperplane through the origin orthogonal to u. For fixed $c_1 \ldots, c_n \in H$ the function*

$$v \colon \mathbb{R}^n \to [0, \infty), \quad z \mapsto [c_1 + z_1 u, \ldots, c_n + z_n u]_1$$

is convex.

To prove a Busemann-Petty centroid inequality for $p > 1$ by the same strategy as for $p = 1$ we need convexity of the function

$$v_p \colon \mathbb{R}^N \to [0, \infty), \quad z \mapsto [c_1 + z_1 u, \ldots, c_N + z_N u]_p,$$

where u is contained in the unit sphere of $\mathbb{R}^n$ and $c_1, \ldots, c_N \in H = u^\perp$. But this function is in general not convex as the following example for $p = n = 2$ shows.

Example 6.19. Let $u = e_1, c_1 = e_2, c_2 = c_3 = -e_2$. Then for $x = (2, 2, 4), y = (0, 0, 4)$ we have

$$\begin{aligned}
&v(\tfrac{1}{2}x + \tfrac{1}{2}y) = v((1,1,4)) = [e_2 + e_1, -e_2 + e_1, -e_2 + 4e_1]_2 \\
&\quad = \operatorname{vol}\left(\operatorname{conv}\{0, e_1^{\odot 2} + 2e_1 \odot e_2 + e_2^{\odot 2}, e_1^{\odot 2} - 2e_1 \odot e_2 + e_2^{\odot 2} e_1, 16e_1^{\odot 2} - 8e_1 \odot e_2 + e_2^{\odot 2}\}\right) \\
&\quad = |\det(T)| \cdot \operatorname{vol}\left(\operatorname{conv}\{0, e_1^{\odot 2}, e_1 \odot e_2, e_2^{\odot 2}\}\right),
\end{aligned}$$

where T is the linear map defined by

$$\begin{aligned}
T(e_1^{\odot 2}) &= e_1^{\odot 2} + 2e_1 \odot e_2 + e_2^{\odot 2} \\
T(e_1 \odot e_2) &= e_1^{\odot 2} - 2e_1 \odot e_2 + e_2^{\odot 2} \\
T(e_2^{\odot 2}) &= 16e_1^{\odot 2} - 8e_1 \odot e_2 + e_2^{\odot 2}.
\end{aligned}$$

Its determinant is given by

$$\det(T) = \begin{vmatrix} 1 & 1 & 16 \\ 2 & -2 & -8 \\ 1 & 1 & 1 \end{vmatrix} = 60.$$

Hence

$$v(\tfrac{1}{2}x + \tfrac{1}{2}y) = 60 \cdot \operatorname{vol}\left(\operatorname{conv}\{0, e_1^{\odot 2}, e_1 \odot e_2, e_2^{\odot 2}\}\right).$$

Finally note that the volume of the right-hand side is positive since it is the convex hull of 4 affinely independent points. On the other hand we have

$$\begin{aligned}\tfrac{1}{2}v(x)+\tfrac{1}{2}v(y) &= \tfrac{1}{2}(v(2,2,4)+v(0,0,4)) \\ &= \tfrac{1}{2}\cdot[e_2+2e_1,-e_2+2e_1,-e_2+4e_1]_2+\tfrac{1}{2}\cdot[e_2,-e_2,-e_2+4e_1]_2\end{aligned}$$

The second summand equals the volume of the convex hull of the points $0, e_2^{\odot 2}, e_2^{\odot 2}, 16e_1^{\odot 2}-8e_1\odot e_2+e_2^{\odot 2}$. Note that these points are not affinely independent. Hence the second sum vanishes. The same calculation as before yields

$$\tfrac{1}{2}\cdot|\det(S)|\operatorname{vol}\left(\operatorname{conv}\{0,e_1^{\odot 2},e_1\odot e_2,e_2^{\odot 2}\}\right)$$

for the right-hand side, where S is given by

$$\begin{pmatrix}4 & 4 & 16\\ 4 & -4 & -8\\ 1 & 1 & 1\end{pmatrix}.$$

Hence

$$\tfrac{1}{2}v(x)+\tfrac{1}{2}v(y)=48\cdot\operatorname{vol}\left(\operatorname{conv}\{0,e_1^{\odot 2},e_1\odot e_2,e_2^{\odot 2}\}\right)$$

and finally

$$v(\tfrac{1}{2}x+\tfrac{1}{2}y)>\tfrac{1}{2}v(x)+\tfrac{1}{2}v(y).$$

Thus, this method can not be used to classify the equality cases in the generalized Busemann-Petty centroid inequality.

