A Fuzzy Inference Model Based on an Uncertainty Forward Propagation Approach

Luis M. de Campos and Antonio González

Departamento de Ciencias de la Computación e Inteligencia Artificial
Universidad de Granada, Spain

ABSTRACT

The management of uncertainty and imprecision is becoming more and more important in knowledge-based systems. Fuzzy logic provides a systematic basis for representing and inferring with this kind of knowledge. This paper describes an approach for fuzzy inference based on an uncertainty forward propagation method and a change in the granularity of the elements involved. The proposed model is able to handle very general kinds of facts and rules, and it also verifies the most usual properties required by a fuzzy inference model.

KEYWORDS: ***Fuzzy logic, fuzzy inference, uncertainty management, upper and lower probabilities***

1. INTRODUCTION

The facts and/or the rules to be represented in knowledge-based systems may be often uncertain or imprecise. Different models for deductive reasoning are based on mathematical models like Dempster/Shafer's theory of belief functions [1], [2], possibility theory [3], [4], among other alternatives to the standard Bayesian model. In particular, the problem of inference from vague or fuzzy premises [5] will be our main concern.

Address correspondence to Antonio González Muñoz, Computer Science and A. I. Departmeut, Universidad de Granada, 18071 Granada, Spain.

Received May 1, 1992; accepted March 10, 1993.

This work was supported by the CICYT under Project TIC92-0665.

International Journal of Approximate Reasoning 1993; 9:139–164

655 Avenue of the Americas, New York, NY 10010 0888-613X/93/$6.00

The basic problem may be stated as follows:

$$\begin{array}{l} \text{If } X \text{ is } A \text{ then } Y \text{ is } B \\ X \text{ is } A^* \\ \hline \text{then } Y \text{ is } B^*. \end{array} \tag{1}$$

where X and Y are variables on reference sets U_1, U_2, respectively. A, B, A^* are fuzzy sets of the respective reference sets. These fuzzy sets may be considered as fuzzy information or soft restrictions on each variable. B^* is also a fuzzy set, representing a soft restriction of Y obtained from the soft restriction A^* on X and the fuzzy rule.

This method was initially introduced by Zadeh [6], who proposed the following solution for the basic problem stated in (1)

$$\mu_B*(s) = \sup_r \{ \mu_A*(r) \wedge (1 - \mu_A(r) + \mu_B(s)) \}$$

Thus, U_1 being the antecedent domain and U_2 the consequent domain, the membership function of the predicate B^* is given by the projection on U_2 of the intersection of the implication relation μ_H, defined by $\mu_H(r,s) = 1 \wedge (1 - \mu_A(r) + \mu_B(s))$, and the cylindrical extension on $U_1 \times U_2$ of the membership function μ_A*.

In contrast to the classical modus ponens, the above rule allows us to use fuzzy predicates A, B, and A^*. Moreover, A^* is not required to be identical with A. When $A = A^*$ and the predicates are crisp, then (1) becomes the classical modus ponens.

A more general version of this generalized modus ponens (GMP) can be obtained if we replace the Min operator $\wedge$ by an alternative t-norm $*$, and the particular implication relation $\mu_H(r,s)$ by another $\mu_{A \to B}$, thus obtaining

$$\mu_B*(s) = \sup_r \{ \mu_A*(r) * \mu_{A \to B}(r,s) \}. \tag{2}$$

Several authors have investigated this approach [3], [7], [8]. We are interested in a more general model within which the uncertain knowledge can be included. In this work, we propose a fuzzy inference model based on an uncertainty forward propagation approach [9].

The paper is arranged in nine sections. Section 2 describes the main ideas on which the proposed model is based, and section 3 outlines the propagation approach used to develop a fuzzy inference device. The basic inference model is introduced in section 4, the propagation approach is applied to a simple inference problem. The performance of this model is studied in section 5. Next, in sections 6 and 7, this basic inference model is extended to include uncertainty degrees in the facts and rules, as well as more general kinds of rules. Section 8 includes an example that illustrates the use of the inference model presented, and section 9 comments on the performance and flexibility of our approach, and also points out future lines of research.

2. OUTLINE OF THE INFERENCE MODEL

Fuzzy logic provides a systematic basis for representing and inferring from imprecise knowledge. The main goal of this work is to develop a fuzzy inference model able to handle a wide class of fuzzy facts and rules. The simplest case would include fuzzy propositions such as

Rule: if X is low then Y is high

Fact: X is very low,

but we would like also to include uncertain knowledge and more general kinds of facts and rules, as for example,

X is low with certainty degree 0.7,

if X is high then Y is very low with certainty degree 0.5, etc.

Thus, we are interested in easy-to-implement and computationally efficient models able to do inference with the different kinds of fuzzy propositions (Zadeh [10]), and moreover, verifying the most usual properties required by a fuzzy inference model.

The proposed model is based on two main ideas [11]:

- The fuzzy rule "If X is A then Y is B" defines a relation among the elements of the sets $U_A = \{A, \neg A\}$ and $U_B = \{B, \neg B\}$.
- This relation is interpreted as a conditioning, that we represent by means of an uncertainty measure.

With respect to the first idea, the models based on the GMP usually assume that the rule "If X is A then Y is B" defines a relation on the cartesian product of the reference sets of X and $Y, U_1 \times U_2$. By contrast, we suppose that the level of granularity in this relation is similar to the granularity of the elements involved, that is to say, the rule does not establish how each element of U_1 and each element from U_2 are related; it only establishes the relation between the concepts represented by $A, \neg A$ and $B, \neg B$. On the other hand, the second idea differs from other models that interpret the rule as a material implication ($\neg A$ or B).

A general input A^*, does not usually match any of the antecedent items. Thus, taking into account the above considerations, the fuzzy inference model should translate the information contained in A^* to information about A and $\neg A$. This translation can be easily done through a compatibility degree between the input and the antecedent of the rule. These degrees (the values α_1, α_2 in Figure 1) will be interpreted as an uncertainty measure generated by the current input A^*, on the set U_A.

This uncertainty measure will be transferred from the set U_A to the set U_B through the fuzzy rule by means of an uncertainty propagation model.

So far, the answer of the inference model is an uncertainty measure on U_B. Finally, by combining the membership functions of B and $\neg B$, and their uncertainty values (the values β_1, β_2 in Figure 1), we will obtain a single output B^* (see Figure 1).

