Symmetrical Basis for Faddeev Equations.

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Summary. — It is shown that the Faddeev equations can be treated in the frame of a three-body basis that takes into account the proper symmetries of the three-nucleon states.

The angular-momentum reduction of the Faddeev equations (1) can be carried out following different approaches. If the frame of the SU_3 representation of the three-particle states (2) is used, it is known that the three-body problem reduces to solving a coupled set of integral equations in only one variable. The analysis of this reduction for spinless particles has been carried out by LEE (3). In this orbital-momentum case an intermediate step through the introduction of the Omnès basis is explicitly used for the reduction in order to obtain a symmetrical description. In ref. (3) the way for including the spin and isospin of the particles is also given.

The main purpose of this note is to present the generalization of an alternative treatment of the orbital-angular-momentum analysis, for any value of l, of the three-body equations in the SU_3 representation, without the above-mentioned intermediate step. This generalization permits us to take into account

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(1) L. D. FADDEEV: *Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory* (Jerusalem, 1965).

(2) A. DRAGT: *Journ. Math. Phys.*, **6**, 533 (1965); Y. SIMONOV: *Sov. Journ. Nucl. Phys.*, **3**, 461 (1966); M. FABRE DE LA RIPELLE: preprint IPNO/TH 157 (1969).

(3) S. P. Y. LEE: *The three-nucleon problem*, Thesis, University of London (unpublished). We wish to thank S. P. Y. LEE for sending us this reference.

the internal degrees of freedom of the particles preserving the symmetry properties of the basis.

Let us start constructing a spin basis for three particles, symmetrical under the interchange of the particle indices. Consider first the usual coupled basis

$$(1)\qquad |SM_S;\, S_{12}\rangle \equiv |(S_1S_2)S_{12}SM_S\rangle = \sum_{\{m\}} |S_1m_1S_2m_2S_3m_3\rangle \cdot \\ \cdot \langle S_1m_1S_2m_2|S_1S_2S_{12}m_{12}\rangle \cdot \langle S_{12}m_{12}S_3m_3|S_{12}S_3SM_S\rangle\,,$$

where the notation used is self-evident.

Our first step is to impose the spin states (1) to be invariant under the action of the symmetric group of three objects, S_3. We can build up the elements of this group from the transpositions $P_{12} \equiv (12)(3)$ and the cyclic permutation $C \equiv (123)$. The operator P_{12} acting on the basis (1) changes it in a phase factor $(-)^{S_1+S_2-S_{12}}$, *i.e.* P_{12} is diagonal in this basis. On the contrary, the matrix elements of the cyclic permutation C are

$$(2)\qquad \langle SM_S;\, S_{12}|C|SM_S;\, S'_{12}\rangle = [(2S_{12}+1)(2S'_{12}+1)]^{\frac{1}{2}} \cdot \\ \cdot (-)^{S_1+S_{12}-S}\, W(S_2S_1SS_3;\, S'_{12}S_{12})\,,$$

W being a Racah coefficient.

Since C is a unitary operator, it can be diagonalized through a unitary transformation U. Thus, we take for our spin basis the eigenvectors of matrix (2), *i.e.*

$$(3)\qquad |SM_S;\, \nu\rangle = \sum_{\Sigma} U^{\nu}_{\Sigma} |SM_S;\, \Sigma\rangle\,,$$

where ν stands for a set of quantum numbers that labels the basis unambiguously. For the most interesting case of three spin-$\frac{1}{2}$ particles coupled to $S=\frac{1}{2}$, the symmetrical basis is determined by

$$(4)\qquad \begin{cases} |SM_S;\, \nu=1\rangle \;\;\;= 2^{-\frac{1}{2}}(|SM_S;\; \Sigma=0\rangle + i|SM_S;\, \Sigma=1\rangle)\,, \\ |SM_S;\, \nu=-1\rangle = 2^{-\frac{1}{2}}(i|SM_S;\; \Sigma=0\rangle + |SM_S;\; \Sigma=1\rangle) \end{cases}$$

with the vectors satisfying (*)

$$(5a)\qquad P_{12}|SM_S;\nu\rangle = i\nu|SM_S;\, -\nu\rangle\,,$$

$$(5b)\qquad C|SM_S;\, \nu\rangle \;= \exp\left[\frac{2\pi i\nu}{3}\right]|SM_S;\, \nu\rangle\,.$$

(*) Clearly the complete symmetry is obtained when the usual procedure of Young tableaux is applied to the basis. We do not need it explicitly in our calculations.

