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Convolution of Ultradistributions, Field Theory, Lorentz Invariance and Resonances *

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Abstract

In this work, a general definition of convolution between two arbitrary Ultradistributions of Exponential type (UET) is given. The

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product of two arbitrary UET is defined via the convolution of its corresponding Fourier Transforms. Some examples of convolution of two UET are given.

Expressions for the Fourier Transform of spherically symmetric (in Euclidean space) and Lorentz invariant (in Minkowskian space) UET in term of modified Bessel distributions are obtained (Generalization of Bochner's theorem).

The generalization to UET of dimensional regularization in configuration space is obtained in both, Euclidean and Minkowskian spaces

As an application of our formalism, we give a solution to the question of normalization of resonances in Quantum Mechanics.

General formulae for convolution of even, spherically symmetric and Lorentz invariant UET are obtained and several examples of application are given.

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1 Introduction

The question of the product of distributions with coincident point singularities is related in Field Theory, to the asymptotic behavior of loop integrals of propagators. From a mathematical point of view, the question reduces to define a product in a ring with zero-factors. As is known, the usual definitions lead to limitations on the set of distributions that can be multiplied together to give another distribution of the same kind. The properties of tempered ultradistributions (ref.[1, 2]) are well adapted for their use in Field Theory. In this respect we have shown (ref.[3, 4, 5]) that it is possible to define the convolution of any pair of tempered Ultradistributions, giving as a result another tempered Ultradistribution.

Ultradistributions also have the advantage of being representable by means of analytic functions. So that, in general, they are easier to work with them and, as we shall see, have interesting properties. One of those properties is that Schwartz's tempered distributions are canonical and continuously injected into tempered Ultradistributions and as a consequence the Rigged Hilbert Space with tempered distributions is canonical and continuously included in the Rigged Hilbert Space with tempered Ultradistributions.

A further step is to consider Ultradistributions of Exponential Type (ref.[1, 6]), and to define a convolution product between any pair of them. This is made in this paper together some examples and applications. In this case Schwartz's tempered dsitributions and Sebastiao e Silva's tempered Ultradistributions are canonical and continuously injected into Ultradistributions of Exponential Type and as a consequence the Rigged Hilbert Spaces with tempered distributions and tempered Ultradistributions are canonical and continuously included in the Rigged Hilbert Space with Ultradistributions of Exponential Type.

Furthermore, Ultradistributions of Exponential Type are adequates to describe Gamow States and exponentially increasing fields in Quantun Field Theory as we show in this paper.

This paper is organized as follow: in section 2 we define the Ultradistributions of Exponential Type and their Fourier transform. They are part of a Guelfand's Triplet (or Rigged Hilbert Space [7]) together with their respective duals and a "middle term" Hilbert space. In section 3 we give a general expression for the Fourier transform of a spherically symmetric Ultraditributions of Exponential Type (UET) and some examples ot it. In section 4 we obtain the expression for the Fourier transform of Lorentz invariant UET and we give some examples of their use. In section 5 we give the generalization to UET of "dimensional regularization in configuration space" obtained in ref.[5, 8]. In section 6 we give the general formula for the convolution of two UET and some examples. In section 7 we give a solution to the question of normalization of Gamow states. We give as example the l=0 state of Square-Well potential. In section 8 we present the formula for the convolution of two even UET, and various examples. In section 9, we give the general formula for the convolution of two Spherically Symmetric UET and several examples. In section 10 we treat the final topic of this paper: the formula for the convolution of two Lorentz Invariant Ultradistributions of Exponential Type. We obtain it and give examples of their use. Finally, section 11 is reserved for a discussion of the principal results.

2 Ultradistributions of Exponential Type

Let $\mathcal{S}$ be the space of Schwartz of test functions rapidly decreasing. Let Λ_j be the region of the complex plane defined as:

$$\Lambda_j = \{z \in \mathbb{C} : |\Im(z)| < j : j \in \mathbb{N}\} \tag{2.1}$$

According to ref.[1, 6] the space of test functions $\hat{\phi} \in \mathcal{V}_j$ is constituted by all entire analytic functions of $\mathcal{S}$ for which

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{(j|\Re(z)|)} |\hat{\phi}^{(k)}(z)| \right] \right\} \tag{2.2}$$

is finite.

The space $\mathcal{Z}$ is then defined as:

$$\mathcal{Z} = \bigcap_{j=0}^{\infty} \mathcal{V}_j \tag{2.3}$$

It is a complete countably normed space with the topology generated by the system of semi-norms $\{||\cdot||_j\}_{j\in\mathbb{N}}$. The dual of $\mathcal{Z}$, denoted by $\mathcal{B}$, is by definition the space of ultradistributions of exponential type (ref.[1, 6]). Let $\mathcal{S}$ the space of rapidly decreasing sequences. According to ref.[7] $\mathcal{S}$ is a nuclear space. We consider now the space of sequences $\mathcal{P}$ generated by the Taylor development of $\hat{\phi} \in \mathcal{Z}$

$$\mathcal{P} = \left\{ \mathcal{Q} : \mathcal{Q}\left(\hat{\phi}(0), \hat{\phi}'(0), \frac{\hat{\phi}''(0)}{2}, ..., \frac{\hat{\phi}^{(n)}(0)}{n!}, ... \right) : \hat{\phi} \in \mathcal{Z} \right\} \tag{2.4}$$

The norms that define the topology of $\mathcal{P}$ are given by:

$$||\hat{\phi}||'_p = \sup_n \frac{n^p}{n} |\hat{\phi}^n(0)| \tag{2.5}$$

$\mathcal{P}$ is a subespace of $\mathcal{S}$ and therefore is a nuclear space. As the norms $\|\cdot\|_j$ and $\|\cdot\|'_p$ are equivalent, the correspondence

$$\mathcal{Z} \Longleftrightarrow \mathcal{P} \tag{2.6}$$

is an isomorphism and therefore $\mathcal{Z}$ is a countably normed nuclear space. We can define now the set of scalar products

$$< \hat{\phi}(z), \hat{\psi}(z) >_n = \sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|z|} \overline{\hat{\phi}^{(q)}}(z) \hat{\psi}^{(q)}(z) \, dz =$$

$$\sum_{q=0}^{n} \int_{-\infty}^{\infty} e^{2n|x|} \overline{\hat{\phi}^{(q)}}(x) \hat{\psi}^{(q)}(x) \, dx \tag{2.7}$$

This scalar product induces the norm

$$\|\hat{\phi}\|''_n = [< \hat{\phi}(x), \hat{\phi}(x) >_n]^{\frac{1}{2}} \tag{2.8}$$

The norms $\|\cdot\|_j$ and $\|\cdot\|''_n$ are equivalent, and therefore $\mathcal{Z}$ is a countably hilbertian nuclear space. Thus, if we call now $\mathcal{Z}_p$ the completion of $\mathcal{Z}$ by the norm p given in (2.8), we have:

$$\mathcal{Z} = \bigcap_{p=0}^{\infty} \mathcal{Z}_p \tag{2.9}$$

where

$$\mathcal{Z}_0 = \mathbf{H} \tag{2.10}$$

is the Hilbert space of square integrable functions.

As a consequence the "nested space"

$$\mathcal{U} = (\mathcal{Z}, \mathsf{H}, \mathcal{B}) \tag{2.11}$$

is a Guelfand's triplet (or a Rigged Hilbert space=RHS. See ref.[7]).

Any Guelfand's triplet $\mathcal{G} = (\boldsymbol{\Phi}, \mathsf{H}, \boldsymbol{\Phi}')$ has the fundamental property that a linear and symmetric operator on $\boldsymbol{\Phi}$, admitting an extension to a self-adjoint operator in H, has a complete set of generalized eigen-functions in $\boldsymbol{\Phi}'$ with real eigenvalues.

$\mathcal{B}$ can also be characterized in the following way (refs.[1],[6]): let $\mathcal{E}_\omega$ be the space of all functions $\hat{F}(z)$ such that:

I- $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C} : |Im(z)| > p\}$.

II- $\hat{F}(z)e^{-p|\Re(z)|}/z^p$ is bounded continuous in $\{z \in \mathbb{C} : |Im(z)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.

Let $\mathcal{N}$ be: $\mathcal{N} = \{\hat{F}(z) \in \mathcal{E}_\omega : \hat{F}(z)$ is entire analytic$\}$. Then $\mathcal{B}$ is the quotient space:

III- $\mathcal{B} = \mathcal{E}_\omega/\mathcal{N}$

Due to these properties it is possible to represent any ultradistribution as (ref.[1, 6]):

$$\hat{F}(\hat{\phi}) = < \hat{F}(z), \hat{\phi}(z) > = \oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\, dz \tag{2.12}$$

where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z) < -\zeta$, $-\zeta < -p$. (Γ surrounds all the singularities of $\hat{F}(z)$).

Formula (2.12) will be our fundamental representation for a tempered ultradistribution. Sometimes use will be made of "Dirac formula" for exponential ultradistributions (ref.[1]):

$$\hat{F}(z) \equiv \frac{1}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{t-z}\, dt \equiv \frac{\cosh(\lambda z)}{2\pi i}\int\limits_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t-z)\cosh(\lambda t)}\, dt \tag{2.13}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint\limits_{\Gamma} \hat{F}(z)\hat{\phi}(z)\, dz = \int\limits_{-\infty}^{\infty} \hat{f}(t)\hat{\phi}(t)\, dt \tag{2.14}$$

(2.13) should be used carefully in this case. While $\hat{F}(z)$ is analytic on Γ, the density $\hat{f}(t)$ is in general singular, so that the r.h.s. of (2.14) should be interpreted in the sense of distribution theory.

Another important property of the analytic representation is the fact that on Γ, $\hat{F}(z)$ is bounded by a exponential and a power of z (ref.[1, 6]):

$$|\hat{F}(z)| \leq C|z|^p e^{p|\Re(z)|} \tag{2.15}$$

where C and p depend on $\hat{F}$.

