DISCONTINUITY IN MULTIPOLAR EXPANSIONS

Sofía A. Cora and Juan C. Muzzio

FCAG - UNLP and PROFOEG - CONICET

ABSTRACT. We show that softening does not avoid discontinuities in some multipolar expansions, but they do not affect the results of numerical integrations of the N-body problem.

White (1983) proposed a multipolar expansion that was extensively used thereafter by L.A. Aguilar (e.g., Aguilar and White, 1985; Aguilar and Merritt, 1990) who kindly let us use a copy of his code. Although White softened the potential, a discontinuity is present in the resulting force and it is important to check whether it could affect the results obtained with the code.

In the present work we limited the expansion to the monopolar terms and considered a spherical system; in this way, all the relevant effects are present, but the work is considerably simplified, particularly in the experiments that demand rather ellaborate modifications of the original code.

In the monopolar approximation the potential produced by a distribution of particles of mass m_j, with positions $\vec{r}_j$ ordered with increasing distance to the center of the expansion, at the position of the i particle is:

$$\phi(\vec{r}_i) = -G \sum_{j>i} \frac{m_j}{p_j} - \frac{G}{p_i} \sum_{j<i} m_j \tag{1}$$

where G is Newton's gravitational constant and $p_j = (r_j^2 + \varepsilon^2)^{1/2}$, i.e., the radius "softened" by the softening parameter ε. The corresponding force is, thus:

$$\vec{F}(\vec{r}_i) = -G \frac{\vec{r}_i}{p_i^3} \sum_{j<i} m_j \tag{2}$$

Here we see clearly the intrinsic discontinuity that affects the computation of forces in this approach: if we continuously increase r_i, there will be a continuous change due to the change of $\vec{r}_i$ and p_i, but also discontinuous jumps every time r_i exceeds a particular r_j and the corresponding m_j has to be included in the summation. Instead of taking infinitesimally thin shells (represented by White, 1983, with Dirac delta functions), we

took shells extending in radius from $r_j - \delta$ to $r_j + \delta$ and with a volume density decreasing with r^{-2} within that region. The potential adopts now the formidable form:

$$\phi(\vec{r}_i) = -G \left\{ \frac{1}{p_i} \sum_{r_j < s_i} m_j + \sum_{r_j > a_i} \frac{m_j}{2\delta} \ln \left[\frac{a_j + (a_j^2 + \varepsilon^2)^{1/2}}{s_j + (s_j^2 + \varepsilon^2)^{1/2}} \right] + \right.$$
$$\left. + \sum_{r_j > s_i}^{r_j < a_i} \frac{m_j}{2\delta} \left[\frac{(r_i - s_j)}{p_i} + \ln \left(\frac{a_j + (a_j^2 + \varepsilon^2)^{1/2}}{r_i + (r_i^2 + \varepsilon^2)^{1/2}} \right) \right] \right\} \qquad (3)$$

where $a_j = r_j + \delta$ and $s_j = r_j - \delta$. The force due to this potential is then:

$$\vec{F}(\vec{r}_i) = -G \left[\frac{1}{p_i^3} \sum_{r_j < s_i} + \sum_{r_j > s_i}^{r_j < a_i} \frac{m_j}{2\delta} \frac{r_i - s_j}{p_i^3} \right] \vec{r}_i \qquad (4)$$

which has no discontinuities. Alternatively, the exquisite simplicity of the White - Aguilar method, which makes it extremely fast, is lost and the code using equations 3 and 4 runs 10 to 100 times slower (depending on the number of bodies and the parameters chosen) than the original one that uses equations 1 and 2. Therefore, although our version offers an useful check of the effect of the discontinuities it is certainly not adequate for frequent use!

Following Hernquist and Barnes (1990) we centered our analysis on the binding energies (per unit mass) of the individual particles. We compared the differences, ΔE_j, between the energies of particle j at two different times (the initial one characterized by the energy E_j^i) and computed, as Hernquist and Barnes did, the mean change, $\overline{\Delta E}$, the standard deviation, σ, and the mean absolute deviation, A.

We investigated the evolution of a system of 4998 bodies initially distributed following an isotropic King model (King, 1966) with central potential $\Psi(0) = 3\sigma^2$. The gravitational constant and the total mass were taken as unity and the total energy as -0.5; the crossing time is, thus, also equal to 1. We followed the evolution of the systems over 4 time units to insure that equilibrium had been reached and, then, for another 4 time units to derive the changes in the binding energies of the particles; in other words, we compared the binding energies at times 4 and 8. The timestep was usually taken as 0.125 and the total energy was conserved to better than 0.5% in all cases; runs with shorter time steps showed little or no improvement.

Table 1 presents our results: the first column identifies the run; the second column gives the value of the parameter ε of equations (1) through (4); run 1 employed the code based in the usual equations (1) and (2), and it is denoted by the zero value of the parameter δ in the third column, while the non-zero δ value of run 2 is the one used with equations (3) and (4) for that run; the last three columns provide the results obtained for the parameters that measure the energy changes. The results of run 1 are essentially equal to those obtained by Hernquist and Barnes (1990) in their own run 3. The fact that our run 2 (using a code that had eliminated the force discontinuity) did

not improve the results, shows that the discontinuities have no ill effect on the results; on the contrary, the marginally poorer energy conservation of the "improved" code might be due to the longer computations it demands and the corresponding accumulation of round-off errors.

Table 1. Fractional changes in particle binding energies.

Run	ϵ	δ	$\overline{\Delta E}$	σ	A
1	0.05	0.00	0.00275	0.0501	0.0396
2	0.05	0.05	0.00108	0.0539	0.0428

ACKNOWLEDGMENTS. We are very grateful to S. D. Abal de Rocha, L. A. Aguilar, L. P. Bassino, D. D. Carpintero, M. Rabolli and F. C. Wachlin for their assistance and help. This work was supported by grants from the Fundación Antorchas and from the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

REFERENCES

Aguilar, L. A., & Merritt, D. 1990, *Astrophys. J.* **354**, 33.
Aguilar, L. A., & White, S. D. M. 1985.*Astrophys. J.* **295**, 374.
Hernquist, L., & Barnes, J. E. 1990. *Astrophys. J.* **349**, 562.
King, I. R. 1966.*Astron. J.* **71**, 64.
White, S. D. M. 1983.*Astron. J.* **274**, 53.