In order to make these general ideas more specific, first we need to choose a formalism to represent uncertainty measures, and a propagation model. The next section is devoted to these topics.

3. THE PROPAGATION MODEL

The formalism we will use to represent pieces of uncertain information is by means of a class of fuzzy measures, namely representable measures (also called lower and upper probabilities). This is a very general framework of representation, which includes probabilities, possibilities [4], belief

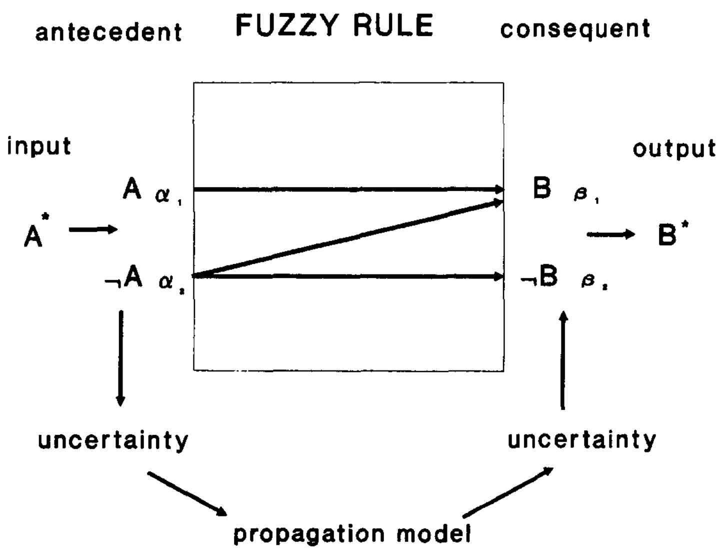


Figure 1. Fuzzy inference model.

functions [1], [2], and Choquet capacities of order two [12] [13] as particular cases.

Let us very briefly introduce the concept of lower–upper probabilities: Let $\boldsymbol{P}$ be a family of probability measures on a referential D_x. We may associate a pair of lower–upper probabilities with $\boldsymbol{P}$, (l, u), given by

$$l(A) = \inf_{P \in \mathbf{P}} P(A) \quad \forall A \subseteq D_x$$

$$u(A) = \sup_{P \in \mathbf{P}} P(A) \quad \forall A \subseteq D_x$$

This defines a pair of ordered fuzzy measures in Sugeno's sense (see [14]).

Now, let us suppose that we have two variables X and Y that can take on values in the sets $D_x = \{x_1, x_2, \ldots, x_n\}$ and $D_y = \{y_1, y_2, \ldots, y_m\}$, respectively and a pair of representable measures, $((l_x(A), u_x(A)), A \subseteq D_x)$, representing our knowledge about the values of X. We also have conditional information about the values of Y, given that we know the true value of X. The problem is to propagate the information from X to Y, through the conditional relationships.

So, we want to obtain on Y another pair of lower and upper probabilities representing the knowledge about the value of the variable Y that we can infer from our knowledge about the value of the variable X and the relationships between X and Y. We model the conditional information on Y given some value x_i by also using conditional representable measures $(l(B/x_i), u(B/x_i)), B \subseteq D_y), x_i \in D_x$.

In [9] the following general solution for this problem was obtained: to calculate the upper measure of any subset of D_y, we must solve this linear programming problem

$$u_Y(B) = \max \sum_{i=1}^{n} u(B/x_i) h_i$$

subject to

$$\sum_{x_i \in A} h_i \leq u_x(A), \forall A \subseteq D_x \tag{4}$$

Several particular cases of this problem can be directly solved without using any optimization techniques (see [15], [9]). Precisely one of these particular cases will be needed in the fuzzy inference model. Further details about the concrete formulation can be found in the appendix.

The following example illustrates how the propagation model works.

Example 1: Let us consider two variables X and Y, which stand for the color and the weight of a set of objects, respectively. Suppose that the values these variables can take on are Black (B) and White (W) for X, and Heavy (H) and Light (L) for Y. So, the domains for the variables X and Y are $D_x = \{B, W\}$, $D_y = \{H, L\}$ respectively.

Let us also suppose that we have the following partial information about the color of the objects and about the relationship between color and weight:

- 70% of the objects are Black, 10% are White, and 20% can be either Black or White.
- 80% of the Black objects are also heavy, and 20% can be either Heavy or Light.
- 30% of the White objects are also heavy, 60% are Light, and 10% can be either Heavy or Light.

We want information about the weight of the objects in the light of the information about the color, and the weight given that we know the color.

These pieces of information can be represented as Dempster–Shafer measures (a particular case of lower and upper probabilities) as follows:
Information about the color:

$$u_x(B) = 0.9, \qquad u_x(W) = 0.3$$
$$l_x(B) = 0.7, \qquad l_x(W) = 0.1$$

Conditional information about the weight given the color:

- If the color is Black:
$$u(H/B) = 1, \qquad u(L/B) = 0.2$$
$$l(H/B) = 0.8, \qquad l(L/B) = 0$$
- If the color is White:
$$u(H/W) = 0.4, \qquad u(L/W) = 0.7$$
$$l(H/W) = 0.3, \qquad l(L/W) = 0.6$$

By applying the propagation model to these measures, we obtain (see the appendix for details) the following measures l_y and u_y on D_y:

$$u_y(H) = 0.94, \qquad u_y(L) = 0.35$$
$$l_y(H) = 0.65 \qquad l_y(L) = 0.06$$

that correspond to the following partial information about the weight of the objects:

65% of the objects are heavy, 6% are Light and 29%, can be either Heavy or Light. ■

4. THE FUZZY INFERENCE MODEL

In order to develop a fuzzy inference model, we are going to express, within the context of the above propagation model, the following fuzzy rule:

$$\text{If } X \text{ is } A \text{ then } Y \text{ is } B \tag{5}$$

where X and Y are variables on the reference sets U_1, U_2, and A and B are fuzzy sets on U_1 and U_2, respectively.

Let $U_A = \{A, \neg A\}$ and $U_B = \{B, \neg B\}$ be two fuzzy partitions of U_1 and U_2. The basic idea to develop the inference model consists in replacing U_1 and U_2 by the fuzzy partitions U_A and U_B respectively, and then to consider uncertainty measures on these.