The representation (2.12) implies that the addition of any entire function $\hat{G}(z \in \mathcal{N})$ to $\hat{F}(z)$ does not alter the ultradistribution:

$$\oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz + \oint_{\Gamma} \hat{G}(z)\hat{\phi}(z)\ dz$$

But:

$$\oint_{\Gamma} \hat{G}(z)\hat{\phi}(z)\ dz = 0$$

as $\hat{G}(z)\hat{\phi}(z)$ is entire analytic (and rapidly decreasing),

$$\therefore \quad \oint_{\Gamma} \{\hat{F}(z) + \hat{G}(z)\}\hat{\phi}(z)\ dz = \oint_{\Gamma} \hat{F}(z)\hat{\phi}(z)\ dz \tag{2.16}$$

Another very important property of $\mathcal{B}$ is that $\mathcal{B}$ is reflexive under the Fourier transform:

$$\mathcal{B} = \mathcal{F}_c\{\mathcal{B}\} = \mathcal{F}\{\mathcal{B}\} \tag{2.17}$$

where the complex Fourier transform $F(k)$ of $\hat{F}(z) \in \mathcal{B}$ is given by:

$$F(k) = \Theta[\Im(k)] \int_{\Gamma_+} \hat{F}(z)e^{ikz}\ dz - \Theta[-\Im(k)] \int_{\Gamma_-} \hat{F}(z)e^{ikz}\ dz =$$

$$\Theta[\Im(k)] \int_{0}^{\infty} \hat{f}(x)e^{ikx}\ dx - \Theta[-\Im(k)] \int_{-\infty}^{0} \hat{f}(x)e^{ikx}\ dx \tag{2.18}$$

Here Γ_+ is the part of Γ with $\Re(z) \geq 0$ and Γ_- is the part of Γ with $\Re(z) \leq 0$

Using (2.18) we can interpret Dirac's formula as:

$$F(k) \equiv \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(s)}{s-k}\, ds \equiv \mathcal{F}_c \left\{ \mathcal{F}^{-1}\{f(s)\} \right\} \tag{2.19}$$

The treatment for ultradistributions of exponential type defined on $\mathbb{C}^n$ is similar to the one-variable. In this case

$$\Lambda_j = \{z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n : |\Im(z_k)| \leq j \quad 1 \leq k \leq n\} \tag{2.20}$$

$$||\hat{\phi}||_j = \max_{k \leq j} \left\{ \sup_{z \in \Lambda_j} \left[e^{j\left[\sum\limits_{p=1}^{n} |\Re(z_p)|\right]} \left| D^{(k)} \hat{\phi}(z) \right| \right] \right\} \tag{2.21}$$

where $D^{(k)} = \partial^{(k_1)} \partial^{(k_2)} \cdots \partial^{(k_n)} \quad k = k_1 + k_2 + \cdots + k_n$

$\mathcal{B}$ is characterized as follows. Let $\mathcal{E}_\omega$ be the space of all functions $\hat{F}(z)$ such that:

I$'$ - $\hat{F}(z)$ is analytic for $\{z \in \mathbb{C}^n : |Im(z_1)| > p, |Im(z_2)| > p, ..., |Im(z_n)| > p\}$.

II$'$ - $\hat{F}(z) e^{-\left[p \sum\limits_{j=1}^{n} |\Re(z_j)|\right]} / z^p$ is bounded continuous in $\{z \in \mathbb{C}^n : |Im(z_1)| \geqq p, |Im(z_2)| \geqq p, ..., |Im(z_n)| \geqq p\}$, where $p = 0, 1, 2, ...$ depends on $\hat{F}(z)$.

Let $\mathcal{N}$ be: $\mathcal{N} = \left\{ \hat{F}(z) \in \mathcal{E}_\omega : \hat{F}(z) \right.$ is entire analytic at minus in one of the variables $z_j \quad 1 \leq j \leq n\}$ Then $\mathcal{B}$ is the quotient space:

III$'$ - $\mathcal{B} = \mathcal{E}_\omega / \mathcal{N}$ We have now

$$\hat{F}(\hat{\phi}) = < \hat{F}(z), \hat{\phi}(z) > = \oint_{\Gamma} \hat{F}(z) \hat{\phi}(z)\ dz_1\ dz_2 \cdots dz_n \tag{2.22}$$

$\Gamma = \Gamma_1 \cup \Gamma_2 \cup ... \Gamma_n$ where the path Γ_j runs parallel to the real axis from $-\infty$ to ∞ for $Im(z_j) > \zeta$, $\zeta > p$ and back from ∞ to $-\infty$ for $Im(z_j) < -\zeta$, $-\zeta < -p$. (Again Γ surrounds all the singularities of $\hat{F}(z)$). The n-dimensional Dirac's formula is

$$\hat{F}(z) = \frac{1}{(2\pi i)^n} \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{(t_1 - z_1)(t_2 - z_2)...(t_n - z_n)} \, dt_1 \, dt_2 \cdots dt_n \tag{2.23}$$

where the "density" $\hat{f}(t)$ is such that

$$\oint_{\Gamma} \hat{F}(z)\hat{\phi}(z) \, dz_1 \, dz_2 \cdots dz_n = \int_{-\infty}^{\infty} f(t)\hat{\phi}(t) \, dt_1 \, dt_2 \cdots dt_n \tag{2.24}$$

and the modulus of $\hat{F}(z)$ is bounded by

$$|\hat{F}(z)| \leq C|z|^p e^{\left[p \sum\limits_{j=1}^{n} |\Re(z_j)|\right]} \tag{2.25}$$

where C and p depend on $\hat{F}$.

3 The Fourier Transform in Euclidean Space

We define a spherically symmetric ultradistribution of exponential type $\hat{F}(z)$ as a ultradistribution of exponential type such that $\hat{f}(t)$ in (2.23) is spherically symmetric (Note that a spherically symmetric ultradistribution is not necessarily spherically symmetric in an explicit way). In this case we can use for the Fourier transform of $\hat{f}(t)$ the formula obtained in ref.[5]:

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \left\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \right\} dx +$$

$$\frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\, dx \tag{3.1}$$

When $\nu = 2n$, n an integer number $\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}$ is null. In fact

$$\rho^{\frac{2-\nu}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}} = \sum_{m=0}^{\frac{\nu-4}{2}} \frac{(\frac{\nu}{2}-m)!}{m!} 4^{\frac{\nu-2-4m}{4}} x^{\frac{4m+2-\nu}{4}} \rho^{\frac{2m+2-\nu}{2}} = 0 \tag{3.2}$$

that is null in the complex variable ρ in a space of dimension $\nu = 2n$. Thus in this case the second integral in (3.1) vanishes and it becomes in:

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \hat{f}(x) x^{\frac{\nu-2}{4}} \left[\Theta[\Im(\rho)] e^{-i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) \right.$$

$$\left. -\Theta[-\Im(\rho)] e^{i\frac{\pi}{4}\nu} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \right] dx \tag{3.3}$$

In the next section we shall see that formulae (3.2), (3.3) can be generalized to Minkowskian space.

When $\hat{f}$ is a spherically symmetric , we can use (3.3) to define its Fourier transform. In addition for $\nu = 2n$ we can follow the treatment of ref.[9] to define the Fourier transform. Thus we have

$$\int_{-\infty}^\infty f(\rho)\phi(\rho)\rho^{\frac{\nu-2}{2}}\, d\rho = (2\pi)^\nu \int_0^\infty \hat{f}(x)\hat{\phi}(x) x^{\frac{\nu-2}{2}}\, dx \tag{3.4}$$

The corresponding ultradistribution of exponential type in the one-dimensional complex variable ρ is obtained in the following way: let $\hat{g}(t)$ defined as:

$$\hat{g}(t) = \frac{1}{(2\pi)^{\nu}} \int\limits_{-\infty}^{\infty} f(\rho) e^{-i\rho t} \, d\rho \tag{3.5}$$

Then:

$$F(\rho) = \Theta[\Im(\rho)] \int\limits_{0}^{\infty} \hat{g}(t) e^{i\rho t} \, dt - \Theta[-\Im(\rho)] \int\limits_{-\infty}^{0} \hat{g}(t) e^{i\rho t} \, dt \tag{3.6}$$

or if we use Dirac's formula

$$F(\rho) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(t)}{t-\rho} \, dt \tag{3.7}$$

The inversion formula ($\nu = 2n$) for $F(\rho)$ is given by

$$\hat{f}(x) = \frac{\pi}{(2\pi)^{\frac{\nu+2}{2}} x^{\frac{\nu-2}{4}}} \oint\limits_{\Gamma} F(\rho) \rho^{\frac{\nu-2}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2}) \, d\rho \tag{3.8}$$

Note that the part of the integrand that multiplies to $F(\rho)$ is an entire function of ρ for $\nu = 2n$. In this case the first term of (4.13) take the form:

$$\oint\limits_{\Gamma} F(\rho) \phi(\rho) \rho^{\frac{\nu-2}{2}} \, d\rho = (2\pi)^{\nu} \int\limits_{0}^{\infty} \hat{f}(x) \hat{\phi}(x) x^{\frac{\nu-2}{2}} \, dx \tag{3.9}$$

We give now same examples of the use of Fourier transform.

Examples

As a first example we calculate the Fourier transform of

$$2^{-\nu} \Theta[\Im(z_1)] \Theta[\Im(z_2)] \cdots \Theta[\Im(z_{\nu})] \cosh\left(a\sqrt{z_1^2 + z_2^2 + \cdots + z_{\nu}^2}\right) \tag{3.10}$$

where a is a complex number for $\nu = 2n$. From (3.3)

$$F(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} \int_0^\infty \cosh(ax^{\frac{1}{2}}) x^{\frac{\nu-2}{4}} \left\{ \Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2})\, dx \right\} \tag{3.11}$$

Now:

$$\int_0^\infty e^{ax^{1/2}} x^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) = 2\sqrt{\pi}\, e^{\frac{i\pi(\nu+2)}{4}} \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2} - ia)} \times \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a - i\rho^{1/2}}{a + i\rho^{1/2}}\right) \quad \Im(\rho) > 0$$

$$\int_0^\infty e^{ax^{1/2}} x^{\frac{\nu-2}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) = 2\sqrt{\pi}\, e^{-\frac{i\pi(\nu+2)}{4}} \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \frac{\rho^{\frac{\nu-2}{4}}}{(\rho^{1/2} + ia)} \times \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a + i\rho^{1/2}}{a - i\rho^{1/2}}\right) \quad \Im(\rho) < 0 \tag{3.12}$$

To obtain (3.12) we have used **6.621**, (3) of ref.[10] (Here $\mathbf{F}$ is the hypergeometric function). Then we have:

$$F(\rho) = (4\pi)^{\frac{\nu-2}{2}} i \frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})} \left\{ \frac{1}{(\rho^{1/2} - ia)} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a - i\rho^{1/2}}{a + i\rho^{1/2}}\right) + \frac{1}{(\rho^{1/2} + ia)} \mathbf{F}\left(\nu, \frac{\nu-1}{2}, \frac{\nu+3}{2}, \frac{a + i\rho^{1/2}}{a - i\rho^{1/2}}\right) \right\} \tag{3.13}$$

As a second example we evaluate the Fourier transform of

$$2^{-\nu}\Theta[\Im(z_1)]\Theta[\Im(z_2)]\cdots\Theta[\Im(z_\nu)] \frac{\pi\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu+2}{2}}} (z_1^2 + z_2^2 + \cdots + z_\nu^2)^{\frac{2-\nu}{2}} \times$$

$$\mathcal{J}_{\frac{\nu-2}{2}}[\mu(z_1^2+z_2^2+\cdots+z_\nu^2)^{\frac{1}{2}}] \tag{3.14}$$

We take into account that for ν even $\mathcal{J}_{\frac{\nu-2}{2}} = e^{\frac{i\pi(\nu-2)}{2}}\mathcal{J}_{\frac{2-\nu}{2}}$. Thus:

$$F(\rho) = \frac{\mu^{\frac{\nu-2}{2}}}{4\pi} e^{\frac{i\pi(\nu-2)}{2}} \rho^{\frac{2-\nu}{4}} \int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \left\{\Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \right\} \, dx \tag{3.15}$$

