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# On the Central Limit Theorem for Nonuniform $\phi$-Mixing Random Fields

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The partial-sum processes, indexed by sets, of a stationary nonuniform $\phi$-mixing random field on the $d$-dimensional integer lattice are considered. A moment inequality is given from which the convergence of the finite-dimensional distributions to a Brownian motion on the Borel subsets of $[0, 1]^d$ is obtained. A Uniform CLT is proved for classes of sets with a metric entropy restriction and applied to certain Gibbs fields. This extends some results of Chen[5] for rectangles. In this case and when the variables are bounded a simpler proof of the uniform CLT is given.

**KEY WORDS:** Random fields on integer lattice; partial-sum process; Brownian motion; uniform central limit theorem; nonuniform $\phi$-mixing; metric entropy; Gibbs fields.

## 1. INTRODUCTION

If $\mathbf{a} = (a_1,..., a_d)$ and $\mathbf{b} = (b_1,..., b_d)$ belong to $\mathbb{R}^d$ (throughout the paper we fix an integer $d \geqslant 2$), let $(\mathbf{a}, \mathbf{b}] = \prod_{i=1}^{d} (a_i, b_i]$. We say that $\mathbf{a} \leqslant \mathbf{b}$ when $a_i \leqslant b_i$ for all $i$. Let $\mathscr{G} = \{(\mathbf{a}, \mathbf{b}] : \mathbf{a}, \mathbf{b} \in [0, 1]^d\}$, $\mathscr{R}$ be the class of sets which are finite unions of elements of $\mathscr{G}$ and $\mathscr{B}$ be the class of Borel subsets of $[0, 1]^d$.

Given $\mathscr{A} \subseteq \mathscr{B}$, a process $\{W(A)\}_{A \in \mathscr{A}}$ is a *Brownian motion* (or *Wiener process*) *on $\mathscr{A}$ with parameter* $\sigma$ if its finite dimensional laws are gaussian with $E(W(A)) = 0$ and $E(W(A)\, W(B)) = \sigma^2 |A \cap B|$ for all $A, B \in \mathscr{A}$. When $\sigma = 1$ we have a standard Brownian motion.

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Following the work of Goldie and Greenwood,[10] we consider on $\mathscr{B}$ the metric $d_L(A, B) = |A \triangle B|$, where $|\cdot|$ is the Lebesgue measure and we identify the sets $A_1$ and $A_2$ when $|A_1 \triangle A_2| = 0$.

Let $\mathscr{A} \subset \mathscr{B}$ be a totally bounded set. Since its closure $\bar{\mathscr{A}}$ is compact, the metric space $CA(\bar{\mathscr{A}})$ of continuous additive real functions $f$ on $\bar{\mathscr{A}}$ with the norm $\|f\|_{\bar{\mathscr{A}}} = \sup_{A \in \bar{\mathscr{A}}} |f(A)|$ is complete and separable ($f$ is additive if $f(A \cup B) = f(A) + f(B) - f(A \cap B)$ whenever $A, B, A \cap B, A \cup B \in \bar{\mathscr{A}}$). The existence of a *continuous Brownian motion on* $\bar{\mathscr{A}}$ (that is a Brownian motion with sample paths in $CA(\bar{\mathscr{A}})$) requires that $\mathscr{A}$ satisfies a metric entropy condition which is stronger than totally boundedness (see Dudley[7]). We say that $\mathscr{A}$ is *totally bounded with inclusion* with *exponent of metric entropy* $r$ if for all $\varepsilon > 0$ there is a finite set $\mathscr{N}(\mathscr{A}, \varepsilon) \subseteq \mathscr{A}$ with minimal cardinality $e^{H(\varepsilon)}$ such that for all $A \in \mathscr{A}$ there exists $A^-, A^+ \in \mathscr{N}(\mathscr{A}, \varepsilon)$ satisfying $A^- \subseteq A \subseteq A^+$, $|A^+ - A^-| \leqslant \varepsilon$ and $r$ is defined by

$$r = \limsup_{\varepsilon \searrow 0} \frac{\log(H(\varepsilon))}{\log(\varepsilon^{-1})}$$

Let $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ be a random field with finite means. For each $n \in \mathbb{Z}_+$ (positive integers) and $\mathbf{k} = (k_1, ..., k_d) \in \mathbb{Z}^d$ such that $\bar{1} \leqslant \mathbf{k} \leqslant \bar{n}$ (denote $\bar{x} = (x, ..., x)$) we define the "$n$-cells"

$$C_{n, \mathbf{k}} = \frac{1}{n}(\mathbf{k} - \bar{1}, \mathbf{k}] = \prod_{i=1}^{d} \left(\frac{k_i - 1}{n}, \frac{k_i}{n}\right]$$

and for each $n \in \mathbb{Z}_+$ and $A \in \mathscr{B}$ let

$$Z_n(A) = \frac{1}{n^{d/2}} \sum_{\bar{1} \leqslant \mathbf{k} \leqslant \bar{n}} \frac{|A \cap C_{n, \mathbf{k}}|}{|C_{n, \mathbf{k}}|} (\xi_{\mathbf{k}} - E\xi_{\mathbf{k}}) \tag{1.1}$$

For each $n$, $Z_n$ is a process with additive continuous sample paths on $\mathscr{B}$; it is the partial-sum process of $n$th-level of $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$. For the dependence of the variables $\xi_{\mathbf{k}}$ we consider, as in Nahapetian[12] and Chen,[5] the *nonuniform $\phi$-mixing* condition: there exists $\phi: [0, +\infty) \to [0, +\infty)$ with $\lim_{t \to +\infty} \phi(t) = 0$ such that for every pair $A_1$, $A_2$ of finite subsets of $\mathbb{Z}^d$

$$\sup_{E \in \sigma_1,\, F \in \sigma_2,\, P(F) > 0} |P(E \mid F) - P(E)| \leqslant \#(A_1)\, \phi(d(A_1, A_2))$$

where $\sigma_i = \sigma(\{\xi_{\mathbf{k}} : \mathbf{k} \in A_i\})$ is the $\sigma$-algebra generated by $\{\xi_{\mathbf{k}} : \mathbf{k} \in A_i\}$, $d(A_1, A_2) = \min\{\|\mathbf{k}_1 - \mathbf{k}_2\| : \mathbf{k}_i \in A_i\}$ and $\|\cdot\|$ is the euclidean norm); without loss of generality we will assume that $\phi$ is nonincreasing and $\phi(0) = 1$. It is known that some Gibbs random fields satisfy this condition but not the uniform $\phi$-mixing condition, in which the factor $\#(A_1)$ is absent (see Nahapetian,[12] p. 533). Nevertheless, in Section 6 we point out that our results continue to hold under other mixing assumptions; for example, Theorem 3 remains valid under a nonuniform absolute regularity condition.

We obtain for stationary random fields the finite-dimensional convergence of $Z_n$ on $\mathscr{B}$ to a Brownian motion (Theorem 1 later) under the same moment and mixing rate assumptions as Nahapetian[12] [Thm. 1] (which is a central limit theorem for an increasing sequence of rectangles). This extends [Chen,[5] Thm. 2.1] to Borel sets under weaker hypotheses.

Lemma 1 presents a fourth-moment inequality which is a key component in our work. Our technique of proof allows to use the criteria by Goldie and Greenwood[9] and does not depend on Nahapetian.[12]

Theorem 2 shows the convergence in distribution of $Z_n$ in $CA(\mathscr{G})$ to a continuous Brownian motion for the case of a stationary random field of bounded variables. The proof of tightness uses Lemma 1 and the results of Bickel and Wichura[2] for the Skorohod space, following the lines in Deo.[6] It simplifies in this case the proof given by Chen[5] [Thm. 1.1]; also, our condition on $\phi$ is weaker.

Theorem 3 is a central limit theorem in $CA(\bar{\mathscr{A}})$, for $\mathscr{A} \subset \mathscr{B}$ satisfying a metric entropy restriction, and is analogous to Goldie and Greenwood,[10] [Thm. 1.1], where the mixing is uniform. It extends Chen[5] [Thm. 1.1] which is a CLT in $CA(\mathscr{G})$, and thus (Corollary 1 later) the uniform CLT for the Gibbs fields of the Ising model given by Chen[5] [Cor. 4.2], to substantially larger classes of sets.