The conditional information comes from a semantic interpretation of the rule (5) in the following sense: This rule generates two conditional representable measures on U_B, $(l(./A), u(./A))$, $(l(./\neg A), u(./\neg A))$ defined by

$$l(B/A) = 1 \qquad u(B/A) = 1$$
$$l(\neg B/A) = 0 \qquad u(\neg B/A) = 0 \tag{6}$$
$$l(B/\neg A) = 0 \qquad u(B/\neg A) = 1$$
$$l(\neg B/\neg A) = 0 \qquad u(\neg B/\neg A) = 1 \tag{7}$$

So, we are interpreting the rule "if X is A then Y is B" as a conditioning (instead of a material implication), that is, if we know that A is true then we can assert that B is also true (this is modelized as the total certainty measure (6)), but if we know that A is false then we cannot infer anything about the truth of B (and it is modelized as the total ignorance measure (7)).

Remark As the representable measures we are using are always dual, in the following we will use only the upper measure. The results of the lower measures can be obtained by duality ($l(H) = 1 - u(\neg H)$).

Moreover, we also need an upper probability measure on U_A, in order to propagate it on Y through the conditional information (the rule). This measure will be obtained from a matching process between the input A^* and each value in U_A, that is, between A^* and A and between A^* and $\neg A$. For this purpose we will use a particular matching based on a compatibility degree between two fuzzy sets, F and G, through a t-norm $*$:

$$c(F,G) = \sup_r \{\mu_F(r) * \mu_G(r)\} \tag{8}$$

Although we could use any t-norm in (8), we believe the Lukasiewicz t-norm to be adequate because it satisfies the noncontradiction law ($a^*(1 - a) = 0$) and thus the compatibility degree between two complementary fuzzy sets is zero:

$$c(F, \neg F) = \sup_r \{\max(\mu_F(r) + \mu_{\neg F}(r) - 1, 0)\} = 0$$

Therefore we will use the Lukasiewicz t-norm in (8) because it makes the elements in each fuzzy partition U_A and U_B incompatible.

Remark Although this choice may be controversial, in our context the fuzzy sets A and $\neg A$, B and $\neg B$ act as the elements of two (crisp) sets U_A and U_B. Then the use of the Lukasiewicz t-norm guarantees having nonoverlapping elements in these sets.

So, we define the upper measure on U_A induced by the input A^* as

$$u_x(A/A^*) = c(A, A^*) = \sup_r \{\max(\mu_A(r) + \mu_A*(r) - 1, 0)\}$$

$$u_x(\neg A/A^*) = c(\neg A, A^*) = \sup_r \{\max(\mu_A*(r) - \mu_A(r), 0)\} \quad (9)$$

It can be easily proved that

$$c(A, A^*) + c(\neg A, A^*) \geq 1 \quad (10)$$

if we only impose the input A^* to be normalized ($\exists r \in U_1/\mu_A*(r) = 1$). We will always suppose that A^* verifies this property in the rest of the paper. From (10) we may interpret u_x as an upper probability measure.

By using the above propagation model and this upper measure on U_A, we get an upper measure on U_B. In this case the solution of (4) is very easy (it may be calculated directly without using an optimization technique: we need only to calculate a Choquet integral (see [12], [13]) with respect to the measure u_x; see appendix for details). The solution is

$$u_Y(B) = 1$$

$$u_Y(\neg B) = c(\neg A, A^*) \quad (11)$$

We may consider this upper measure as the first interesting answer of the fuzzy inference:

Rule	if X is A then Y is B	
Input	X is A^*	
Output	Y is B is $[1 - \lambda, 1]$	(12)
	Y is $\neg B$ is $[0, \lambda]$,	

where $\lambda = c(\neg A, A^*)$ and "Y is C is α" means a proposition at degree α [10]. In our case the uncertainty degree α is an interval $[\alpha_{\text{inf}}, \alpha_{\text{sup}}]$ representing the lower and upper probability respectively.

This answer generates certainty values for the results of B and $\neg B$. Observe that if the compatibility between A^* and $\neg A$ is zero then the result is unambiguously B. This only happens if $\mu_A*(r) \leq \mu_A(r)\ \forall r$, that is, when the input A^* is included in A. When the compatibility between A^* and $\neg A$ is maximum ($\exists r \in U_1/\mu_A*(r) = 1$ and $\mu_A(r) = 0$), for instance if $A^* = \neg A$, then we obtain an uncertainty measure representing the total ignorance, in agreement with the conditional interpretation of the fuzzy rule we have made.

The kind of solution given by (12) is similar to that proposed by other authors (see [16]) in the sense that we obtain an uncertainty distribution on the possible answers.

In some cases this solution may not be appropriate enough (e.g., in fuzzy control problems). Thus, if we want to obtain an answer B^* as output, we could combine the two pieces of information that we have: the upper measure on U_B and the membership functions of B and $\neg B$. The idea in carrying out this combination is to produce the result B^* as an expected value of B and $\neg B$, weighted by their upper measures, through a fuzzy integral.

In the above process, we obtained the answer $u_Y(.)$ from $u_X(.)$ and the conditional relation $A \to B$ as an average of $u(./.)$ weighted by the measure u_x. In a similar way, we could obtain B^* (μ_B*) from $u_Y(.)$ and $\{\mu_B(s), \mu_{\neg B}(s)\}$ because we interpret the membership function of a fuzzy set C, $\mu_C(s)$, as the conditional possibility (a particular case of upper probability measure) of s given that C is true: $\mu_C(s) = \pi(s/C)$. Then again we have an upper measure $u_Y(.)$ on U_B and two conditional upper measures $\pi(s/B) = \mu_B(s)$ and $\pi(s/\neg B) = \mu_{\neg B}(s) = 1 - \mu_B(s)$. The model of forward propagation, when it is applied to this case, produces (using again the Choquet integral) the following membership function for the output B^*:

$$\mu_B{}^*(s) = u(\{s\}) = \begin{cases} \mu_B(s) & \text{if } \mu_B(s) \geq 1/2 \\ \mu_B(s)(1 - 2\lambda) + \lambda & \text{if } \mu_B(s) \leq 1/2 \end{cases}$$

or equivalently

$$\mu_B*(s) = (1 - \lambda)\mu_B(s) + \lambda \max(\mu_B(s), 1 - \mu_B(s)), \qquad (13)$$

where $\lambda = c(\neg A, A^*)$.

