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The logic of equilibrium and abelian lattice ordered groups

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Abstract. We introduce a deductive system $\mathcal{B}al$ which models the logic of balance of opposing forces or of balance between conflicting evidence or influences. "Truth values" are interpreted as deviations from a state of equilibrium, so in this sense, the theorems of $\mathcal{B}al$ are to be interpreted as balanced statements, for which reason there is only one distinguished truth value, namely the one that represents equilibrium.

The main results are that the system $\mathcal{B}al$ is algebraizable in the sense of [5] and its equivalent algebraic semantics BAL is definitionally equivalent to the variety of abelian lattice ordered groups, that is, the categories of the algebras in BAL and of ℓ–groups are *isomorphic* (see [10], Ch.4, 4). We also prove the deduction theorem for $\mathcal{B}al$ and we study different kinds of semantic consequence associated to $\mathcal{B}al$. Finally, we prove the co-NP-completeness of the tautology problem of $\mathcal{B}al$.

1. Introduction

In this paper we want to model some aspects of arguments in which conflicting pieces of evidence, such as those that appear in a police investigation, credit records, political influences, and even in scientific research, are confronted. In all these cases, the pieces of evidence can be assigned a degree of relevance to the case at hand. It appears natural then to assign positive degrees to some bits of evidence and negative degrees to those that oppose them. Needless to say, the positiveness is arbitrary, it is simply a fixed privileged direction.

Evidence can pile up, but it is obvious that no amount of it will prove that some fact is either true or false, so we should not talk about truth or falsehood, instead our statements talk about potentially unbounded evidence for or against some fact. In this scenario, it is natural to pay attention to those statements which do not provide evidence that will incline us in any direction. This is precisely what our system models, our theorems are the balanced statements, those that do not favor any fact. So it is only natural that we have a single distinguished "truth" value, that represents equilibrium, and an unbounded set of truth values to both sides of this point.

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In Section 2 we introduce the system $\mathcal{B}al$and prove several technical results. In Section 3, we prove that $\mathcal{B}al$is algebraizable in the sense of [5]. In Section 4 we characterize the equivalent algebraic semantics BAL and prove that it is definitionally equivalent to the variety of abelian lattice ordered groups. In other words, the categories of the algebras in BAL and of ℓ–groups are *isomorphic* (see [10], ch.4, 4). In Section 5 we prove the deduction theorem for $\mathcal{B}al$ and we study different kinds of semantic consequence associated to $\mathcal{B}al$. Section 6 is devoted to the proof of co-NP-completeness of the tautology problem of $\mathcal{B}al$.

2. The logic $\mathcal{B}al$

Throughout this paper we are going to use the following notations and concepts as defined in [5]. A *propositional language* $\mathcal{L}$ is a set of logical connectives, each one with an associated arity. The set of formulas or terms $\mathcal{F}_\mathcal{L}$ of $\mathcal{L}$ is built recursively as usual from the set of connectives and a denumerable set $Var = \{X_1, X_2, \ldots, X_n, \ldots\}$ of variables. A *deductive system* $\mathcal{S}$ over a propositional language $\mathcal{L}$ is determined by a set of axioms and inference rules. The *consequence relation on* $\mathcal{S}$ denoted by $\vdash_\mathcal{S}$ is the usual relation between a set Γ of formulas of $\mathcal{L}$ and a formula φ of $\mathcal{L}$ that holds when φ is obtained from Γ by means of a finite number of applications of deduction rules. We consider $\mathcal{S}$ to be the pair $\langle \mathcal{L}, \vdash_\mathcal{S} \rangle$. The *consequence operator* $Con_\mathcal{S}$ of $\mathcal{S}$ is the function from the power set of $\mathcal{F}_\mathcal{L}$ into $\mathcal{F}_\mathcal{L}$ assigning to each set Γ of formulas the set $Con_\mathcal{S}(\Gamma) = \{\varphi \in \mathcal{F}_\mathcal{L} : \Gamma \vdash_\mathcal{S} \varphi\}$. An $\mathcal{S}$–theory is a set Θ of formulas closed under $\vdash_\mathcal{S}$, that is $\Theta \vdash_\mathcal{S} \varphi \quad \Rightarrow \quad \varphi \in \Theta$.

2.1. Language

Let us consider the language $\mathcal{L}_{\mathcal{B}al} = \{\rightarrow, ^+\}$ of type $(2, 1)$. The formulas are built recursively as usual from a denumerable set of propositional variables.

Given our intuition that the formulas φ and ψ stand for pieces of evidence, and that asserting φ means "the evidence provided by φ is in equilibrium or balanced", we should give the corresponding intuitions about the meaning of the formulas $\varphi \rightarrow \psi$ and φ^+.

The statement $\varphi \rightarrow \psi$ should be interpreted as "the amount of evidence needed to go from the state described by φ to the state described by ψ". So to assert $\varphi \rightarrow \psi$ means that the evidence φ and the evidence ψ are balanced.

The statement φ^+ should be interpreted as "the positive part of the evidence described by statement φ". As we pointed out before, positive is an arbitrary direction.

2.2. Axioms

(B) $(\varphi \rightarrow \psi) \rightarrow ((\theta \rightarrow \varphi) \rightarrow (\theta \rightarrow \psi))$,

(C) $(\varphi \rightarrow (\psi \rightarrow \theta)) \rightarrow (\psi \rightarrow (\varphi \rightarrow \theta))$,

(N) $((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow \varphi$,

(P) $\varphi^{++} \rightarrow \varphi^+$,

(O) $((\psi \rightarrow \varphi)^+ \rightarrow (\varphi \rightarrow \psi)^+) \rightarrow (\varphi \rightarrow \psi)$.

2.3. *Inference rules*

$$\text{(MP)} \quad \frac{\varphi\ ,\ \varphi \to \psi}{\psi}, \qquad \text{(G)} \quad \frac{\varphi,\ \psi}{\varphi \to \psi},$$

$$\text{(PI)} \quad \frac{\varphi}{\varphi^+}, \qquad \text{(MI)} \quad \frac{(\varphi \to \psi)^+}{(\varphi^+ \to \psi^+)^+}.$$

We will say that φ is a syntactical consequence of the set Γ, denoted $\Gamma \vdash_{\mathcal{B}al} \varphi$ (or more briefly $\Gamma \vdash \varphi$) if there is a finite sequence of formulas $\psi_1, \ldots, \psi_k$ such that for each $i = 1, \ldots, k$, ψ_i is an instance of an axiom or else it belongs to Γ or is obtained from the previous formulas in the sequence by application of some inference rule.

2.4. *About the axioms and rules*

Given that we are trying to model a certain logic, the proposed deductive system should be justified or, at least, the intuitions behind certain axioms and rules should be explained.

Axiom (B) states that balance is not affected if equal information is added to existing pieces of evidence. (See axiom (N) below.) Axiom (C) states that the order in which evidence is provided does not affect the outcome. Axiom(N) is non–standard. It should be noted that this is the crucial *Axiom of Relativity* in Meyer & Slaney's Abelian logic $\mathcal{A}$, (axiom A9 in [11].) They interpret $\varphi \to \psi$ as a *negation of* φ *relative to* ψ, and write $\varphi \to \psi = \sim_\psi \varphi$, so the axiom reduces to

$$\sim_\psi \sim_\psi \varphi \to \varphi,$$

that is, the axiom of double (relative) negations. There is another interesting consequence. Axiom (B), prefixing, is equivalent with suffixing, $(\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))$, which if we interpret implication as relative negation, yields

$$(\varphi \to \psi) \to (\sim_\theta \psi \to \sim_\theta \varphi),$$

that is, (B) is a relativized version of contraposition.