It will not escape the reader's attention that, in the case of higher spin, an additional set of operators will be needed in order to break up all the possible degeneracies ([4]).

It is clear that a similar analysis can be carried out in connection with the isospin. We call the resulting isospin basis $|II_z;\, \varrho\rangle$.

Having at our disposal the proper spin basis, our next step is to couple it with the orbital SU_3 basis obtained from Dragt's ([2]) results, to get the total-angular-momentum basis. Formally

$$(6)\qquad |P^2\lambda^\alpha\nu;\, LSJM\rangle = \sum_{M_L, M_S} \langle LM_L SM_S|LSJM\rangle |P^2\lambda^\alpha,\, LM_L\rangle |SM_S;\, \nu\rangle\,,$$

where $\lambda^\alpha \equiv (\lambda, \mu, \omega)$ are the eigenvalues of the Casimir operators of SU_3 and the additional indices required to break down all the degeneracies; L is the orbital angular momentum and $P^2 \equiv \bar{q}_1^2 + \bar{q}_2^2 + \bar{q}_3^2 = \bar{p}^{(1)2} + \bar{p}^{(2)2}$ the kinetic energy in the centre-of-momentum frame of reference.

When projected in momentum space, the three-body wave function is written in terms of P^2, the Dalitz-Fabri co-ordinates and the Euler angles. These variables describe the size, shape and orientation of the triangle defined by the tips of the momentum vectors of the particles (henceforth called the momentum triangle). The body-fixed reference frame is chosen with the z-axis being normal to the momentum triangle plane, and the x- and y-axis lying along the principal directions of the inertia tensor of the momentum triangle ([2]).

The transformation brackets from the momentum basis to the SU_3 basis with spin can be written as follows:

$$(7)\qquad \langle \bar{p}^{(1)}\bar{p}^{(2)};\, S_\beta m_\beta|P^2\lambda^\alpha\nu;\, JM\rangle = A\,\frac{\delta\left(P^2 - \sum\limits_{i=1}^{2}\bar{p}^{2(i)}\right)}{P^2}\sum_K G^{JK}_{\lambda^\alpha\nu}(\varrho\varphi)\mathscr{D}^{*J}_{MK}(R)\,,$$

where explicit use of the group properties of rotation matrices has been done and where we have defined

$$(8)\qquad G^{JK}_{\lambda^\alpha\nu}(\varrho\varphi) = \sum_{K_L K_S} g^{LK_L}_{\lambda^\alpha}(\varrho\varphi)\langle S_\beta m_\beta|SK_S;\, \nu\rangle\, \langle LK_L SK_S|LSJK\rangle\,,$$

$$(9)\qquad \langle S_\beta m_\beta|SK_S;\, \nu\rangle = \sum_\Sigma \langle S_\beta m_\beta|(S_1S_2)\Sigma SK_S\rangle\,.$$

In eq. (8), $g^{LK_L}_{\lambda^\alpha}(\varrho, \varphi)$ stands for the (ϱ, φ)-component of the normalized « angular » eigenfunctions introduced in ref. ([2]). The normalization factor A is chosen such that

$$(10)\qquad \langle P^2\lambda^\alpha\nu,\, LSJM|Q^2\lambda^{\alpha'}\nu';\, L'S'J'M'\rangle = \delta(P^2 - Q^2)\,\delta_{\lambda^\alpha\lambda^{\alpha'}}\delta_{\nu\nu'}\delta_{LL'}\delta_{SS'}\delta_{JJ'}\delta_{MM'}\,.$$

([4]) J. M. LEVY-LEBLOND and M. LEVY-NAHAS: *Journ. Math. Phys.*, **6**, 1372 (1965).

We shall need later a rotated fixed-body system, such that its z-axis lies along the momentum $\boldsymbol{p}^{(2)}$ (which is proportional to $\boldsymbol{q}_3$). In this case the wave function can be written as

$$(11)\qquad \langle \overline{p}^{(1)}\overline{p}^{(2)};\, S_\beta m_\beta | P^2\lambda^\alpha\nu,\, JM\rangle = A\,\frac{\delta\left(P^2-\sum_i \overline{p}^{2(i)}\right)}{P^2}\sum_K G^{JK}_{\lambda^\alpha\nu}(\varrho\varphi)\sum_{K'}\mathscr{D}^{*J}_{MK'}(\hat{\xi})\mathscr{D}^{*J}_{K'K}(R),$$

$\hat{\xi} = (\xi^{(2)}, \pi/2, 0)$ being the necessary rotation. Certainly, similar rotations on the other momentum vectors can be performed if needed.

