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ON RAYLEIGH WAVES IN ANISOTROPIC MEDIA

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ABSTRAC

The propagation of elastic body waves in anisotropic media has been studied extensively in the past century, and some attention has been also devoted to a kind of surface waves of the type, but not really Rayleigh waves. The used methods used were cumbersome; therefore only particular cases were considered. Here we consider true Rayleigh waves defined as in the case of isotropic media, and give an exact easy method for studying them in any medium. Applying it to some orthorhombic, transversely, isotropic and thorough isotropic medium we found that the Rayleigh wave are possible only in the last of them.
Keywords: Rayleigh waves - anisotropic media.

RESUMEN

La propagación en medios anisótropos de ondas elásticas internas ha sido estudiada extensamente en el siglo pasado y alguna atención se le ha prestado también a ciertas ondas superficiales parecidas pero no propiamente ondas Rayleigh. El método que se usó para ello es engorroso, por lo que se trató solo casos particulares. En el presente trabajo consideramos ondas Rayleigh definidas del mismo modo que cuando se las estudia en medios isótropos. Damos un método sencillo y exacto aplicable a esas ondas en cualquier medio anisótropo ó no. Cálculos efectuados con él relativos a un medio particular ortorrómbico, a otro transversalmente isótropo, y a uno isótropo, mostraron posibilidades de ondas Rayleigh solo en el medio isótropo.
Palabras claves: Ondas Rayleigh - medio anisótropo.

INTRODUCTION

Research on surface waves in anisotropic media goes back to the first decades of the past century. Rudzki (1912) started studying them in transversely isotropic media. Stoneley (1955) did it later in cubic media; also in orthorhombic (1963). The way followed by these authors is cumbersome; they restraint therefore the considerations only to particular cases. Furthermore the waves they considered were not true Rayleigh waves. Another way, which is easy, simple and efficient, will be given here. It will be applied to the last waves, using their classical definitions (Bullen 1965, Gershanik 1996). It will be used to explore whether they are possible in some orthorhombic and some transversely isotropic medium. That is interesting since the upper mantle contains much olivine, which is orthorhombic and even being isotropic may become transversely isotropic by its own weight. To test the efficiency of the method we will apply it also to a thorough isotropic medium, known to allow the waves.

BASIC EXPRESSIONS

Since the Rayleigh wave are confined in vertical planes their displacements S is expressed by

$$s = \bar{e}_r\, u_r + \bar{e}_3\, u_3 \qquad (1)$$

$\vec{e}_r$ and $\vec{e}_3$ being versors respectively after an horizontal line r, and after the vertical direction x_3 ; u_r and u_3 being the components of S .By Helmholtz decomposition is

$$u_r = \frac{\partial\varphi}{\partial r} + \frac{\partial\psi}{\partial x_3} \qquad u_3 = \frac{\partial\varphi}{\partial x_3} - \frac{\partial\psi}{\partial r} \qquad (1)$$

We suppose, like in the isotropic case:

$$\begin{aligned} \varphi &= A\, e^{ikv_1} & \psi &= B\, e^{ikv_2} \\ v_1 &= ipx_3 + r - Vt & v_2 &= iqx_3 + r - Vt \end{aligned} \qquad (2)$$

p y q attenuation parameters.

In an elastic medium the displacement satisfy the equations of equilibrium

$$\frac{\partial Thi}{\partial x_i} - \rho \frac{\partial^2 u_r}{\partial t^2} = 0 \qquad (h,i = 1,2,3) \qquad (3)$$

ρ density, T_{hi} elements of the tensor of tensions.

$$\begin{aligned} T_{11} &= \frac{1}{2}\frac{\partial W}{\partial \xi_1} & T_{22} &= \frac{1}{2}\frac{\partial W}{\partial \xi_2} & T_{33} &= \frac{1}{2}\frac{\partial W}{\partial \xi_3} \\ T_{12} &= \frac{1}{2}\frac{\partial W}{\partial \xi_4} & T_{13} &= \frac{1}{2}\frac{\partial W}{\partial \xi_5} & T_{23} &= \frac{1}{2}\frac{\partial W}{\partial \xi_6} \end{aligned} \qquad (4)$$

$$T_{hi} = T_{ih}$$

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1/2 W elastic energy

$$\frac{1}{2}W=\frac{1}{2}\sum_{n,m}(2-\delta_{nm})\,C_{nm}\,\xi_n\,\xi_m \qquad (n,m=1,2...6) \tag{5}$$

