Energy distribution of 2+1 dimensional black holes with nonlinear electrodynamics

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Abstract

The energy distributions for a black hole solution resulting from coupling electrodynamics and gravity in 2+1 dimensions are obtained. This solution considers the correction for a 2+1 static charged black hole from the first contribution of the weak field limit of one loop QED in 2+1 dimensions. The Einstein and Møller energy-momentum prescriptions are used to evaluate the energy distributions associated with the mentioned 2+1 dimensional black hole and other 2+1 black hole solutions coupled with nonlinear electrodynamics. A relation that connects the coefficients of both prescriptions is established.

1 Introduction

Due to general covariance of the theory, there is not a single way to define an energy-momentum tensor in the context of general relativity. Thus, since the introduction of the Einstein energy-momentum complex (or pseudotensor) [1], many other energy-momentum complexes have been introduced [2, 3, 4, 5, 6, 7, 8]; all of them restricted to evaluate the energy distribution in quasi-Cartesian coordinates. Møller [9] proposed an expression for an energy-momentum complex which could be used to calculate the energy distribution inside a spherical surface for the gravitational field in any coordinates system and not only in quasi-Cartesian coordinates. In the recent years, numerous researches have been accomplished on evaluating the energy distributions of different space-time solutions using the

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energy-momentum complexes (see for example Refs. [10] and [11] and references cited therein).

Some authors have studied the energy localization in 2+1 dimensional models using energy-momentum complexes. Virbhadra obtained the energy-momentum complexes in quasi-Cartesian coordinates for two exact solutions of the Einstein equation [12]. Yang and Radinschi calculated energy distributions of different black hole solutions using Einstein and Møller energy-momentum complexes [13] and also the Landau-Lifshitz and Weinberg energy-momentum complexes [14]. The Møller energy-momentum complex was also used to obtain the energy distribution of a radial magnetic field in AdS [15] and of an Einstein-Klein-Gordon system [16] respectively. Vagenas evaluated energy distributions for non-static spinless [17] and rotating BTZ black holes [18]. Besides other formulations have been applied to calculate the gravitational energy of black holes in 2+1 dimensions: the Brown-York method [19] (see also Refs. [20] and [21]) and the gravitational teleparallelism [22].

In this paper we investigate the static charged BTZ metric [23] with nonlinear electromagnetic field which arises from the correction of the first contribution of the weak field limit of one loop QED in 2+1 dimensions. We use energy-momentum prescriptions of Einstein and Møller to investigate the energy distributions of the obtained solution in this work. It is also interesting to consider other 2+1 dimensional electrodynamic theories of nonlinear type coupled to the static charged 2+1 black hole with cosmological constant given in Refs. [24] and [25] to get the corresponding energy distributions. Thus, it is shown there is a relation that connects the "coefficients" of the expression of the Einstein energy distribution which is of the form $E(r) = \sum \alpha_n^{(E)} r^{-n}$ with those of the expression of Møller energy distribution, i.e. $E(r) = \sum \alpha_n^{(M)} r^{-n}$, where in both cases we can define $\alpha_0^{(E)}$ and $\alpha_0^{(M)}$ as functions depending of r. We have to mention that Vagenas obtained the analog connection in 3+1 dimensions (see Ref. [10]). However, as was shown by Vagenas in Ref. [26], such a relation does not hold in 1+1 dimensions since he found that the Einstein energy-momentum complex is zero, thus it is not possible to establish a relation.

The paper is organized as follows. In Section 2 we calculate for the static BTZ solution considering QED in 2+1 dimensions, the counterpart of the correction for the Reissner-Nordström metric from the first contribution of the weak field limit of one loop QED in 3+1 dimensions. In the next two sections we find the energy distributions of the obtained metric. Sections 3 and 4 show the results obtained for the energy distribution using the Einstein and Møller prescriptions, respectively. The same calculations were performed for other kinds of static black hole solutions in 2+1 dimensions coupled with nonlinear electrodynamics. The results are showed in Section 5. Finally, in Section 6 we summarize the results.

2 The minimal coupling between gravitation and nonlinear electrodynamics in 2+1 dimensions

In this section we perform the coupling between electrodynamics and gravity in 2+1 dimensions. We consider the correction for the static charged black hole with cosmological constant from the first contribution of the weak field limit of one loop QED in 2+1 dimensions.