## 2. A MOMENT INEQUALITY AND CONVERGENCES OF FINITE DIMENSIONAL DISTRIBUTIONS

It is well known the existence of positive constant $L_1, L_2, K_1, K_2$ such that:

$$L_1 r^d \leqslant \#(\{\mathbf{k} \in \mathbb{Z}^d : \|\mathbf{k}\| \leqslant r\}) \leqslant K_1 r^d \tag{2.1}$$

$$L_2 r^{d-1} \leqslant \#(\{\mathbf{k} \in \mathbb{Z}^d : r \leqslant \|\mathbf{k}\| \leqslant r+1\}) \leqslant K_2 r^{d-1} \tag{2.2}$$

If $\varphi: [0, +\infty) \to [0, +\infty)$ is nonincreasing, it can be proved, using (2.1) and (2.2), that:

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \varphi(\|\mathbf{k}\|) < +\infty \Leftrightarrow \sum_{n=1}^{\infty} n^{d-1}\varphi(n) < +\infty \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{d-1}\varphi(n) < +\infty \Rightarrow \lim_{t \to +\infty} t^d \varphi(t) = 0 \tag{2.4}$$

In our results we will consider $\varphi = \phi$ or $\varphi = \phi^{1/2}$.

**Lemma 1.** Let $(\eta_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ be a random field with the nonuniform-$\phi$-mixing condition with $h=\sup_{t\geqslant 0} t^{2d}\phi(t)<+\infty$. Suppose that there exists $C>0$ such that $|\eta_{\mathbf{k}}(\omega)|\leqslant C$ for every $\mathbf{k}$ and $\omega$ and $E\eta_{\mathbf{k}}=0$ for all $\mathbf{k}$. Then, for all finite sets $M\subset\mathbb{Z}^d$:

$$E\left(\sum_{\mathbf{k}\in M}\eta_{\mathbf{k}}\right)^4\leqslant K_\phi(\#(M))^2\,C^4 \tag{2.5}$$

with $K_\phi=4!(1+4(\sum_{\mathbf{w}\in\mathbb{Z}^d}\phi(\|\mathbf{w}\|))^2+6K_1^2 9^d h)$ where $K_1$ is the constant in inequality (2.1).

*Proof.* First, we bound the value of $|E(\eta_{\mathbf{i}}\eta_{\mathbf{j}}\eta_{\mathbf{k}}\eta_{\boldsymbol{l}})|$ in the different cases in which the set $T=\{\mathbf{i},\mathbf{j},\mathbf{k},\boldsymbol{l}\}$ can be presented. We have $E\eta_{\mathbf{i}}^4\leqslant C^4$ when $\#(T)=1$. If $\#(T)\geqslant 2$ let $r=r(T)$ be the greatest distance between two nonempty sets $A$ and $B$ such that $\{A,B\}$ is a partition of $T$. Suppose, for instance, that $\|\mathbf{i}-\mathbf{j}\|=r$ with $\mathbf{i}\in A$ and $\mathbf{j}\in B$. We claim that $A\subseteq\{\mathbf{x}:\|\mathbf{x}-\mathbf{i}\|\leqslant 3r\}$ and $B\subseteq\{\mathbf{x}:\|\mathbf{x}-\mathbf{j}\|\leqslant 3r\}$. If $\#(T)=2$ there is nothing to prove. Assume $\#(T)=3$ and that $\mathbf{k}\in A$ and $\|\mathbf{k}-\mathbf{i}\|>3r$; then $\|\mathbf{k}-\mathbf{j}\|>r$ and $\{\{\mathbf{k}\},\{\mathbf{i},\mathbf{j}\}\}$ would be better than $\{A,B\}$. If $\#(T)=4$, $\mathbf{k}\in A$ and $\|\mathbf{k}-\mathbf{i}\|>3r$ then we take the partition $\{\{\mathbf{k}\},\{\mathbf{i},\mathbf{j},\boldsymbol{l}\}\}$ if $\|\boldsymbol{l}-\mathbf{k}\|>r$ or $\{\{\boldsymbol{l},\mathbf{k}\},\{\mathbf{i},\mathbf{j}\}\}$ if $|\boldsymbol{l}-\mathbf{k}\|\leqslant r$. Hence we have the announced inclusions of $A$ and $B$. Now we apply an inequality similar to Billingsley,[3] [Lemma 2, p. 171] (considering the cardinality), that is:

$$|E(\gamma_1\gamma_2)-E(\gamma_1)\,E(\gamma_2)|\leqslant 2\#(A_1)\,C_1C_2\phi(d(A_1,A_2))$$

where $A_i\subset\mathbb{Z}^d$, $\gamma_i\in\sigma(\{\eta_{\mathbf{k}},\mathbf{k}\in A_i\})$ and $|\gamma_i|\leqslant C_i$. If $\#(T)\geqslant 2$ we have two cases:

(a) $\|\mathbf{i}-\mathbf{j}\|=r$, $\|\mathbf{k}-\mathbf{i}\|\leqslant 3r$, $\|\boldsymbol{l}-\mathbf{j}\|\leqslant 3r$. $|E(\eta_{\mathbf{i}}\eta_{\mathbf{k}})(\eta_{\mathbf{j}}\eta_{\boldsymbol{l}})|\leqslant|E(\eta_{\mathbf{i}}\eta_{\mathbf{k}})\,E(\eta_{\mathbf{j}}\eta_{\boldsymbol{l}})|+2.2.C^2.C^2\phi(r)\leqslant 4C^4(\phi(\|\mathbf{k}-\mathbf{i}\|)\,\phi(|\boldsymbol{l}-\mathbf{j}\|)+\phi(r))$.

(b) $\|\mathbf{i}-\mathbf{j}\|=r$, $\|\mathbf{k}-\mathbf{i}\|\leqslant 3r$, $\|\boldsymbol{l}-\mathbf{i}\|\leqslant 3r$. $|E(\eta_{\mathbf{i}}\eta_{\mathbf{k}}\eta_{\boldsymbol{l}})(\eta_{\mathbf{j}})|\leqslant 2C^3.C\phi(r)=2C^4\phi(r)$.

Finally, taking into account (with excess) the possible permutations of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, $\boldsymbol{l}$:

$$\begin{aligned}E\left(\sum_{\mathbf{k}\in M}\eta_{\mathbf{k}}\right)^4\leqslant{}&\#(M)\,C^4+4!.4C^4\sum_{\mathbf{i}\in M}\sum_{\mathbf{j}\in M}\\&\times\left(\sum_{\|\mathbf{k}-\mathbf{i}\|,\,\|\boldsymbol{l}-\mathbf{j}\|\leqslant 3\,\|\mathbf{i}-\mathbf{j}\|}(\phi(\|\mathbf{k}-\mathbf{i}\|)\,\phi(\|\boldsymbol{l}-\mathbf{j}\|)+\phi(\|\mathbf{i}-\mathbf{j}\|))\right)\\&+4!.2C^4\sum_{\mathbf{i}\in M}\sum_{\mathbf{j}\in M}\left(\sum_{\|\mathbf{k}-\mathbf{i}\|,\,\|\boldsymbol{l}-\mathbf{i}\|\leqslant 3\,\|\mathbf{i}-\mathbf{j}\|}\phi(\|\mathbf{i}-\mathbf{j}\|)\right)\end{aligned}$$

$$\leqslant \#(M)\, C^4 + 4!.4C^4 \sum_{\mathbf{i}\in M} \sum_{\mathbf{v}+\mathbf{i}\in M} \sum_{\|k'\|,\, \|l'\| \leqslant 3\,\|\mathbf{v}\|} \phi(\|k'\|)\, \phi(\|l'\|)$$

$$+4!.6C^4 \sum_{\mathbf{i}\in M} \sum_{\mathbf{v}+\mathbf{i}\in M} \sum_{\|k'\|,\, \|l'\| \leqslant 3\,\|\mathbf{v}\|} \phi(\|\mathbf{v}\|)$$

$$\leqslant \#(M)\, C^4 + 4!.4C^4 \left( \#(M) \sum_{\mathbf{w}\in \mathbb{Z}^d} \phi(\|\mathbf{w}\|)\right)^2$$

$$+4!\, 6C^4 \#(M) \sum_{i\in M} \sup_{v}(9^d K_1^2\, \|\mathbf{v}\|^{2d}\, \phi(\|\mathbf{v}\|))$$

$$\leqslant K_\phi(\#(M))^2\, C^4.$$

Note that the convergence of $\sum \phi(\|\mathbf{v}\|)$ is a consequence of $h < +\infty$ and (2.3). □

**Remark 1.** The preceding result extends [Deo,[6] Lemma 4] where $(\xi_{\mathbf{k}})$ is stationary, the mixing is uniform and $M$ is a rectangle in $\mathbb{Z}^d$.