Having obtained the appropriate symmetrical basis for three particles with spin, we shall show how the Faddeev equations look in terms of the angular-momentum reduction we have just presented.

We shall restrict ourselves to the case of identical particles in order to simplify the algebraic manipulations. The generalization to nonidentical particles follows the same steps in a straighforward manner. In the present case, the Faddeev equations reduce to only one operator equation

$$(12)\qquad \hat{T}(z) = 3\hat{t}(z) + 2\hat{t}(z)G_0(z)\,\hat{T}(z)\,,$$

where $\hat{T}(z)$ is the total three-body scattering operator, $G_0(z)$ is the resolvent and $\hat{t}(z)$ is the two-body scattering operator. When matrix elements of eq. (12) are considered, the main problem is the evaluation of the quantity

$$(13)\qquad O^{\nu\nu'}_{\lambda^\alpha\lambda^{\alpha'}} = \langle P^2\lambda^\alpha\nu;\, JM|\hat{t}(z)|Q^2\lambda^{\alpha'}\nu';\, J'M'\rangle\,.$$

Now we use the well-known expression for the two-body $\hat{t}$-matrix in the three-particle space

$$(14)\qquad \langle \boldsymbol{p}^{(1)}\boldsymbol{p}^{(2)};\, S_\beta m_\beta|\hat{t}(z)|\boldsymbol{q}^{(1)}\boldsymbol{q}^{(2)};\, S_\beta m'_\beta\rangle =$$
$$= \delta(\boldsymbol{p}^{(2)}-\boldsymbol{q}^{(2)})\langle S_3 m_3|S_3 m'_3\rangle\langle \boldsymbol{p}^{(1)};\, S_\alpha m_\alpha|t(z-\overline{p}^{2(2)})|\boldsymbol{q}^{(1)};\, S_\alpha m'_\alpha\rangle \qquad (\alpha = 1, 2)\,.$$

If we define

$$\mathrm{d}\Delta = \varrho\,\mathrm{d}\varrho\,\mathrm{d}\varphi\,,\qquad \mathrm{d}R = \sin\beta\,\mathrm{d}\beta\,\mathrm{d}\alpha\,\mathrm{d}\gamma\,,\qquad \{K\} = \{K, K', \overline{K}, \overline{K}'\}\,,$$

and perform the integrations over P_1^2 and Q_1^2, we obtain

$$(15)\qquad O^{\nu\nu'}_{\lambda^\alpha\lambda^{\alpha'}} = \sum_{m_\alpha, m'_\alpha}\left(\frac{A}{8}\cdot\frac{A^*}{8}\right)\frac{P^2Q^2}{4}\int_\Omega \mathrm{d}\Delta\,\mathrm{d}R\int_{\Omega'}\mathrm{d}\Delta'\,\mathrm{d}R'\,\delta(\overline{p}^{(2)}-\overline{q}^{(2)})\langle S_3m_3|S_3m'_3\rangle\cdot$$
$$\cdot\sum_{\{K\}} G^{*JK}_{\lambda^\alpha,\nu}(\varrho,\varphi)G^{J'K'}_{\lambda^{\alpha'},\nu'}(\varrho',\varphi')\mathscr{D}^J_{M\overline{K}}(R)\,\mathscr{D}^{*J'}_{M'\overline{K}'}(R')\mathscr{D}^J_{\overline{K}K}(\hat{\xi})\mathscr{D}^{*J'}_{\overline{K},K'}(\hat{\xi}')\cdot$$
$$\cdot\langle\overline{p}^{(1)};\, S_\alpha m_\alpha|t(z' = z-p^{(2)})|\overline{q}^{(1)};\, S_\alpha m'_\alpha\rangle\,,$$

where we have used explicitly the transformation bracket (11).