Now:

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) \, dx = e^{\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2} \; ; \; \Im(\rho) > 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \, dx = e^{-\frac{i\pi(6-\nu)}{4}} \mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2} \; ; \; \Im(\rho) < 0 \tag{3.16}$$

where we have used **6.576,** (3) of ref.[10]. Then we have:

$$F(\rho) = -\frac{1}{2\pi i(\rho-\mu^2)} \tag{3.17}$$

4 The Fourier Transform in Minkowskian Space

We define a Lorentz invariant ultradistribution of exponential type $\hat{F}(z)$ as an ultradistribution of exponential type such that $\hat{f}(t)$ in (2.23) is Lorentz invariant (Note that a Lorentz invariant ultradistribution is not necessarily Lorentz invariant in an explicit way). In this case we can use for the Fourier transform of $\hat{f}(t)$ the formula obtained in ref.[5]:

$$F(\rho) = (2\pi)^{\frac{\nu-2}{2}} \int_{-\infty}^{\infty} \hat{f}(x) \left\{ \Theta[\Im(\rho)] e^{\frac{i\pi(\nu-2)}{4}} \frac{(x+i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}] - \right.$$

$$\left. \Theta[-\Im(\rho)] e^{\frac{i\pi(2-\nu)}{4}} \frac{(x-i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}] \right\} dx \qquad (4.1)$$

Here we have taken $\rho = \gamma + i\sigma$ and

$$\rho^{1/2} = \sqrt{\frac{\gamma + \sqrt{\gamma^2+\sigma^2}}{2}} + i\mathrm{Sgn}(\sigma)\sqrt{\frac{-\gamma + \sqrt{\gamma^2+\sigma^2}}{2}} \qquad (4.2)$$

When $\hat{f}$ is Lorentz invariant , we can use (4.1) or adopt the following treatment: starting from

$$\iiiint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0)\ d^4k = (2\pi)^{\nu} \iiiint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0) d^4x \qquad (4.3)$$

we can deduce the equality:

$$\iint_{-\infty}^{\infty} f(\rho)\phi(\rho, k^0)(k_0^2 - \rho)_+^{\frac{\nu-3}{2}}\ d\rho\ dk^0 =$$

$$\iint_{-\infty}^{\infty} \hat{f}(x)\hat{\phi}(x, x^0)(x - x_0^2)_+^{\frac{\nu-3}{2}}\ dx\ dx^0 \qquad (4.4)$$

Let $g(t)$ be defined as:

$$\hat{g}(t) = \frac{1}{(2\pi)^{\nu}} \int_{-\infty}^{\infty} f(\rho) e^{-i\rho t}\ d\rho \qquad (4.5)$$

Then:

$$F(\rho) = \Theta[\Im(\rho)] \int_0^{\infty} \hat{g}(t) e^{i\rho t}\ dt - \Theta[-\Im(\rho)] \int_{-\infty}^{0} \hat{g}(t) e^{i\rho t}\ dt \qquad (4.6)$$

or if we use Dirac's formula

$$F(\rho) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(t)}{t-\rho}\, dt \tag{4.7}$$

The inverse of the Fourier transform can be evaluated in the following way: we define

$$\hat{G}(x,\Lambda) = \frac{1}{(2\pi)^{\frac{\nu+2}{2}}} \oint\limits_{\Gamma} F(\rho) \left\{ e^{\frac{i\pi(\nu-2)}{4}} \frac{(\rho+\Lambda)^{\frac{\nu-2}{4}}}{(x+i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}(\rho+\Lambda)^{1/2}] + \right.$$

$$\left. + e^{\frac{i\pi(2-\nu)}{4}} \frac{(\rho-\Lambda)^{\frac{\nu-2}{4}}}{(x-i0)^{\frac{\nu-2}{4}}} \mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}(\rho-\Lambda)^{1/2}] \right\} d\rho \tag{4.8}$$

then

$$\hat{f}(x) = \hat{G}(x, i0^+) \tag{4.9}$$

We give now same examples of the use of Fourier transform in Minkowskian space.

Examples

As a first example we consider the Fourier transform of the ultradistribution

$$2^{\nu}\Theta;\Im(z_0)]\Theta[\Im(z_2)]\cdots\Theta[\Im(z_{\nu-1})]\left[\cosh\left(a\sqrt{|z_0^2-z_1^2-\cdots-z_{\nu-1}^2|}\right)\right.$$

$$\left. + \cos\left(a\sqrt{|z_0^2-z_1^2-\cdots-z_{\nu-1}^2|}\right)\right] \tag{4.10}$$

where a is a complex number.

The cut along the real axis of (4.10) is:

$$2^{-1}\left[e^{a\sqrt{|x_0^2-r^2|}}+e^{-a\sqrt{|x_0^2-r^2|}}+e^{ia\sqrt{|x_0^2-r^2|}}+e^{-ia\sqrt{|x_0^2-r^2|}}\right] \qquad (4.11)$$

The Fourier transform is:

$$F(\rho)=(2\pi)^{\frac{\nu-2}{2}}\int\limits_{-\infty}^{\infty} e^{|x|^{\frac{1}{2}}}2^{-1}\left[e^{a\sqrt{|x_0^2-r^2|}}+e^{-a\sqrt{|x_0^2-r^2|}}+e^{ia\sqrt{|x_0^2-r^2|}}+e^{-ia\sqrt{|x_0^2-r^2|}}\right]$$

$$\left\{\Theta[\Im(\rho)]e^{\frac{i\pi(\nu-2)}{4}}\frac{(x+i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}]-\right.$$

$$\left.\Theta[-\Im(\rho)]e^{\frac{i\pi(2-\nu)}{4}}\frac{(x-i0)^{\frac{\nu-2}{4}}}{\rho^{\frac{\nu-2}{4}}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}]\right\}\,dx \qquad (4.12)$$

Now:

$$e^{\frac{i\pi(\nu-2)}{4}}\int\limits_{-\infty}^{\infty} e^{a|x|^{\frac{1}{2}}}(x+i0)^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}[-i(x+i0)^{1/2}\rho^{1/2}]=$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}-ia)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)-$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)\quad \Im(\rho)>0 \qquad (4.13)$$

$$e^{\frac{i\pi(2-\nu)}{4}}\int\limits_{-\infty}^{\infty} e^{a|x|^{\frac{1}{2}}}(x-i0)^{\frac{\nu-2}{4}}\mathcal{K}_{\frac{\nu-2}{2}}[i(x-i0)^{1/2}\rho^{1/2}]=$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{-\frac{i\pi\nu}{2}}}{(\rho^{1/2}+ia)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)-$$

$$2^{\frac{\nu}{2}}\sqrt{\pi}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\frac{e^{\frac{i\pi\nu}{2}}}{(\rho^{1/2}+a)^{\nu}}F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)\quad \Im(\rho)<0 \qquad (4.14)$$

For to obtain (4.13) and (4.14) we have used **6.621**, (3) of ref.[10]. With these results we have:

$$F(\rho) = \frac{(4\pi)^{\frac{\nu-1}{2}}}{2}\frac{\Gamma(\nu)}{\Gamma(\frac{\nu+3}{2})}\left\{\Theta[\Im(\rho)]e^{\frac{i\pi\nu}{2}}\left[\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^\nu}+\right.\right.$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^\nu}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}+a)^\nu}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}-a)^\nu}-$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^\nu}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}-a)^\nu}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^\nu}-$$

$$\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^\nu}\right]-\Theta[-\Im(\rho)]e^{-\frac{i\pi\nu}{2}}\left[\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^\nu}+\right.$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^\nu}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}-a)^\nu}+\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}+a)^\nu}-$$

$$\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+\rho^{1/2}}{a-\rho^{1/2}}\right)}{(\rho^{1/2}+a)^\nu}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-\rho^{1/2}}{a+\rho^{1/2}}\right)}{(\rho^{1/2}-a)^\nu}-\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a-i\rho^{1/2}}{a+i\rho^{1/2}}\right)}{(\rho^{1/2}+ia)^\nu}-$$

$$\left.\left.\frac{F\left(\nu,\frac{\nu-1}{2},\frac{\nu+3}{2},\frac{a+i\rho^{1/2}}{a-i\rho^{1/2}}\right)}{(\rho^{1/2}-ia)^\nu}\right]\right\} \tag{4.15}$$

As a second example we evaluate the Fourier transform of the ultradistribution ($\nu = 2n$):

$$\hat{F}(z) = -\frac{(-1)^{\frac{\nu}{2}}i\mu^{\nu-2}}{2^\nu\pi^{\frac{\nu-2}{2}}}\Theta[\Im(z_0)]\Theta[\Im(z_2)]\cdots-\Theta[\Im(z_{\nu-1})]\times$$

$$\sum_{k=0}^{\infty}\frac{(-1)^k\mu^{2k}(z_0^2-z_1^2-\cdots-z_{\nu-1}^2)^{kk}}{2^{2k}(k)!\Gamma(\nu+k)} \tag{4.16}$$

The cut along the real axis of $\hat{F}(z)$ is:

$$\hat{f}(x) = \hat{f}_\mu(x_+) - \hat{f}_\mu(x_-) \tag{4.17}$$

where

$$\hat{f}_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}} x^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \tag{4.18}$$

Observe that

$$\hat{f}_\mu(x_+) = w_\mu(x) = -\frac{i\pi}{2}\frac{\mu^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}} x_+^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{2-\nu}{2}}(\mu x_+^{1/2}) \tag{4.19}$$

is the complex mass Wheeler's propagator. Then according to (4.1)

$$F(\rho) = -\frac{i(\mu)^{\frac{\nu-2}{2}}}{4}\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu x^{1/2}) \left\{\frac{\Theta[\Im(\rho)]}{\rho^{\frac{\nu-2}{4}}}\left[e^{\frac{i\pi(\nu-2)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) + e^{\frac{i\pi(\nu-2)}{2}}\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\right] - \Theta[-\Im(\rho)]\left[e^{\frac{i\pi(2-\nu)}{4}}\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) + e^{\frac{i\pi(2-\nu)}{2}}\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2})\right]\right\} \, dx \tag{4.20}$$

Taking into account that (See **6.576,** (3), ref.[10]):

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(-ix^{1/2}\rho^{1/2}) \, dx = 2\mu^{\frac{2-\nu}{2}} e^{\frac{i\pi(6-\nu)}{4}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2} \;\Im(\rho) > 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(ix^{1/2}\rho^{1/2}) \, dx = 2\mu^{\frac{2-\nu}{2}} e^{\frac{i\pi(\nu-6)}{4}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho-\mu^2} \;\Im(\rho) < 0$$

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(x^{1/2})\mathcal{K}_{\frac{\nu-2}{2}}(x^{1/2}\rho^{1/2}) \, dx = 2\mu^{\frac{2-\nu}{2}} \frac{\rho^{\frac{\nu-2}{4}}}{\rho+\mu^2} \tag{4.21}$$

we obtain:

$$F(\rho) = \frac{i}{2} Sgn[\Im(\rho)] \left[\frac{1}{\rho - \mu^2} + \frac{\cosh \pi(\frac{\nu-2}{2})}{\rho + \mu^2} \right] \tag{4.22}$$