Axiom (P) says that positive evidence is balanced with its positive part. Together with (O), this axiom implies that the positive part of some piece of evidence does not carry negative evidence. Axiom (O) is the most obscure of the axioms. It implies that all evidence is composed by a positive part and a negative part. We have chosen this form in order to stress the relation of $\mathcal{B}al$ and Chang's logic L^* of [6].

The inclusion of modus ponens (MP) is straightforward. If $\varphi \to \psi$ is balanced then no evidence is necessary to go from state φ to state ψ, so if φ is balanced, so must be ψ. Rule (G), sometimes known as Gödel's rule, is very simple. If φ and ψ are in equilibrium, then no evidence is needed to go from state φ to state ψ, that is, they are balanced. This rule is essential for our definition of negation. The rule

(G) is equivalent with the following derived rule $\varphi \vdash_{\mathcal{B}al} \neg\varphi$, where negation is a derived operation that we will define later on. In $\mathcal{B}al$this is not at all surprising since, if an assertion is balanced, its negation, the statement that carries the opposite information, should also be balanced. The rule (PI) is also straightforward, the positive part of a balanced statement is balanced. Finally, the rule (MI) looks very technical but is simple in spirit. Following the intuitions developed so far, it is not difficult to see that $\vdash_{\mathcal{B}al} (\varphi \to \psi)^+$, asserts that $\psi \leq \varphi$. The rule says that the operation $^+$ is monotone.

We observe that if we define $\Box\varphi = \varphi^+$, then some of the axioms and rules have a modal connotation.

$$
\begin{array}{ll}
\text{(P')} & \Box\Box\varphi \to \Box\varphi\,, \\
\text{(O')} & (\Box(\psi \to \varphi) \to \Box(\varphi \to \psi)\,) \to (\varphi \to \psi)\,, \\
\text{(PI')} & \frac{\varphi}{\Box\varphi}\,, \\
\text{(MI')} & \frac{\Box(\varphi\to\psi)}{\Box(\Box\varphi\to\Box\psi)}\,.
\end{array}
$$

We will not pursue this issue any further in this paper.

2.5. *Preliminary results*

Theorem 1. *The following hold in* $\mathcal{B}al$.

1. $\vdash \varphi \to \varphi\,,$ (I),
2. $\vdash (\varphi \to \psi) \to ((\psi \to \theta) \to (\varphi \to \theta))\,,$
3. $\vdash (\varphi \to \varphi) \to (\psi \to \psi)\,,$
4. $\varphi \vdash \varphi \to (\psi \to \psi)\,,$
5. $\vdash (\varphi \to (\psi \to \psi)) \to (\varphi \to (\theta \to \theta))\,,$
6. $\varphi \to \psi \vdash \psi \to \varphi\,,$
7. $\vdash \varphi \to ((\varphi \to (\psi \to \psi)) \to (\psi \to \psi))\,,$
8. $\vdash ((\psi \to \psi) \to \varphi) \to \varphi\,,$
9. $\varphi \to \psi\ ,\ u \to v\ \vdash\ (\varphi \to u) \to (\psi \to v)\,,$
10. $\varphi \to \psi \vdash \varphi^+ \to \psi^+.$

Proof. The proofs are very simple, we only indicate the main axioms and rules involved in each case. *1.*, the principle of identity (I), is obtained from (N) *2.*, suffixing, is obtained from (B) prefixing, using (C) and (MP). It should be noted that prefixing can be obtained from suffixing in a similar way. *3.* and *4.* follow from (I) and (G). *5.* follows from *3.*, (B) and (MP). *6.* follows from *2.*, *4.*, (N) and (MP). *7.* is a special case of the converse of axiom (N), it is immediate, and so are *8.* and *9.*. Item *10.* uses the “order rules and axiom”, (PI), (SI) and (O). □

3. From $\mathcal{B}al$ to the quasivariety BAL

In this section we study the class of algebras that belong to the equivalent algebraic semantics BAL for the deductive system $\mathcal{B}al$.

3.1. Background

Given a language $\mathcal{L}$, an $\mathcal{L}$–algebra is an algebra $\boldsymbol{A} = \langle A; (\alpha^A)\rangle$, where for each n–ary connective α of $\mathcal{L}$, α^A is an n–ary operation on A. For example, the set $\mathcal{F}_{\mathcal{L}}$ of all terms, together with the connectives as formal operations constitute an $\mathcal{L}$–algebra, called the *absolutely free* algebra over the set Var of propositional variables.

Let $\mathcal{K}$ be a class of algebras. We define the relation $\models_{\mathcal{K}}$ between a set of *equations* (pairs of terms (φ, ψ) written as $\varphi \approx \psi$), and an equation as follows. $\Gamma \models_{\mathcal{K}} \varphi \approx \psi$ if and only if for every $\boldsymbol{A} \in \mathcal{K}$ and for every interpretation of the variables of $\Gamma \cup \{\varphi \approx \psi\}$, if the equations in Γ hold in that interpretation, then $\varphi \approx \psi$ also holds in that interpretation. This relation is called *equational consequence determined by* $\mathcal{K}$. We observe that the operator $\models_{\mathcal{K}}$ is not necessarily finitary.

We say that the class of algebras $\mathcal{K}$ is an *equivalent algebraic semantics* for a deductive system $\mathcal{S}$ if $\vdash_{\mathcal{S}}$ can be interpreted in $\models_{\mathcal{K}}$ in the following way. There exists a finite set of equations $\delta_i(x) \approx \varepsilon_i(x)$ for $i < n$, the *defining equations*, and a finite set $\Delta_k(x, y)$, for $k < m$, the *equivalence formulas*, such that for $\Gamma \cup \{\varphi\} \subseteq \mathcal{F}_{\mathcal{L}}$,

(1) $\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \{\delta_i(\psi) \approx \varepsilon_i(\psi) : i < n, \psi \in \Gamma\} \models_{\mathcal{K}} \delta_j(\varphi) \approx \varepsilon_j(\varphi)$,

for each $j < n$, and for every equation $\varphi \approx \psi$ of $\mathcal{K}$

(2') $\varphi \approx \psi \models_{\mathcal{K}} \delta_i(\varphi \Delta_k \psi) \approx \varepsilon_i(\varphi \Delta_k \psi)$ and
(2") $\{\delta_i(\varphi \Delta_k \psi) \approx \varepsilon_i(\varphi \Delta_k \psi) : i < n \quad \text{and} \quad k < m\} \models_{\mathcal{K}} \varphi \approx \psi$.

A deductive system $\mathcal{S}$ is *algebraizable* if it has an equivalent algebraic semantics $\mathcal{K}$. For all unexplained details we refer the reader to [5].

3.2. Results

The following theorem is an application of the algebraizability criterion given in Corollary 4.8 in [5].

Theorem 2. *The system $\mathcal{B}al$ is algebraizable by means of the single defining equation $\varphi \approx \varphi \to \varphi$ and the single equivalence formula $\varphi \to \psi$.*

Proof.

1. By (I), that is, Thm. 1 (1), $\vdash \varphi \to \varphi$.
2. By Thm. 1 (6), $\varphi \to \psi \vdash \psi \to \varphi$.
3. By Thm. 1 (2) and (MP), $\varphi \to \psi\ ,\ \psi \to \theta \vdash \varphi \to \theta$.
4. By Thm. 1 (9), $\varphi_1 \to \psi_1\ ,\ \varphi_2 \to \psi_2 \vdash (\varphi_1 \to \varphi_2) \to (\psi_1 \to \psi_2)$ and by Thm. 1 (10), $\varphi \to \psi \vdash \varphi^+ \to \psi^+$, so $\to$ defines a congruence on the algebra of formulas of $\mathcal{B}al$.