**Lemma 2.** Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ be a random field whose variables $\xi_{\mathbf{k}}$ have all the same distribution with $E\xi_{\mathbf{k}} = 0$, $E\xi_{\mathbf{k}}^2 < +\infty$ and satisfying the non-uniform-$\phi$-mixing condition with

$$S(\phi) := \sum_{\mathbf{v}\in\mathbb{Z}^d} \phi^{1/2}(\|\mathbf{v}\|) < +\infty$$

If, for each finite set $M \subset \mathbb{Z}^d$ and each collection $\lambda = (\lambda_{\mathbf{j}})_{\mathbf{j}\in M}$ of real numbers such that $0 \leqslant \lambda_{\mathbf{j}} \leqslant 1$ $\forall \mathbf{j}$, we define $T_{M,\lambda} = \sum_{\mathbf{j}\in M} \lambda_{\mathbf{j}} \xi_{\mathbf{j}}$ then the family $(T_{M,\lambda}^2/\#(M))_{M,\lambda}$ is uniformly integrable.

*Proof.* We follow Billingsley,[3] [p. 176] where the one-dimensional case is considered. Fix $M$ and $\lambda$. Let $u > 0$; for each $\mathbf{k}\in M$ define:

$$\eta_{\mathbf{k},u}^{(1)} = \begin{cases} \lambda_{\mathbf{k}}\xi_{\mathbf{k}} & \text{if } |\xi_{\mathbf{k}}| \leqslant u \\ 0 & \text{if } |\xi_{\mathbf{k}}| > u, \end{cases} \qquad \eta_{\mathbf{k},u}^{(2)} = \begin{cases} 0 & \text{if } |\xi_{\mathbf{k}}| \leqslant u \\ \lambda_{\mathbf{k}}\xi_{\mathbf{k}} & \text{if } |\xi_{\mathbf{k}}| > u \end{cases}$$

$$\eta'_{\mathbf{k},u} = \eta_{\mathbf{k},u}^{(1)} - E\eta_{\mathbf{k},u}^{(1)}, \qquad \eta''_{\mathbf{k},u} = \eta_{\mathbf{k},u}^{(2)} - E\eta_{\mathbf{k},u}^{(2)}$$

$$Y_{M,u} = \sum_{\mathbf{k}\in M} \eta'_{\mathbf{k},u}, \qquad D_{M,u} = \sum_{\mathbf{k}\in M} \eta''_{\mathbf{k},u}$$

We have $T_{M,\lambda} = Y_{M,u} + D_{M,u}$. Now Lemma 1 applied to $(\eta'_{\mathbf{k},u})_{\mathbf{k}\in M}$ (the condition $h < +\infty$ holds by (2.3) and (2.4)) gives for each $\gamma > 0$

$$E_\gamma\left(\frac{Y_{M,u}^2}{\#(M)}\right) \leqslant \frac{1}{\gamma} E\left(\frac{Y_{M,u}^4}{(\#(M))^2}\right) \leqslant \frac{K_\phi(2u)^4}{\gamma} \tag{2.6}$$

(we denote $E_\gamma(X) = E(XI\{x \geq \gamma\})$). By an inequality similar to Billingsley,[3] [Lemma 1, p. 170] (note that the cardinalities have value 1 and that $\phi(0) = 1$)

$$\begin{aligned} |E(\eta''_{\mathbf{j},u}\eta''_{\mathbf{k},u})| &\leq 2\phi^{1/2}(\|\mathbf{k}-\mathbf{j}\|)\, E^{1/2}((\eta''_{\mathbf{j},u})^2)\, E^{1/2}((\eta''_{\mathbf{k},u})^2) \\ &\leq 8\phi^{1/2}(\|\mathbf{k}-\mathbf{j}\|)\, E_{u^2}(\xi_{\mathbf{0}}^2) \end{aligned}$$

Then

$$E(D^2_{M,u}) \leq \sum_{\mathbf{j}\in M}\left(\sum_{\mathbf{k}\in M} |E(\eta''_{\mathbf{j},u}\eta''_{\mathbf{k},u})|\right) \leq 8\,\#(M)\, E_{u^2}(\xi_{\mathbf{0}}^2)\, S(\phi) \tag{2.7}$$

By (2.6), (2.7) and the inequality $T^2_{M,\lambda} \leq 2Y^2_{M,u} + 2D^2_{M,u}$:

$$E_\gamma\left(\frac{T^2_{M,\lambda}}{\#(M)}\right) \leq 4E_{\gamma/4}\left(\frac{Y^2_{M,u}}{\#(M)}\right) + 4E\left(\frac{D^2_{M,u}}{\#(M)}\right) \leq K'_\phi\left(\frac{u^4}{\gamma} + E_{u^2}(\xi_{\mathbf{0}}^2)\right)$$

where $K'_\phi$ depends only on $\phi$ and $d$.

Given $\varepsilon > 0$, take $u$ such that $K'_\phi E_{u^2}(\xi_{\mathbf{0}}^2) < \varepsilon/2$; then if $\gamma > 2K'_\phi u^4/\varepsilon$, we obtain $E_\gamma(T^2_{M,\lambda}/\#(M)) < \varepsilon$ for any $M$ and $\lambda$. This ends the proof. □

For each $n \in \mathbb{Z}_+$ let $H_n$ be the class of subsets of $[0, 1]^d$ which are union of $n$-cells $C_{n,\mathbf{k}}$. Observe that for each $A \in \mathscr{B}$, $V(n, A) = \bigcup_{C_{n,\mathbf{k}} \cap A \neq \varnothing} C_{n,\mathbf{k}}$ is the smallest element of $H_n$ containing $A$.

**Lemma 3.** The family $(Z^2_n(A)/|V(n, A)|)_{A\in\mathscr{B}, n\in\mathbb{Z}_+}$, under the same conditions of Lemma 2, is uniformly integrable.

*Proof.* Take $M = M_A = \{\mathbf{j} : C_{n,\mathbf{j}} \cap A \neq \varnothing\}$ and for each $\mathbf{j} \in M$ let $\lambda_{\mathbf{j}} = \lambda_{A,\mathbf{j}} = |A \cap C_{n,\mathbf{j}}|/|C_{n,\mathbf{j}}|$. Observe that $Z^2_n(A) = n^{-d}T^2_{M,\lambda}$, $|V(n, A)| = n^{-d}\#(M)$ and apply Lemma 2.

**Lemma 4.** Let $\mathscr{C} = \{\prod_{i=1}^d (a_i, b_i] \in \mathscr{G} : \exists L > 0 \text{ with } b_i - q_i = L\ \forall i\}$ and $F = \{(n, A): n \in \mathbb{Z}_+,\ A \in \mathscr{C},\ |A| \geq n^{-d}\}$. Then, under the same conditions of Lemma 2, the family $(Z^2_n(A)/|A|)_{(n,A)\in F}$ is uniformly integrable.

*Proof.* Use Lemma 3 and the fact that $(n, A) \in F$ implies $|V(n, A)| \leq 3^d\,|A|$.

**Lemma 5.** Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ be a random field with all the $\xi_{\mathbf{k}}$ equally distributed, $E\xi_{\mathbf{0}}^2 < +\infty$ and satisfying the nonuniform-$\phi$-mixing condition with $S(\phi) < +\infty$. Let $f: \mathbb{R} \to \mathbb{R}$ be a Borel-measurable function such that $|f(x)| \leq |x|$ for all $x$.