5 The generalization of Dimensional regularization to Ultradistributions of Exponential Type

Let $\hat{F}(z)$ and $\hat{G}(z)$ be ultradistributions of exponential type such that their cuts along the real axis are $\hat{f}(x)$ and $\hat{g}(x)$. We suppose that $\hat{F}(z)$ and $\hat{G}(z)$ are spherically symmetric in Euclidean case or Lorentz Invariant in Minkowskian space. If we use the dimension ν as a regularizing parameter we can define the convolution of $F(\rho)$ and $G(\rho)$ as:

$$F(\rho, \nu) * G(\rho, \nu) = (2\pi)^\nu \mathcal{F} \left\{ \hat{f}(x, \nu) \hat{g}(x, \nu) \right\} \tag{5.1}$$

The Euclidean Case

As an example of use of (5.1) in Euclidean space we consider an ultradistribution of exponential type $\hat{F}(z)$ such that $\hat{f}(x)$ is defined in the point $a > 0$ of the real axis and take the value $\hat{f}(a)$, with the ultradistribution $\hat{G}(z)$ whose cut along the real axis is $\delta(x-a)$ According to (3.1) we have

$$\mathcal{F}\left\{\hat{F}\right\}(\rho) = F(\rho) \tag{5.2}$$

$$\mathcal{F}\{\delta(x-a)\} = G(\rho) = \frac{(2\pi)^{\frac{\nu-2}{2}}}{\rho^{\frac{\nu-2}{4}}} a^{\frac{\nu-2}{4}} \left\{\Theta[\Im(\rho)] e^{-\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(-i a^{1/2}\rho^{1/2}) - \Theta[-\Im(\rho)] e^{\frac{i\pi\nu}{4}} \mathcal{K}_{\frac{\nu-2}{2}}(i a^{1/2}\rho^{1/2})\right\} + \frac{2\pi^{\frac{\nu-2}{2}}}{\Gamma(\frac{\nu-2}{2})\rho^{\frac{\nu-2}{4}}} a^{\frac{\nu-2}{4}} \mathcal{S}_{\frac{\nu-4}{2},\frac{\nu-2}{2}}(a^{1/2}\rho^{1/2}) \tag{5.3}$$

Due to

$$\mathcal{F}\left\{\hat{f}(x)\delta(x-a)\right\} = \hat{f}(a)\mathcal{F}\{\delta(x-a)\} = \hat{f}(a)G(\rho) \tag{5.4}$$

we have

$$F(\rho) * G(\rho) = (2\pi)^{\nu}\hat{f}(a)G(\rho) \tag{5.5}$$

The Minkowskian case

We consider

$$F(\rho) = G(\rho) = \frac{i}{2} Sgn[\Im(\rho)] \left[\frac{1}{\rho-\mu^2} + \frac{\cosh\pi(\frac{\nu-2}{2})}{\rho+\mu^2}\right] \tag{5.6}$$

From (4.17) we have

$$\hat{f}(x) = \hat{g}(x) = -\frac{i\pi}{2}\frac{(-\mu)^{\frac{\nu-2}{2}}}{(2\pi)^{\frac{\nu}{2}}} \left[x_+^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\mu x_+^{1/2}) - x_-^{\frac{2-\nu}{4}} \mathcal{J}_{\frac{\nu-2}{2}}(\mu x_-^{1/2})\right] \tag{5.7}$$

Then

$$F(\rho) * G(\rho) = \frac{(2\pi)^{\frac{\nu+1}{2}}}{2^{\frac{3\nu-1}{2}}}\Gamma\left(\frac{3-\nu}{2}\right) e^{i\pi(\frac{\nu-2}{2})}\rho^{\frac{2-\nu}{2}} Sgn[\Im(\rho)]\times$$

$$\left[(\rho^2-2\rho\mu^2)^{\frac{\nu-3}{2}}+(\rho^2+2\rho\mu^2)^{\frac{\nu-3}{2}}\right] \tag{5.8}$$

To obtain (5.8) we use

$$\int\limits_0^\infty \mathcal{J}_{\frac{2-\nu}{2}}(\mu_1 x)\mathcal{J}_{\frac{2-\nu}{2}}(\mu_2 x)\mathcal{K}_{\frac{\nu-2}{2}}(xz)\ dx =$$

$$-\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3-\nu}{2}\right)}{2^{\frac{3\nu-6}{2}}}\frac{z^{\frac{2-\nu}{2}}}{(\mu_1\mu_2)^{\frac{\nu-2}{2}}}\left[(z^2+\mu_1^2+\mu_2^2)^2-4\mu_1^2\mu_2^2\right]^{\frac{\nu-3}{2}} \tag{5.9}$$

and to deduce (5.9) we have used:

$$\mathcal{K}_{\frac{\nu-2}{2}}(xz)=\frac{1}{2}\left(\frac{zx}{2}\right)^{\frac{\nu-2}{2}}\int\limits_0^\infty t^{-\frac{\nu}{2}}e^{-t-\frac{z^2x^2}{4t}}dt \tag{5.10}$$

(See **8.432**(6) of ref.[10]).

We proceed now to the calculation of the convolution of two ultradistributions of exponential type.

6 The Convolution of two Ultradistributions of Exponential Type

The convolution of two ultraditributions of exponential type can be defined with a change in the formula obtained in ref.([4]) for tempered oltradistributions Let here be

$$H_{\gamma\lambda}(k) = \frac{i}{2\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{[2\cosh(\gamma k_1)]^{-\lambda} F(k_1) [2\cosh(\gamma k_2)]^{-\lambda} G(k_2)}{k - k_1 - k_2} dk_1 \, dk_2 \quad (6.1)$$

$$| \Im(k) | > | \Im(k_1) | + | \Im(k_2 | \; ; \; \gamma < \min \left(\frac{\pi}{2 \mid \Im(k_1) \mid}; \frac{\pi}{2 \mid \Im(k_2) \mid} \right)$$

With this value of γ the hyperbolic cosines has not singularities in the integration zone. Again we have the Laurent (or Taylor) expansion:

$$H_{\gamma\lambda}(k) = \sum_n H_\gamma^{(n)}(k) \lambda^n \quad (6.2)$$

where the sum might have terms with negative n. We now define the convolution product as the λ-independent term of (6.2)

$$(F * G)(k) = H(k) = H_\gamma^{(0)}(k) = H^{(0)}(k) \quad (6.3)$$

that is γ-independent

To see this we consider a typical integral term in (6.1)

$$I = \int_c^\infty \frac{F(k + i\sigma)}{[\cosh \gamma (k + i\sigma)]^\lambda} \, dk \quad (6.4)$$

with

$$|F(k)| \leq A \, |k|^p \, e^{p |\Re(k)|} \quad (6.5)$$

Then I has the value

$$I = e^{i(p+1)} \left[\sum_{n=1;n\neq(p-1,p+1)}^{\infty} a_n(p,\sigma) \frac{e^{-c(n-p-1)}}{n-p-1} - a_{p-1}(p,\sigma)\frac{e^{2c}}{2} \right.$$

$$\left. + \frac{a_{p+1}(p,\sigma)}{\lambda\gamma} \right] \tag{6.6}$$

Thus the λ-independent term of I does not depend of γ. As (6.1) is composed by sums and products of integrals of the type (6.4) we conclude that (6.3) is true.

Examples

As a first example we consider the convolution of two exponentials. Let be:

$$F(k) = Sgn[\Im(k)]\frac{e^{ak}}{2} \quad ; \quad G(k) = Sgn[\Im(k)]\frac{e^{bk}}{2} \tag{6.7}$$

(a and b complex). Then

$$H_{\gamma\lambda}(k) = \frac{i}{8\pi} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{Sgn[\Im(k_1)]e^{ak_1} Sgn[\Im(k_2)]e^{bk_2}}{[2\cosh(\gamma k_1)]^{\lambda}[2\cosh(\gamma k_2)]^{\lambda}(k-k_1-k_2)} dk_1\, dk_2 =$$

$$\frac{1}{2\pi} \iint_{-\infty}^{\infty} \frac{e^{ak_1} e^{bk_2}}{[2\cosh(\gamma k_1)]^{\lambda}[2\cosh(\gamma k_2)]^{\lambda}(k-k_1-k_2)} dk_1\, dk_2 \tag{6.8}$$

or

$$H_{\gamma\lambda}(k) = \frac{\Theta[\Im(k)]}{2\pi} \int_0^{\infty} \iint_{-\infty}^{\infty} \frac{e^{ak_1} e^{bk_2} e^{i(k-k_1-k_2)t}}{(e^{\gamma k_1} + e^{-\gamma k_1})^{\lambda}(e^{\gamma k_2} + e^{-\gamma k_2})^{\lambda}} \, dk_1\, dk_2\, dt$$

$$-\frac{\Theta[-\Im(k)]}{2\pi}\int\limits_{-\infty}^{0}\iint\limits_{-\infty}^{\infty}\frac{e^{ak_1}e^{bk_2}e^{i(k-k_1-k_2)t}}{(e^{\gamma k_1}+e^{-\gamma k_1})^{\lambda}(e^{\gamma k_2}+e^{-\gamma k_2})^{\lambda}}\,dk_1\,dk_2\,dt \qquad (6.9)$$

To evaluate (6.9) we take into account that

$$\int\limits_{-\infty}^{\infty}\frac{e^{(a-it)k_1}}{(e^{\gamma k_1}+e^{-\gamma k_1})^{\lambda}}\,dk_1=\frac{1}{2\gamma}\int\limits_{0}^{\infty}\frac{y^{\frac{\gamma\lambda+a-it}{2\gamma}-1}}{(1+Y)^{\lambda}}\,dy=$$

$$\frac{1}{2\gamma}\frac{\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)}{\Gamma(\lambda)} \qquad (6.10)$$

Then

$$H_{\gamma\lambda}(k)=\frac{1}{8\pi\gamma^2\Gamma^2(\lambda)}\Bigg\{\Theta[\Im(k)]\int\limits_{0}^{\infty}\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)\times$$

$$\Gamma\left(\frac{\gamma\lambda+b-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-b+it}{2\gamma}\right)e^{ikt}\,dt\;-$$

$$\Theta[-\Im(k)]\int\limits_{-\infty}^{0}\Gamma\left(\frac{\gamma\lambda+a-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-a+it}{2\gamma}\right)\times$$

$$\Gamma\left(\frac{\gamma\lambda+b-it}{2\gamma}\right)\Gamma\left(\frac{\gamma\lambda-b+it}{2\gamma}\right)e^{ikt}\,dt\Bigg\} \qquad (6.11)$$

and using the equality ([10], **3.381**, 4)

$$\int\limits_{0}^{\infty}x^{\nu-1}e^{-\mu x}\,dx=\mu^{-\nu}\Gamma(\nu) \qquad (6.12)$$

and performing the integral in the variable t, we have for (6.11)

$$H_{\gamma\lambda}(k)=-\frac{1}{8\pi i\gamma^2\Gamma^2(\lambda)}\iiiint\limits_{0}^{\infty}s_1^{\frac{\gamma\lambda+a}{2\gamma}-1}e^{-s_1}s_2^{\frac{\gamma\lambda-a}{2\gamma}-1}e^{-s_2}s_3^{\frac{\gamma\lambda+b}{2\gamma}-1}e^{-s_3}s_4^{\frac{\gamma\lambda-b}{2\gamma}-1}e^{-s_4}\times$$

$$\frac{1}{k+\frac{1}{2\gamma}\ln(\frac{s_2 s_4}{s_1 s_3})}\ ds_1\ ds_2\ ds_3\ ds_4 \tag{6.13}$$