By [5], Thm. 4.8, the system $\mathcal{B}al$ is algebraizable since both modus ponens and rule G hold in $\mathcal{B}al$. □

Theorem 3. *Let $\mathcal{F}_{\mathcal{B}al}$ be the set of terms of $\mathcal{B}al$. Let the binary relation $\equiv$ over $\mathcal{F}_{\mathcal{B}al}$ be defined by $\varphi \equiv \psi$ if and only if $\vdash_{\mathcal{B}al} \varphi \to \psi$. Then $\equiv$ is a congruence on the absolutely free (term) algebra $\langle \mathcal{F}_{\mathcal{B}al}, {}^{+}, \to \rangle$.*

Proof. From the previous theorem. □

Notation. For every formula φ we shall denote by $[\varphi]$ its equivalence class. The equivalence class of $\psi \to \psi$ shall be denoted by $\mathbb{O}$.

Before giving an axiomatization for BAL we will prove

Lemma 4. *The following identities hold in* BAL.

$$x \to x \approx y \to y\,,$$
$$x \to (y \to y) \approx x \to (z \to z)\,.$$

Proof. Recall that if BAL is the algebraic semantics equivalent to $\mathcal{B}al$, by [5], Def. 2.8, (which is summarized in (1), (2') and (2") above.)

$$\vdash x \to y \qquad \text{if and only if} \qquad \models_{\mathcal{K}} x \approx y\,. \tag{*}$$

Our lemma follows from (*) and Thm. 1 (3) and (5), respectively. □

Notation. We can now use the abbreviations

$$\mathbf{0} := x \to x$$
$$-x := x \to \mathbf{0}$$
$$x \,\&\, y := -x \to y = (x \to \mathbf{0}) \to y$$

which by Lemma 4 are well defined.

Theorem 5. *The algebraic semantics equivalent to $\mathcal{B}al$is the quasi–variety* BAL *defined by the following identities and quasi–identity.*

$\mathsf{BAL}_{(AA)}$	$(x \to x) \approx \mathbf{0}\,,$
$\mathsf{BAL}_{(B)}$	$(x \to y) \approx (z \to x) \to (z \to y)\,,$
$\mathsf{BAL}_{(C)}$	$x \to (y \to z) \approx y \to (x \to z)\,,$
$\mathsf{BAL}_{(N)}$	$(x \to y) \to y \approx x\,,$
$\mathsf{BAL}_{(P)}$	$x^{++} \approx x^{+}\,,$
$\mathsf{BAL}_{(O)}$	$(y \to x)^{+} \to (x \to y)^{+} \approx x \to y\,,$
$\mathsf{BAL}_{(MP)}$	$\mathbf{0} \to x \approx x\,,$
$\mathsf{BAL}_{(PI)}$	$\mathbf{0}^{+} \approx \mathbf{0}\,,$
$\mathsf{BAL}_{(AQ)}$	$x \,\&\, (x \to y) \approx y\,,$
$\mathsf{BAL}_{(SI)}$	$(x \to y)^{+} \approx \mathbf{0} \implies (x^{+} \to y^{+})^{+} \approx \mathbf{0}.$

Proof. We can get an axiomatization of BAL from [5], Thm. 2.17. For each $\mathcal{B}al$–axiom σ there is a BAL–identity $\delta(\sigma) \approx \varepsilon(\sigma)$. In our case, this means $\sigma \approx \mathbf{0}$. A quasi–identity is associated with each $\mathcal{B}al$–rule.

There is also an extra identity that arises in the process. In our case this is

$$x \to x \approx (x \to x) \to (x \to x)\,. \tag{AA}$$

Finally, there is a quasi–identity associated to the algebraization process, namely

$$x \to y \approx \mathbf{0} \quad \Longrightarrow \quad x \approx y \qquad (AQ).$$

We will now check all axioms and rules of $\mathcal{B}al$ and establish their counterparts in BAL. We will prove that not all of them are necessary and that the axiomatization given in this theorem is equivalent and simpler.

(AA) The extra axiom (AA) reduces to $x \to x \approx \mathbf{0}$.
(B) From $\mathcal{B}al$–axiom (B) and (*), it is immediate that $x \to y \approx (z \to x) \to (z \to y)$ holds in BAL.
(C) From $\mathcal{B}al$–axiom (C) and (*), $x \to (y \to z) \approx y \to (x \to z)$ holds in BAL.
(N) From $\mathcal{B}al$–axiom (N) and (*), $(x \to y) \to y \approx x$ holds in BAL.
(P) From $\mathcal{B}al$–axiom (P) and (*), $x^{++} \approx x^+$.
(O) From $\mathcal{B}al$–axiom (O) and (*), $(y \to x)^+ \to (x \to y)^+ \approx x \to y$ holds in BAL.
(MP) $\mathcal{B}al$–rule (MP) translates into $x \approx \mathbf{0}\,,\ x \to y \approx \mathbf{0} \implies y \approx \mathbf{0}$. This quasi–identity can be replaced by $\mathbf{0} \to x \approx x$.
The identity follows from Thm. 1 (7) and (*). In order to show that the quasi–identity can be deduced, assume $\mathbf{0} \to y \approx y$. Then if $x \approx \mathbf{0}$ and $x \to y \approx \mathbf{0}$, we get $y \approx \mathbf{0} \to y \approx x \to y \approx \mathbf{0}$, so the quasi–identity holds.
(G) $\mathcal{B}al$–rule (G) translates into $x \approx \mathbf{0}\,,\ y \approx \mathbf{0} \implies x \to y \approx \mathbf{0}$ and can be replaced by $\mathbf{0} \to \mathbf{0} \approx \mathbf{0}$ which is an instance of ($\mathsf{BAL}_{\mathrm{AA}}$).
(PI) $\mathcal{B}al$–rule (PI) translates into $x \approx \mathbf{0} \implies x^+ \approx \mathbf{0}$, which is equivalent to the identity $\mathbf{0}^+ \approx \mathbf{0}$.
(AQ) We first prove that the identity $y \to (x \& (x \to y)) \approx \mathbf{0}$ holds in BAL.

$$\begin{aligned} y \to (x\&(x \to y)) &\approx y \to ((x \to \mathbf{0}) \to (x \to y))\,, && \text{def.},\\ &\approx (x \to \mathbf{0}) \to (y \to (x \to y))\,, && (\mathsf{BAL}_{\mathrm{C}}),\\ &\approx (x \to \mathbf{0}) \to (x \to (y \to y))\,, && (\mathsf{BAL}_{\mathrm{C}}),\\ &\approx (x \to \mathbf{0}) \to (x \to \mathbf{0})\,, && (\mathsf{BAL}_{\mathrm{I}}),\\ &\approx \mathbf{0}\,, && (\mathsf{BAL}_{\mathrm{I}}). \end{aligned}$$

Using (*), the quasi–identity (AQ) implies $y \approx x + (x \to y)$.
Assume ($\mathsf{BAL}_{\mathrm{AQ}}$). If we also assume that $x \to y \approx \mathbf{0}$, then by two applications of ($\mathsf{BAL}_{\mathrm{AQ}}$),

$$x \approx x + (x \to x) \approx x + \mathbf{0} \approx x + (x \to y) \approx y\,,$$

thus proving (AQ).
(MI) The quasi–identity ($\mathsf{BAL}_{\mathrm{MI}}$) is a direct translation of $\mathcal{B}al$–rule (MI).