6.2 An example for $\mathrm{Sym}^p V\otimes \mathrm{Sym}^q V^*$

6.2.1 The normal cycle

In this section, we want to discuss another tool to define valuations using the integral. Let $V=\mathbb{R}^n$ be equipped with the standard euclidean structure. The *sphere bundle* is the $(2n-1)$-dimensional manifold $S\mathbb{R}^n := \mathbb{R}^n\times S^{n-1}$.

Definition 6.20 (Normal cycle)**.** Let $K\subset\mathbb{R}^n$ be a convex body. The *normal cycle* of K is the set

$$\operatorname{nc}(K)=\{(x,v)\in S\mathbb{R}^n \mid \langle v,y-x\rangle\geq 0,\ \forall y\in K\}.$$

The notion of the normal cycle goes back to Wintgen [96] and Zähle [98, 99]. It was also heavily studied by Fu [34, 35, 36]. The normal cycle is a naturally oriented $(n-1)$-dimensional Lipschitz submanifold of $S\mathbb{R}^n$ [34]. It is well-known that the normal cycle can be used to define valuations. More precisely, regarded as a current, the normal cycle satisfies the valuation property and is continuous [35, 36]. The concept of the normal cycle can be generalized to manifolds. For connections with valuations on manifolds see for example [11, 12].

Let us regard the sphere bundle as a subset of $\mathbb{R}^n \times (\mathbb{R}^n)^*$. Then $\operatorname{nc}(K) \subset \mathbb{R}^n \times (\mathbb{R}^n)^*$ and $\mathrm{GL}(n)$ acts on the normal cycle as follows. We define $\phi \cdot (x, v) = (\phi x, \frac{1}{|\phi v|}\phi v)$, where ϕv is as usual defined by $v \circ \phi^{-1}$. In this situation, the normal cycle is $\mathrm{GL}(n)$ equivariant in the following sense.

Lemma 6.21. *Let $K \in \mathcal{K}(\mathbb{R}^n)$ and $\phi \in \mathrm{GL}(n)$. Then*

$$\operatorname{nc}(\phi K) = \phi \operatorname{nc}(K).$$

Proof. We have

$$\begin{aligned}\phi \operatorname{nc}(K) &= \{(\phi x, \tfrac{1}{|\phi v|}\phi v)) \in S\mathbb{R}^n \mid \langle v, y - x\rangle \geq 0,\ \forall y \in K\}\\ &= \{(\phi x, \tfrac{1}{|\phi v|}\phi v) \in S\mathbb{R}^n \mid \langle \tfrac{1}{|\phi v|}\phi v, \phi y - \phi x\rangle \geq 0,\ \forall y \in K\}\\ &= \{(z, w) \in S\mathbb{R}^n \mid \langle z, \phi y - w\rangle \geq 0,\ \forall y \in K\} = \operatorname{nc}(\phi K).\end{aligned}$$

□

6.2.2 Definition and properties

Let $K \subset \mathbb{R}^n$ be a convex body containing the origin in its interior. For $p, q \in \mathbb{N}$ consider the map $F_{p,q}\colon (\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*)^* \to \mathbb{R}$ given by

$$F^K_{p,q}(z) = \int\limits_{\operatorname{nc}(K)} |\langle z, x^{\odot p} \otimes u^{\odot q}\rangle| \langle x, u\rangle^{-q} i_x \operatorname{vol}. \tag{6.7}$$

Here we identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$. In (6.7) the form vol is the usual volume form and $(x, u) \in \operatorname{nc}(K)$. Then the natural pairing is indeed defined. Since $0 \in \operatorname{int}(K)$ the quantity $\langle x, u\rangle$ is positive. Finally $F^K_{p,q}$ is clearly sublinear and therefore it is a support function for a convex body

$$G_{p,q}K \in \mathcal{K}\left(\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*\right).$$

Further, by the discussion in Section 6.2.1 the map $K \mapsto F^K_{p,q}(z)$ is a continuous real valuation. Hence

$$G_{p,q}\colon \mathcal{K}_{(0)}(\mathbb{R}^n) \to \mathcal{K}\left(\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*\right)$$

is a continuous Minkowski valuation. As for the generalized moment, body we show that $G_{p,q}$ commutes with the action of $\mathrm{SL}(n)$ using similar methods.