As

$$\frac{1}{k+\frac{1}{2\gamma}\ln(\frac{s_2 s_4}{s_1 s_3})} = \sum_{n=0}^{\infty}\frac{(ik_I)^n}{n!}\frac{\partial^n}{\partial k_R^n}\delta\left[k_R+\frac{1}{2\gamma}\ln\left(\frac{s_2 s_4}{s_1 s_3}\right)\right] \tag{6.14}$$

where $(k = k_R + ik_I)$ and

$$\delta\left[k_R+\frac{1}{2\gamma}\ln\left(\frac{s_2 s_4}{s_1 s_3}\right)\right] = \frac{s_1 s_3}{s_2}e^{-2\gamma k_R}\delta\left(s_4-\frac{s_1 s_3}{s_2}e^{-2\gamma k_R}\right) \tag{6.15}$$

we obtain

$$H_{\gamma\lambda}(k) = -\sum_{n=0}^{\infty}\frac{(ik_I)^n}{n!}\frac{\partial^n}{\partial k_R^n}\frac{e^{k_R(b-\gamma\lambda)}}{4\pi i\gamma\Gamma^2(\lambda)}\iiiint_0^\infty s_1^{\frac{2\gamma\lambda+a-b}{2\gamma}-1}e^{-s_1}\ \times$$

$$s_2^{\frac{b-a}{2\gamma}-1}e^{-s_2}s_3^{\lambda-1}e^{-s_3}e^{-\left(\frac{s_1 s_3}{s_2}e^{-2\gamma k_R}\right)}\ ds_1\ ds_2\ ds_3\ ds_4 \tag{6.16}$$

After to evaluate the four-fold integral $H_{\gamma\lambda}$ take the form:

$$H_{\gamma\lambda}(k) = -\frac{e^{k(b+\gamma\lambda)}}{4\pi i\gamma\Gamma(2\lambda)}\Gamma\left(\frac{a-b+2\gamma\lambda}{2\gamma}\right)\Gamma\left(\frac{b-a+2\gamma\lambda}{2\gamma}\right)\times$$

$$F\left(\lambda+\frac{b-a}{2\gamma},\lambda,2\lambda;1-e^{2\gamma k}\right) \tag{6.17}$$

When $a \neq b$

$$\lim_{\lambda\to 0}H_{\gamma\lambda}(k) = 0 \tag{6.18}$$

When $a = b$

$$\lim_{\lambda\to 0}H_{\gamma\lambda}(k) = \frac{ke^{ka}}{2\pi i} \equiv 0 \tag{6.19}$$

and then $H(k)$ is the null ultradistribution Thus we have finally:

$$Sgn[\Im(k)]\frac{e^{ak}}{2} * Sgn[\Im(k)]\frac{e^{bk}}{2} = 0$$

and Fourier antitransforming

$$\delta(z-a)\delta(z-b) = 0 \tag{6.20}$$

As a second example we consider the convolution of two complex Dirac's deltas:

$$F(k) = -\frac{1}{2\pi i(k-a)} \quad ; \quad G(k) = -\frac{1}{2\pi i(k-b)} \tag{6.21}$$

We have

$$H_{\gamma\lambda}(k) = \frac{i}{2\pi}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2} \frac{i}{2\pi(k_1-a)}\frac{i}{2\pi(k_2-b)}\times$$

$$\frac{[2\cosh(\gamma k_1)]^{-\lambda}[2\cosh(\gamma k_2)]^{-\lambda}}{k-k_1-k_2}dk_1\,dk_2 = \tag{6.22}$$

$$\frac{1}{2\pi}\frac{[2\cosh(\gamma a)]^{-\lambda}[2\cosh(\gamma b)]^{-\lambda}}{k-a-b} \tag{6.23}$$

and as a consequence

$$H(k) = -\frac{1}{2\pi i}\frac{1}{k-a-b} \tag{6.24}$$

or

$$\delta(k-a) * \delta(k-b) = \delta(k-a-b) \tag{6.25}$$

and in the configuration space:

$$\frac{Sgn[\Im(z)]}{2}e^{az}\ \frac{Sgn[\Im(z)]}{2}e^{bz} = \frac{Sgn[\Im(z)]}{2}e^{(a+b)z} \tag{6.26}$$

Formula (6.1) can be generelized to ν dimensions:

$$H_{\gamma\lambda}(k) = \frac{i^\nu}{(2\pi)^\nu} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{\prod_{j=1}^{\nu} [2\cosh(\gamma_j k_{1j}]^{-\lambda_j} [2\cosh(\gamma_j k_{2j})]^{-\lambda_j}}{\prod_{j=1}^{\nu} (k_j - k_{1j} - k_{2j})} \times$$

$$F(k_1) G(k_2) \ \ d^\nu k_1 \ d^\nu k_2 \tag{6.27}$$

$$\gamma_j < \min\left(\frac{\pi}{2 \mid \Im(k_{1j}) \mid}; \frac{\pi}{2 \mid \Im(k_{2j}) \mid} \right)$$

As in the one-dimensional case

$$H_{\gamma\lambda}(k) = \sum_{n_1, n_2, .., n_\nu} \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_\nu^{n_\nu} H^{(n_1 + n_2 + \cdots + n_\nu)}(k) \tag{6.28}$$

and again

$$(F * G)(k) = H(k) = H^{(0)}(k) \tag{6.29}$$

7 Solution to the question of the normalization of Gamow States in Quantum Mechanics

As an application of the results of section 6, we give in this section a solution to the question of normalization of Gamow states in Quantum Mechanics. If we have a Gamow state that depends on $l+m$ variables $\phi(k_1, k_2, ..., k_l; \rho_1, \rho_2, ...\rho_m)$, and we wish to calculate

$$I(k_1, k_2, .., k_l) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} |\phi|^2(k_1, k_2, .., k_l; \rho_1, \rho_2, .., \rho_m)\, d\rho_1\, d\rho_2 \cdot\cdot d\rho_m \quad (7.1)$$

we define

$$\Phi(k_1, k_2, ..., k_l; z_1, z_2, ..., z_m) = \frac{1}{(2\pi i)^m} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \times$$

$$\frac{|\phi|^2(k_1, k_2, ..., k_l; \rho_1, \rho_2, ..., \rho_m)}{(\rho_1 - z_1)(\rho_2 - z_2) \cdots (\rho_m - z_m)}\, d\rho_1\, d\rho_2 \cdots d\rho_m \quad (7.2)$$

and

$$H_{\gamma_1 \gamma_2 ... \gamma_m \lambda_1 \lambda_2 ... \lambda_m}(k_1, k_2, ..., k_l) = \oint_{\Gamma_1} \cdots \oint_{\Gamma_m} \times$$

$$\frac{\Phi(k_1, k_2, ..., k_l; z_1, z_2, ..., z_m)}{[\cosh(\gamma_1 z_1)]^{\lambda_1} [\cosh(\gamma_2 z_2)]^{\lambda_2} ... [\cosh(\gamma_m z_m)]^{\lambda_m}}\, dz_1\, dz_2 \cdots dz_m \quad (7.3)$$

We have again the Laurent's expansion:

$$H_{\gamma_1 \gamma_2 ... \gamma_m \lambda_1 \lambda_2 ... \lambda_m}(k_1, k_2, ..., k_l) = \sum_{n_1, n_2, ... n_m} \times$$

$$\lambda_1^{n_1} \lambda_2^{n_2} ... \lambda_m^{n_m} H^{(n_1 + n_2 + ... + n_m)}_{\gamma_1 \gamma_2 ... \gamma_m}(k_1, k_2, ..., k_l) \quad (7.4)$$

and as a consequence of section 6 we define:

$$I(k_1, k_2, ..., k_l) = H(k_1, k_2, ..., k_l) = H^{(0)}(k_1, k_2, ..., k_l) =$$

$$H^{(0)}_{\gamma_1 \gamma_2 ... \gamma_m}(k_1, k_2, ..., k_l) \quad (7.5)$$

As an example of appilcation of (7.1-7.5) we evaluate

$$I(k) = \int_0^\infty \phi_0^2(k,r)\, dr = \int_{-\infty}^\infty \Theta(r)\phi_0^2(k,r)\, dr \tag{7.6}$$

where $\phi_0(k,r)$ is the $l = 0$ function corresponding to the Square-Well potential used in ref.[11]:

$$\phi_0(k,r) = \begin{cases} \frac{\sin(qr)}{q} & \text{if } r < a \\ \frac{\sin(qa)}{q} e^{ik(a-r)} & \text{if } r > a \end{cases} \tag{7.7}$$

Here q is given by:

$$q^2 = \frac{2m}{\hbar^2}[E - V(r)] = k^2 - \frac{2m}{\hbar^2}V(r) \tag{7.8}$$

and

$$V(r) = \begin{cases} 0 & \text{if } r > a \\ -V_0 & \text{if } r \geq a \end{cases} \tag{7.9}$$

We can write:

$$\phi(k,r) = [\Theta(r) - \Theta(r-a)]\,\frac{\sin(qr)}{q} + \Theta(r-a)\frac{\sin(qa)}{q}e^{ik(a-r)} \tag{7.10}$$

$$\phi^2(k,r) = [\Theta(r) - \Theta(r-a)]\,\frac{\sin^2(qr)}{q^2} + \Theta(r-a)\frac{\sin^2(qa)}{q^2}e^{2ik(a-r)} \tag{7.11}$$

and according to (7.2)

$$\Phi(k,z) = \frac{1}{2\pi i}\,[\ln(a-z) - \ln(z)]\,\frac{\sin^2(qz)}{q^2} \quad -$$

$$\frac{1}{2\pi i}\frac{\sin^2(qa)}{q^2}\ln(a-z)e^{2ik(a-z)} \tag{7.12}$$

Thus we have:

$$H_{\gamma\lambda}(k) = \frac{1}{2\pi i q^2}\oint_\Gamma \frac{\ln(a-z)-\ln(z)}{[\cosh(\gamma z)]^\lambda}\sin^2(qz)\ dz \quad -$$

$$\frac{\sin^2(qa)}{2\pi i q^2}e^{2ika}\oint_\Gamma \frac{\ln(a-z)}{[\cosh(\gamma z)]^\lambda}e^{-2ikz}\ dz \ =$$

$$\frac{1}{q^2}\int_0^a \frac{\sin^2(qr)}{[\cosh(\gamma r)]^\lambda}\ dr + \frac{\sin^2(qa)}{q^2}e^{2ika}\int_a^\infty \frac{e^{-2ikr}}{[\cosh(\gamma r)]^\lambda}\ dr \tag{7.13}$$