□

Theorem 6. *The system* $\mathcal{F}/_{\equiv} = \langle \mathcal{F}_{\mathcal{B}al}/_{\equiv}\,; \to, ^+ \rangle$ *is an algebra in* BAL*, where* $\to$ *and* $^+$ *are defined by*

$[x] \to [y] = [x \to y]$
$[x]^+ = [x^+]$

The algebra $\mathcal{F}/_{\equiv}$ *is free in* BAL *with the set of free generators* $\{[X_1], [X_2], \ldots\}$*, where* $X_1, X_2, \ldots$ *are the propositional variables.*

Proof. By the previous theorem, it suffices to prove the identities corresponding to the axioms (which are immediate), those associated to the rules MP and PI, the identity (BAL_{AQ}) and the quasi–identity (BAL_{SI}). We can deduce (BAL_{MP}) from Thm. 1, 8 and (BAL_{PI}) follows from rule (PI) and (*),with $x = \mathbf{0}$. In order to prove (BAL_{AQ}) we need to prove $\vdash_{\mathcal{B}al}$ $((x \to (u \to u)) \to (x \to y)) \to y$. This last assertion follows from axiom (B), (BAL_{MP}) and hypothetical syllogism. Finally, the proof of (BAL_{SI}) is straightforward, as well as the fact that the algebra $\mathcal{F}/_{\equiv}$ is freely generated by $\{[X_1], [X_2], \ldots\}$. □

4. From BAL to ℓ–groups

In this section we establish a categorical isomorphism between BAL and ℓ–groups (lattice–ordered abelian groups). As a key tool, we shall show that the quasivariety BAL is in fact a variety.

Theorem 7. *Let $\mathbf{A} = \langle A; \to, ^{+} \rangle$ be an algebra in* BAL*. Then $\mathbf{A}^* = \langle A;\ \&\,, -, \mathbf{0}\rangle$ is an abelian group. Moreover, $x \to y \approx y - x$.*

Proof. Recall that by definition and axiom ($\mathsf{BAL}_{\mathrm{N}}$)

$$- - x \approx -x \to \mathbf{0} \approx (x \to \mathbf{0}) \to \mathbf{0} \approx x\,.$$

Replacing x, y and z by $-x$, $-y$ and $\mathbf{0}$, respectively, in axiom ($\mathsf{BAL}_{\mathrm{C}}$),

$$-x \to (-y \to \mathbf{0}) \approx -y \to (-x \to \mathbf{0})\,,$$

so by the definition of $\&$,

$$x \,\&\, y \approx -x \to y \approx -x \to - - y \approx -y \to - - x \approx -y \to x \approx y \,\&\, x\,,$$

so $\&$ is commutative.

By axiom ($\mathsf{BAL}_{\mathrm{C}}$) again, replacing x, y and z by $-x$, $-z$ and y, respectively, we get

$$-x \to (-z \to y) \approx -z \to (-x \to y)\,,$$

which by definition of $\&$ is

$$x \,\&\, (z \,\&\, y) \approx z \,\&\, (x \,\&\, y)\,,$$

so by commutativity, associativity holds.

Next, by ($\mathsf{BAL}_{\mathrm{AQ}}$)

$$x \,\&\, \mathbf{0} \approx x \,\&\, (x \to x) \approx x\,,$$

proving that $\mathbf{0}$ is an identity for $\&$.

Finally, by axiom ($\mathsf{BAL}_{\mathrm{AA}}$),

$$x \,\&\, (-x) \approx -x \to -x \approx \mathbf{0}$$

thus proving that $-x$ is the additive inverse of x.

It is immediate that

$$x \to y \approx - - x \to y \approx -x \,\&\, y \,. \qquad \square$$

Lemma 8. *The following identities and quasi–identities hold in* BAL.

1. $x \approx x^+ - (-x)^+$,
2. $(-(x^+))^+ \approx \mathbf{0}$,
3. $(x^+ \to x)^+ \approx \mathbf{0}$,
4. $(x \to y)^+ \approx \mathbf{0}$, $(z \to x)^+ \approx \mathbf{0} \implies (z \to y)^+ \approx \mathbf{0}$,
5. $x^+ \approx \mathbf{0} \implies (y^+ \to x)^+ \approx \mathbf{0}$,
6. $(x^+ \,\&\, y^+)^+ \approx x^+ \,\&\, y^+$.

Proof. We point out the main axioms used in each proof. *1.* follows from ($\mathsf{BAL_O}$). *2.* uses ($\mathsf{BAL_P}$). *3.* follows from *2.* Item *4.* uses ($\mathsf{BAL_B}$), ($\mathsf{BAL_{MI}}$), ($\mathsf{BAL_{MP}}$) and ($\mathsf{BAL_P}$). *5.* follows from *4.* replacing x, y and z by $\mathbf{0}$, x and y^+, respectively. Item *6.* follows from *5.* using *2.* and *1.* □

Theorem 9. *The group* $\boldsymbol{A}^*$ *defined in Thm. 7 is a p.o.–group whose positive cone is given by*

$$P = \{a \in A : a^+ = a\} \,.$$

Proof. If $a \in P$ and $-a \in P$, $a = a^+$ and $-a = (-a)^+$, so $a = -(-a)^+ = -(-(a^+))^+$ and by Lemma 8 (2), $x = -\mathbf{0} = \mathbf{0}$, and thus $P \cap -P = \{\mathbf{0}\}$.

Let $a \in P$ and $b \in P$. Then by Lemma 8 (6), $(a^+ \,\&\, b^+)^+ = a^+ \,\&\, b^+$, so $P \& P \subseteq P$.

Finally, since $\boldsymbol{A}^*$ is abelian, for any $a \in A$, $a \,\&\, P \,\&\, (-a) \subseteq P$.

By [9], Chapter II, Thm. 2, $\boldsymbol{A}^*$ is a p.o.–group. □

Lemma 10. *The order defined by the positive cone* P *is*

$$x \le y \quad \textit{if and only if} \quad x \to y = y - x \in P \quad \textit{if and only if} \quad (y \to x)^+ = \mathbf{0} \,.$$

Proof. The latter is obtained as follows. If $y - x \in P$ then $(x - y)^+ = (-(y - x))^+ = (-(y - x)^+)^+ = \mathbf{0}$, by Lemma 8 (2).

On the other hand, if $(x - y)^+ = \mathbf{0}$, by Lemma 8 (1), $x - y = -(-(x - y))^+$, so $y - x = (y - x)^+$, and $y - x \in P$. □

Lemma 11. *Let* $\le$ *be the partial order defined in the previous lemma. Then for all* $a \in A$, $a^+ = l.u.b.\{a, \mathbf{0}\}$.

Proof. From the definition of P and Lemma 8 (2), for any $a \in A$, we have $\mathbf{0} \le a^+$. Also, by Lemmas 8 (3), and 10, $a \le a^+$.

Let b be any upper bound of both $\mathbf{0}$ and a. Then $b = b^+$ and $b^+ \ge a$, that is $(b^+ \to a)^+ = \mathbf{0}$, so by quasi–identity ($\mathsf{BAL_{SI}}$), $(b^+ \to a^+)^+ = \mathbf{0}$, that is $b = b^+ \ge a^+$, so $a^+ = l.u.b.\{a, \mathbf{0}\}$. □

Lemma 12. *Let* $\boldsymbol{A} \in \mathsf{BAL}$. *Then*

$$a \le_{\boldsymbol{A}} b \implies a^+ \le_{\boldsymbol{A}} b^+ \,.$$

Theorem 13. *The p.o.–group defined in Thms. 7 and 9 is an ℓ–group.*

Proof. Define

$$a \vee b := (b \to a)^{+} \,\&\, b\,,$$
$$a \wedge b := -(-a \vee -b)\,.$$

By [9], F), page 67, or [8], Thm. 3.3, the algebra $\boldsymbol{A}^{**} = \langle A,\ \&\,, -, \mathbf{0}, \vee, \wedge\rangle$ is an ℓ–group. □

The class BAL is defined by a set of identities and a quasi–identity, so in principle, it is a quasi–variety. Nevertheless, we will prove that BAL is closed under homomorphic images. Since it is already closed under subalgebras and direct products, BAL is a variety. As in the previous section, for any $\boldsymbol{A} \in$ BAL, the associated ℓ–group will be denoted $\boldsymbol{A}^{*}$.