The angles R and R' and the kets $|S_3 m_3\rangle$ and $|S_3 m_3'\rangle$ in eq. (15) are related by means of a rotation of the body-fixed system through the two-body scattering angle. Introducing now the partial-wave decomposition

$$(16)\qquad \langle\bar{p}^{(1)};\, S_\alpha m_\alpha|t(z')|\bar{q}^{(1)};\, S_\alpha m_\alpha'\rangle = \sum_{\{\Omega\}} Y_{lm}(\hat{p}^{(1)})\, Y^*_{l'm'}(\hat{q}^{(1)})\mathscr{T}^{S_{12},j}_{ll'}(p^{(1)}, q^{(1)}; z')\cdot$$
$$\cdot\langle S_1 S_2 S_{12} m_{12}|S_1 m_1 S_2 m_2\rangle\cdot\langle S_1 S_2 S_{12} m_{12}'|S_1 m_1' S_2 m_2'\rangle\cdot$$
$$\cdot\langle lm\, S_{12} m_{12}|l S_{12} j j_z\rangle\,\langle l' S_{12} j j_z|l' m' S_{12} m_{12}'\rangle$$

with $\{\Omega\} = \{l, l', m, m', S_{12}, m_{12}, j\}$, we can carry out the integration over the Euler angles, showing explicitly the conservation of the total angular momentum. We then obtain

$$(17)\qquad O^{\nu\nu'}_{\lambda\alpha,\lambda\alpha'} = \left(\frac{A}{8}\cdot\frac{A^*}{8}\right)\frac{2\pi\cdot 8\pi^2\cdot P^2 Q^2}{4(2J+1)}\,\delta_{JJ'}\,\delta_{MM'}\int_0^1 \varrho\,\mathrm{d}\varrho\int_0^{2\pi}\mathrm{d}\varphi\int_0^1 \varrho'\,\mathrm{d}\varrho'\int_0^{2\pi}\mathrm{d}\varphi'\cdot$$
$$\cdot 4\sqrt{2}\,\frac{\delta[P^2(1+\varrho\cos\varphi)-Q^2(1+\varrho'\cos\varphi')]}{P(1+\varrho\cos\varphi)^{\frac{1}{2}}}\sum_{l,l',S_{12},j}\mathscr{T}^{S_{12},j}_{l,l'}(p^{(1)}, q^{(1)}; z')\cdot$$
$$\cdot\, U^{*\nu}_{S_{12}}\, U^{\nu'}_{S_{12}}\sum_{m,m'} Y_{l,m}(\beta^{(1)};\, 0)\, Y^*_{l',m'}(\beta'^{(1)};\, 0)\sum_{K_L,K_S,\bar{K}',K'} g^{*LK_L}_{\lambda\alpha}(\varrho, \varphi)\, g^{LK_L'}_{\lambda\alpha'}(\varrho', \varphi')\cdot$$
$$\cdot\,\Xi\cdot\mathscr{D}^J_{K'+m-m',K_L+K_S}(\xi;\, \pi/2;\, 0)\,\mathscr{D}^{*J}_{\bar{K}',K'}(\xi';\, \pi/2;\, 0)\,,$$

where $K_L' = K' - K_S + m' - m$ and

$$\Xi = \langle S_{12}(K_S - K' + m'),\, S_3(K' - m')|S_{12} S_3 S K_S\rangle\cdot$$
$$\cdot\langle S_{12} S_3 S(K_S + m - m')|S_{12}(K_S - K' + m),\, S_3(K' - m')\rangle\cdot$$
$$\cdot\langle lm, S_{12}(K_S - K' + m')|l S_{12} j(K_S - K' + m + m')\rangle\cdot\langle LK_L S K_S|LSJ(K_L + K_S)\rangle\cdot$$
$$\cdot\langle l' S_{12} j(K_S - K' + m + m')|l' m',\, S_{12}(m + K_S - K')\rangle\cdot$$
$$\cdot\langle LSJK'|L(K' - K_S + m' - m),\, S(K_S + m - m')\rangle\,.$$

The delta-function that appears in the above equations is eliminated when the integration over the Dalitz-Fabri co-ordinates is performed. In order to show this explicitly and, at the same time, recast eq. (17) in a more symmetric way, we introduce Cartesian co-ordinates in the Dalitz plot through