The Einstein theory coupled with the nonlinear electrodynamics in 2+1 dimensions arises from the following action

$$S = \int \sqrt{-g} \left[\frac{1}{2\kappa}(R - 2\Lambda) + L(F) \right] d^3x \ , \tag{1}$$

where $\kappa = 8\pi$, g is the determinant of the metric tensor $g_{\mu\nu}$, Λ is the cosmological constant and $L(F)$ is an electromagnetic lagrangian depending on $F \equiv F^{\mu\nu}F_{\mu\nu}$. Varying this action with respect to gravitational field we get the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{2}$$

where the corresponding stress-energy tensor is given by

$$T_{\mu\nu} = g_{\mu\nu}L(F) - 4 \, L_{,F} \, F_{\mu\alpha}F_\nu{}^\alpha \tag{3}$$

and $L_{,F}$ denotes the derivative of $L(F)$ with respect to F. The electromagnetic field equations are

$$\nabla_\mu(F^{\mu\nu}L_{,F}) = 0 \ . \tag{4}$$

For the static and spherically symmetric 2+1 dimensional solution, the metric is given by

$$ds^2 = -A(r)dt^2 + \frac{1}{A(r)}dr^2 + r^2d\theta^2 \ , \tag{5}$$

where $A(r)$ is a metric function to be determined.

Considering only the non zero component, the electromagnetic tensor becomes $F^{tr} = f(r)$. Then, Eqs. (2-4) implies

$$\frac{1}{2r}\frac{\partial A(r)}{\partial r} = \kappa[L(F) + 4f^2(r)L_{,F}] - \Lambda \tag{6}$$

$$f(r)L_{,F} = -\frac{Q}{8\pi r} \ . \tag{7}$$

From the analysis of the weak field limit of the complete one-loop approximation of QED in 2+1 dimensions (see Ref. [27], where it was analyzed the case of four photon scattering in any dimension $d \geq 2$), the effective lagrangian is given by

$$L = -\frac{1}{4\pi}F + \frac{\mu}{4\pi}F^2 \ , \tag{8}$$

where $\mu = e^4/(480\,\pi^2 m_e^5)$. The invariant $F = -2f(r)^2$ and Eq. (6), allows us to write

$$A(r) = -M - \Lambda r^2 - 2\kappa \int \left[\frac{r}{2\pi}f^2(r) + \frac{3}{\pi}r\mu f^4(r) \right] dr \ . \tag{9}$$

Moreover, from Eqs. (7) and (8) result

$$f(r) = \frac{Q}{2r} - \frac{\mu Q^3}{2r^3} \ , \tag{10}$$

which permits to obtain an explicit expression for the metric function

$$A(r) = -M - \Lambda r^2 - 2Q^2 \ln r - \frac{\mu Q^4}{2r^2} + O(\mu^2) \ . \tag{11}$$

Note that for the limit case $\mu = 0$ in Eq. (11) the static charged BTZ solution is obtained. The case of QED to one loop in 3+1 dimensions for the Reissner-Nordström metric has been considered in Ref. [28].

In the next two sections we use the Einstein and Møller prescriptions to calculate the energy distributions of the modified static charged BTZ metric.

3   Energy distribution under the Einstein prescription

In general relativity the conservation law must be valid for all frames of reference, thus if $T_\mu{}^\nu$ is the symmetric energy-momentum tensor of matter, it satisfies

$$T_\mu{}^\nu{}_{;\nu} = 0 \ , \tag{12}$$

The quantity that satisfies the conservation law in the usual sense is called energy-momentum complex which is given by

$$\Theta_\mu{}^\nu = \sqrt{-g} \left(T_\mu{}^\nu + t_\mu{}^\nu \right) \ , \tag{13}$$

where $t_\mu{}^\nu$ is an energy-momentum pseudotensor for the gravitational field. This complex can be written as the divergence of an antisymmetric superpotential [29]

$$\Theta_\mu{}^\nu = \frac{1}{2\kappa} H_\mu{}^{\nu\lambda}{}_{,\lambda} \ , \tag{14}$$

where the superpotential is of the form

$$H_\mu{}^{\nu\lambda} = \frac{g_{\mu\sigma}}{\sqrt{-g}} [-g(g^{\nu\sigma} g^{\lambda\rho} - g^{\lambda\sigma} g^{\nu\rho})]_{,\rho} \ . \tag{15}$$

The Einstein energy-momentum complex satisfies the local conservation equation

$$\Theta_\mu{}^\nu{}_{,\nu} = 0 \ . \tag{16}$$

The energy and the momentum components in quasi-Cartesian coordinates in 2+1 dimensions are given by

$$P_\mu = \frac{1}{2\kappa} \int H_\mu^{0i} \, n_i dl \ , \tag{17}$$

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where $i = 1, 2$ and n_i is the normal vector to the closed line l, and using the Gauss theorem we obtain the energy component

$$E_E(r) = \frac{1}{2\kappa} \int H_0^{0i} \, n_i dl \ . \tag{18}$$

In this prescription, the nonzero components H_0^{0i} of the Einstein energy-momentum complex are

$$H_0^{01} = \frac{x}{r^2}(-M - \Lambda r^2 - 2Q^2 \ln r - \mu \frac{Q^4}{2r^2} - 1) \ , \tag{19}$$

$$H_0^{02} = \frac{y}{r^2}(-M - \Lambda r^2 - 2Q^2 \ln r - \mu \frac{Q^4}{2r^2} - 1) \ . \tag{20}$$