We can evaluate the second integral in (7.13):

$$\int_a^\infty \frac{e^{-2ikr}}{[\cosh(\gamma r)]^\lambda}\ dr \ =$$

$$\frac{e^{-a(\gamma\lambda+2ik)}}{\gamma\lambda+2ik}F\left(\lambda, \frac{\gamma\lambda+2ik}{2\gamma}, \frac{\gamma\lambda+2ik}{2\gamma}+1; -e^{4\gamma a}\right) \tag{7.14}$$

Taking into account that:

$$\lim_{\lambda\to 0}\frac{1}{\gamma\lambda+2ik} = \begin{cases} \frac{-i}{2(k-i0)} & \text{if } \Im(k)=0 \\ \\ \frac{-i}{2k} & \text{if } \Im(k)\neq 0 \end{cases} \tag{7.15}$$

we obtain:

$$I(k) = H(k) = \frac{1}{q^2}\int_0^a \sin^2(qr)\ dr + \frac{\sin^2(qa)}{2ikq^2} =$$

$$\frac{a}{2q^2} - \frac{\sin(2qa)}{4q^3} + \frac{\sin^2(qa)}{2ikq^2} \tag{7.16}$$

Using the equality:

$$\cos(qa) = -i\frac{k}{q}\sin(qa) \tag{7.17}$$

we have:

$$I(k) = \frac{1 + ika}{2ik}\ \frac{q^2 - k^2}{q^4}\ \sin^2(qa) \tag{7.18}$$

This result coincides with the result obtained in ref.[11].

8 The Convolution of four-dimensional even Ultradistributionsof Exponential Type

The convolution of two even ultraditributions of exponential type can be defined with a change of the formula obtained in ref.([4]) for tempered ultradistributions Let here be

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \frac{1}{4\pi\rho}\oint\limits_{\Gamma_1^0}\oint\limits_{\Gamma_2^0}\oint\limits_{\Gamma_1}\oint\limits_{\Gamma_2}\frac{[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0}[2\cosh(\gamma\rho_1)]^{-\lambda}F(k_1^0,\rho_1)}{k^0 - k_1^0 - k_2^0}\times$$

$$[2\cosh(\gamma_0 k_2^0)]^{-\lambda_0}[2\cosh(\gamma\rho_2)]^{-\lambda}G(k_2^0,\rho_2)\ \times$$

$$\ln[\rho^2 - (\rho_1 + \rho_2)^2]\rho_1\rho_2\ d\rho_1\ d\rho_2\ dk_1^0\ dk_2^0 \tag{8.1}$$

$$|\ \Im(k^0)\ |>|\ \Im(k_1^0\ | + |\ \Im(k_2^0)\ |\ ;\ |\ \Im(\rho)\ |>|\ \Im(\rho_1\ | + |\ \Im(\rho_2)\ |$$

$$\gamma_0 < \min\left(\frac{\pi}{2\ |\ \Im(k_1^0)\ |};\frac{\pi}{2\ |\ \Im(k_2^0)\ |}\right)\ ;\ \gamma < \min\left(\frac{\pi}{2\ |\ \Im(\rho_1)\ |};\frac{\pi}{2\ |\ \Im(\rho_2)\ |}\right)$$

The difference between

$$\int \frac{2\rho}{\rho^2 - (\rho_1 + \rho_2)^2}\, d\rho \qquad \text{and} \qquad \ln[\rho^2 - (\rho_1 + \rho_2)^2]$$

is an entire analytic function. With this substitution in (8.1) we obtain

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0, \rho) = \frac{1}{2\pi\rho} \int \rho\, d\rho \oint_{\Gamma_1^0} \oint_{\Gamma_2^0} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{F(k_1^0, \rho_1) G(k_2^0, \rho_2)}{k^0 - k_1^0 - k_2^0} \times$$

$$[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0} [2\cosh(\gamma\rho_1)]^{-\lambda} [2\cosh(\gamma_0 k_2^0)]^{-\lambda_0} [2\cosh(\gamma\rho_2)]^{-\lambda} \times$$

$$\frac{1}{\rho^2 - (\rho_1 + \rho_2)^2} \quad \rho_1\rho_2\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0 \tag{8.2}$$

We can again perform the Laurent expansion :

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0, \rho) = \sum_{mn} H_{\gamma_0\gamma}^{(m,n)}(k^0, \rho) \lambda_0^m \lambda^n \tag{8.3}$$

and define the convolution product as the (λ_0, λ)-independent term of (8.3).

$$H(k) = H(k^0, \rho) = H_{\gamma_0\gamma}^{(0,0)}(k^0, \rho) = H^{(0,0)}(k^0, \rho) \tag{8.4}$$

If we define:

$$L_{\gamma_0\gamma\lambda_0\lambda}(k^0, \rho) = \oint_{\Gamma_1^0} \oint_{\Gamma_2^0} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{F(k_1^0, \rho_1) G(k_2^0, \rho_2)}{k^0 - k_1^0 - k_2^0} \times$$

$$[2\cosh(\gamma_0 k_1^0)]^{-\lambda_0} [2\cosh(\gamma\rho_1)]^{-\lambda} [2\cosh(\gamma_0 k_2^0)]^{-\lambda_0} [2\cosh(\gamma\rho_2)]^{-\lambda} \times$$

$$\frac{1}{\rho^2 - (\rho_1 + \rho_2)^2} \quad \rho_1\rho_2\, d\rho_1\, d\rho_2\, dk_1^0\, dk_2^0 \tag{8.5}$$

then

$$H_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho) = \frac{1}{2\pi\rho}\int L_{\gamma_0\gamma\lambda_0\lambda}(k^0,\rho)\ \rho\ d\rho \tag{8.6}$$

Now we show that the cut on the real axis of (8.1) $h_{\lambda_0\lambda}(k^0,\rho)$ is an even function of k^0 and ρ. It is explicitly odd in ρ. For the variable k^0 we take into account that $e^{i\pi\lambda_0\{Sgn[\Im(k_1^0)]+Sgn[\Im(k_2^0)]\}} = 1$ and as a consequence (8.1) is odd in k^0 too. We consider now the parity in variable ρ.

$$\oint_{\Gamma_0}\oint_{\Gamma} H_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\ dk^0\ d\rho = -\iint_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,-\rho)\phi(k^0,\rho)\ dk^0\ d\rho =$$

$$-\oint_{\Gamma_0}\oint_{\Gamma} H_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\ dk^0\ d\rho = -\iint_{-\infty}^{+\infty} h_{\lambda_0\lambda}(k^0,\rho)\phi(k^0,\rho)\ dk^0\ d\rho \tag{8.7}$$

Thus we have

$$h_{\lambda_0\lambda}(k^0,-\rho) = h_{\lambda_0\lambda}(k^0,\rho) \tag{8.8}$$

The proof for the variable k^0 is similar.

Examples

As a first example we shall calculate the convolution between $F(k_0,\rho) = \delta(k_0^2-a^2)\delta(\rho-b)$ and $G(k_0,\rho) = \delta(k_0^2-c^2)\delta(\rho-d)$. We have:

$$H(k_0,\rho) = \frac{bd}{16\pi|a||b|\rho}\left(\frac{1}{k_0-a-c}+\frac{1}{k_0-a+c}+\frac{1}{k_0+a-c}+\frac{1}{k_0+a+c}\right)\times$$

$$\ln[\rho^2-(b+d)^2] \tag{8.9}$$

and simplifying the last expression:

$$H(k_0, \rho) = \frac{bd}{8\pi|a||b|\rho}\left[\frac{k_0 - a}{(k_0 - a)^2 - c^2} + \frac{k_0 + a}{(k_0 + a)^2 - c^2}\right] \times$$

$$\ln[\rho^2 - (b + d)^2] \tag{8.10}$$

As a second example we evaluate the convolution of $F(k_0, \rho) = \delta(k_0)\delta(\rho - a)$ and $G(k_0, \rho) = \frac{1}{2}Sgn[\Im(k_0)]e^{ibk_0}\delta(\rho - c)$

We have:

$$H_{\gamma_0\gamma\lambda_0\lambda}(k_0, \rho) = \frac{ac}{8\pi\rho}\frac{\ln[\rho^2 - (a + c)^2]}{[\cosh(\gamma a]^\lambda[\cosh(\gamma c)]^\lambda}\oint\limits_{\Gamma_{02}}\frac{Sgn[\Im(k_{02})]e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0}(k_0 - k_{02})}\, dk_{02}$$

$$= \frac{ac}{4\pi\rho}\frac{\ln[\rho^2 - (a + c)^2]}{[\cosh(\gamma a]^\lambda[\cosh(\gamma c)]^\lambda}\int\limits_{-\infty}^{\infty}\frac{e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0}(k_0 - k_{02})}\, dk_{02} \tag{8.11}$$

Taking into account that

$$\lim \lambda_0 \to 0 \int\limits_{-\infty}^{\infty}\frac{e^{ibk_{02}}}{[\cosh(\gamma_0 k_{02})]^{\lambda_0}(k_0 - k_{02})}\, dk_{02} =$$

$$-\pi i Sgn[\Im(k_0)]e^{ibk_0} \tag{8.12}$$

we obtain

$$H(k_0, \rho) = \frac{ac}{4\pi i\rho}Sgn[\Im(k_0)]e^{ibk_0}\ln[\rho^2 - (a + d)^2] \tag{8.13}$$

9 The Convolution of Spherically Symmetric Ultradistributions of Exponential Type in Euclidean Space

The convolution of two spherically symmetric Ultraditributions of Exponential Type can be defined with a change of the formula obtained in ref.([5]) for tempered ultradistributions Let here be

$$H_{\gamma\lambda}(\rho) = \frac{i\pi}{4\rho} \oint\limits_{\Gamma_1} \oint\limits_{\Gamma_2} [2\cosh(\gamma\rho_1)]^{-\lambda} F(\rho_1) [2\cosh(\gamma\rho_2)]^{-\lambda} G(\rho_2) \times$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}\right] d\rho_1 \, d\rho_2 \tag{9.1}$$

Again we have the Laurent expansion:

$$H_{\gamma\lambda}(\rho) = \sum_{n=-m}^{\infty} H_{\gamma}^{(n)}(\rho)\lambda^n \tag{9.2}$$

We now define the convolution product as the λ-independent term of (9.2):

$$H(\rho) = H_{\gamma}^{(0)}(\rho) = H^{(0)}(\rho) \tag{9.3}$$

Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier antitransform of $H_{\gamma\lambda}(\rho)$:

$$\hat{H}_{\gamma\lambda}(x) = \sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n \tag{9.4}$$

If we define:

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{[2\cosh(\gamma\rho)]^{-\lambda}F(\rho)\}$$

$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{[2\cosh(\gamma\rho)]^{-\lambda}G(\rho)\} \tag{9.5}$$

then

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4 \hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x) \tag{9.6}$$

and with the use of the Laurent's developments of $\hat{f}$ and $\hat{g}$:

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}_{\gamma}^{(n)}(x)\lambda^n$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}_{\gamma}^{(n)}(x)\lambda^n \tag{9.7}$$

we can write:

$$\sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \right) \lambda^n \tag{9.8}$$

$(m = m_f + m_g)$

and as a consequence:

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \tag{9.9}$$

We shall give now some examples.