$$x = \varrho\cos\varphi\,, \qquad y = \varrho\sin\varphi\,.$$

Thus the final expression for the matrix element (13) takes the form

$$(18)\qquad O^{\nu\nu'}_{\lambda\alpha\lambda\alpha'} = \frac{16\sqrt{2}\pi^3}{2J+1}\left(\frac{A\cdot A^*}{64}\right)\delta_{JJ'}\,\delta_{MM'}\,P^2 Q^2\sum_{S_{12}} U^{*\nu}_{S_{12}}\, U^{\nu'}_{T_{12}}\sum_{\substack{l,l',j\\ m,m'}}\;\sum_{\substack{K_L,K_S\\ K',\bar{K}'}}\Xi\cdot$$
$$\cdot\int_{-1}^{+1}\mathrm{d}x\int_{-1}^{+1}\mathrm{d}x'\,\frac{\delta[P^2(1+x)-Q^2(1+x')]}{[P^2(1+x)]^{\frac{1}{2}}}\,S^{l,\lambda\alpha}_J(x)\,S^{*l',\lambda\alpha'}_J(x')\,\mathscr{T}^{S_{12},j}_{l,l'}(p^{(1)}(x), q^{(1)}(x), z'(x)),$$

where we have introduced

$$S_J^{l,\lambda^\alpha}(x) = \int\limits_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \mathrm{d}y\, \mathscr{D}^J_{K'+m-m',K_L+K_S}(\xi^{(2)}, \pi/2;\, 0)\, g^{*LK_L}_{\lambda^\alpha}(x, y)\cdot Y_{lm}(\beta^{(1)};\, 0)\,, \tag{19}$$

$$S^{*l',\lambda^{\alpha'}}_{J'}(x') = \int\limits_{-\sqrt{1-x'^2}}^{+\sqrt{1-x'^2}} \mathrm{d}y'\, \mathscr{D}^{*J'}_{\bar{K}',K'}(\xi'^{(2)};\, \pi/2;\, 0)\, g^{LK'_L}_{\lambda^{\alpha'}}(x', y')\, Y^*_{l'm'}(\beta'^{(1)};\, 0)\,. \tag{20}$$

We note that in eq. (18) the mentioned δ-function only implies a restriction on the (x, x')-integration. It is also worth mentioning that the integration involved in the definitions of $S_J^{l\lambda^\alpha}(x)$ and $S^{*l'\alpha'}_{J'}(x')$, eqs. (19) and (20) respectively, can be performed in many cases analytically. This means a considerable simplification in computing the kernel and the inhomogeneous term of the Faddeev equation, that follows immediately from eq. (18). As we have already mentioned, the inclusion of the isospin does not offer any difficulties: we have to consider the direct product of our totally symmetric $|P^2\lambda^\alpha\nu;\, JM\rangle$ basis with the above-mentioned isospin basis.

The awe-inspiring set of channels that could be present in eq. (18) is not a serious computational trouble, at least for the case of identical particles. Indeed, the number of channels is considerably reduced by the requirement of antisymmetry of the three-nucleon states. This last condition requires

$$P_{12}|\lambda^\alpha, \nu, \varrho\rangle_A = (-)|\lambda^\alpha, \nu, \varrho\rangle_A \tag{21}$$

and

$$C|\lambda^\alpha, \nu, \varrho\rangle_A = |\lambda^\alpha, \nu, \varrho\rangle_A\,. \tag{22}$$

From eq. (5b) and relation (22) we can obtain the restriction

$$\mu + \nu + \varrho \equiv 0 \qquad (\text{mod } 3) \tag{23}$$

on the symmetric quantum numbers defining our state. This means a reduction of the number of channels appearing in eq. (18). At the same time, condition (21) together with (5a) implies, for the spin and isospin case,

$$|\lambda\mu\nu\varrho\rangle_A = 2^{-\frac{1}{2}}[|\lambda\mu\nu\varrho\rangle + \nu\varrho|\lambda - \mu - \nu - \varrho\rangle]\,, \tag{24}$$

which halves the number of physical channels to be considered. Similar restrictions hold for the spin (or isospin)-zero case.

If we are interested in the calculation of three-nucleon low-energy parameters, we shall find that only small values of λ are important. This means that the rate of convergence found for this symmetrical basis is extremely good ([5]).

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([5]) S. HOCHBERG, S. P. Y. LEE and R. SIBBEL: *Nucl. Phys.*, **28** B, 308 (1971); J. L. ALESSANDRINI: Thesis, Universidad de La Plata (1971) (unpublished).

● RIASSUNTO (*)

Si dimostra che le equazioni di Faddeev possono essere trattate nel quadro di una base di tre corpi che tiene conto delle simmetrie proprie degli stati di tre nucleoni.

(*) *Traduzione a cura della Redazione.*

Резюме не представлено.