This method has the following drawback: let us suppose that $\lambda = u_Y(\neg B) = 1$, that is, $\neg B$ is as credible as B. This happens, for instance, when the input A^* is equal to $\neg A$. In this case, and taking into account that we are interpreting the rule $A \to B$ as a conditioning, the reasonable output should be B^* such that $\mu_B *(s) = 1\ \forall s$, that is to say, we do not know anything about the universe U_2 (the variable Y has no restrictions). However, the output produced by (13) is B^* such that

$$\mu_B *(s) = \mu_B(s) \vee (1 - \mu_B(s)) \neq 1$$

So, the output B^* corresponds to the disjunction "B or $\neg B$". This result is coherent at the semantic level (if the possible results are B and $\neg B$ and we do not know anything, all we can say is "B or $\neg B$") but it is not coherent at a membership function level. The problem arises because the integral we are using is based on the maximum operator which does not make the sets B and $\neg B$ exhaustive ($B \vee \neg B \neq U_2$).

One way to solve this problem is to consider a suitable modification of the integral we are using in which the max operator is replaced by the bounded sum. The reason for this replacement is similar to the reason why we chose the Lukasiewicz t-norm. In both cases, we need t-norm and t-conorm such that, in the first case, A and $\neg A$ are incompatible and, in the second case, B or $\neg B$ are exhaustive. To do this, it is necessary to take into account that the Choquet integral with respect to an upper measure g, defined on a finite set W, coincides with the upper Dempster integral (see [1]) when the upper measure g is a plausibility function with basic probability assignment (BPA) m:

$$\mathbf{E}_g(f) = \sum_{D \subseteq W} m(D) \max_{w \in D} \{f(w)\} \tag{14}$$

In our case the upper measure, u_Y, is a plausibility function on U_B with BPA $m_Y(\{B\}) = 1 - \lambda$, $m_Y(\{\neg B\}) = 0$ and $m_Y(\{B, \neg B\}) = \lambda$. Then we propose to use the following modified Dempster (or Choquet) integral:

$$\sum_{D \subseteq W} m(D) \min\left(\sum_{w \in D} f(w), 1 \right) \tag{15}$$

This integral, when it is applied to our problem, produces the following result B^*:

$$\begin{aligned} \mu_B *(s) &= (1 - \lambda)\mu_B(s) + \lambda \min(\mu_B(s) + 1 - \mu_B(s), 1) \\ &= (1 - \lambda)\mu_B(s) + \lambda. \end{aligned}$$

It is obvious that $\mu_B *$ can be written as

$$\mu_B *(s) = \mu_B(s) + \lambda - \mu_B(s)\lambda \tag{16}$$

and therefore the output of the inference model is

$$\mu_B *(s) = \mu_B(s) \oplus \lambda \tag{17}$$

where $\lambda = c(\neg A, A^*)$ and $\oplus$ is the probabilistic sum t-conorm $(a \oplus b = a + b - ab)$.

Example 2: Let us consider the following rule:

If X Very Low Then Y is Medium

where the labels Very Low and Medium have been represented by the following fuzzy sets:

$$\text{Very Low} = (0, 0, 25)$$

$$\text{Medium} = (50, 25, 25)$$

with $A = (m, a, b)$ representing the following parameters associated to a triangular fuzzy number A: m the modal value of A, and a and b the left and right spreads, respectively. Now, given the input

$$A^* = \text{"Around 10"} = (10, 5, 5),$$

the output B^* corresponds to Figure 2 with $B = (50, 25, 25)$ and $\lambda = c(\neg A, A^*) = 0.4$. ■

5. PERFORMANCE OF THE MODEL

As we mentioned previously, we are interested in fuzzy inference models that are easy to implement and computationally efficient, and in verifying the most usual properties required by several authors for these models.

With respect to the first question, (17) shows how easy the model is to use: it only needs to calculate the compatibility between the input A^* and $\neg A$, and to operate this value with the membership function of B by

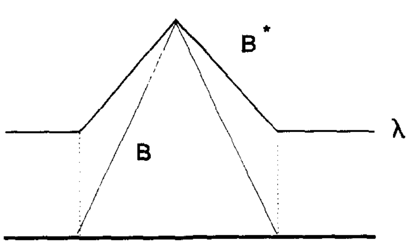


Figure 2. The output B^* in the proposed model.

means of the probabilistic sum t-conorm (see Figure 2). Obviously, this simple case can be interpreted as a fuzzy modus ponens [17], [6].

With respect to the properties, our model verifies the following intuitive properties, which are usually considered as a test for the performance of a fuzzy modus ponens:

(i) If $A^* = A$ then $B^* = B$:

In effect, when $A^* = A$, the compatibility between A^* and $\neg A$ is zero, and $\mu_B*(.) = \mu_B(.) \oplus 0 = \mu_B(.)$. So, this model extends the classical modus ponens.

(ii) If $A^* \subseteq \neg A$ then $B^* = U_2$:

In this case, the compatibility between A^* and $\neg A$ is equal to one, and then $\mu_B*(.) = \mu_B(.) \oplus 1 = 1$. So, for inputs completely different from A, the inference process gives no information, that is, all the elements of the consequent's domain are maximally possible.

(iii) If $A^* \subseteq A$ then $B^* = B$:

When the input A^* is a subset of A, the compatibility between A^* and $\neg A$ is zero, and we obtain the result in a similar way as in (i). When the input is more precise than the antecedent, the rule can only infer the consequent, without adding information not contained in the rule (the rule only says "if X is A then Y is B," but it does not say, for instance, "the more X is A then the more Y is B").

In addition to these basic properties, it would also be interesting to study how this model works with respect to the chaining of rules. Let us suppose the knowledge base contains two such rules as

$$\begin{array}{l} \text{If } X \text{ is } A \text{ then } Y \text{ is } B \\ \text{If } Y \text{ is } B \text{ then } Z \text{ is } C \end{array} \tag{18}$$

In classical logic, from the input A one would expect to obtain the conclusion C by chaining the two rules. We want to study what kind of output C^* is obtained from an input A^*.