δ_{nm} Kronecker function,

$$\xi_1=\frac{\partial u_1}{\partial x_1} \qquad \xi_2=\frac{\partial u_2}{\partial x_2} \qquad \xi_3=\frac{\partial u_3}{\partial x_3}$$
$$\xi_4=\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1} \qquad \xi_5=\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1} \qquad \xi_6=\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2} \tag{6}$$

u_h *(h= 1,2,3)* components of S in the system $(x_1x_2x_3)$ *s* must satisfy the border conditions

$$T_{r3}=T_{t3}=T_{33}=0 \tag{7}$$

when X_3= 0.
Being θ the angle between the planes $(x_3\ r_3)$ and $(x_1\ x_3)$

$$T_{r3}=l\ T_{13}+m\ T_{23}$$
$$T_{p3}=m\ T_{13}+l\ T_{23} \tag{8}$$
$$l=\cos\theta \qquad m=\sin\theta$$

T_{r3} tensor after r, T_{p3} tensor perpendicular to r.
From (4) and (5) we have

$$T_{31}=\sum_n C_{n5}\,\xi_n \quad T_{32}=\sum_n C_{n6}\,\xi_n \quad T_{33}=\sum_n C_{n3}\,\xi_n \quad (n=1,2...6) \tag{9}$$

Moreover it is

$$u_1=l\ u_r \qquad u_2=m\ u_r \qquad r=l\ x_1+m\ x_2$$

then;

$$\frac{\partial u_1}{\partial x_1}=l^2\ \frac{\partial u_r}{\partial r} \qquad \frac{\partial u_2}{\partial x_1}=lm\ \frac{\partial u_r}{\partial r} \qquad \frac{\partial u_3}{\partial x_1}=l\ \frac{\partial u_5}{\partial r}$$
$$\frac{\partial u_1}{\partial x_2}=lm\ \frac{\partial u_r}{\partial r} \qquad \frac{\partial u_2}{\partial x_2}=m^2\ \frac{\partial u_r}{\partial r} \qquad \frac{\partial u_3}{\partial x_2}=m\ \frac{\partial u_3}{\partial r} \tag{10}$$
$$\frac{\partial u_1}{\partial x_3}=l\ \frac{\partial u_r}{\partial x_3} \qquad \frac{\partial u_2}{\partial x_3}=m\ \frac{\partial u_r}{\partial x_3}$$

From (2) and (1) is

$$u_r=ik\left(A\ e^{iv_1}+Biq\ e^{iv_3}\right) \qquad u_3=ik\left(Aip\ e^{iv_1}-B\ e^{iv_2}\right) \tag{11}$$

and when $X_3=0$.

$$u_r=ik\ e^{iv}\left(A+iBq\right) \qquad u_3=ik\ e^{iv}\left(iAp-B\right) \qquad v=r-Vt \tag{12}$$

Equations (7) result in a system of three homogeneous equations in A and B, with complex coefficients f_{ij}, functions of *p, q, l,* when introducing (8) (9) (6) and (11) in them. The same happens to equations (3), but with coefficients $F_{ij}\pm\rho V^2$, F_{ij}, also being functions of *p, q, l.*

Joining separately real and imaginary terms a system of twelve equations in the unknown *p, q, l A/B* $(\rho\ V^2)$, is formed that can be solved by the least square method. If the system has a solution, the minimum of the sum E of the errors will be null.