Lemma 6.22. *The valuation $G_{p,q}$ is $\mathrm{SL}(n)$ equivariant.*

Proof. Let $\phi \in \mathrm{SL}(V)$. Note that any representation preserves the natural pairing by definition. Further it is $\phi_* i_{\phi x} \operatorname{vol} = i_x \operatorname{vol}$. It follows

$$\begin{aligned}
h_{G_{p,q}\phi K}(z) &= \int\limits_{\mathrm{nc}(\phi K)} |\langle z, x^{\odot p} \otimes u^{\odot q}\rangle| \langle x, u\rangle^{-q} i_x \operatorname{vol} \\
&= \int\limits_{\phi\,\mathrm{nc}(K)} |\langle z, x^{\odot p} \otimes u^{\odot q}\rangle| \langle x, u\rangle^{-q} i_x \operatorname{vol} \\
&= \int\limits_{\mathrm{nc}(K)} |\langle z, (\phi x)^{\odot p} \otimes (\tfrac{1}{|\phi u|}\phi u)^{\odot q}\rangle| \langle \phi x, \tfrac{1}{|\phi u|}\phi u\rangle^{-q} \phi_* i_{\phi x} \operatorname{vol} \\
&= \int\limits_{\mathrm{nc}(K)} \tfrac{1}{|\phi u|^q} |\langle z, \phi(x^{\odot p} \otimes u^{\odot q})\rangle| \tfrac{1}{|\phi u|^{-q}} \langle \phi x, \phi u\rangle^{-q} i_x \operatorname{vol} \\
&= \int\limits_{\mathrm{nc}(K)} |\langle \phi z, x^{\odot p} \otimes u^{\odot q}\rangle| \langle x, u\rangle^{-q} i_{\phi x} \operatorname{vol} \\
&= h_{G_{p,q}K}(\phi z) = h_{\phi G_{p,q}K}(z).
\end{aligned}$$

□

Proposition 6.23. *For $K \in \mathcal{K}_{(o)}(\mathbb{R}^n)$ we have that $G_{p,q}K$ has non-empty interior.*

Proof. The proof is exactly the same as for Proposition 6.6. □

6.2.3 A decomposition of tensor products

As we saw in Section 3.4.3 the $\mathrm{SL}(n)$-representation $\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*$ is not irreducible for $p, q > 0$ and $n > 1$. Therefore it decomposes into a sum of irreducible representations W_i. If π_i denotes the projection of $\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*$ onto W_i, then

$$\pi_i \circ G_{p,q} \colon \mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W_i)$$

is a non-trivial continuous $\mathrm{SL}(n)$ equivariant map satisfying the valuation property. The goal of this section is to describe the irreducible representations W_i in terms of highest weights.

Lemma 6.24. *The spaces occuring in the decomposition of $\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*$ into irreducible $\mathrm{SL}(n)$-representations are exactly the spaces of highest weight*

$$(p + q - 2i)\varepsilon_1 + (q - i)(\varepsilon_2 + \cdots + \varepsilon_{n-1}), \quad i = 0, \ldots, \min\{p, q\}.$$

Each of these spaces occurs exactly once. See Section 3.3.2 for notation.

Proof. Note that $\operatorname{Sym}^p \mathbb{R}^n = \mathbb{S}_\lambda \mathbb{R}^n$ for the Young tableau $\lambda = (p, 0, \ldots, 0)$ and $(\operatorname{Sym}^q \mathbb{R}^n)^* = \mathbb{S}_\mu \mathbb{R}^n$ for the Young tableau $\mu = (q, \ldots q, 0)$, where q occurs in $n - 1$ slots. Now by Theorem 3.56 we have

$$\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^* = \bigoplus_\nu (\mathbb{S}_\nu \mathbb{R}^n)^{N^\nu_{\lambda,\mu}}.$$