Given an input A^*, from the first rule we obtain on U_B the measure

$$u_Y(B) = 1, \qquad u_Y(\neg B) = \lambda \quad (\lambda = c(\neg A, A^*))$$

and then by combining this measure with the membership function of B and $\neg B$ we get B^* such that

$$\mu_B*(s) = \mu_B(s) \oplus \lambda$$

Now, the input to the second rule is the output of the first one. Starting from B^*, we calculate the compatibility degree of B^* with B and $\neg B$ and then we apply the inference process to the second rule again. So, let us see what the compatibility degrees $c(\neg B, B^*)$ and $c(B, B^*)$ are. If we suppose that the fuzzy set B satisfies

$$\exists s_1, s_2 \in U_2 / \mu_B(s_1) = 1 \quad \text{and} \quad \mu_B(s_2) = 0$$

that is to say, B and $\neg B$ are both normal fuzzy sets, then

a)
$$\mu_B(s_1) = 1, \qquad \mu_B{*}(s_1) = \mu_B(s_1) \oplus \lambda = 1$$
$$\Rightarrow \max(\mu_B{*}(s_1) + \mu_B(s_1) - 1, 0) = 1,$$

and therefore

$$c(B, B^*) = 1.$$

b)
$$\mu_B(s_2) = 0, \qquad \mu_B{*}(s_2) = \mu_B(s_2) \oplus \lambda$$
$$= \lambda \Rightarrow \max(\mu_B{*}(s_2) - \mu_B(s_2), 0) = \lambda.$$

So, $c(\neg B, B^*) \geq \lambda$. Moreover, as the t-norm $\oplus$ verifies that $a \oplus b \leq a + b$ $\forall a, b \in [0,1]$, then

$$\mu_B(s) \oplus \lambda \leq \mu_B(s) + \lambda \, \forall s \Rightarrow \mu_B(s) \oplus \lambda - \mu_B(s)$$
$$\leq \lambda \, \forall s \Rightarrow \max(\mu_B{*}(s) - \mu_B(s), 0) \leq \lambda \, \forall \lambda.$$

Thus, $c(\neg B, B^*)$ is also less than or equal to λ, and then

$$c(\neg B, B^*) = \lambda.$$

Therefore, once we apply the inference process to the second rule, we obtain the following measure on U_C:

$$u_Z(C) = 1, \qquad u_Z(\neg C) = \lambda \quad (\lambda = c(\neg A, A^*)) \tag{19}$$

and the final result is the fuzzy set C^* with membership function

$$\mu_C{*}(t) = \mu_C(t) \oplus \lambda \tag{20}$$

This result coincides with that obtained by the rule "if X is A then Z is C" and the output A^*.

It is interesting to note that the imprecision in the result does not increase through the chaining: $c(\neg A, A^*) = \lambda$, $\mu_B{*}(.) = \mu_B(.) \oplus \lambda$, $\mu_C{*}(.) = \mu_C(.) \oplus \lambda$, which seems reasonable to us because the two rules are supposed to be absolutely certain, and the only source of uncertainty is the partial matching between the input and the antecedent of the first rule. (This situation will change when we consider uncertain rules.)

As we mentioned before, the output of the inference model could be either an uncertainty measure u_Y on U_B (see (11)), or a fuzzy set B^* obtained by integration of the above measure and the membership functions of B and $\neg B$ (see (17)). In the previous study of chaining, we have used the second kind of output. Now, let us see how the same result is obtained if we use the first kind of output.

In effect, the measure u_Y obtained from u_X and the first rule will be the input to the second rule. By directly applying the forward propagation mechanism to u_Y and to the second rule we get a measure u_Z on U_C:

$$u_Z(C) = 1, \qquad u_Z(\neg C) = \lambda$$

which coincides with (19) and therefore C^*, obtained from integration of u_Z coincides with (20).

In this way, the outputs of the fuzzy modus ponens given by (12) and (17), are equivalent, that is, we can infer either an uncertainty measure on U_B or a fuzzy set B^*, and any of them will produce the same result when they are used by the inference model.

To end this section, we show how our model verifies another property suggested by Magrez and Smets [8]. These authors proposed four properties that any fuzzy modus ponens should verify. The properties (i), (ii), and (iii) above, that our model satisfies, are the first three properties in [8]. The fourth one is the following:

(iv) If $A \subseteq A^*$ then $\exists a \in [0, 1]$ such that $\forall s \in U_2$ $\mu_B*(s) = \mu_B(s) \perp a$, where $\perp$ is a t-conorm. This last property imposes two requirements in a model: first, whenever an indetermination should appear on the output, each of the elements of the consequent domain receives the same degree of indetermination, and second, the shapes of the output B^* and the input A^* are independent.

Obviously, our model also verifies the property (iv) for the probabilistic sum t-conorm. The model proposed by Magrez and Smets [8] is

$$\mu_B*(s) = \min(\mu_B(s) + (1 - v), 1)$$

where $v = 1 - \sup_r\{1 - \mu_A(r))^*\mu_A*(r))$, and $*$ is the Lukasiewicz t-norm. So, their model verifies the fourth property by taking the bounded sum as t-conorm. Although both models are obtained from different approaches, the results are similar, differing only in the t-conorm used.

6. MANAGEMENT OF UNCERTAINTY IN FACTS AND RULES

The propositions we have been using so far do not contain any kind of uncertainty different from fuzziness, that is, the facts and rules considered are fuzzy but there is not any doubt about their certainty. For instance, when we say "X is a young man", we are not sure about the exact age of X, but the information conveyed by this fuzzy proposition is supposed to be completely certain.

Zadeh in [10] divided the propositions, in a knowledge base, into four principal categories:

1. An unconditional, unqualified proposition: X is A.
2. An unconditional, qualified proposition: X is A is α.
3. A conditional, unqualified proposition: If X is A then Y is B.
4. A conditional, qualified proposition: If X is A then Y is B is α.

Categories 1 and 3 are precisely those we have previously considered. In this section we are going to study the incorporation of categories 2 and 4 that correspond to uncertain facts and rules in the inference model.

Uncertain facts

An uncertain fact can be written as

$$X \text{ is } A \text{ is } \alpha \tag{21}$$

The degree α may be interpreted in several ways (linguistic probability, numerical value, ...). One way to interpret this value is as a necessity degree of the proposition "X is A." Therefore, following the formalism given in (12), the proposition (21) will be rewritten as

$$\begin{aligned} &X \text{ is } A \text{ is } [\alpha, 1] \\ &X \text{ is } \neg A \text{ is } [0, 1-\alpha] \end{aligned} \tag{22}$$

that is to say, the proposition (21) is translated in the inference model as a pair of uncertainty measures on U_A given by

$$\begin{aligned} l_x(A) &= \alpha & u_x(A) &= 1 \\ l_x(\neg A) &= 0, & u_x(\neg A) &= 1-\alpha \end{aligned} \tag{23}$$

This pair of measures obtained from (21) is obviously a pair of necessity and possibility measures. The formalism we are using to represent pieces of uncertain information includes, as a particular case, necessity and possibility measures, and even allows us to represent a more general kind of uncertainty. So, the kind of uncertain facts that could be interesting to include in this model are described by

$$\begin{array}{l} X \text{ is } A \text{ is } [\alpha, \beta] \\ X \text{ is } \neg A \text{ is } [1-\beta, 1-\alpha] \end{array} \tag{24}$$

where $\alpha, \beta \in [0,1]$, $\alpha \leq \beta$, and the intervals [.,.] represent lower and upper probabilities. When we take $\beta = 1$ we obtain (22), and from $\alpha = 1$ (and then $\beta = 1$, too) we get the true facts included in category 1.