PARTICULAR MEDIA

The problem considered is much simplified when the number of elastic parameters of the media is small. Such is the case of the orthorhombic, the transversely isotropic, the cubic and the isotropic media. Following, we refer to thema) Orthorhombic media.
The expression of the elastic energy is in them:

$$\frac{1}{2}W=\frac{1}{2}C_{11}\xi_1^2+C_{12}\xi_1\xi_2+C_{13}\xi_1\xi_3+\frac{1}{2}C_{22}\xi_2^2+C_{23}\xi_2\xi_3+\frac{1}{2}C_{33}\xi_3^2+..$$
$$\frac{1}{2}C_{44}\xi_4^2+\frac{1}{2}C_{55}\xi_5^2+\frac{1}{2}C_{66}\xi_6^2 \tag{13}$$

Consequently the equations of elastic equilibrium are

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2}+C_{12}\frac{\partial^2 u_2}{\partial x_1\partial x_2}+C_{13}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+C_{44}\left(\frac{\partial^2 u_1}{\partial x_2^2}+\frac{\partial^2 u_2}{\partial x_1\partial x_2}\right)+...$$
$$C_{55}\left(\frac{\partial^2 u_1}{\partial x_3^2}+\frac{\partial^2 u_3}{\partial x_1\partial x_3}\right)-\rho\frac{\partial^2 u_1}{\partial t^2}=0$$
$$C_{44}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_2}+\frac{\partial^2 u_2}{\partial x_1^2}\right)+C_{12}\frac{\partial^2 u_1}{\partial x_2\partial x_1}+C_{22}\frac{\partial^2 u_2}{\partial x_2^2}+C_{23}\frac{\partial^2 u_3}{\partial x_3\partial x_2}+... \tag{14}$$
$$C_{66}\left(\frac{\partial^2 u_2}{\partial x_3^2}+\frac{\partial^2 u_3}{\partial x_2\partial x_3}\right)-\rho\frac{\partial^2 u_2}{\partial t^2}=0$$
$$C_{55}\left(\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\frac{\partial^2 u_3}{\partial x_1^2}\right)+C_{66}\left(\frac{\partial^2 u_2}{\partial x_3\partial x_2}+\frac{\partial^2 u_3}{\partial x_2^2}\right)+C_{13}\frac{\partial^2 u_1}{\partial x_1\partial x_3}+...$$
$$C_{23}\frac{\partial^2 u_2}{\partial x_2\partial x_3}+C_{33}\frac{\partial^2 u_3}{\partial x_3^2}-\rho\frac{\partial^2 u_3}{\partial t^2}=0$$

and the equation of border condition:

$$T_{r3}=l\ C_{55}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)+m\ C_{66}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)=0$$

$$T_{r3}=l\ C_{55}\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right)+m\ C_{66}\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right)=0$$

$$T_{33}=\ C_{13}\frac{\partial u_1}{\partial x_3}+\ C_{23}\frac{\partial u_2}{\partial x_2}+\ C_{33}\frac{\partial u_3}{\partial x_3}=0$$

By (10) these equations become

$$l^2\left(C_{55}+m^2\ C_{66}\right)\left(\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial x_r}\right)=0$$

$$lm\left(C_{55}+C_{66}\right)\left(\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial x_r}\right)=0$$

$$\left(C_{13}\ l^2+C_{23}\ m^2\right)\frac{\partial u_r}{\partial r}+C_{33}\frac{\partial u_3}{\partial x_3}=0$$

The first two are equivalent consequently the set reduces to

$$\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial r}=0$$
$$\left(l^2\ C_{13}+m^2\ C_{23}\right)\frac{\partial u_r}{\partial r}+C_{33}\frac{\partial u_3}{\partial x_3}=0 \qquad (15)$$

and with (11)

$$2iAp-B\left(1+q^2\right)=0$$
$$A\left(l^2\ C_{13}+m^2\ C_{23}-C_{33}\ p^2\right)+iBq\left(l^2\ C_{13}+m^2\ C_{23}-C_{33}\right)=0 \qquad (16)$$