Lemma 14. *The following inequality holds in any ℓ–group.*
$\mathbf{0} \leq (a^{+} \to b^{+})^{+} \leq (a \to b)^{+}$.

Proof. As we know, the variety of ℓ–groups is generated by $\mathbb{Z}$, so it is enough to check that this identity holds in the integers and this is straightforward. □

Theorem 15. *The quasi–identity (*$\mathsf{BAL_{SI}}$*)*

$$(y \to x)^{+} \approx \mathbf{0} \quad \Longrightarrow \quad (y^{+} \to x^{+})^{+} \approx \mathbf{0}$$

is preserved under homomorphic images of algebras in BAL.

Proof. Let $\boldsymbol{A} \in$ BAL and $h : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ be an epimorphism. Then $\boldsymbol{B}^{*}$ is an ℓ–group and thus by Lemma 14, for any $a, b \in B$, if $(a \to b)^{+} = \mathbf{0}$, then $(a^{+} \to b^{+})^{+} = \mathbf{0}$, so the quasi–identity holds. □

Lemma 16. *The quasi–identity (*$\mathsf{BAL_{SI}}$*) can be replaced by the following identity* $((y \to x)^{+} \to (y^{+} \to x^{+})^{+})^{+} \approx \mathbf{0}$.

Proof. It is clear that if $((y \to x)^{+} \to (y^{+} \to x^{+})^{+})^{+} \approx \mathbf{0}$, the quasi–identity ($\mathsf{BAL_{SI}}$) holds.

On the other hand, since with the defined operations, BAL is the variety of ℓ–groups and this identity holds in $\mathbb{Z}$, it holds in any algebra in BAL. □

Notation. In the following theorem we shall denote with the same symbols the variety BAL and the category of algebras in BAL and BAL–homomorphisms. We shall also write LG for the variety of ℓ–groups and the category LG of ℓ–groups and ℓ–group–homomorphisms.

Theorem 17. *The algebraic semantics equivalent to $\mathcal{B}al$ is a variety* BAL *definitionally equivalent to the variety* LG *of ℓ–groups. Thus, the category* BAL *is isomorphic to* LG.

Proof. The first part is proved by the previous theorem. We shall now deal with the categorical isomorphism.

Let $\langle A, \rightarrow, ^+ \rangle$ be an object of BAL. Define the following operations over A :

$$\begin{aligned} \mathbf{0} &:= x \rightarrow x \\ -x &:= x \rightarrow \mathbf{0} \\ x \,\&\, y &:= (x \rightarrow \mathbf{0}) \rightarrow y \\ x \vee y &:= (x \rightarrow y)^+ \,\&\, x \\ x \wedge y &:= -(-x \vee -y) \end{aligned}$$

Then by Thm. 13, $\langle A, \&, -, \mathbf{0}, \vee, \wedge \rangle$ is an ℓ–group.

Conversely, let $\langle G, +, -, 0, \vee, \wedge \rangle$ be an ℓ–group. Then $\langle G, \rightarrow, ^+ \rangle$ belongs to BAL, if we define the operations as follows.

$$\begin{aligned} x \rightarrow y &:= y - x \\ x^+ &:= x \vee 0 \,. \end{aligned}$$

Checking that the identities (BAL_{AA}), (BAL_B), (BAL_C), (BAL_N), (BAL_P), (BAL_{MP}), (BAL_{PI}) and (BAL_{AQ}) hold is trivial. We deduce (BAL_O) from Lemma 8 (1). (See [1]). The quasi–identity (BAL_{SI}) is proved as follows.

Suppose $y - x \leq 0$. Then, $y - x^+ \leq 0$. Therefore, $y^+ - x^+ = y - x^+ \vee -x^+ \leq 0$. □

Notation. In what follows we will use without distinction the symbols corresponding to BAL and to LG respectively assuming the definitions used so far. For instance x^+ to mean $x \vee 0$ and $(-x)^+$ or x^- to mean $-x \vee 0$, etc. Also, $|x|$ will be $x^+ \,\&\, (-x)^+$ (respectively $x^+ + x^-$).

Corollary 18. *The system $\langle \mathcal{F}/_\equiv; +, -, \mathbb{O}, \vee, \wedge, \rangle$ is an ℓ–group, where the operations are defined by*

$-[x] = [x \rightarrow \mathbb{O}]$
$[x] + [y] = -[x] \rightarrow [y]$
$[x] \vee [y] = [x \rightarrow y]^+ + [x]$
$[x] \wedge [y] = -(-[x] \vee -[y])$

Proof. From Theorem 6. □

We assume the reader to be familiar with the notion of ℓ–group term (see e.g., [7], Ch 2, 2.5) in the variables $X_1, \ldots, X_n$. For any such term $\tau = \tau(X_1, \ldots, X_n)$ one defines the associated term function $f_\tau : \mathbb{R}^n \longrightarrow \mathbb{R}$ in the usual way.

Notation. We let $Term_n^{\mathbb{R}}$ denote the set of ℓ–group term functions over $\mathbb{R}^n$.

Theorem 19. *For every $n \geq 1$, $Term_n^{\mathbb{R}}$ with pointwise ℓ–group operations is the free ℓ–group over the generating set $\{[X_1], \ldots, [X_n]\}$ of the canonical projection functions. $Term_n^{\mathbb{R}}$ is the ℓ–group of all piecewise linear homogeneous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ with integer coefficients. $\mathcal{F}_n/_\equiv$ is the same set of functions equipped with the operations $\rightarrow$, $^+$.*

Proof. From [1], 6.3, 17 and the fact that $\mathcal{F}_n/_\equiv$ is free (see Thm. 6.) □

5. Deduction theorem and $\mathcal{B}al$-consequence

By the categorical isomorphism of the previous section, the underlying sets of objects and morphisms in BAL and in LG coincide. In this section we further explore kernels of morphisms to prove the appropriate deduction theorem for $\mathcal{B}al$.

Let $\boldsymbol{A} = \langle A; \to, ^+ \rangle$ be an algebra in BAL and let $\boldsymbol{A}^* = \langle A;\ \&, -, \vee, \wedge, \mathbf{0}\rangle$ be the associated ℓ–group.

Recall that a $\mathcal{B}al$–filter of $\boldsymbol{A}$ is a subset F of A that contains all interpretations in $\boldsymbol{A}$ of the axioms of $\mathcal{B}al$and is closed under all the rules of $\mathcal{B}al$. This means that

1. $\mathbf{0} \in F$ (since all axioms take the value $\mathbf{0}$ when interpreted in A.)
2. If a and $a \to b \in F$, then $b \in F$.
3. If a and $b \in F$, then $a \to b \in F$.
4. If $a \in F$, then $a^+ \in F$.
5. If $(a \to b)^+ \in F$, then $(a^+ \to b^+)^+ \in F$.

In this section we shall use standard definitions from ℓ–groups theory. See, for example, [1] [4] or [9]. An *ℓ–subgroup* H of an ℓ–group G is a sublattice and a subgroup of G. If H is *convex*, that is, if $x, z \in H$ and $x < y < z$ implies $y \in H$, then H is called an *ℓ–ideal*. If H is an ℓ–ideal, the quotient G/H has a structure of ℓ–group. An ℓ–ideal is called *prime* if G/H is a totally ordered ℓ–group. A *principal* ideal is a finitely generated ℓ–ideal.