In order to simplify the notation, from now on we will represent (24) only by writing the first expression

$$X \text{ is } A \text{ is } [\alpha, \beta] \tag{25}$$

because it is obvious that once we know (25) then we know (24) too.

In this way, the inference problem for uncertain facts can be written as

$$\begin{array}{c} \text{If } X \text{ is } A \text{ then } Y \text{ is } B \\ X \text{ is } A^* \text{ is } [\alpha, \beta] \end{array} \tag{26}$$

The way in which we solved the basic inference model given by (12) may be applied again, we only need to get a measure u_X on U_A. To do that we could transfer to U_A the measure we have on U_A*, which is,

$$\begin{array}{ll} l_x(A^*) = \alpha, & u_x(A^*) = \beta \\ l_x(\neg A^*) = 1 - \beta, & u_x(\neg A^*) = 1 - \alpha \end{array} \tag{27}$$

This transference will be made through a fictitious, uncertain rule, such as

$$\text{if } X \text{ is } A^* \text{ then } X \text{ is } A$$

where the conditional measures that define this rule are the compatibilities among elements of U_A and U_A*,

$$\begin{array}{ll} u(A/A^*) = c(A, A^*) & u(\neg A/A^*) = c(\neg A, A^*) \\ u(A/\neg A^*) = c(A, \neg A^*) & u(\neg A/\neg A^*) = c(\neg A, \neg A^*) \end{array} \tag{28}$$

Then, by chaining this rule with the real rule, we obtain a measure on U_B, or equivalently, a fuzzy output B^*. Next, we are going to develop this process.

From the measure (27), the conditional measures (28) and the propagation model, we get the following measure on U_A:

$$\begin{aligned}
&\text{If } u(A/A^*) \leq u(A/\neg A^*) \Rightarrow u_x(A) = \alpha u(A/A^*) \\
&\qquad + (1-\alpha)u(A/\neg A^*) \\
&\text{If } u(A/A^*) \geq u(A/\neg A^*) \Rightarrow u_x(A) = \beta u(A/A^*) \\
&\qquad + (1-\beta)u(A/\neg A^*) \\
&\text{If } u(\neg A/A^*) \leq u(\neg A/\neg A^*) \Rightarrow u_x(\neg A) = \alpha u(\neg A/A^*) \\
&\qquad + (1-\alpha)u(\neg A/\neg A^*) \\
&\text{If } u(\neg A/A^*) \geq u(\neg A/\neg A^*) \Rightarrow u_x(\neg A) = \beta u(\neg A/A^*) \\
&\qquad + (1-\beta)u(\neg A/\neg A^*)
\end{aligned} \tag{29}$$

Finally, by propagating (29) to U_B through the rule we obtain

$$\mu_B * (s) = \mu_B(s) \oplus \gamma$$

with

$$\gamma = \begin{cases} \alpha c(\neg A, A^*) + (1-\alpha)c(\neg A, \neg A^*) & \text{if } c(\neg A, A^*) \leq c(\neg A, \neg A^*) \\ \beta c(\neg A, A^*) + (1-\beta)c(\neg A, \neg A^*) & \text{if } c(\neg A, A^*) \geq c(\neg A, \neg A^*) \end{cases} \tag{30}$$

This last equation can be easily simplified in many cases. Since $c(\neg A, \neg A^*) = 1$ when $\exists r_0/\mu_A(r_0) = \mu_A * (r_0) = 0$ (for instance, this condition is obviously verified for fuzzy quantities with a bounded support set), (30) is written in this case as $\gamma = \alpha c(\neg A, A^*) + 1 - \alpha = c(\neg A, A^*) \oplus (1 - \alpha)$, and therefore the resultant output is

$$\mu_B * (s) = \mu_B(s) \oplus (\lambda \oplus (1 - \alpha)) \tag{31}$$

with $\lambda = c(\neg A, A^*)$.

By considering this last value of $\gamma = \lambda \oplus (1 - \alpha)$, several particular cases of uncertain facts can be studied:

- If $\alpha = 1$ then $\mu_B * (s) = \mu_B(s) \oplus \lambda$. That is, when the fact does not present any uncertainty, the basic inference model given by (17) is obtained again.
- If $\alpha = 0$ then $\mu_B * (s) = \mu_B(s) \oplus 1 = 1$. So, the ignorance of the facts makes every value of the consequent domain completely possible.
- If $A^* \subseteq A$ then $\lambda = 0$ and $\mu_B * (s) = \mu_B(s) \oplus (1 - \alpha)$. In this case the uncertainty of the conclusion only comes from the uncertainty of the fact.
- If $A^* \subseteq \neg A$ then $\lambda = 1$ and $\mu_B * (s) = \mu_B(s) \oplus 1 = 1$. In this way, facts completely opposed to the antecedent, even being uncertain, give rise to ignorance of the consequent.

The equation (31) shows how the output B^* does not depend on the upper probability β. Curiously, we decided to represent the uncertainty of a fact in the formalism of the representable measures, but the inference model only uses the lower probability, and therefore to represent uncertain facts it suffices to use (21) and (22). That is to say, the possibilistic interpretation of (21) is enough for the purpose of this inference model.