With (11) and (10) the equations (14) are

$$A\left[l^2C_{11}+m^2C_{12}-\left(C_{13}+2C_{55}\right)p^2+2m^2C_{44}-\rho V^2\right]+...$$
$$iqB\left[l^2C_{11}+m^2C_{12}-C_{13}+2m^2C_{44}-\left(q^2+1\right)C_{55}-\rho V^2\right]=0$$
$$A\left[m^2C_{22}+l^2C_{12}-\left(C_{23}+2C_{66}\right)p^2-2l^2C_{44}-\rho V^2\right]+...$$
$$iqB\left[m^2C_{22}+l^2C_{12}-C_{23}+2l^2C_{44}-\left(q^2+1\right)C_{66}-\rho V^2\right]=0 \qquad (17)$$
$$iAp\left[2\left(l^2C_{55}+m^2C_{66}\right)+C_{13}l^2+C_{23}m^2-C_{33}p^2-\rho V^2\right]-...$$
$$B\left[\left(l^2C_{55}+m^2C_{66}\right)\left(q^2+1\right)+\left(l^2C_{13}+m^2C_{23}-C_{33}\right)q-\rho V^2\right]=0$$

Equations (16) are satisfied if

$$\begin{vmatrix} 2ip & -\left(1+q^2\right) \\ l^2C_{13}+m^2C_{23}-C_{33}p^2 & iq\left(l^2C_{13}+m^2C_{23}-C_{33}\right)\end{vmatrix}=0 \qquad (18)$$

from this we obtain

$$q=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad (19)$$

where

$$a=l^2C_{13}+m^2C_{23}-C_{33}p^2$$
$$b=-2p\left(l^2C_{13}+m^2C_{23}-C_{33}\right)$$
$$c=a$$

Attenuation with $x_3>0$ happens if q is real and positive, i.e. if $b^2>4\ ac$ and the sign of $-b$ and a are the same.

b) Transversely isotropic, cubic and isotropic media. For them are valid the same formulae as in the orthorhombic media; only that in the case of the transversely isotropic media is

$$C_{11}=C_{22} \quad C_{13}=C_{23} \quad C_{55}=C_{66} \quad C_{12}=C_{11}-2C_{44}$$

in the case of the cubic and isotropic is

$$C_{11}=C_{22}=C_{33} \quad C_{12}=C_{13}=C_{23} \quad C_{44}=C_{55}=C_{66}$$

but in the case of isotropic is

$$C_{11}=\lambda+2\mu \quad C_2=\lambda \quad C_{44}=\mu$$

RESULTS WITH THE PROPOSED METHOD

By the reason explained in the introduction the proposed method was applied to establish whether the Rayleigh waves are possible in a medium rich in olivine defined after Verma (Stoneley, 1963) by

$$C_{11}=32.4 \quad C_{44}=6.67 \quad C_{12}=5.9$$
$$C_{22}=19.8 \quad C_{55}=8.10 \quad C_{13}=7.9$$
$$C_{33}=24.9 \quad C_{66}=7.93 \quad C_{23}=7.8$$

in beryllium which is transversely isotropic, defined after Love (1934) by

$$C_{11}=27.46 \quad C_{44}=8.83 \quad C_{13}=6.74$$
$$C_{33}=24.09 \quad C_{55}=6.66$$

and in the layer A of Bullen's model A (Bullen, 1965) supposed to be isotropic, defined by

$$C_{44}=6.3 \quad C_{12}=7.4$$

We restrained us to q and $p<1$, a condition to be fulfilled in the case of isotropic media.

Following results were obtained:

Orthorhombic medium							
l	p	q	η	ρV^2	E	F	Possibility
0.395	0.763	0.276	-0.705	7.02	3.511	$1.39\text{x}10^{-5}$	No

Transversely isotropic medium						
0.879	0.478	-0.699	8.79	52.28	4.67x10^{-4}	No

Isotropic medium						
0.855	0.386	-0.671	5.363	1.2x10^{-13}	2.38x10^{-6}	Yes

$E = \Sigma \varepsilon_i^2$
F=Gradient of E

The system is solved when F=0, and the Rayleigh waves are possible if E=0.

CONCLUSIONS

The results obtained show that the proposed method is efficient, and by the way that the Rayleigh waves are not allowed in the anisotropic media considered while p and q are less than 1.