An ℓ–group is called *semisimple* if the intersection of its maximal ideals is $\{0\}$.

Theorem 20. *Let $\boldsymbol{A}$ be an algebra in* BAL *and $F \subseteq A$. Then F is a $\mathcal{B}al$–filter of $\boldsymbol{A}$ if and only if F is an ℓ–ideal of $\boldsymbol{A}^*$.*

Proof. Let F be a $\mathcal{B}al$–filter of $\boldsymbol{A}$. Since $\mathbf{0} \in F$, F is not empty, furthermore, if a, $b \in F$, by condition 3, $b - a = a \to b \in F$, so F is a subgroup of $\boldsymbol{A}^*$.

Next, if $a, b \in F$, by conditions 2 and 3, $(a \to b)^+ \in F$ and since F is a subgroup, $(a \to b)^+ \& a = a \vee b \in F$, so F is closed under suprema, and thus also closed under infima, so it is an ℓ–group.

To prove F is convex, let $a, b \in F$ and let $a \le c \le b$. Then

$a \to b \in F$,	(3),
$(c \to a) \to (c \to b) \in F$,	(B), (2),
$((c \to a) \to (c \to b))^+ \in F$,	(4),
$((c \to a)^+ \to (c \to b)^+)^+ \in F$,	(5),
$((c \to b)^+ \to (c \to a)^+)^+ \in F$,	similarly,
$(c \to a)^+ \to (c \to b)^+ \in F$,	(O), (2),
$(-c \to (c \to a)^+) \to (-c \to (c \to b)^+) \in F$,	(B), (2),
$(c \,\&\, (c \to a)^+)) \to (c \,\&\, (c \to b)^+) \in F$,	definition,
$(c \vee a) \to (c \vee b) \in F$,	definition,
$c \to b \in F$,	hypothesis,
$b \to c \in F$,	Thm. 1 6,
$c \in F$,	(2),

so F is convex.

Assume now that F is an ℓ–ideal of $\boldsymbol{A}^*$. Then obviously $\mathbf{0} \in F$ and F verifies condition 3 and 4.

If a and $a \to b \in F$, then $a \,\&\, (a \to b) = b \in F$, so condition 2 holds.

Finally, by Lemma 14, $\mathbf{0} \leq (a^+ \to b^+)^+ \leq (a \to b)^+$, so if $(a \to b)^+ \in F$, by convexity, $(a^+ \to b^+)^+ \in F$, thus proving condition 5, so F is a $\mathcal{B}al$–filter. □

Corollary 21. *The $\mathcal{B}al$–filter generated by a nonempty set E is the set $I(E) = \{a \in A : \exists e_1, \ldots, e_k \in E, |a| \leq |e_1| \,\&\, \ldots \,\&\, |e_n|\}$.*

Proof. See [9] page 79. □

It is easy to see that $I(E) = I(E^+)$, where E^+ is the set of positive elements of E. Thus we can assume without loss of generality that a set generating an ideal contains only positive elements.

Theorem 22. *There is an isomorphism between the class of $\mathcal{B}al$–filters of F_ω(BAL) and the class of theories of $\mathcal{B}al$.*

Proof. Straightforward. □

Theorem 23. *Let f be an element of F_ω(BAL), the free algebra in* BAL *with countable many free generators. Let E be a nonempty subset of positive elements of F_ω(BAL). Then $f \in I(E)$ if and only if there exist elements $g_1, \ldots, g_t \in E$ such that*

$$f^+ \leq g_1 \,\&\, \ldots \,\&\, g_t \qquad \textit{and} \quad f^- \leq g_1 \,\&\, \ldots \,\&\, g_t$$

Proof. In view of the isomorphism between BAL and LG, this is just a reformulation of the definition of ideal generated by E. □

The following is the deduction theorem for $\mathcal{B}al$.

Theorem 24. *Let $\Gamma \subseteq \mathcal{F}_{\mathcal{B}al}$, φ a formula such that $\Gamma \vdash_{\mathcal{B}al} \varphi$. Then there exist formulas $\alpha_1, \ldots, \alpha_k \in \Gamma$ such that $\vdash_{\mathcal{B}al} ((\alpha_1 \,\&\, \ldots \,\&\, \alpha_k) \to \varphi^+)^+$ and $\vdash_{\mathcal{B}al} ((\alpha_1 \,\&\, \ldots \,\&\, \alpha_k) \to \varphi^-)^+$. The converse is also obviously true.*

Proof. It follows from the definition of order in an algebra of BAL: $x \leq y$ if and only if $(y \to x)^+ = 0$. □

Theorem 25. (See [1] Cor. 4.1.2. or [4] Cor. 4.1.8.) *Every ℓ–group is a subdirect product of totally ordered abelian groups.*

Theorem 26. (See [4] Cor. 2.5.5.) *Every ℓ–ideal of an ℓ–group is an intersection of all prime ideals containing it.*

The following well known proposition is analogous to Wójcicki's theorem for MV–algebras (see [7], Chap. 3, Thm. 3.6.9.). We give a sketch of proof in our context.

Theorem 27. *Every quotient of a free ℓ–group by a principal ℓ–ideal is semisimple.*

In Benyon (see [3], Thm. 3.1) it is proved that every n–generated projective ℓ–group G is the quotient of the free n–generated ℓ–group $Term_n^{\mathbb{R}}$ by some principal ideal J (that is, an ideal generated by some function f.) As a corollary (see page 251) we can deduce that a zero–set of a function in $Term_n^{\mathbb{R}}$ is the quotient just mentioned (see also [2]). Thus, $Term_n^{\mathbb{R}}/J$ is the group of restrictions of the functions of $Term_n^{\mathbb{R}}$ to the zero–set of f. So, since $Term_n^{\mathbb{R}}/J$ is a group of real–valued functions, it is semisimple. Briefly, every finitely generated projective ℓ–group is semisimple and thus, every quotient of a free n generated ℓ–group by a principal ℓ–ideal is semisimple. Free ℓ–groups in general are direct limits of finitely generated free ℓ–groups.

Let $\mathbb{F}$ be any one of the totally ordered abelian groups $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$. It is well known that the variety $V(\mathbb{F})$ generated by $\mathbb{F}$ is LG. This is a direct consequence of the fact that every free ℓ–group is a subdirect product of copies of $\mathbb{Z}$. (See for instance [4], Cor. A.1.6 and Cor. A.1.7.)

Definition 1. *An $\mathbb{F}$–valuation is a function $v : Var \longrightarrow \mathbb{F}$. A valuation v can be extended recursively to all formulas, $\mathcal{F}$. Define $\overline{v} : \mathcal{F} \longrightarrow \mathbb{F}$, as follows.*

1. $\overline{v}(X) = v(X)$,
   for any propositional variable $X \in Var$,
2. $\overline{v}(\varphi \to \psi) = \overline{v}(\psi) - \overline{v}(\varphi)$,
3. $\overline{v}(-\varphi) = -\overline{v}(\varphi)$,
4. $\overline{v}(\varphi^+) = max\{\overline{v}(\varphi), 0\}$.

We say that a valuation v *satisfies* φ if $\overline{v}(\varphi) = 0$. The formula φ is $\mathbb{F}$–*valid* if $\overline{v}(\varphi) = 0$ for any $\mathbb{F}$–valuation v. Similarly, given a set Γ of formulas and a formula φ, we say φ is a $\mathbb{F}$–*consequence* of Γ if any $\mathbb{F}$– valuation that satisfies all formulas of Γ also satisfies φ. We then write $\Gamma \models_{\mathbb{F}} \varphi$.

By the remarks before the definition of valuation, a formula φ is $\mathbb{Z}$–valid if and only if it is $\mathbb{Q}$–valid if and only if it is $\mathbb{R}$–valid.