Uncertain rules

An uncertain rule can be written as

$$\text{If } X \text{ is } A \text{ then } Y \text{ is } B \text{ is } [\alpha, \beta] \tag{32}$$

In a similar way as we interpreted the uncertainty in facts by means of lower and upper probabilities, we are going to interpret the uncertain rule (32) as the following conditional lower and upper measures

$$\begin{aligned} l(B/A) &= \alpha & u(B/A) &= \beta \\ l(\neg B/A) &= 1 - \beta & u(\neg B/A) &= 1 - \alpha \end{aligned} \tag{33}$$

$$\begin{aligned} l(B/\neg A) &= 0 & u(B/\neg A) &= 1 \\ 1(\neg B/\neg A) &= 0 & u(\neg B/\neg A) &= 1 \end{aligned} \tag{34}$$

Here, by propagating the measure (9) through these conditional measures we obtain

$$\begin{aligned} u_Y(B) &= \beta(1 - c(\neg A, A^*)) + c(\neg A, A^*) \\ &= \beta \oplus c(\neg A, A^*) = \beta \oplus \lambda \\ u_Y(\neg B) &= (1 - \alpha)(1 - c(\neg A, A^*)) + c(\neg A, A^*) \\ &= (1 - \alpha) \oplus c(\neg A, A^*) = (1 - \alpha) \oplus \lambda \end{aligned} \tag{35}$$

where $\lambda = c(\neg A, A^*)$.

The output B^* produced by combining (35) and the membership function of B and $\neg B$ is

$$\mu_B * (s) = (\mu_B(s) \oplus \lambda \oplus (1 - \alpha)) - (\mu_B(s)(1 - \beta)(1 - \lambda)) \tag{36}$$

This equation defines an unnormalized fuzzy set for $\beta \neq 1$. The lack of normalization does not create any problems when B^* is a final output of the inference model. On the contrary, when the output may be used as input for another rule, unnormalization blocks the way of B^* through the second rule. To prevent this kind of problem, we impose the condition $\beta = 1$ to the rules. Thus, we are restricting the kind of uncertainty in the rules to the possibilistic interpretation, that is,

$$\text{If } X \text{ is } A \text{ then } Y \text{ is } B \text{ is } \alpha \tag{37}$$

with

$$l(B/A) = \alpha \qquad u(B/A) = 1$$
$$l(\neg B/A) = 0 \qquad u(\neg B/A) = 1 - \alpha \tag{38}$$

and preserving (34). Therefore, when the system uses uncertain rules, the output generated is

$$\mu_B*(s) = \mu_B(s) \oplus \lambda \oplus (1 - \alpha) \tag{39}$$

Note how uncertainty in rules and facts (at the same degree) produces the same output. This coincidence shows how the system uniformly manages the uncertainty throughout their different components.

When uncertain facts and rules appear together, that is, when we have a rule "if X is A then Y is B is α" and a fact "X is A^* is α'", then the output B^* is

$$\mu_B*(s) = \mu_B(s) \oplus \lambda \oplus (1 - \alpha) \oplus (1 - \alpha') \tag{40}$$

where $\lambda = c(\neg A, A^*)$.

7. MANAGEMENT OF CONJUNCTIONS AND DISJUNCTIONS IN RULES

The kind of rules contained in a real knowledge base are not always as simple as the basic rule (1). In this section, we extend the inference model to include conjunctions and disjunctions in rules.

Conjunctions in premises

Let us consider the following kind of fuzzy inference

$$\begin{array}{l} \text{If } X_1 \text{ is } A_1 \text{ and } X_2 \text{ is } A_2 \text{ and } \ldots \text{ and } X_n \text{ is } A_n \text{ then } Y \text{ is } B \\ \quad X_1 \text{ is } A_1^* \text{ and } X_2 \text{ is } A_2^* \text{ and } \ldots \text{ and } X_n \text{ is } A_n^* \\ \hline \hfill Y \text{ is } B^* \end{array} \tag{41}$$

where X_i are variables on reference sets U_i, Y is a variable on reference set V, A_i are fuzzy sets on U_i, and B is a fuzzy set on V. A_i^* are the inputs and B^* is the output of the inference model.

The expression (41) can be rewritten in the following way

$$\begin{array}{c} \text{If } X \text{ is } A \text{ then } Y \text{ is } B \\ X \text{ is } A^* \end{array} \tag{42}$$

where $X = (X_1, X_2, \ldots, X_n)$, $A = (A_1, A_2, \ldots, A_n)$ and $A^* = (A_1^*, A_2^*, \ldots, A_n^*)$.

Thus, by replacing (41) by (42) we get a similar situation to the previous one, and the only problem is to define a measure on U_A and a conditional measure to represent this rule. For this purpose, we need to define the compatibility between the conjunction of A_i,'s and the conjunction of A_i^*'s. The greater all the compatibilities between each A_i and A_i^* ($c(A_i, A_i^*)$) are, the greater the compatibility between the vectors A and A^* ($c(A, A^*)$) is, because these vectors denote a conjunction of facts. For the same reason, as soon as an i exists such that the compatibility between A_i and A_i^* is low, the compatibility between A and A^* should be low, too. Thus, it seems appropriate to modelize $c(A, A^*)$ as a function of $c(A_i, A_i^*)$ through a conjunctive operator as a t-norm, and by taking the minimum t-norm, the compatibility between A and A^* is defined by

$$c(A, A^*) = \min_i \{c(A_i, A_i^*)\} \tag{43}$$

where $c(A_i, A_i^*)$ was defined in (8). The negation of the vector A can be interpreted as a disjunction of the negations of each A_i. So, the compatibility between $\neg A$ and A^* will be defined through a disjunctive operator, as a t-conorm, and by taking the maximum t-conorm, $c(\neg A, A^*)$ is defined by

$$c(\neg A, A^*) = \max_i \{c(\neg A_i, A_i^*)\} \tag{44}$$

From (43) and (44) we can also define the measure on the antecedent domain as

$$\begin{aligned} u_x(A/A^*) &= c(A, A^*) \\ u_x(\neg A/A^*) &= c(\neg A, A^*) \end{aligned} \tag{45}$$

It can be easily proved that $u_x(A/A^*) + u_x(\neg A/A^*) \geq 1$ since $c(A_i, A_i^*) + c(\neg A_i, A_i^*) \geq 1 \ \forall i$, and so, we can interpret u_x as an upper probability measure.

By using the above measure, the conditional measure defined by (6) and (7) for certain rules or defined by (34) and (38) for uncertain rules, and the propagation model, the result of the inference model for conjunctions in premises is direct:

$$\mu_B *(s) = \mu_B(s) \oplus \lambda \tag{46}$$

for certain rules and

$$\mu_B *(s) = \mu_B(s) \oplus \lambda \oplus (1 - \alpha) \tag{47}$$

for uncertain rules, with $\lambda = \max_i \lambda_i$, $\lambda_i = c(\neg A_i, A_i^*)$ and α is the uncertainty of the rule.