Then, the energy distribution inside a circle with radius r becomes

$$E_E(r) = \frac{\pi}{\kappa}(-M - \Lambda r^2 - 2Q^2 \ln r - \mu \frac{Q^4}{2r^2} - 1) \ . \tag{21}$$

4   Energy distribution under the Møller prescription

The Møller energy-momentum complex differs from the one corresponding to the Einstein prescription by an added quantity $S_\mu{}^\nu$ which verify $S_\mu{}^\nu{}_{,\nu} = 0$. The definition of Møller energy-momentum complex is given by [9]

$$M_\mu{}^\nu = \frac{1}{\kappa} \chi_\mu{}^{\nu\lambda}{}_{,\lambda} \ , \tag{22}$$

where $\chi_\mu{}^{\nu\lambda}$ is an antisymmetric superpotential of the form

$$\chi_\mu{}^{\nu\lambda} = \sqrt{-g} \, (g_{\mu\sigma,\rho} - g_{\mu\rho,\sigma})g^{\nu\rho}g^{\lambda\sigma} \ . \tag{23}$$

The local conservation equation has the form

$$M_\mu{}^\nu{}_{,\nu} = 0 \ . \tag{24}$$

The energy and momentum components calculated using the Møller energy-momentum complex are

$$P_\mu = \frac{1}{\kappa} \int \chi_\mu{}^{0i}{}_{,i} \, d^2x \ , \tag{25}$$

where $i = 1, 2$. Through the Gauss theorem the corresponding Møller energy result

$$E_M(r) = \frac{1}{\kappa} \int \chi_0{}^{0i}{}_{,i} \, drd\theta \ . \tag{26}$$

At last, the nonzero component of the superpotential is

$$\chi_0^{01} = 2\Lambda r^2 + 2Q^2 - \mu \frac{Q^4}{r^2} \ . \tag{27}$$

Solving the integral in (26) considering (27) the energy distribution in a circular region of radius r is calculated

$$E_M(r) = \frac{\pi}{\kappa}(4\Lambda r^2 + 4Q^2 - 2\mu\frac{Q^4}{r^2}) \ . \tag{28}$$

Just as we expected, if $\mu \to 0$ the results expressed in (21) and (28) are in agreement with the energy distributions in both prescriptions for the 2+1 dimensional charged black hole as was evaluated in Ref. [13].

5 Energy distribution for another 2+1 dimensional static black hole solutions coupled to nonlinear electric field

In the following paragraphs we consider two different 2+1 dimensional static black-hole solutions, both coupled to nonlinear electric field and where the metrics have the form of Eq. (5). As in the above case, we calculate the corresponding distributions of energy under both considered prescriptions.

Let us consider the solution 2+1 dimensional black hole coupled to the Born-Infeld electrodynamics [24]

$$A(r) = -M - (\Lambda - b^2)r^2 - 2b^2 r\sqrt{r^2 + \frac{Q^2}{b^2}} - 2Q^2 \ln(r + \sqrt{r^2 + \frac{Q^2}{b^2}}) \ , \tag{29}$$

which can be rewritten as

$$A(r) = -M - (\Lambda - b^2)r^2 + 4b^2 \int_r^s \sqrt{u^2 + \frac{Q^2}{b^2}} \, du - C_1 \ , \tag{30}$$

where Λ is the cosmological constant, b is the Born-Infeld parameter, s is a constant and $C_1 = 2b^2 s\sqrt{s^2 + \frac{Q^2}{b^2}} + 2Q^2 \ln(s + \sqrt{s^2 + \frac{Q^2}{b^2}})$. Then

$$E_E(r) = \frac{\pi}{\kappa}\left[-M - (\Lambda - b^2)r^2 + 4b^2 \int_r^s \sqrt{u^2 + \frac{Q^2}{b^2}} \, du - C_1 - 1\right] \ , \tag{31}$$

$$E_M(r) = \frac{\pi}{\kappa}\left[4(\Lambda - b^2)r^2 + 8b^2 r\sqrt{r^2 + \frac{Q^2}{b^2}}\right] \ . \tag{32}$$

The second considered solution [25] is given by

$$A(r) = -M - \Lambda r^2 - Q^2 \ln(r^2 + a^2) \ , \tag{33}$$

where M, a, Q and Λ are free parameters. Rewriting, we have

$$A(r) = -M - \Lambda r^2 + Q^2 \int_r^s \frac{2u}{(u^2 + a^2)} du - Q^2 \ln(s^2 + a^2) \ . \tag{34}$$