Then, for all $n \in \mathbb{Z}_+$ and $A \in \mathscr{B}$:

$$\operatorname{Var}\left(\frac{1}{n^{d/2}} \sum_{\bar{1} \leqslant \mathbf{k} \leqslant \bar{n}} \frac{|A \cap C_{n,\mathbf{k}}|}{|C_{n,\mathbf{k}}|} f(\xi_{\mathbf{k}})\right) \leqslant 2S(\phi)\, E\xi_{\mathbf{0}}^2\, |A|$$

*Proof.* For each $\mathbf{k}$ denote $\eta_{\mathbf{k}} = f(\xi_{\mathbf{k}})$ and $\mu_{\mathbf{k}} = |A \cap C_{n,\mathbf{k}}|$. Recalling that $|C_{n,\mathbf{k}}| = n^{-d}$ and $\phi(0) = 1$ we have

$$\begin{aligned}
&\operatorname{Var}\left(\frac{1}{n^{d/2}} \sum_{\bar{1} \leqslant \mathbf{k} \leqslant \bar{n}} \frac{|A \cap C_{n,\mathbf{k}}|}{|C_{n,\mathbf{k}}|} \eta_{\mathbf{k}}\right)\\
&\quad = \operatorname{Var}\left(n^{d/2} \sum_{\mathbf{k}} \mu_{\mathbf{k}} \eta_{\mathbf{k}}\right)\\
&\quad = n^d E\left(\sum_{\mathbf{k}} \mu_{\mathbf{k}} \eta_{\mathbf{k}}\right)^2 - n^d E^2\left(\sum_{\mathbf{k}} \mu_{\mathbf{k}} \eta_{\mathbf{k}}\right)\\
&\quad = n^d \sum_{\mathbf{i}} \sum_{\mathbf{j}} \mu_{\mathbf{i}} \mu_{\mathbf{j}} (E(\eta_{\mathbf{i}} \eta_{\mathbf{j}}) - E\eta_{\mathbf{i}} E\eta_{\mathbf{j}})\\
&\quad \leqslant n^d \max_{\mathbf{k}} \{\mu_{\mathbf{k}}\} \sum_{\mathbf{i}} \mu_{\mathbf{i}} \sum_{\mathbf{j}} 2\phi^{1/2}(\|\mathbf{i} - \mathbf{j}\|)\, E\xi_{\mathbf{0}}^2\\
&\quad \leqslant 2S(\phi)\, E\xi_{\mathbf{0}}^2 \sum_{\mathbf{i}} \mu_{\mathbf{i}} = 2S(\phi)\, E\xi_{\mathbf{0}}^2\, |A|
\end{aligned}$$

**Proposition 1.** Let $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ be a strictly stationary random field with $E\xi_{\mathbf{0}} = 0$, $E\xi_{\mathbf{0}}^2 < +\infty$ and satisfying the nonuniform-$\phi$-mixing condition with $S(\phi) < +\infty$. Then, $\sum_{\mathbf{k} \in \mathbb{Z}^d} E(\xi_{\mathbf{0}} \xi_{\mathbf{k}})$ converges absolutely to a value $\sigma^2 \geqslant 0$. If $\sigma^2 > 0$, the partial-sum processes $\{Z_n(A)\}_{A \in \mathscr{R}}$ converge, in the sense of finite dimensional distributions, to the Brownian motion on $\mathscr{R}$ with parameter $\sigma$.

*Proof.* The summability of $\sum E(\xi_{\mathbf{0}} \xi_{\mathbf{k}})$ holds by the facts $S(\phi) < +\infty$ and $|E(\xi_{\mathbf{0}} \xi_{\mathbf{k}})| \leqslant 2\phi^{1/2}(\|\mathbf{k}\|)\, E\xi_{\mathbf{0}}^2$. The argument by Goldie and Greenwood,[10] which next proves (ii), shows that the sum cannot be negative. Since $\sigma^2 > 0$ we suppose without loss of generality that $\sigma^2 = 1$. By using Goldie and Greenwood,[9] [Thm. 2.2] it suffices to show that the following conditions are verified:

(i) $\lim_{n \to \infty} EZ_n(C) = 0$. $\forall C \in \mathscr{G}$.

(ii) $\lim_{n \to \infty} EZ_n^2(C) = |C|$ $\forall C \in \mathscr{G}$.

(iii) If $C_1, \ldots, C_k \in \mathcal{G}$ satisfy $d(C_i, C_j) > 0$ for $i \neq j$ and $z_1, \ldots, z_k \in \mathbb{R}$, then

$$P(Z_n(C_1) \leqslant z_1, \ldots, Z_n(C_k) \leqslant z_k) - \prod_{i=1}^{k} P(Z_n(C_i) \leqslant z_i) \to 0$$

when $n \to \infty$.

(iv) $\forall \varepsilon > 0 \ \lim_{m \to \infty} \limsup_{n \to \infty} \sum_{\bar{1} \leqslant \mathbf{j} \leqslant \bar{m}} P(|Z_n(C_{m,\mathbf{j}})| \geqslant \varepsilon) = 0$.

(v) $(Z_n^2(C))_{n \in \mathbb{Z}_+}$ is uniformly integrable for each $C \in \mathcal{G}$.

(i) is immediate and the proof of (ii) is essentially the same as Goldie and Greenwood,[10] [proof of Cor. 1.4]. For (iii) we prove the case $k = 2$ (the general case is analogous). Let $r = d(C_1, C_2)$, $A_n = \{Z_n(C_1) \leqslant z_1\}$ and $B_n = \{Z_n(C_2) \leqslant z_2\}$. By the nonuniform-$\phi$-mixing condition, (2.3) and (2.4):

$$|P(A_n \cap B_n) - P(A_n)\, P(B_n)| \leqslant n^d \phi(nr - 2d^{1/2}) \to 0 \quad \text{when} \quad n \to \infty$$

Now, we prove (iv). Let $\varepsilon, \gamma > 0$. By Lemma 4 there exists $p \in \mathbb{Z}_+$ such that for all $(n, A) \in F$

$$E_{p^d \varepsilon^2}\left(\frac{Z_n^2(A)}{|A|}\right) \leqslant \varepsilon^2 \gamma$$

Given $m \geqslant p$, for all $n \geqslant m$ and all $\mathbf{j}$ it is $(n, C_{m,\mathbf{j}}) \in F$. Then:

$$\begin{aligned}
\sum_{\bar{1} \leqslant \mathbf{j} \leqslant \bar{m}} P(|Z_n(C_{m,\mathbf{j}})| \geqslant \varepsilon) &\leqslant \sum_{\bar{1} \leqslant \mathbf{j} \leqslant \bar{m}} P\left(\frac{Z_n^2(C_{m,\mathbf{j}})}{|C_{m,\mathbf{j}}|} \geqslant \varepsilon^2 m^d\right) \\
&\leqslant \frac{1}{m^d \varepsilon^2} \sum_{\bar{1} \leqslant \mathbf{j} \leqslant \bar{m}} E_{p^d \varepsilon^2}\left(\frac{Z_n^2(C_{m,\mathbf{j}})}{|C_{m,\mathbf{j}}|}\right) \\
&\leqslant \frac{1}{m^d \varepsilon^2} \sum_{\bar{1} \leqslant \mathbf{j} \leqslant \bar{m}} \varepsilon^2 \gamma = \gamma
\end{aligned}$$

Finally, (v) is immediate by Lemma 3.

Note that in Chen,[5] [Section 2] the proof of the convergence of the finite dimensional distributions requires $E\,|\xi_{\mathbf{0}}|^{2+\delta} < +\infty$ with $\delta > 0$.

**Theorem 1.** Under the same conditions of Proposition 1 the partial-sum processes $\{Z_n(A)\}_{A \in \mathcal{B}}$ converge, in the sense of finite dimensional distributions, to the Brownian motion on $\mathcal{B}$ with parameter $\sigma$.