Tie to the classical method for isotropic media

From the equations (14) is obtained in the case of isotropic media:

$$(\lambda+2\mu)\frac{\partial^2 u_r}{\partial r^2}+(\lambda+\mu)\frac{\partial^2 u_3}{\partial r\partial x_3}-\mu\frac{\partial^2 u_r}{\partial x_3^2}-\rho\frac{\partial^2 u_r}{\partial t^2}=0$$
$$(\lambda+\mu)\frac{\partial^2 u_3}{\partial x_3^2}+(\lambda+\mu)\frac{\partial^2 u_r}{\partial r\partial x_3}+\mu\frac{\partial^2 u_3}{\partial r^2}-\rho\frac{\partial^2 u_3}{\partial x_3}=0 \qquad (20)$$

and from (15)

$$\frac{\partial u_r}{\partial x_3}+\frac{\partial u_3}{\partial r}=0 \qquad \lambda\frac{\partial u_r}{\partial r}-(\lambda+2\mu)\frac{\partial u_3}{\partial x_3}=0$$

with (11) this is

$$2ipA-(q^2+1)B=0$$
$$\lambda-(\lambda+2\mu)p^2A-i\mu qB=0$$

Hence

$$\begin{vmatrix} 2ip & -(q^2+1) \\ \lambda-(\lambda+2\mu)p^2 & -2i\mu q \end{vmatrix}=0 \qquad (21)$$

and from (20)

$$A(F_{11}-\rho V^2)+B(F_{21}-\rho V^2)iq=0$$
$$ipA(F_{12}-\rho V^2)-B(F_{22}-\rho V^2)=0$$
$$F_{11}=(\lambda+2\mu)(1-p^2) \qquad F_{21}=\mu(1-q^2) \qquad (22)$$
$$F_{12}=(\lambda+2\mu)(1-p^2) \qquad F_{22}=-\mu(1-q^2)$$

(22) is satisfied if

$$\begin{vmatrix} F_{11}-\rho V^2 & (F_{21}-\rho V^2)iq \\ (F_{12}-\rho V^2)ip & F_{22}+\rho V^2 \end{vmatrix}=0 \qquad (23)$$

(22) shows that

$$F_{12}=F_{11} \qquad F_{22}=-F_{21}$$

We obtain therefore

$$(F_{11}-\rho V^2)(F_{22}-\rho V^2)(1-qp)=0$$

Being 1 - $qp \neq 0$ this will be satisfied if either of its factors is null. But then the other should also be null to satisfy (22). That means that if

$$(\lambda+2\mu)(1-p^2)-\rho V^2=0 \qquad (24)$$

also

$$\mu(1-q^2)-\rho V^2=0 \qquad (25)$$

i.e.

$$p^2=1-\frac{\rho V^2}{\lambda+2\mu} \qquad q^2=1-\frac{\rho V^2}{\mu} \qquad (26)$$

From (24) and (25) is obtained

$$\lambda-\beta^2(\lambda+2\mu)=-(1+q^2)\mu$$

Then instead of (21) we have the Rayleigh determinant

$$\begin{vmatrix} 2ip & -(1+q^2) \\ -(1+q^2)\mu & -2i\mu q \end{vmatrix}=0$$

Introducing (26) in it we obtain the well know cubic equation on $\rho V^2/\mu$

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REFERENCES

Bullen K. 1965. An Introduction to the theory of Seismology. Cambridge University Press, London.
Gershanik S. 1996. Sismología - Universidad Nacional de La Plata, Facultad de Ciencias

Astronómicas y Geofísicas - Argentina.
Love A.E.H. 1934. Mathematical theory of elasticity, Cambridge- University Press.
Rudzki M. F. 1912. Sur la propagation de una onda elastique superficielle dans un milieu transversalment isotropic. Bulletin del l'Academie des Sciences de Cracovia.
Stoneley R. 1955. The propagation of surface waves in cubic cristal. Proc. Royal Society London A-232, p447-457.
Stoneley R. 1963. Propagation of surface waves in an elastic orthorhombic symmetry. Geophysics J. R. Astron. Soc. 8 p176-186.