Examples

The first example that we shall give is the convolution between $F[\rho] = \delta(\rho - a)$ and $G(\rho) = \delta(\rho - b)$ We have:

$$H_{\gamma\lambda}(\rho) = \frac{i\pi}{4\rho} \oint_{\Gamma_1} \oint_{\Gamma_2} [2\cosh(\gamma\rho_1)]^{-\lambda} \delta(\rho_1 - a) [2\cosh(\gamma\rho_2)]^{-\lambda} \delta(\rho_2 - b) \times$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_2)^2 - 4\rho_1\rho_2}\right] d\rho_1 \, d\rho_2 \qquad (9.10)$$

whose result is

$$H(\rho) = \frac{i\pi}{4\rho} \left[\rho - a - b - \sqrt{(\rho - a - b)^2 - 4ab}\right] \qquad (9.11)$$

When a and b are real numbers, from (9.11) we obtain in the real ρ-axis

$$h(\rho) = \frac{\pi}{2\rho} \left[(\rho - a - b)^2 - 4ab\right]_+^{\frac{1}{2}} \qquad (9.12)$$

As a second example we evaluate the convolution between $F(\rho) = E_i(-ia\rho)e^{ia\rho}/2\pi i$ and $G(\rho) = \delta'(\rho) = (2\pi i \rho^2)^{-1}$, where $E_i(z)$ is the Exponential Integral Function. We have

$$H_{\gamma\lambda}(\rho) = \frac{1}{8\rho} \oint_{\Gamma_1} \oint_{\Gamma_2} \frac{E_i(-ia\rho_1)e^{ia\rho_1}\delta'(\rho_2)}{[2\cosh(\gamma\rho_1)]^{\lambda}[2\cosh(\gamma\rho_2)]^{\lambda}}$$

$$\left[\rho - \rho_1 - \rho_2 - \sqrt{(\rho - \rho_1 - \rho_1)^2 - 4\rho_1\rho_2}\right] \, d\rho_1 \, d\rho_2 \qquad (9.13)$$

After integration in the variable ρ_2 we have

$$H_{\gamma\lambda}(\rho) = \frac{1}{4\rho} \oint_{\Gamma_1} \frac{\rho_1 E_i(-ia\rho_1)e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^{\lambda}(\rho_1 - \rho)} \, d\rho_1 =$$

$$\frac{1}{4\rho}\int\limits_0^\infty \frac{\rho_1 e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda(\rho_1-\rho)}\,d\rho_1 =$$

$$\frac{1}{4\rho}\int\limits_0^\infty \frac{e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda}\,d\rho_1 + \frac{1}{4}\int\limits_0^\infty \frac{e^{ia\rho_1}}{[2\cosh(\gamma\rho_1)]^\lambda(\rho_1-\rho)}\,d\rho_1 \tag{9.14}$$

After to evaluate the integrals in (9.15) ($\lambda \to 0$) we obtain:

$$H(\rho) = \frac{i}{4a\rho} - \frac{i}{8\pi}e^{ia\rho}E_i(-ia\rho) \tag{9.15}$$

10 The Convolution of two Lorentz invariant Ultradistributions of Exponential Type in Minkowskian space

For Lorentz invariant ultradistributions of exponential type, following ref.([5]) we have:

$$H_{\gamma\lambda}(\rho,\Lambda) = \frac{1}{8\pi^2\rho}\int\limits_{\Gamma_1}\int\limits_{\Gamma_2}[2\cosh(\gamma\rho_1)]^{-\lambda}F(\rho_1)[2\cosh(\gamma\rho_2)]^{-\lambda}G(\rho_2)$$

$$\{\Theta[\Im(\rho)]\,\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)]\times$$

$$[\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}\right\}-$$

$$\Theta[-\Im(\rho)]\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2+2\Lambda)^2}-i(\rho-\rho_1-\rho_2+2\Lambda)}{2\sqrt{(\rho_1-\Lambda)(\rho_2-\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}\times$$

$$\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2-2\Lambda)^2}-i(\rho-\rho_1-\rho_2-2\Lambda)}{2\sqrt{(\rho_1+\Lambda)(\rho_2+\Lambda)}}\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\ln\left[\frac{\sqrt{4(\rho_1-\Lambda)(\rho_2+\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1-\Lambda)(\rho_2+\Lambda)}}\right]\right\}+$$

$$[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$\left\{\frac{i\pi}{2}\left[\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)\right]+\right.$$

$$\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}\times$$

$$\left.\left.\ln\left[\frac{\sqrt{4(\rho_1+\Lambda)(\rho_2-\Lambda)-(\rho-\rho_1-\rho_2)^2}-i(\rho-\rho_1-\rho_2)}{2i\sqrt{-(\rho_1+\Lambda)(\rho_2-\Lambda)}}\right]\right\}\right\}-\frac{i}{2}\times$$

$$\{[\ln(-\rho_1+\Lambda)-\ln(-\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(i\sqrt{\frac{\rho_1+\Lambda}{\rho_2+\Lambda}}\right)+\ln\left(-i\sqrt{\frac{\rho_1-\Lambda}{\rho_2-\Lambda}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(\rho_2+\Lambda)-\ln(\rho_2-\Lambda)]\times$$

$$(\rho_1-\rho_2)\left[\ln\left(-i\sqrt{\frac{\Lambda-\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(i\sqrt{\frac{\Lambda+\rho_1}{\Lambda+\rho_2}}\right)\right]+$$

$$[\ln(\rho_1+\Lambda)-\ln(\rho_1-\Lambda)][\ln(-\rho_2+\Lambda)-\ln(-\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)+\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right)\right]+\right.$$

$$\frac{(\rho_1-\rho_2)}{2}[\ln(-\rho_1-\rho_2+\Lambda)-\ln(-\rho_1-\rho_2-\Lambda)-$$

$$\ln(\rho_1+\rho_2+\Lambda)+\ln(\rho_1+\rho_2-\Lambda)]+\rho_2[\ln(-\rho_1-\rho_2+\Lambda)-$$

$$\ln(-\rho_1-\rho_2-\Lambda)] + \rho_1 \left[\ln(\rho_1+\rho_2+\Lambda) - \ln(\rho_1+\rho_2-\Lambda)\right]\}$$

$$[\ln(-\rho_1+\Lambda) - \ln(-\rho_1-\Lambda)][\ln(\rho_2+\Lambda) - \ln(\rho_2-\Lambda)]\times$$

$$\left\{(\rho_1-\rho_2)\left[\ln\left(\sqrt{\frac{\Lambda-\rho_1}{\Lambda+\rho_2}}\right) + \ln\left(\sqrt{\frac{\Lambda+\rho_1}{\Lambda-\rho_2}}\right)\right] + \right.$$

$$\frac{(\rho_1-\rho_2)}{2}\left[\ln(\rho_1+\rho_2+\Lambda) - \ln(\rho_1+\rho_2-\Lambda) - \right.$$

$$\ln(-\rho_1-\rho_2+\Lambda) + \ln(-\rho_1-\rho_2-\Lambda)] + \rho_1\left[\ln(-\rho_1-\rho_2+\Lambda) - \right.$$

$$\ln(-\rho_1-\rho_2-\Lambda)] + \rho_2\left[\ln(\rho_1+\rho_2+\Lambda) - \ln(\rho_1+\rho_2-\Lambda)\right]\}\}\}\ d\rho_1\, d\rho_2 \tag{10.1}$$

$$|\Im(\rho)| > \Im(\Lambda) > |\Im(\rho_1)| + |\Im(\rho_2)|\ ;\ \gamma < \min\left(\frac{\pi}{2\mid\Im(\rho_1)\mid};\frac{\pi}{2\mid\Im(\rho_2)\mid}\right)$$

We define

$$H(\rho) = H^{(0)}(\rho, i0^+) = H_\gamma^{(0)}(\rho, i0^+) \tag{10.2}$$

$$H_{\gamma\lambda}(\rho, i0^+) = \sum_{-m}^{\infty} H_\gamma^{(n)}(\rho, i0^+)\lambda^n \tag{10.3}$$

If we take into account that singularities (in the variable Λ) are contained in a horizontal band of width $|\sigma_0|$ we have:

$$H_{\gamma\lambda}(\rho, i0^+) = \sum_{-m}^{\infty} H_{\gamma\lambda}^{(n)}(\rho, i\sigma)\frac{(-i\sigma)^n}{n!} \qquad \sigma > |\sigma_0| \tag{10.4}$$

As in the other cases we define now

$$\{F * G\}(\rho) = H(\rho) \tag{10.5}$$

as the convolution of two Lorentz invariant ultradistributions of exponential type.

Let $\hat{H}_{\gamma\lambda}(x)$ be the Fourier antitransform of $H_{\gamma\lambda}(\rho, i0^+)$:

$$\hat{H}_{\gamma\lambda}(x) = \sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n \tag{10.6}$$

If we define:

$$\hat{f}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{F_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}F(\rho)\}$$

$$\hat{g}_{\gamma\lambda}(x) = \mathcal{F}^{-1}\{G_{\gamma\lambda}(\rho)\} = \mathcal{F}^{-1}\{[\cosh(\gamma\rho)]^{-\lambda}G(\rho)\} \tag{10.7}$$

then

$$\hat{H}_{\gamma\lambda}(x) = (2\pi)^4 \hat{f}_{\gamma\lambda}(x)\hat{g}_{\gamma\lambda}(x) \tag{10.8}$$

and taking into account the Laurent's developments of $\hat{f}$ and $\hat{g}$:

$$\hat{f}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{f}_{\gamma}^{(n)}(x)\lambda^n$$

$$\hat{g}_{\gamma\lambda}(x) = \sum_{n=-m_f}^{\infty} \hat{g}_{\gamma}^{(n)}(x)\lambda^n \tag{10.9}$$

we can write:

$$\sum_{n=-m}^{\infty} \hat{H}_{\gamma}^{(n)}(x)\lambda^n = (2\pi)^4 \sum_{n=-m}^{\infty} \left(\sum_{k=-m}^{n} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \right) \lambda^n \tag{10.10}$$

$(m = m_f + m_g)$

and as a consequence:

$$\hat{H}^{(0)}(x) = \sum_{k=-m}^{0} \hat{f}_{\gamma}^{(k)}(x)\hat{g}_{\gamma}^{(n-k)}(x) \tag{10.11}$$