*Proof.* By Goldie and Greenwood,[9] [Thm. 3.1] it is sufficient to prove:

(i) $\{Z_n(A)\}_{A \in \mathscr{R}}$ converges, in the sense of finite-dimensional distributions, to the Brownian motion on $\mathscr{R}$ with parameter $\sigma$;

(ii) If $A \in \mathscr{B}$ and $(C_l)$ is a nonincreasing sequence of sets of $\mathscr{R}$ such that $|\bigcap C_l - A| = 0$ then, for all $\varepsilon > 0$

$$\lim_{l \to \infty} \limsup_{n \to \infty} P(|Z_n(A) - Z_n(C_l)| \geqslant \varepsilon) = 0$$

Proposition 1 gives (i). Condition (ii) is a consequence of the inequality

$$P(|Z_n(A) - Z_n(C_l)| \geqslant \varepsilon) \leqslant \frac{2S(\phi)\, E\xi_0^2\, |A - C_l|}{\varepsilon^2}$$

which is valid by Chebyschev inequality, the additivity of $Z_n$ and Lemma 5.

## 3. A UNIFORM CLT FOR RECTANGLES

Let $\mathscr{A}$ be a totally bounded subset of $\mathscr{B}$ such that there exists a Brownian motion with paths in $CA(\bar{\mathscr{A}})$. In order to show that $(Z_n)$ converges in distribution in $CA(\bar{\mathscr{A}})$ to a Brownian motion it suffices to prove the convergence of the finite-dimensional distributions and the following tightness condition:

$$\forall \varepsilon > 0 \qquad \lim_{\alpha \searrow 0} \limsup_{n \to \infty} P(\|Z_n\|_{\mathscr{A}_\alpha} \geqslant \varepsilon) = 0 \tag{3.1}$$

where $\mathscr{A}_\alpha = \{A - B : A, B \in \mathscr{A} : |A - B| \leqslant \alpha\}$ (see Goldie and Greenwood,[10] [5.7 and (5.3.4)]).

In this section we consider $\mathscr{G}$, which satisfies $\bar{\mathscr{G}} = \mathscr{G}$ and whose exponent of metric entropy $r = 0$ guarantees the existence of a continuous Brownian motion. In this case (3.1) is equivalent to

$$\forall \varepsilon > 0 \qquad \lim_{\alpha \searrow 0} \limsup_{n \to \infty} P\Big(\sup_{A \in \mathscr{G},\, |A| \leqslant \alpha} |Z_n(A)| \geqslant \varepsilon\Big) = 0 \tag{3.2}$$

(note that $A - B$ is the union of at most $2d$ disjoint sets of $\mathscr{G}$ when $A, B \in \mathscr{G}$).

Now, given a process $X$ (with sample paths) in $CA(\mathscr{G})$ we define the process $\hat{X}$, indexed by the points of $[0,1]^d$, in the Skorohod space $\mathfrak{D}([0,1]^d)$ (see Bickel and Wichura,[2] [p. 1662]):

$$\hat{X}(\mathbf{t}) = \hat{X}(t_1,..., t_d) = \begin{cases} 0 & \text{if some } \quad t_i = 0, \\ X((\mathbf{0}, \mathbf{t}]) & \text{otherwise} \end{cases}$$

The sample paths of $\hat{X}$ are in the subspace of $\mathfrak{D}([0,1]^d)$ whose elements are the continuous functions vanishing in the points with at least one null coordinate. If $B = \prod_{i=1}^d (s_i, t_i]$ we have

$$X(B) = \sum_{i_1=0}^{1} \cdots \sum_{i_d=0}^{1} (-1)^{d-\sum_p i_p} \hat{X}(s_1 + i_1(t_1 - s_1),..., s_d + i_d(t_d - s_d)) \tag{3.3}$$

Given a random field $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with zero means, besides the smoothed partial sum processes $\hat{Z}_n$ we consider its non-smoothed versions

$$T_n(\mathbf{t}) = \frac{1}{n^{d/2}} \sum_{\bar{1} \leqslant \mathbf{j} \leqslant [n\mathbf{t}]} \xi_{\mathbf{j}}$$

where $[n\mathbf{t}] = ([nt_1],..., [nt_d])$, which also have sample paths in $\mathfrak{D}([0,1]^d)$. Proposition 2 will be used for the comparison of $Z_n$ and $T_n$.

**Proposition 2.** Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ be a random field with $E\xi_{\mathbf{k}} = 0$ for all $\mathbf{k}$. Let $n \in \mathbb{Z}_+$, $A = \prod_{i=1}^d (u_i, v_i] \in \mathscr{G}$. Then, there exist $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^d$ with $|u_i - (p_i/n)| \leqslant 1/n$ and $|v_i - (q_i/n)| \leqslant 1/n$ for all $i$, such that $B = \prod_{i=1}^d (p_i/n, q_i/n]$ satisfies $|Z_n(B)| \geqslant |Z_n(A)|$.

*Proof.* Take, for each $i$, integers $r_i$, $s_i$ such that $0 \leqslant r_i/n \leqslant u_i \leqslant (r_i+1)/n \leqslant 1$ and $0 \leqslant s_i/n \leqslant v_i \leqslant (s_i+1)/n \leqslant 1$. We claim that there exist $p_d \in \{r_d, r_d+1\}$ and $q_d \in \{s_d, s_d+1\}$ with $p_d < q_d$, such that if we take $B_1 = (\prod_{i=1}^{d-1} (u_i, v_i]) \times (p_d/n, q_d/n]$ then $|Z_n(B_1)| \geqslant |Z_n(A)|$. To prove this fact define $f\colon [r_d/n, (s_d+1)/n] \to \mathbb{R}$ as $f(\lambda) = Z_n((\prod_{i=1}^{d-1} (u_i, v_i]) \times (r_d/n, \lambda])$ when $\lambda > r_d/n$ and $f(r_d/n) = 0$. We will prove first that $f$ is a polygonal function whose vertices are $(m/n, f(m/n))$ with $m \in \mathbb{Z}$ and $r_d \leqslant m \leqslant s_d + 1$. It is sufficient to see that for all $m$ there exists $c_m$ such that for every $\lambda \in (m/n, (m+1)/n]$, $f(\lambda) - f(m/n) = c_m(\lambda - m/n)$. By the additivity of $Z_n$, $f(\lambda) - f(m/n) = Z_n(E_{\lambda,m})$ where $E_{\lambda,m} = (\prod_{i=1}^{d-1} (u_i, v_i]) \times (m/n, \lambda]$.

For the following calculations we observe that $L_m = \{\mathbf{k} : C_{n,\mathbf{k}} \cap E_{\lambda,m} \neq \varnothing\}$ does not depend on $\lambda$. We have $\forall \mathbf{k} = (k_1,..., k_d) \in L_m$

$$|C_{n,\mathbf{k}} \cap E_{\lambda,m}| = \left(\lambda - \frac{m}{n}\right) w_k \qquad \text{where} \quad w_k = \prod_{i=1}^{d-1} \left| \left(\frac{(k_i - 1)}{n}, \frac{k_i}{n}\right] \cap (u_i, v_i] \right|$$

Then

$$\begin{aligned} Z_n(E_{\lambda,m}) &= n^{-d/2} \sum_{\mathbf{k} \in L_m} \frac{|E_{\lambda,m} \cap C_{n,\mathbf{k}}|}{|C_{n,\mathbf{k}}|} \xi_{\mathbf{k}} \\ &= n^{d/2} \sum_{\mathbf{k} \in L_m} |E_{\lambda,m} \cap C_{n,\mathbf{k}}| \, \xi_{\mathbf{k}} \\ &= \left(\lambda - \frac{m}{n}\right) n^{d/2} \sum_{\mathbf{k} \in L_m} w_k \end{aligned}$$

We obtain $c_m = n^{d/2} \sum_{\mathbf{k} \in L_m} w_k$. Now by a simple property of the polygonal functions, there exist $p_d \in \{r_d, r_d + 1\}$, $q_d \in \{s_d, s_d + 1\}$ with $p_d < q_d$ such that

$$Z_n(A) = |f(v_m) - f(u_m)| \leqslant \left| f\left(\frac{q_d}{n}\right) - f\left(\frac{p_d}{n}\right) \right| = Z_n(B_1)$$