Theorem 28. *For every (possibly uncountable[1]) set Γ of formulas together with any formula φ we have*

1. $\Gamma \vdash_{\mathcal{B}al} \varphi \quad \Rightarrow \quad \Gamma \models_{\mathbb{R}} \varphi$.
2. $\Gamma \models_{\mathbb{R}} \varphi \quad \Rightarrow \quad \Gamma \models_{\mathbb{Q}} \varphi$.
3. $\Gamma \models_{\mathbb{Q}} \varphi \quad \Leftrightarrow \quad \Gamma \models_{\mathbb{Z}} \varphi$.

Proof. *1.* Let v be an $\mathbb{R}$–valuation satisfying all formulas ψ in Γ. Then the correctness of the axioms and rules of $\mathcal{B}al$ ensures that v satisfies all formulas obtained from the various steps on the proof of φ.

2. Let v be a $\mathbb{Q}$–valuation that satisfies every ψ in Γ. Then v is a $\mathbb{R}$–valuation, and by hypothesis it satisfies φ.

3. One direction is proven as above. The other direction follows from the representation theorem of free algebras in BAL (BAL isomorphic to LG) as continuous real valued piecewise linear homogeneous functions with integer coefficients defined on $\mathbb{R}^n$. □

[1] With obvious modifications concerning the set of propositional variables

The following theorem shows that, in the finite case, the notion of consequence $\vdash_{\mathcal{B}al}$ (having a syntactic algorithmic nature) has a semantic counterpart.

Theorem 29. *Let $\Gamma \cup \{\varphi\}$ be a finite set of formulas. Then,*

$$\Gamma \vdash_{\mathcal{B}al} \varphi \qquad \textit{if and only if} \qquad \Gamma \models_{\mathbb{F}} \varphi\,.$$

Proof. The proof is based on Thm. 27 taking into account the isomorphism between algebras in BAL and in LG (section 4).

Observe that if Γ is finite, the assertion $\{\gamma \approx \mathbf{0} \,:\, \gamma \in \Gamma\} \models_{\mathsf{BAL}} \varphi \approx \mathbf{0}$, where BAL is the variety of ℓ–groups, is equivalent to say that the quasi–identity

$$\bigwedge_{\gamma\in\Gamma} \gamma \approx \mathbf{0} \qquad \Longrightarrow \qquad \varphi \approx \mathbf{0}$$

holds in the variety of ℓ–groups. Now since the variety of ℓ–groups is generated by $\mathbb{F}$, this implies that

$$\{\gamma \approx \mathbf{0} \,:\, \gamma \in \Gamma\} \models_{\mathsf{BAL}} \varphi \approx \mathbf{0} \quad \text{if and only if} \quad \{\gamma \approx \mathbf{0} \,:\, \gamma \in \Gamma\} \models_{\mathbb{F}} \varphi \approx \mathbf{0},$$

and the last assertion is a different way of writing $\Gamma \models_{\mathbb{F}} \varphi$. So by the algebraization theorem of $\mathcal{B}al$, if Γ is finite, then $\Gamma \vdash \varphi$ if and only if $\Gamma \models_{\mathbb{F}} \varphi$. □

When Γ is infinite the situation is more delicate.

Theorem 30. *1. There exists a set of formulas Γ and a formula φ such that $\Gamma \models_{\mathbb{R}} \varphi$ and $\Gamma \not\vdash_{\mathcal{B}al} \varphi$*
2. There exists a set of formulas Γ and a formula φ such that $\Gamma \models_{\mathbb{Q}} \varphi$ and $\Gamma \not\models_{\mathbb{R}} \varphi$

Proof. 1. In order to prove that $\Gamma \models_{\mathbb{R}} \varphi$ does not imply $\Gamma \vdash_{\mathcal{B}al} \varphi$, we prove that $\models_{\mathbb{R}}$ is infinitary.

For each positive integer n let $\gamma_n = (x \to ny)^+$, where $1y = y$ and $(n+1)y = ny + y$.

Each formula γ_n, interpreted under all possible valuations, defines a function $g_n : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$

$$g_n(x, y) = \begin{cases} 0 & \text{if } ny - x \leq 0\,, \\ ny - x & \text{if } ny - x > 0. \end{cases}$$

Let $\Gamma = \{\gamma_n : n \text{ is a positive integer}\}$ and let v be a valuation such that for all n, $v(\gamma_n) = 0$. Then it is easy to check that $v(y) \leq 0$. For, if $v(y) > 0$, then $v(x) > 0$, so for some positive integer m, $mv(y) - v(x) > 0$, contradicting the assumption that $v(\gamma_m) = 0$. This observation implies that $\Gamma \models_{\mathbb{R}} y^+$. On the other hand, for every finite set $\Gamma' \subseteq \Gamma$ it is not the case that $\Gamma' \models_{\mathbb{R}} y^+$.

As a matter of fact, letting $\Gamma' = \{\gamma_{n_1}, \ldots, \gamma_{n_t}\}$ with $n_1 \leq \cdots \leq n_t$, let the valuation v be such that $v(x) = 2n_t$ and $v(y) = 1$. Then $v(\gamma_n) = 0$ for all $n \leq n_t$ and in particular $v(\gamma_{n_i}) = 0$ for $i = 1, \ldots, t$. Since $v(y^+) \neq 0$ we see that $\Gamma' \models_{\mathbb{R}} y^+$ and $\models_{\mathbb{R}}$ is infinitary, as required.

On the other hand, $\vdash_{\mathcal{B}al}$ is finitary, and the first implication of proposition 28 can not be reversed.

2. We shall construct an infinite set Γ of formulas in the two variables X, Y such that no $\mathbb{Q}$–valuation v satisfies the formulas $\psi \in \Gamma$. Then a fortiori: $\Gamma \models_{\mathbb{Q}} \varphi$, for every φ. But some $\mathbb{R}$–valuation satisfies Γ and not φ.

Let L be an irrational line in $\mathbb{R}^2$ through the origin ($ax + by = 0$ with $\frac{a}{b} \notin \mathbb{Q}$). Let $(x^*, y^*) \in L \cap (\mathbb{R}^+)^2$. Let v^* the $\mathbb{R}$–valuation $v^*(X) = x^*$, $v^*(Y) = y^*$. Let Γ be the set of all formulas $\psi(X, Y)$ such that $v^*(\psi) = 0$. In other words, the term function f_ψ associated to ψ is 0 over the line L.

A $\mathbb{R}$–valuation v satisfies Γ if and only if $v(X) = \lambda x^*$, $v(Y) = \lambda y^*$. So, no $\mathbb{Q}$–valuation satisfies Γ. Then, for all φ, $\Gamma \models_{\mathbb{Q}} \varphi$.

Take now $\varphi = |X| + |Y|$, where $|Z| = Z^+ + (-Z)^+$. By definition, v^* satisfies all $\psi \in \Gamma$. But v^* does not satisfies φ because $v^*(\varphi) = x^* + y^* > 0$.

Then $\Gamma \not\models_{\mathbb{R}} \varphi$. □

To obtain a purely semantic notion of consequence coincident with $\vdash_{\mathcal{B}al}$ we (routinely) proceed as follows.

Definition 2. *Let $\mathbb{T}$ be a totally ordered abelian group (o–group, for short). A $\mathbb{T}$–valuation is a function v from the variables into $\mathbb{T}$. This is canonically extended to a $\mathbb{T}$–valuation $\overline{v}$ to the formulas.*

Let Γ be a set of formulas, φ a formula. We say that φ is an *o–group–consequence* of Γ and denote $\Gamma \models^* \varphi$ if and only if for every totally ordered group $\mathbb{T}$, for every $\mathbb{T}$–valuation w, if w satisfies all $\psi \in \Gamma$ then w satisfies φ.