To compute the Littlewood-Richardson coefficient $N^{\nu}_{\lambda,\mu}$ we have to extend μ by λ according to the Littlwood-Richardson rule. In this case, the Littlewood-Richardson rule simplifies as follows. We have to add p boxes to the Young tableau μ, not two in the same column, such that the resulting scheme is again a Young tableau. It is easy to see that there are exactly $\min\{p, q\}$ ways to extend μ in this way, and the resulting Young tableau is given by $\nu = (q + p - i, q \dots, q, i)$. The extensions are illustrated as

$$\mu = \begin{array}{cccc} \boxed{} & \boxed{} & \cdots & \boxed{} \\ \vdots & \vdots & & \vdots \\ \boxed{} & \boxed{} & \cdots & \boxed{} \end{array}, \quad \lambda = \boxed{}\,\boxed{} \cdots \boxed{} \quad \rightsquigarrow \quad \nu = \begin{array}{ccccccccc} \boxed{} & \cdots & \boxed{} & \cdots & \boxed{} & \boxed{1} & \cdots & \boxed{1} \\ \vdots & & \vdots & & \vdots & & & \\ \boxed{} & \cdots & \boxed{} & \cdots & \boxed{} & & & \\ \boxed{1} & \cdots & \boxed{1} & & & & & \end{array}.$$

In other words, the Littlewood-Richardson coefficient $N^{\nu}_{\lambda,\mu}$ is zero unless $\nu = (q + p - i, q \dots, q, i)$, where q occurs $n - 2$ times and i ranges from 0 to $\min\{p, q\}$. If ν is of the latter type the Littlewood-Richardson coefficient equals 1 by Theorem 3.25. As discussed in Section 3.3.2 the representation $\mathbb{S}_\nu \mathbb{R}^n$ has highest weight

$$(q + p - i)\varepsilon_1 + q(\varepsilon_2 + \cdots + \varepsilon_{n-1}) + i\varepsilon_n.$$

Since for representations of $\mathrm{SL}(V)$ there is the relation $\varepsilon_1 + \cdots + \varepsilon_n = 0$ we can write the highest weight as

$$(q + p - 2i)\varepsilon_1 + (q - i)(\varepsilon_2 + \cdots + \varepsilon_{n-1}).$$

$\square$

Using this decomposition we can show that there is a bunch of irreducible $\mathrm{SL}(n)$-representations W such that a continuous $\mathrm{SL}(n)$ equivariant valuation $\mathcal{K}_{(0)}(\mathbb{R}^n) \to \mathcal{K}(W)$ exists.

Theorem 6.25. *Let W be the irreducible $\mathrm{SL}(n)$-representation with highest weight $(p + q - 2i)\varepsilon_1 + (q - i)(\varepsilon_2 + \cdots + \varepsilon_{n-1})$, where $p, q \in \mathbb{N}$ and $0 \le i \le \min\{p, q\}$. There is a non-trivial $\mathrm{SL}(n)$ equivariant continuous valuation $\mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W)$.*

Proof. By Lemma 6.24 we have $W \subset \operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*$. Consider the valuation

$$G_{p,q} \colon \mathcal{K}_{(o)} \to \mathcal{K}(\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*).$$

Since $G_{p,q}K$ has full dimension the map

$$\pi \circ G_{p,q} \colon \mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W),$$

where π denotes the projection onto W, is non-trivial. Since $G_{p,q}$ is a continuous $\mathrm{SL}(n)$ equivariant valuation so is $\pi \circ G_{p,q}$. $\square$

Since any finite-dimensional $\mathrm{SL}(V)$-representation is uniquely determined by a partition of size $n - 1$ we have the following corollary.

Corollary 6.26. *Let $n \le 3$ and W be a representation of $\mathrm{SL}(V)$. Then there is a non-trivial continuous $\mathrm{SL}(V)$ equivariant Minkowski valuation $\mathcal{K}_{(o)}(V) \to \mathcal{K}(W)$.*

6.3 Modifications of M^p and $G_{p,q}$

6.3.1 Polar valuations

In this section, we want to mention that we can build more examples from the previous Minkowski valuations.

Definition 6.27. Let $Z\colon \mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W)$ be a valuation. Then we define the *polar valuation* by

$$Z^*\colon \mathcal{K}_{(o)}((\mathbb{R}^n)^*) \to \mathcal{K}(W), \quad Z^*(K) = Z(K^\circ)$$

The following lemma is an easy consequence of the definition (see also Section 2.1).