This proves the existence of $B_1$. Apply now the same argument successively for $k = 2,..., d-1$, obtaining sets $B_k$ and integers $p_{d-k+1}$, $q_{d-k+1}$ such that:

$$B_k = \left( \prod_{i=1}^{d-k} (u_i, v_i] \right) \times \left( \prod_{i=d-k+1}^{d} \left( \frac{p_i}{n}, \frac{q_i}{n} \right] \right) \quad \text{and} \quad |Z_n(B_k)| \geqslant |Z_n(B_{k-1})|$$

Finally, $B = B_d = \prod_{i=1}^{d} (p_i/n, q_i/n]$ satisfies $|Z_n(B)| \geqslant |Z_n(B_{d-1})| \geqslant |Z_n(A)|$. □

Now consider the following tightness condition for a given sequence of processes $(Y_n)$ in $\mathfrak{D}([0, 1]^d)$:

$$\forall \varepsilon > 0 \qquad \lim_{\alpha \searrow 0} \limsup_{n \to \infty} P\left( \sup_{\|\mathbf{t} - \mathbf{s}\| \leqslant \alpha} |Y_n(\mathbf{t}) - Y_n(\mathbf{s})| \geqslant \varepsilon \right) = 0 \tag{3.4}$$

**Lemma 6.** Let $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ a random field with $E\xi_{\mathbf{k}} = 0$ for all $\mathbf{k}$. Then, the following facts are equivalent:

(a) $(Z_n)$ satisfies (3.2).

(b) $(\hat{Z}_n)$ satisfies (3.4).

(c) $(T_n)$ satisfies (3.4).

*Proof.* (a) ⇒ (b). If $\alpha > 0$ and $\|\mathbf{s} - \mathbf{t}\| \leqslant \alpha$, by a property similar to the mentioned after (3.1) and the additivity of $Z_n$, we have $|\hat{Z}_n(\mathbf{t}) - \hat{Z}_n(\mathbf{s})| \leqslant \sum_{i=1}^{p} |Z_n(A_i)|$, where $p \leqslant 2d$ and $A_i \in \mathcal{G}$, $|A_i| \leqslant \alpha$ for all $i$.

Then $\sup_{\|\mathbf{t}-\mathbf{s}\| \leqslant \alpha} |\hat{Z}_n(\mathbf{t}) - \hat{Z}_n(\mathbf{s})| \leqslant 2d \sup_{A \in \mathscr{G}, |A| \leqslant \alpha} |Z_n(A)|$.

(b) $\Rightarrow$ (c). Given $\alpha > 0$ take $n \geqslant 2d^{1/2}/\alpha$. Then, for every $\mathbf{s}$, $\mathbf{t}$ with $\|\mathbf{t}-\mathbf{s}\| \leqslant \alpha$, taking $\mathbf{t}_n = [n\mathbf{t}]/n$ and $\mathbf{s}_n = [n\mathbf{s}]/n$, we have $T_n(\mathbf{t}) = \hat{Z}_n(\mathbf{t}_n)$, $T_n(\mathbf{s}) = \hat{Z}_n(\mathbf{s}_n)$ and $\|\mathbf{t}_n - \mathbf{s}_n\| \leqslant 2\alpha$. This shows that $\sup_{\|\mathbf{t}-\mathbf{s}\| \leqslant \alpha} |T_n(\mathbf{t}) - T_n(\mathbf{s})| \leqslant \sup_{\|\mathbf{t}-\mathbf{s}\| \leqslant 2\alpha} |\hat{Z}_n(\mathbf{t}) - \hat{Z}_n(\mathbf{s})|$.

(c) $\Rightarrow$ (a). Assume $\alpha > 0$ and $n \geqslant 2/\alpha^{1/d}$. If $A = \prod_{i=1}^d (u_i, v_i] \in \mathscr{G}$ with $|A| \leqslant \alpha$, Proposition 2 gives $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^d$ such that, defining $s_i = p_i/n$, $t_i = q_i/n$ for all $i$ and $B = \prod_{i=1}^d (s_i, t_i]$, we have $|s_i - u_i| \leqslant 1/n$, $|t_i - v_i| \leqslant 1/n$ for all $i$ and $|Z_n(B)| \geqslant |Z_n(A)|$. Since $|A| \leqslant \alpha$ and $n \geqslant 2/\alpha^{1/d}$, there exists $j$ satisfying $|t_j - s_j| \leqslant 2\alpha^{1/d}$; suppose without loss of generality that $j = 1$. For each $\mathbf{m} = (m_2,..., m_d) \in \{0, 1\}^{d-1}$ we consider

$$\mathbf{x_m} = (s_1, s_2 + m_2(t_2 - s_2),..., s_d + m_d(t_d - s_d))$$

and

$$\mathbf{y_m} = (t_1, s_2 + m_2(t_2 - s_2),..., s_d + m_d(t_d - s_d))$$

Since $\|\mathbf{y_m} - \mathbf{x_m}\| \leqslant 2\alpha^{1/d}$ and $|Z_n(B)| \leqslant \sum_{\mathbf{m}} |T_n(\mathbf{y_m}) - T_n(\mathbf{x_m})|$ (use (3.3)):

$$\sup_{A \in \mathscr{G}, |A| \leqslant \alpha} |Z_n(A)| \leqslant 2^{d-1} \sup_{\|\mathbf{t}-\mathbf{s}\| \leqslant 2\alpha^{1/d}} |T_n(\mathbf{t}) - T_n(\mathbf{s})|$$

**Theorem 2.** Let $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ be a strictly stationary field of bounded random variables with $E\xi_{\mathbf{0}} = 0$ and satisfying the nonuniform-$\phi$-mixing condition with $S(\phi) < +\infty$. Suppose that $\sum_{\mathbf{k} \in \mathbb{Z}^d} E(\xi_{\mathbf{0}} \xi_{\mathbf{k}}) = \sigma^2 > 0$. Then the sequence $(Z_n)$ of partial-sum processes converges in distribution in $CA(\mathscr{G})$ to the continuous Brownian motion with parameter $\sigma$.

*Sketch of proof.* By Proposition 1 we have the convergence of the finite-dimensional distributions. Apply now Lemma 1 when $M$ is a product of finite integer intervals. As in Deo[6] [Lemma 5], the inequality (2.5) and the results of Bickel and Wichura[2] imply (3.4) for $T_n$. Conclude applying Lemma 6.

**Remark 2.** It is easy to prove, under the conditions of Theorem 2, the convergence of the finite dimensional laws of $(\hat{Z}_n)$ and $(T_n)$ thus obtaining their convergence in distribution in $\mathfrak{D}([0, 1]^d)$. This last case gives a version of Deo[6] [Thm. 1] for nonuniform-$\phi$-mixing when the variables are bounded.

## 4. A UNIFORM CLT THEOREM FOR MORE GENERAL CLASSES OF SETS

The conditions imposed on the class $\mathscr{A}$ in the following result, guarantee the existence of a continuous Brownian motion on $\bar{\mathscr{A}}$ (see Dudley[7]).

**Theorem 3.** Let $(\xi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ be a strictly stationary random field with $E\xi_{\mathbf{0}}=0$ and $\mathscr{A}\subset\mathscr{B}$ a class of sets which is totally bounded with inclusion with exponent of metric entropy $r$. Suppose that there exists a real number $\delta>0$ satisfying $E\,|\xi_{\mathbf{0}}|^{2+\delta}<+\infty$, $r<1/1+\delta$ and that $(\xi_{\mathbf{k}})$ is nonuniform-$\phi$-mixing with $\lim_{t\to+\infty} t^{2d(1+\delta^{-1})}\phi(t)=0$. Then, if $\sum_{\mathbf{k}\in\mathbb{Z}^d} E(\xi_{\mathbf{0}}\xi_{\mathbf{k}})=\sigma^2>0$, the sequence $(Z_n)$ of partial-sum processes converges in distribution in $CA(\bar{\mathscr{A}})$ to the continuous Brownian motion with parameter $\sigma$.