The energy distributions become

$$E_E(r) = \frac{\pi}{\kappa}[-M - \Lambda r^2 + Q^2 \int_r^s \frac{2u}{(u^2 + a^2)}du - Q^2 \ln(s^2 + a^2) - 1] \qquad (35)$$

$$E_M(r) = \frac{\pi}{\kappa}\left[4\Lambda r^2 + \frac{4Q^2 r^2}{(r^2 + a^2)}\right] \quad . \qquad (36)$$

The above results suggest we can establish for the $2 + 1$ case, similar to Ref. [10] for $3 + 1$ dimensional static and spherically symmetric black holes (also see Ref. [11]), a connection between the coefficients $\alpha_n^{(E)}$ of the energy distribution under the Einstein prescription

$$E_E(r) = \sum_{n=-2}^{\infty} \alpha_n^{(E)} r^{-n} \qquad (37)$$

and the coefficients $\alpha_n^{(M)}$ of the energy distribution under the Møller prescription

$$E_M(r) = \sum_{n=-2}^{\infty} \alpha_n^{(M)} r^{-n} \quad . \qquad (38)$$

That relation is given by

$$\alpha_n^{(M)} = 2n\alpha_n^{(E)} \quad . \qquad (39)$$

If the the energy distribution has any term proportional to $\int F(u)\,du - C$, it can be written as

$$\alpha_0^{(E)} = \int_r^s F(u)\,du - C \qquad (40)$$

and therefore

$$\alpha_0^{(M)} = 2\,r\,F(r) \quad . \qquad (41)$$

In what follows we establish the relation between Eqs. (21) and (28) having in mind the above relations. To establish it we rewrite the Eq. (11) as

$$A(r) = -M - \Lambda r^2 + 2Q^2 \int_r^s \frac{1}{u}\,du - 2Q^2 \ln(s) - \frac{\mu Q^4}{2r^2} \quad . \qquad (42)$$

Considering Eq. (42), the energy distribution in Einstein prescription is

$$E_E(r) = \frac{\pi}{\kappa}[-M - \Lambda r^2 + 2Q^2 \int_r^s \frac{1}{u}\,du - 2Q^2 \ln(s) - \mu\frac{Q^4}{2r^2} - 1] \quad . \qquad (43)$$

Note that this expression is equivalent to Eq. (21). Now if we use the Eqs. (39-41) we get the following energy distribution

$$E_M(r) = \frac{\pi}{\kappa}[4\Lambda r^2 + 4Q^2 - 2\mu\frac{Q^4}{r^2}] \quad , \qquad (44)$$

which is similar to (28) obtained under the Møller prescription. Additionally, the relation given by (39) is valid for the other considered cases in Ref. [13]. One may also check the relation for other 2+1 dimensional black hole solutions with metric of the form (5). For example, the cases considered in Refs. [30] and [31] show that the relation (39) is fulfilled.

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6 Summary

In the previous sections we have obtained energy distributions for several 2+1 dimensional static black-hole solutions coupled to nonlinear electric field. First, we have presented the metric function (11) obtained from the first contribution of the weak field limit of one loop QED in 2+1 dimensions. We have calculated the energy distributions associated with the obtained metric using Einstein and Møller prescriptions. Similar calculations have been accomplished for another static black hole solutions in 2+1 dimensions coupled with nonlinear electrodynamics [24, 25]. We have established when the energy distributions are of form given by expressions (37) and (38) there is a relation that connects the coefficients α_r of the Einstein prescription with those in the Møller prescription. In summary, we showed that if the metric has the form of Eq. (5), the energy distributions in the Einstein and Møller prescriptions are, respectively

$$E_E(r) = \frac{\pi}{\kappa}[A(r) - 1] \; , \tag{45}$$

$$E_M(r) = -\frac{2\pi}{\kappa} \, r A'(r) \; . \tag{46}$$

Therefore, if we consider the expressions (37) and (45) we obtain

$$A'(r) = \sum_{n=-2}^{\infty} \frac{\kappa}{\pi} \left[(\alpha_n^{(E)})' \, r^{-n} - n \, \alpha_n^{(E)} r^{-(n+1)} \right] \; . \tag{47}$$

Thus, substituting (47) in (46) and considering the form (38) for the Møller energy distribution, we arrive to

$$\alpha_n^{(M)} = 2 \, n \, \alpha_n^{(E)} \tag{48}$$

and

$$\alpha_0^{(M)} = -2 \, r \, (\alpha_0^{(E)})' \; , \tag{49}$$

when $\alpha_0^{(E)} = \int_r^s F(u) du$.

Acknowledgments

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