Lemma 6.28. *Let $Z\colon \mathcal{K}_{(o)}(\mathbb{R}^n) \to \mathcal{K}(W)$ be a valuation. Then Z^* is a valuation. Further*

- *If Z is continuous, then Z^* is continuous.*
- *If Z is homogeneous of degree k, then Z^* is homogeneous of degree $-k$,*
- *If Z is $\mathrm{SL}(n)$ equivariant, so is Z^*,*
- *$Z = (Z^*)^*$.*

The previous lemma implies that the following examples are also continuous and $\mathrm{SL}(n)$ equivariant valuations. Since $G_{p,q}$ is defined on the set of convex bodies containing the origin in its interior we can also define

$$G^*_{p,q}\colon \mathcal{K}_{(o)}(V^*) \to \mathcal{K}(\mathrm{Sym}^p V \otimes \mathrm{Sym}^q V^*), \quad K \mapsto G_{p,q}(K^\circ).$$

Similar we can define

$$(M^p|_{\mathcal{K}_{(o)}(\mathbb{R}^n)})^*\colon \mathcal{K}_{(o)}((\mathbb{R}^n)^\circ) \to \mathcal{K}(\mathrm{Sym}^p \mathbb{R}^n), \quad K \mapsto M^p(K^\circ).$$

Further $(M^p|_{\mathcal{K}_{(o)}(\mathbb{R}^n)})^*$ is homogeneous of degree $-(n+p)$.

6.3.2 The valuations M^p_+ and $(G_{p,q})_+$

Let

$$|\cdot|_+\colon \mathbb{R} \to [0,\infty), \quad |x|_+ = \begin{cases} x & , x \geq 0 \\ 0 & , x < 0. \end{cases}$$

Then the map $M^p_+\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\mathrm{Sym}^p \mathbb{R}^n)$ defined by the support function

$$h_{M^p_+ K}(u) = \int_K |\langle u, x^{\odot p}\rangle|_+ dx$$

is a continuous and $\mathrm{SL}(n)$ equivariant valuation homogeneous of degree $n+p$ (repeat the proofs for M^p). Similar the map $(G_{p,q})_+\colon \mathcal{K}(\mathbb{R}^n) \to \mathcal{K}(\operatorname{Sym}^p \mathbb{R}^n \otimes (\operatorname{Sym}^q \mathbb{R}^n)^*)$ defined by the support function

$$h_{(G_{p,q})_+}(z) = \int_{\mathrm{nc}(K)} |\langle z, x^{\odot p} \otimes u^{\odot q}\rangle|_+ \langle x, u\rangle^{-q} i_x d\,\mathrm{vol}$$

is a continuous and $\mathrm{SL}(n)$ equivariant valuation (repeat the proofs for $G_{p,q}$).

Bibliography

[1] J. Abardia. Difference bodies in complex vector spaces. *arXiv preprint arXiv:1206.0880*, 2012.

[2] J. Abardia and A. Bernig. Projection bodies in complex vector spaces. *Advances in Mathematics*, 227(2):830–846, 2011.

[3] S. Alesker. On P. McMullen's conjecture on translation invariant valuations. *Advances in Mathematics*, 155(2):239–263, 2000.

[4] S. Alesker. Description of translation invariant valuations on convex sets with solution of P. McMullen's conjecture. *Geometric & Functional Analysis GAFA*, 11(2):244–272, 2001.

[5] S. Alesker. Theory of valuations on manifolds. I. Linear spaces. *Israel J. Math.*, 156:311–339, 2006.

[6] S. Alesker. Theory of valuations on manifolds, II. *Advances in Mathematics*, 207(1):420–454, 2006.

[7] S. Alesker. Theory of valuations on manifolds, IV. New properties of the multiplicative structure. In *Geometric aspects of functional analysis*, pages 1–44. Springer, 2007.

[8] S. Alesker. Some conjectures on intrinsic volumes of Riemannian manifolds and Alexandrov spaces. *Arnold Mathematical Journal*, 4(1):1–17, 2018.

[9] S. Alesker and J. Bernstein. Range characterization of the cosine transform on higher Grassmannians. *Advances in Mathematics*, 184(2):367–379, 2004.