*Proof.* We use the criterion mentioned at the beginning of Section 3. Since (2.3) and $\lim_{t\to+\infty} t^{2d(1+(\delta)^{-1})}\phi(t)=0$ imply $S(\phi)<+\infty$, the convergence of the finite-dimensional distribution follows from Theorem 1. For tightness it is sufficient to prove (3.1). We follow Goldie and Greenwood,[10] (where an uniform mixing is considered) and Chen's[5] ideas for nonuniform-$\phi$-mixing. The method is an adaptation of a technique due to Bass.[1] Denote, as in Goldie and Greenwood,[10] $s'=2+\delta$ and $\xi_{n,\mathbf{j}}=n^{-d/2}\xi_{\mathbf{j}}$.

**Step 1:** *Truncation.* Let, for $x\geqslant 0$, $h(x)=E_{x^{s'}}|\xi_{\mathbf{0}}|^{s'}$. If $0\leqslant u\leqslant v\leqslant\infty$, $n\in\mathbb{Z}_+$ and $\bar{1}\leqslant\mathbf{j}\leqslant\bar{n}$ we define

$$\xi_{n,\mathbf{j}}(u,v)=\xi_{n,\mathbf{j}}I\{u\leqslant n^{d(s'-2)/(2(s'-1))}\,|\xi_{n,\mathbf{j}}|<v\}$$

$$Z_n(A,u,v)=\sum_{\bar{1}\leqslant\mathbf{j}\leqslant\bar{n}}\frac{|A\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\,(\xi_{n,\mathbf{j}}(u,v)-E\xi_{n,\mathbf{j}}(u,v))$$

$$U_n(A,u,v)=\sum_{\bar{1}\leqslant\mathbf{j}\leqslant\bar{n}}\frac{|A\cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|}\,|\xi_{n,\mathbf{j}}(u,v)|$$

Using the same argument by Goldie and Greenwood,[10] we arrive to [Ref. 10, (5.4.1)]: it suffices to prove

$$\forall\varepsilon>0\qquad \lim_{a,\alpha\searrow 0}\ \limsup_{n\to\infty} P(\|Z_n(\cdot,0,a)\|_{\mathscr{A}_\alpha}\geqslant\varepsilon)=0 \tag{4.1}$$

**Step 2:** *Blocking.* Define $p_n=[n^{s'/(2(s'-1))}]$ and $m_n=n/(2p_n)$. Divide each cell $C_{p_n,\mathbf{k}}$ $(\mathbf{k}\leqslant\bar{p}_n)$ in $2^d$ sub-cells of the form $C_{2p_n,\mathbf{j}}$. Denote these by $I_{n,\mathbf{k},i}$, $i=1,\ldots,2^d$, indexed the same way in each $C_{p_n,\mathbf{k}}$ (see [Ref. 10, 5.5]). Let, for each $i$,

$$I_{n,i}=\bigcup_{\mathbf{k}\leqslant\bar{p}_n} I_{n,\mathbf{k},i}$$

For all $A$ we have

$$Z_n(A, 0, a) = \sum_{i=1}^{2^d} Z_n(A \cap I_{n,i}, 0, a)$$

Then it is sufficient to prove, in place of (4.1), that for each $i$

$$\forall \varepsilon > 0 \qquad \lim_{a, \alpha \searrow 0} \limsup_{n \to \infty} P(\|Z_n(\cdot \cap I_{n,i}, 0, a)\|_{\mathscr{A}_\alpha} \geqslant \varepsilon) = 0 \tag{4.2}$$

We take now a fixed value of $i$. By the additivity of $Z_n$ we have for each $A$

$$Z_n(A \cap I_{n,i}, 0, a) = \sum_{\mathbf{k} \leqslant \bar{p}_n} V_{n,\mathbf{k}}(A, 0, a)$$

where

$$V_{n,\mathbf{k}}(A, u, v) = Z_n(A \cap I_{n,\mathbf{k},i}, u, v)$$
$$= \sum_{\mathbf{j} \in T(n,\mathbf{k},i)} \frac{|A \cap I_{n,\mathbf{k},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\xi_{n,\mathbf{j}}(u, v) - E\xi_{n,\mathbf{j}}(u, v))$$

with $T(n, \mathbf{k}, i) = \{\mathbf{j} \leqslant \bar{n} : C_{n,\mathbf{j}} \cap I_{n,\mathbf{k},i} \neq \varnothing\}$. Observe that $\mathbf{j}_1 \in T(n, \mathbf{k}_1, i)$, $\mathbf{j}_2 \in T(n, \mathbf{k}_2, i)$ and $\mathbf{k}_1 \neq \mathbf{k}_2$ imply $\|\mathbf{j}_2 - \mathbf{j}_1\| \geqslant m_n - 2$ (recall that we use the euclidean norm). For $n \in \mathbb{Z}_+$, $\mathbf{j} \in \bigcup_{\bar{1} \leqslant \mathbf{k} \leqslant \bar{p}_n} T(n, \mathbf{k}, i)$, we consider random variables $\bar{\xi}_{n,\mathbf{j}}$ such that for each $\mathbf{k}$ the processes $\{\xi_{n,\mathbf{j}}\}_{\mathbf{j} \in T(n,\mathbf{k},i)}$, $\{\bar{\xi}_{n,\mathbf{j}}\}_{\mathbf{j} \in T(n,\mathbf{k},i)}$ have the same distribution and $(\bar{\xi}_{n,\mathbf{j}})_{\mathbf{j} \in T(n,\mathbf{k},i)}$, $\bar{1} \leqslant \mathbf{k} \leqslant \bar{p}_n$, are independent. Define now

$$\bar{\xi}_{n,\mathbf{j}}(u, v) = \bar{\xi}_{n,\mathbf{j}} I\{u \leqslant n^{d(s'-2)/(2(s'-1))} |\bar{\xi}_{n,\mathbf{j}}| < v\}$$

$$\bar{V}_{n,\mathbf{k}}(A, u, v) = \sum_{\mathbf{j} \in T(n,\mathbf{k},i)} \frac{|A \cap I_{n,\mathbf{k},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} (\bar{\xi}_{n,\mathbf{j}}(u, v) - E\bar{\xi}_{n,\mathbf{j}}(u, v))$$

$$\bar{Z}_n(A, u, v) = \sum_{\mathbf{k} \leqslant \bar{p}_n} \bar{V}_{n,\mathbf{k}}(A, u, v)$$

$$W_{n,\mathbf{k}}(A, u, v) = \sum_{\mathbf{j} \in T(n,\mathbf{k},i)} \frac{|A \cap I_{n,\mathbf{k},i} \cap C_{n,\mathbf{j}}|}{|C_{n,\mathbf{j}}|} |\bar{\xi}_{n,\mathbf{j}}(u, v)|$$

$$\bar{U}_n(A, u, v) = \sum_{\mathbf{k} \leqslant \bar{p}_n} W_{n,\mathbf{k}}(A, u, v)$$

We prove that for each $a$ ($L(X)$ denotes the law of the r.v. $X$; $\|\cdot\|_{\mathrm{Var}}$ is the total variation norm)

$$\|L(Z_n(\cdot \cap I_{n,i}, 0, a)) - L(\bar{Z}_n(\cdot, 0, a))\|_{\mathrm{Var}} \to 0 \qquad \text{when} \quad n \to \infty \tag{4.3}$$

As in [Chen,[5] Prop. 3.3] we deduce that $\#(T(n, \mathbf{k}, i)) \leqslant C' n^{d(s'-2)/(2(s'-1))}$, where $C'$ depends only on $d$. Denote, for each $\mathbf{k} \leqslant \bar{p}_n$, $L_{n,\mathbf{k}}$ the law of $V_{n,\mathbf{k}}(\cdot, u, v)$ ($u$ and $v$ are fixed), $L_n^{\mathbf{k}}$ the joint law of all the $V_{n,\mathbf{j}}(\cdot, u, v)$ with $\mathbf{j} \neq \mathbf{k}$ and let $L_n$ be the joint law of all the $V_{n,\mathbf{k}}(\cdot, u, v)$. If for each $\mathbf{k}$, $\mathcal{A}_{\mathbf{k}} = \sigma(V_{n,\mathbf{k}}(\cdot, u, v))$ and $\mathcal{A}^{(\mathbf{k})} = \sigma(\{V_{n,\mathbf{j}}(\cdot, u, v), \mathbf{j} \neq \mathbf{k}\})$, we use [Eberlein,[8] Lemma 3.5] and the nonuniform-$\phi$-mixing condition to obtain

$$\begin{aligned} \|L_n - L_{n,\mathbf{k}} \otimes L_n^{\mathbf{k}}\|_{\mathrm{Var}} &\leqslant 2\#(T(n, \mathbf{k}, i))\, \phi(m_n - 2) \\ &\leqslant 2C' n^{d(s'-2)/(2(s'-1))} \phi(\tfrac{1}{2} n^{(s'-2)/(2(s'-1))} - 2) \end{aligned}$$