Theorem 31. *Let Γ be a set of formulas, φ a formula. Then*

$$\Gamma \vdash_{\mathcal{B}al} \varphi \quad \textit{if and only if} \quad \Gamma \models^* \varphi .$$

Proof. This is just restating the fact that BAL is the equivalent algebraic semantics of $\mathcal{B}al$. □

6. Co-NP-completeness of the tautology problem for $\mathcal{B}al$

The isomorphism between BAL and LG and the well known facts about free ℓ–groups(see theorem 17, and corollary 19) enables us to prove the co-NP-completeness of the tautology problem for $\mathcal{B}al$.

Notation. Let $Var = \{X_1, X_2, \ldots, X_n, \ldots\}$ be the set of variables of $\mathcal{F}$. For a formula $\varphi(X_1, X_2, \ldots, X_n)$ in $\mathcal{F}_n$ we denote f_φ the corresponding function in F_n by the isomorphism of 19.

Theorem 32. *For every formula $\varphi \in \mathcal{F}_n$ and every unit vector $\hat{u} = (u_1, \ldots, u_n)$ in $\mathbb{R}^n$ the directional derivative $\frac{\partial f_\varphi}{\partial \hat{u}}(\vec{x})$ exists at each $\vec{x} \in \mathbb{R}^n$.*

In particular, we are interested in the derivatives in the direction of the positive and negative axis, i.e. $\frac{\partial f_\varphi}{\partial \hat{u}}$ for $\hat{u} = (0, \ldots, 1, \ldots, 0)$ and $\hat{u} = (0, \ldots, -1, \ldots, 0)$, (1 in the i^{th} place.) These will be denoted $\frac{\partial f_\varphi}{\partial x_i^+}$ and $\frac{\partial f_\varphi}{\partial x_i^-}$, respectively.

Theorem 33. (Following [12].) *Let $\varphi(X_1, X_2, \ldots, X_n) \in \mathcal{F}_n$ and let $\|\varphi\|$ be the number of occurrences of variables in φ. Then $|\frac{\partial f_\varphi}{\partial x_i^-}| \leq \|\varphi\|$ and $|\frac{\partial f_\varphi}{\partial x_i^+}| \leq \|\varphi\|$.*

Proof. By induction on the number k of connectives in φ.
Let $k = 0$. Then, φ is a variable and the absolute value of $\frac{\partial f_\varphi}{\partial x_i^\pm}$ is 0 or 1. Therefore $|\frac{\partial f_\varphi}{\partial x_i^\pm}| \leq \|\varphi\| = 1$.

For the induction step here are two cases.,
Case 1. Let φ be ψ^+. Then trivially,

$$|\frac{\partial f_\varphi}{\partial x_i^\pm}| = |\frac{\partial f_\psi}{\partial x_i^\pm}| \leq \|\psi\| = \|\varphi\|.$$

Case 2. Let φ be $\chi \to \psi$. Then,

$$|\frac{\partial f_\varphi}{\partial x_i^\pm}| = |\frac{\partial f_\psi}{\partial x_i^\pm} - \frac{\partial f_\chi}{\partial x_i^\pm}| \leq |\frac{\partial f_\psi}{\partial x_i^\pm}| + |\frac{\partial f_\chi}{\partial x_i^\pm}| \leq \|\psi\| + \|\chi\| = \|\varphi\|.$$

We are now in position to prove our main result in this section. □

Theorem 34. *The tautology problem for $\mathcal{B}al$ is co-NP-complete.*

Proof. Suppose φ is not a tautology. This means that f_φ is not constantly equal to 0.

Consider the n–cube $\mathbf{C}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ |x_1| \leq 1, \ldots, |x_n| \leq 1\}$.

Since f_φ is homogeneous, there is a face Φ of $\mathbf{C}_n$ such that the restriction $f_\varphi \restriction_\Phi \neq 0$. Without loss of generality, all points $(y_1, \ldots, y_n) \in \Phi$ have $y_1 = 1$. Being f_φ continuous, Φ compact, there is a point $(1, z_2, \ldots, z_n)$ of Φ where f_φ attains its maximum value, t. Since f_φ is piecewise linear homogeneous with *integer* coefficients, all the $z_2, \ldots, z_n$, as well as t, are rationals.

The same analysis of [7], chapter 9, shows that the point $(1, z_2, \ldots, z_n)$ is the intersection of n hyperplanes with integer coefficients bounded by $\|\varphi\|$ (see Thm. 33).

An application of Hadamard's theorem shows that the least common denominator d of $z_2, \ldots, z_n$ satisfies the inequality, say, $d \leq \|\varphi\|^{5(\|\varphi\|^3)}$.

When written in binary notation, the point $(1, z_2, \ldots, z_n)$ requires an amount of space bounded by a fixed polynomial in the length of φ.

To conclude, here is a fast nondeterministic procedure to decide if φ is not a tautology: guess such short input number $(1, z_2, \ldots, z_n)$ and quickly check that $f_\varphi(1, z_2, \ldots, z_n) \neq 0$. Indeed, the check is quick because φ involves only a small number of comparisons and subtractions of rational numbers in Φ with the same denominator d.

We have just proved that the complement of the tautology problem in $\mathcal{B}al$is NP. To complete the proof we now show NP–hardness, by reducing to it the satisfiability problem for the infinite-valued propositional Lukasiewicz calculus (see [12]).

We shall describe a polynomial time transformation of each formula $\varphi(X_1, \ldots, X_n)$ of the Lukasiewicz calculus into a formula $\varphi^*(X_1, \ldots, X_n, Y)$ of $\mathcal{B}al$ such that φ is satisfiable (in the sense that the Mc Naughton function $\mathbf{f}_\varphi$ associated to φ is not constantly 0) if and only if φ^* is not a tautology of $\mathcal{B}al$.

The reduction has two stages. We first transform φ into φ' as follows.

$$X'_j = X_j$$
$$(\neg\psi)' = Y - \psi'$$
$$(\psi \oplus \rho)' = Y \wedge (\psi' \ \& \ \rho').$$

Now $\varphi^* = [\varphi' \wedge X_1 \wedge \cdots \wedge X_n \wedge Y \wedge (Y - X_1) \wedge \cdots \wedge (Y - X_n)] \vee 0$.

Claim: $\mathbf{f}_\varphi(x_1, \ldots, x_n)$is not constantly 0 if and only if φ^* is not a $\mathcal{B}al$ tautology (i.e., the term function f_{φ^*} is not constantly 0.)

Proof of the claim: Suppose $x_1, \ldots, x_n \in [0, 1]$ satisfies $\mathbf{f}_\varphi(x_1, \ldots, x_n) \neq 0$. Without loss of generality, by continuity of $\mathbf{f}_\varphi$, we can assume that for each $j = 1, \ldots, n$, $x_j \neq 0$, $x_j \neq 1$. Then by induction on the complexity of φ, one sees that $f_{\varphi^*}(x_1, \ldots, x_n, 1) \neq 0$.

Conversely, suppose f_{φ^*} is not constantly 0. That is, suppose $x_1, \ldots, x_n, y \in \mathbb{R}$ are such that $f_{\varphi^*}(x_1, \ldots, x_n, y) \neq 0$, whence $f_{\varphi^*}(x_1, \ldots, x_n, y) > 0$. By definition of φ^* we have $0 < x_j < 1$ and $y > x_j$. By homogeneity we can safely assume $f_{\varphi^*}(x_1, \ldots, x_n, 1) > 0$. So, by induction on the complexity of φ we see that $\mathbf{f}_\varphi(x_1, \ldots, x_n) > 0$. □

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