[10] S. Alesker and D. Faifman. Convex valuations invariant under the Lorentz group. *Journal of Differential Geometry*, 98(2):183–236, 2014.

[11] S. Alesker, J. H. Fu, E. Gallego, and G. Solanes. *Integral geometry and valuations.* Springer, 2014.

[12] S. Alesker and J. H. G. Fu. Theory of valuations on manifolds. III. Multiplicative structure in the general case. *Trans. Amer. Math. Soc.*, 360(4):1951–1981, 2008.

[13] M. Berger. *Geometry I.* Springer Science & Business Media, 2009.

[14] A. Bernig and D. Faifman. Valuation theory of indefinite orthogonal groups. *Journal of Functional Analysis*, 273(6):2167–2247, 2017.

[15] A. Bernig, D. Faifman, and G. Solanes. Uniqueness of curvature measures in pseudo-riemannian geometry. *The Journal of Geometric Analysis*, 31(12):11819–11848, 2021.

[16] A. Bernig, D. Faifman, and G. Solanes. Curvature measures of pseudo-Riemannian manifolds. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2022.

[17] U. Betke and P. McMullen. Estimating the sizes of convex bodies from projections. *Journal of the London Mathematical Society*, 2(3):525–538, 1983.

[18] W. Blaschke. Über affine Geometrie VII: Neue Extremeigenschaften von Ellipse und Ellipsoid. *Ber. Verh. Sächs. Akad. Wiss. Leipzig, Math.-Phys. Kl*, 69:306–318, 1917.

[19] W. Blaschke. *Vorlesungen über Integralgeometrie*, volume 1. Teubner, 1936.

[20] W. Blaschke. *Kreis und Kugel.* de Gruyter, 2013.

[21] T. Bonnesen and W. Fenchel. Theorie der konvexen Körper, Ergebnisse, vol. 3, 1934.

[22] J. Bourgain and J. Lindenstrauss. Projection bodies. In *Geometric Aspects of Functional Analysis*, pages 250–270. Springer, 1988.

[23] H. Busemann. Volume in terms of concurrent cross-sections. *Pacific Journal of Mathematics*, 3(1):1–12, 1953.

[24] S. Campi. *Reconstructing a convex surface from certain measurements of its projections.* Consiglio Nazionale delle Ricerche. Istituto di Analisi Globale ed Applicazioni, 1985.

[25] I. Chavel. *Isoperimetric inequalities: differential geometric and analytic perspectives*, volume 145. Cambridge University Press, 2001.

[26] A. Colesanti, M. Ludwig, and F. Mussnig. Minkowski valuations on convex functions. *Calculus of variations and partial differential equations*, 56(6):1–29, 2017.

[27] A. Colesanti, M. Ludwig, and F. Mussnig. Valuations on convex functions. *International Mathematics Research Notices*, 2019.

[28] F. Dai and Y. Xu. *Approximation theory and harmonic analysis on spheres and balls*, volume 23. Springer, 2013.

[29] L. Danzer, D. Laugwitz, and H. Lenz. Über das Löwnersche Ellipsoid und sein Analogon unter den einem Eikörper einbeschriebenen Ellipsoiden. *Archiv der Mathematik*, 8(3):214–219, 1957.

[30] M. Dehn. Über den Rauminhalt. *Mathematische Annalen*, 55(3):465–478, 1901.

[31] K. Erdmann and M. J. Wildon. *Introduction to Lie algebras*, volume 122. Springer, 2006.

[32] D. Faifman and T. Wannerer. The Weyl principle on the Finsler frontier. *Selecta Mathematica*, 27(2):1–30, 2021.

[33] B. Fleury, O. Guédon, and G. Paouris. A stability result for mean width of Lp-centroid bodies. *Advances in Mathematics*, 214(2):865–877, 2007.

[34] J. H. Fu. Curvature measures and generalized Morse theory. *Journal of Differential Geometry*, 30(3):619–642, 1989.

[35] J. H. Fu. Convergence of curvatures in secant approximations. *Journal of Differential Geometry*, 37(1):177–190, 1993.

[36] J. H. Fu. Curvature measures of subanalytic sets. *American Journal of Mathematics*, pages 819–880, 1994.

[37] W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[38] R. J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, New York, second edition, 2006.