Applying a result similar to Goldie and Greenwood,[10] [Lemma 5.0]:

$$\begin{aligned} &\left\| L_n - \bigotimes_{\mathbf{k}} L_{n,\mathbf{k}} \right\|_{\mathrm{Var}} \\ &\quad\leqslant (p_n^d - 1)\, 2C' n^{d(s'-2)/(2(s'-1))} \phi(\tfrac{1}{2} n^{(s'-2)/(2(s'-1))} - 2) \\ &\quad\leqslant 2C'((2(K+2)))^{2d(s'-1)/(s'-2)} \phi(K) \\ &\quad\leqslant C'' K^{2d(s'-1)/(s'-2)} \phi(K) \end{aligned}$$

where $K \to \infty$ when $n \to \infty$. Since $\lim_{t \to +\infty} t^{2d(1+\delta^{-1})} \phi(t) = 0$ we deduce (4.3).

Now the following implies (4.2):

$$\forall \varepsilon > 0 \qquad \lim_{a,\alpha \searrow 0} \limsup_{n \to \infty} P(\|\bar{Z}_n(\cdot, 0, a)\|_{\mathscr{A}_\alpha} \geqslant \varepsilon) = 0$$

For its proof we must verify some a.s. bounds which are required in the last step for the application of the Bernstein inequality to sums of independent variables:

$$|\bar{V}_{n,\mathbf{k}}(A, u, v)| \leqslant 2v, \qquad |W_{n,\mathbf{k}}(A, u, v)| \leqslant v$$

and

$$E\bar{U}_n(A, u, v) \leqslant |A|\, u^{-(s'-1)} h(0)$$

hold with the same proof as Goldie and Greenwood.[10] We obtain also Var $\bar{Z}_n(A, u, v) \leqslant C\,|A|$ and Var $\bar{U}_n(A, u, v) \leqslant C\,|A|$ by using our Lemma 5 to prove condition (B) of [Ref. 10, Lemma 5.1] with $C = 2S(\phi)\, E\xi_{\mathbf{0}}^2$.

**Step 3:** *Nesting.* Follow the argument in [Ref. 10, 5.6], taking $r'$ such that $r < r' < 1/(s'-1)$ and observing that we have the entropy bound given in [Ref. 10, (5.3.5)].

## 5. AN APPLICATION

We refer to [Chen,[5] Section 4] for the definition and properties of the Gibbs fields of the Ising model which are needed for the following statement and its derivation from Theorem 3 (we only note that in this example the $\xi_{\mathbf{k}}$'s are bounded, centered and that $\delta$ can be taken arbitrarily small because $\phi$ decreases exponentially to zero).

**Corollary 1.** Suppose the distribution of the random field $(\xi_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^d}$ is the Gibbs field of the potential $\{(\beta\Phi)_\Lambda; \Lambda \subset \mathbb{Z}^d, \#(\Lambda) < \infty\}$ where $0 < \beta < \beta_{cr}$ and let $\mathscr{A} \subset \mathscr{B}$ be totally bounded with inclusion with exponent of metric entropy $r < 1$. Then the sequence of partial-sum processes $(Z_n)$ converges in distribution in $CA(\mathscr{A})$ to the continuous Brownian motion with parameter $\sigma$, where $0 < \sigma^2 := \sum_{\mathbf{k} \in \mathbb{Z}^d} E(\xi_{\mathbf{0}}\xi_{\mathbf{k}}) < \infty$.

We remark that since in the example the random variables are bounded, the case $\mathscr{A} = \mathscr{G}$, that is [Chen,[5] Corol. 4.2], can be obtained from our Theorem 2.

## 6. ABOUT THE MIXING ASSUMPTION

In order to preserve the unity of our work, we assumed throughout this paper the usual $\phi$-mixing condition (1.2) for Gibbs fields [see Künsch,[11] Nahapetian,[12] and Chen[5]]. Taken into account well known relations between mixing coefficients, we give now some alternatives to weaken the mixing assumptions in the preceding results.

Lemma 1 remains valid with an $\alpha$-mixing condition as in Bolthausen:[4]

$$\sup_{t \geqslant 0} t^{2d} \alpha_4(t) < +\infty \tag{6.1}$$

where $\alpha_4(t)$ is the supremum of all the quantities $|P(E \cap F) - P(E)\,P(F)|$ for $E \in \sigma_1$, $F \in \sigma_2$, $d(\Lambda_1, \Lambda_2) \geqslant t$ and $\#(\Lambda_1) + \#(\Lambda_2) \leqslant 4$ (the notations are the same than in (1.2)).

Lemmas 2 to 5 hold under (6.1) and the $\rho$-mixing condition (see also Bolthausen[4])

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} \rho_2(\|\mathbf{k}\|) < +\infty \tag{6.2}$$

where $\rho_2(t)$ is the supremum of $|Corr(X, Y))|$ for $X \in L^2(\sigma(\xi_{\mathbf{i}}))$, $Y \in L^2(\sigma(\xi_{\mathbf{j}}))$, with $\mathbf{i}, \mathbf{j} \in \mathbb{Z}^d$ and $\|\mathbf{i} - \mathbf{j}\| \geq t$.

Proposition 1 and, consequently, Theorems 1 and 2 are valid under (6.2) and the nonuniform $\alpha$-mixing condition:

$$\sup_{t \geq 0} t^{2d} \alpha(t) < +\infty \tag{6.3}$$

where $\alpha(t)$ is the supremum of all the quantities $(\#(A_1))^{-1} |P(E \cap F) - P(E) P(F)|$ for finite sets $A_1$ and $A_2$, with $d(A_1, A_2) \geq t$, $E \in \sigma_1$ and $F \in \sigma_2$. See (iii) in the proof of Proposition 1. Another possibility is to assume the non uniform $\rho$-mixing condition $\sup\{t^{2d}\rho(t) : t \geq 0\} < +\infty$, where $\rho(t)$ is the supremum of $(\#(A_1))^{-1} |Corr(X, Y))|$ for all finite sets $A_1$ and $A_2$ with $d(A_1, A_2) \geq t$, $X \in L^2(\sigma_1)$ and $Y \in L^2(\sigma_2)$. This condition implies (6.2) and (6.3).

Theorem 3 can be proved assuming only the non uniform $\beta$-mixing condition

$$\lim_{t \to \infty} t^{2d(1+\delta^{-1})} \beta(t) = 0 \tag{6.4}$$

with

$$\beta(t) = \sup \frac{1}{\#(A_1)} \left( \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} |P(U_i) P(V_j) - P(U_i \cap V_j)| \right)$$

where the supremum is taken over all pair of finite sets $A_1$, $A_2$ such that $d(A_1, A_2) \geq t$ and all the partitions $(U_i)$, $(V_j)$ of the whole space $\Omega$ such that, for all $i$ and $j$, $U_i \in \sigma_1$, $V_J \in \sigma_2$.

We give a justification of this assertion: (6.4) allows to prove the key point (4.3). Then, it remains to show that we have enough mixing assumptions for Theorem 1. Since $2\alpha(t) \leq \beta(t)$, condition (6.3) holds. By (2.3), (6.3), and (6.4) we have $\sum_{\mathbf{k} \in \mathbb{Z}^d} \alpha^{\delta/2+\delta}(\|\mathbf{k}\|) < +\infty$. This fact and Davydov's inequality (see [Bolthausen,[4] Lemma 1]), using that $E|\xi_{\mathbf{0}}|^{2+\delta} < +\infty$, replace the use of condition (6.2) in the arguments leading to Theorem 1.

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