Examples

As a first example of the use of (10.1) we shall evaluate the convolution product of $\delta(\rho)$ with $\delta(\rho - \mu^2)$ with $\mu = \mu_R + i\mu_I$ a complex number such that: $\mu_R^2 > \mu_I^2$, $\mu_R\mu_I > 0$. Thus from (10.1) we obtain:

$$H_{\gamma\lambda}(\rho, \Lambda) = -i\pi \frac{\ln(-\mu^2+\Lambda) - \ln(-\mu^2+\lambda)}{[2\cosh(\gamma\mu^2)]^{\lambda}} \left\{ \frac{i(\rho-\mu^2)}{8\pi^2\rho} \left[\ln\left(\frac{\rho-\mu^2}{\sqrt{\Lambda(\mu^2+\Lambda)}} \right) + \ln\left(\frac{\mu^2-\rho}{\sqrt{-\Lambda(\mu^2+\Lambda)}} \right) \right] + \frac{\mu^2-\rho}{16\pi\rho} \right\} - i\pi \frac{\ln(-\mu^2+\Lambda) - \ln(-\mu^2+\lambda)}{[2\cosh(\gamma\mu^2)]^{\lambda}} \times \left\{ \frac{-i\mu^2)}{8\pi^2\rho} \left[\ln\left(\sqrt{\frac{\Lambda}{\mu^2+\Lambda}} \right) + \ln\left(\sqrt{\frac{\Lambda}{\Lambda-\mu^2}} \right) \right] - \frac{\mu^2}{16\pi\rho} \right\} \tag{10.12}$$

Simplifying terms and taking the limit $\lambda \to 0$ (10.15) turns into:

$$H^{(0)}(\rho, \Lambda) = -i\pi[\ln(-\mu^2+\Lambda) - \ln(-\mu^2+\lambda)] \left\{ \frac{i(\rho-\mu^2)}{8\pi^2\rho} \left[\ln\left(\rho-\mu^2\right) + \ln\left(\mu^2-\rho\right) \right] + \frac{i\mu^2}{8\pi^2\rho} \left[\ln(\mu^2+\Lambda) + \ln(\mu^2-\Lambda) \right] \right\} \tag{10.13}$$

Now, if

$$F_1(\mu, \Lambda) = \ln(-\mu^2+\Lambda) - \ln(-\mu^2-\Lambda)$$

then

$$F_1(\mu, i0^+) = 2i\pi \ ; \ \ \mu_R^2 > \mu_I^2 \ ; \ \ \mu_R\mu_I > 0$$

And, if

$$F_2(\mu, \Lambda) = \ln(\mu^2 + \Lambda) - \ln(\mu^2 - \Lambda)$$

then

$$F_2(\mu, i0^+) = 0 \ ; \ \ \mu_R^2 > \mu_I^2 \ ; \ \ \mu_R\mu_I > 0$$

Using these results we obtain:

$$H(\rho) = \frac{i(\rho - \mu^2)}{4\rho}\left[\ln(\rho - \mu^2) + \ln(\mu^2 - \rho)\right] + \frac{i\mu^2}{2\rho}\ln(\mu^2) \tag{10.14}$$

As a second example we will evaluate the convolution of $\Theta[\Im(\rho)]e^{ia\rho}$ (a real) with $\delta(\rho)$.

The convolutin can be performed in the real ρ-axis to obtain:

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2^{\lambda+1}\rho}\int_{-\infty}^{\infty} \frac{e^{ia\rho_2}|\rho - \rho_2|}{[2\cosh(\gamma\rho_2)]^\lambda}\, d\rho_2 \tag{10.15}$$

which can be written as:

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2^{\lambda+1}}\left[\frac{i}{\rho}\frac{d}{da}\int_{-\infty}^{\rho} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^\lambda}\, d\rho_2 + \int_{-\infty}^{\rho} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^\lambda}\, d\rho_2 - \right.$$

$$\left. \frac{i}{\rho}\frac{d}{da}\int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^\lambda}\, d\rho_2 - \int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^\lambda}\, d\rho_2\right] \tag{10.16}$$

With the use of the results:

$$\int_{-\infty}^{\rho} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 = \frac{e^{(ia+\gamma\lambda)\rho}}{ia+\gamma\lambda} \times$$

$$F\left(\lambda, \frac{ia+\gamma\lambda}{2\gamma}, \frac{ia+\gamma\lambda}{2\gamma}+1; -e^{-2\gamma\rho}\right) \tag{10.17}$$

$$\int_{\rho}^{\infty} \frac{e^{ia\rho_2}}{[2\cosh(\gamma\rho_2)]^{\lambda}}\, d\rho_2 = \frac{e^{(ia-\gamma\lambda)\rho}}{\gamma\lambda-ia} \times$$

$$F\left(\lambda, \frac{\gamma\lambda-ia}{2\gamma}, \frac{\gamma\lambda-ia}{2\gamma}+1; -e^{2\gamma\rho}\right) \tag{10.18}$$

in the limit $\lambda \to 0$ we obtain:

$$h(\rho) = -\frac{\pi}{a^2}\frac{e^{ia\rho}}{\rho} \tag{10.19}$$

and therefore, in the complex ρ-plane, the corresponding ultradistribution of exponential type is:

$$H(\rho) = -\frac{\pi}{a^2\rho}\left\{\Theta[\Im(\rho)]e^{ia\rho} - \frac{1}{2}\right\} \tag{10.20}$$

As final example we evaluate the convolution between $F(\rho) = (1/2)Sgn[\Im(\rho)]$ $e^{ia\rho}\cosh(\rho^{1/2})$ (a real) and $G(\rho) = \delta(\rho)$. We perform the calculation of the convolution in the real ρ-plane and then we pass tho the complex ρ-plane. By the use of the Taylor's development of $\cosh(\rho^{1/2})$

$$\cosh(\rho^{1/2}) = \sum_{n=0}^{\infty} \frac{\rho^n}{2n!} \tag{10.21}$$

we obtain

$$h_{\gamma\lambda}(\rho) = \frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\int_{-\infty}^{\infty} e^{ia\rho_1}\delta(\rho_2)\times$$

$$\frac{[(\rho-\rho_1-\rho_2)^2-4\rho_1\rho_2]_+^{\frac{1}{2}}}{[\cosh(\gamma\rho_1)]^\lambda[\cosh(\gamma\rho_2)]^\lambda}\,d\rho_1\,d\rho_2 \;=$$

$$\frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\int_{-\infty}^{\infty}\frac{e^{ia\rho_1}|\rho-\rho_1|}{[\cosh(\gamma\rho_1)]^\lambda}\,d\rho_1 \tag{10.22}$$

By means of the use of equations (10.17), (10.18) and in the limit $\lambda \to 0$ we obtain:

$$h(\rho) = -\pi\left(1+\frac{i}{\rho}\frac{\partial}{\partial a}\right)\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\left(\frac{e^{ia\rho}}{a}\right) \tag{10.23}$$

and consequently:

$$H(\rho) = \pi\left[\left(\frac{\Theta[\Im(\rho)]}{\rho}\frac{\partial}{\partial a}-\frac{i}{2}Sgn[\Im(\rho)]\right)\sum_{n=0}^{\infty}\frac{(-i)^n}{2n!}\frac{\partial^n}{\partial a^n}\left(\frac{e^{ia\rho}}{a}\right)\right]+$$

$$\frac{\pi}{2\rho}\sum_{n=0}^{\infty}\frac{i^n}{2n!}\frac{(n+1)!}{a^{n+2}} \tag{10.24}$$

As an example of the use of (10.11) we will evaluate the convolution product of two Dirac's delta: $\delta(\rho)*\delta(\rho)$. In this case we have:

$$F_{\gamma\lambda}(\rho) = -\frac{[\cosh(\gamma\rho)]^\lambda}{2\pi i\rho} = -\frac{1}{2\pi i\rho} \tag{10.25}$$

and as a consequence:

$$f_{\gamma\lambda}(\rho) = \delta(\rho) \tag{10.26}$$

The Fourier antitransform of (10.26) is:

$$\hat{f}_{\gamma\lambda}(x) = \frac{2}{(2\pi)^3} x^{-1} \tag{10.27}$$

Thus we have:

$$\hat{f}^2_{\gamma\lambda}(x) = \frac{4}{(2\pi)^6} x^{-2} \tag{10.28}$$

From (10.28) we obtain:

$$\lim_{\lambda \to 0} \hat{f}^2_{\gamma\lambda}(x) = \frac{4}{(2\pi)^6} x^{-2} \tag{10.29}$$

and taking into account that:

$$\mathcal{F}\{x^{-2}\} = \frac{\pi^3}{2} Sgn(\rho) \tag{10.30}$$

we obtain

$$\delta(\rho) * \delta(\rho) = \frac{\pi}{2} Sgn(\rho) \tag{10.31}$$

11 Discussion

In a first paper [3] we have shown the existence of the convolution of two one-dimensional tempered ultradistributions. In a second paper ref.[4] we have extended these procedure to n-dimensional space. In four-dimensional space we have given a expression for the convolution of two tempered ultradistributions (even) in the variables k^0 and ρ. In a third paper [5]we have defined spherically symmetric and Lorentz invariant tempered ultradistributions and we have given the formulaes for the Fourier transform and four-dimensional convolution of them.

In this paper we have extended these results to ultradistributions of exponential type and, in a intermadiate step of deduction we have obtained the generalization to ultradistributions of exponential type in Euclidean and Minkowskian space of dimensional regularization in configuration space (ref.[5, 8])

Furthermore, as an application of our formalism we have solved the question of normalization of Gamow States and we have given an example .

When we use the perturbative development in Quantum Field Theory, we have to deal with products of distributions in configuration space, or else, with convolutions in the Fourier transformed p-space. Unfortunately, products or convolutions (of distributions) are in general ill-defined quantities. On the contrary, the convolution of ultradistributions is a well-defined quantity and the correspondig product in configuration space is a true product in a ring with zero-factors.

In physical applications one introduces some “regularization” scheme, which allows us to give sense to divergent integrals. Among these procedures we would like to mention the dimensional regularization method (ref. [12, 13]). Essentially, the method consists in the separation of the volume element ($d^\nu p$) into an angular factor ($d\Omega$) and a radial factor ($p^{\nu-1}dp$). First the angular integration is carried out and then the number of dimensions ν is taken as a free parameter. It can be adjusted to give a convergent integral, which is an analytic function of ν.

Our formula (10.1) is similar to the expression one obtains with dimensional regularization. However, the parameter λ is completely independents of any dimensional interpretation.

All ultradistributions provide integrands (in (6.1),(6.27),(8.1),(9.1),(10.1)) that are analytic functions along the integration path. The lambda parameters permit us to control the possible exponential asymptotic behavior (cf. eq.(2.25)). The existence of a region of analyticity for λ, and a subsequent continuation to the point of interest, defines the convolution product.

The properties described below show that ultradistributions provide an appropriate framework for applications to physics. Furthermore, they can "absorb" arbitrary pseudo-polynomials (in the variables $s_i = e^{z_i}$), thanks to eq.(2.16). A property that is interesting for renormalization theory. For this reason we decided to begin this paper (and also for the benefit of the reader), with a summary of the main characteristics of ultradistributions of exponential type and their Fourier transform.

As a final remark we would like to point out that our formulae for convolutions are general definitions and not regularization methods.

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