[39] A. Giannopoulos, G. Paouris, and B.-H. Vritsiou. A remark on the slicing problem. *Journal of Functional Analysis*, 262(3):1062–1086, 2012.

[40] J. A. Green. Polynomial representations of GL(n). In *Algebra Carbondale 1980*, pages 124–140. Springer, 1981.

[41] H. Groemer. *Geometric applications of Fourier series and spherical harmonics*, volume 61. Cambridge University Press, 1996.

[42] P. M. Gruber. *Convex and discrete geometry*, volume 336 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Berlin, 2007.

[43] C. Haberl. Minkowski valuations intertwining the special linear group. *Journal of the European Mathematical Society*, 14(5):1565–1597, 2012.

[44] C. Haberl and L. Parapatits. Moments and valuations. *American Journal of Mathematics*, 138(6):1575–1603, 2016.

[45] C. Haberl and L. Parapatits. Centro-affine tensor valuations. *Advances in Mathematics*, 316:806–865, 2017.

[46] H. Hadwiger. *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

[47] B. C. Hall. Lie groups, Lie algebras, and representations. In *Quantum Theory for Mathematicians*, pages 333–366. Springer, 2013.

[48] D. Hilbert. Mathematical problems. *Bulletin of the American Mathematical Society*, 8(10):437–479, 1902.

[49] N. Jacobson. *Lie algebras*. Number 10. Courier Corporation, 1979.

[50] D. Klain. Even valuations on convex bodies. *Transactions of the American Mathematical Society*, 352(1):71–93, 2000.

[51] D. A. Klain and G.-C. Rota. *Introduction to geometric probability*. Lezioni Lincee. [Lincei Lectures]. Cambridge University Press, Cambridge, 1997.

[52] B. Klartag and E. Milman. Centroid bodies and the logarithmic laplace transform–a unified approach. *Journal of Functional Analysis*, 262(1):10–34, 2012.

[53] A. W. Knapp. *Lie groups beyond an introduction*, volume 140. Springer, 1996.

[54] A. W. Knapp. Representation theory of semisimple groups. In *Representation Theory of Semisimple Groups*. Princeton university press, 2016.

[55] H. Kraft. Klassische Invariantentheorie. In *Algebraische Transformationsgruppen und Invariantentheorie Algebraic Transformation Groups and Invariant Theory*, pages 41–62. Springer, 1989.

[56] H. Kraft and C. Procesi. Classical invariant theory. 1996.

[57] M. Ludwig. *Moment vectors of polytopes*. na, 2002.

[58] M. Ludwig. Projection bodies and valuations. *Advances in Mathematics*, 172(2):158–168, 2002.

[59] M. Ludwig. Minkowski valuations. *Transactions of the American Mathematical Society*, 357(10):4191–4213, 2005.

[60] M. Ludwig. Minkowski areas and valuations. *Journal of Differential Geometry*, 86(1):133–162, 2010.

[61] M. Ludwig and M. Reitzner. A classification of $\mathrm{SL}(n)$ invariant valuations. *Ann. of Math. (2)*, 172(2):1219–1267, 2010.

[62] E. Lutwak. Centroid bodies and dual mixed volumes. *Proceedings of the London Mathematical Society*, 3(2):365–391, 1990.

[63] P. McMullen. Valuations and Euler-type relations on certain classes of convex polytopes. *Proceedings of the London Mathematical Society*, 3(1):113–135, 1977.

[64] P. McMullen. Continuous translation invariant valuations on the space of compact convex sets. *Archiv der Mathematik*, 34(1):377–384, 1980.

[65] E. Michael. Topologies on spaces of subsets. *Transactions of the American Mathematical Society*, 71(1):152–182, 1951.

[66] H. Minkowski. Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs. *Gesammelte Abhandlungen*, 2:131–229, 1911.

[67] H. Minkowski. Volumen und Oberfläche. In *Ausgewählte Arbeiten zur Zahlentheorie und zur Geometrie*, pages 146–192. Springer, 1989.

[68] S. Okada. Applications of minor summation formulas to rectangular-shaped representations of classical groups. *Journal of Algebra*, 205(2):337–367, 1998.

[69] R. Osserman. The isoperimetric inequality. *Bulletin of the American Mathematical Society*, 84(6):1182–1238, 1978.

[70] G. Paouris. Concentration of mass on convex bodies. *Geometric & Functional Analysis GAFA*, 16(5):1021–1049, 2006.

[71] L. Parapatits and T. Wannerer. On the inverse Klain map. *Duke Mathematical Journal*, 162(11):1895–1922, 2013.

[72] C. M. Petty. Centroid surfaces. *Pacific Journal of Mathematics*, 11(4):1535–1547, 1961.

[73] C. M. Petty. Projection bodies. *Proceedings of the Colloquium on Convexity (Copenhagen)*, pages 234–241, 1967.

[74] C. M. Petty. Isoperimetric problems. *Proceedings Conference Convexity and Combinatorial Geometry, University of Oklahoma*, pages 26–41, 1971.

[75] C. Procesi. *Lie groups: an approach through invariants and representations*. Springer Science & Business Media, 2006.

[76] C. A. Rogers and G. C. Shephard. The difference body of a convex body. *Archiv der Mathematik*, 8(3):220–233, 1957.

[77] L. A. Santaló. Un invariante afin para los cuerpos convexos del espacio de n dimensiones. *Portugaliae mathematica*, 8(4):155–161, 1949.

[78] R. Schneider. Zu einem Problem von Shephard über die Projektionen konvexer Körper. *Mathematische Zeitschrift*, 101(1):71–82, 1967.

[79] R. Schneider. On the projections of a convex polytope. *Pacific Journal of Mathematics*, 32(3):799–803, 1970.

[80] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 151 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, expanded edition, 2014.

[81] R. Schneider and F. E. Schuster. Rotation equivariant Minkowski valuations, 2012.

[82] R. Schneider and W. Weil. Über die Bestimmung eines konvexen Körpers durch die Inhalte seiner Projektionen. *Mathematische Zeitschrift*, 116(4):338–348, 1970.

[83] R. Schneider and W. Weil. *Integralgeometrie*. Springer-Verlag, 2013.

[84] F. Schuster and T. Wannerer. GL(n) contravariant Minkowski valuations. *Transactions of the American Mathematical Society*, 364(2):815–826, 2012.

[85] F. Schuster and T. Wannerer. Minkowski valuations and generalized valuations. *J. Eur. Math. Soc. (JEMS)*, 20(8):1851–1884, 2018.

[86] F. E. Schuster and T. Wannerer. Even Minkowski valuations. *American Journal of Mathematics*, 137(6):1651–1683, 2015.

[87] G. Schwab. *Sagen des klassischen Altertums*. Jazzybee Verlag, 2016.

[88] R. P. Stanley. Symmetries of plane partitions. *Journal of Combinatorial Theory, Series A*, 43(1):103–113, 1986.

[89] J. Steiner. Einfache Beweise der isoperimetrischen Hauptsätze. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1838(18):281–296, 1838.

[90] J. Steiner. Über parallele Flächen. *Monatsber. Preuss. Akad. Wiss*, 2:114–118, 1840.

[91] M. Sugiura. Representations of compact groups realized by spherical functions on symmetric spaces. *Proceedings of the Japan Academy*, 38(3):111–113, 1962.

[92] M. Takeuchi. *Modern spherical functions*. American Mathematical Soc., 1994.

[93] T. Wannerer. GL(n) equivariant Minkowski valuations. *Indiana University Mathematics Journal*, pages 1655–1672, 2011.

[94] T. Wannerer. Integral geometry of unitary area measures. *Advances in Mathematics*, 263:1–44, 2014.

[95] H. Weyl. The classical groups. In *The Classical Groups*. Princeton university press, 2016.

[96] P. Wintgen. Normal cycle and integral curvature for polyhedra in Riemannian manifolds. *Differential geometry*, 21, 1982.

[97] V. Zaguskin. Circumscribed and inscribed ellipsoids of extremal volume. *Uspekhi Matematicheskikh Nauk*, 13(6):89–93, 1958.

[98] M. Zähle. Integral and current representation of Federer's curvature measures. *Archiv der Mathematik*, 46(6):557–567, 1986.

[99] M. Zähle. Curvatures and currents for unions of sets with positive reach. *Geometriae Dedicata*, 23(2):155–171, 1987.

[100] G. Zhang. Restricted chord projection and affine inequalities. *Geometriae Dedicata*, 39(2):213–222, 1991.

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