Note that (46) (and similarly (47)) can also be written as

$$\mu_B*(s) = \max_i \{\mu_{B_i}(s)\} = \max_i \{\mu_B(s) \oplus \lambda_i\} \tag{48}$$

where each $\mu_{B_i}(s) = \mu_B(s) \oplus \lambda_i$ would be the output produced by a simple rule "If X_i is A_i then Y is B."

Disjunctions in premises

Now, let us consider the following kind of fuzzy inference

$$\begin{array}{l} \text{If } X_1 \text{ is } A_1 \text{ or } X_2 \text{ is } A_2 \text{ or } \ldots \text{ or } X_n \text{ is } A_n \text{ then } Y \text{ is } B \\ \quad X_1 \text{ is } A_1^* \text{ and } X_2 \text{ is } A_2^* \text{ and } \ldots \text{ and } X_n \text{ is } A_n^* \\ \hline \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad Y \text{ is } B^* \end{array} \tag{49}$$

where X_i are variables on reference sets U_i, Y is a variable on reference set V, A_i are fuzzy sets on U_i, and B is a fuzzy set on V. A_i^* are the inputs and B^* is the output to the inference model.

The solution of this problem can be solved in a similar way to that of conjunctions in premises, by using the following compatibilities

$$c(A_1 \vee A_1 \vee \cdots \vee A_n, A_1^* \wedge A_2^* \wedge \cdots \wedge A_n^*) \\ = \max_i \{c(A_i, A_i^*)\} \tag{50}$$

$$c(\neg A_1 \wedge \neg A_2 \wedge \cdots \wedge \neg A_n, A_1^* \wedge A_2^* \wedge \cdots \wedge A_n^*) \\ = \min_i \{c(\neg A_i, A_i^*)\} \tag{51}$$

that is to say, by defining a measure u_x on $U_A = \{A, \neg A]$, where $\neg A = (\neg A_1, \neg A_2, \ldots, \neg A_n)$, as

$$\begin{aligned} u_x(A/A^*) &= \max_i \{c(A_i, A_i^*)\} \\ u_x(\neg A/A^*) &= \min_i \{c(\neg A_i, A_i^*)\end{aligned} \tag{52}$$

By using the conditional measures (6) and (7) or (34) and (38) again, the propagation model, and using the measure (52) too, we obtain the output

$$\mu_B*(s) = \mu_B(s) \oplus \lambda \tag{53}$$

where $\lambda = \min_i \lambda_i$ and $\lambda_i = c(\neg A_i, A_i^*)\ \forall i$.

Note that (53) may also be written as

$$\mu_B*(s) = \min_i \{\mu_{B_i}(s)\} = \min_i \{\mu_B(s) \oplus \lambda_i\} \tag{54}$$

where again $\mu_{B_i}(s) = \mu_B(s) \oplus \lambda_i$.

In fact, the rule (49) could be considered as equivalent to the set of rules

$$\text{If } X_i \text{ is } A_i \text{ then } Y \text{ is } B, i = 1, \ldots n \tag{55}$$

Therefore, the output generated by the complex rule (49) is equivalent to the conjunction of the outputs from the simpler rules in (55).

8. EXAMPLE

Finally, we are going to show a simple but representative example of the inference method presented. Suppose that we have three rules and three facts in our knowledge base. The fuzzy sets that appear in the rules all belong to the set {Very Low, Low, Medium, High, Very High}, whose elements are the fuzzy numbers represented in Figure 3 and defined by

Very Low VL = (0, 0, 25)
Low L = (25, 25, 25)
Medium M = (50, 25, 25)
High H = (75, 25, 25)
Very High VH = (100, 25, 0)

where the parameters of the triangular fuzzy number $A = (m, a, b)$ have the same meaning as in example 2.

The facts and rules in the knowledge base are the following:

X_1 is A
X_2 is 30
X_4 is B is 0.8

if X_1 is VL and X_2 is L then X_3 is M
if X_3 is M then X_5 is H is 0.9
if X_4 is VH then X_5 is VH

where $A = (10, 10, 20)$ and $B = (95, 1, 1)$.

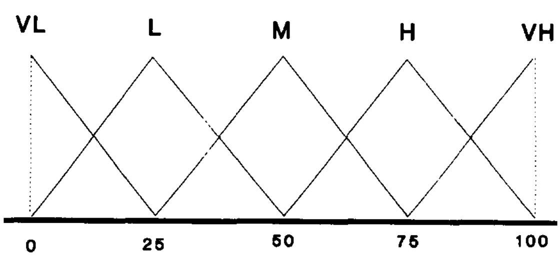


Figure 3. Linguistic labels for antecedent and consequent in the rules.

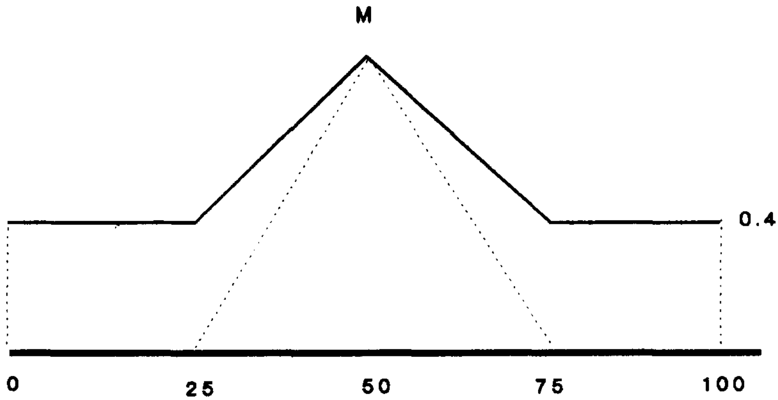
Let us see the inference process:

i) Let the values of X_1 and X_2 be inputs to the first rule. Since $c(A, \neg VL) = 0.4$, $c(30, \neg L) = 0.2$, the output is: X_3 is O_1, with $\mu_{O_1}(x) = \mu_M(x) \oplus 0.4$. If we take this fuzzy quantity as input to the second rule, as $c(O_1, \neg M) = 0.4$, then the new output is X_5 is O_2, with $\mu_{O_2}(x) = \mu_H(x) \oplus (0.4 \oplus 0.1) = \mu_H(x) \oplus 0.46$.

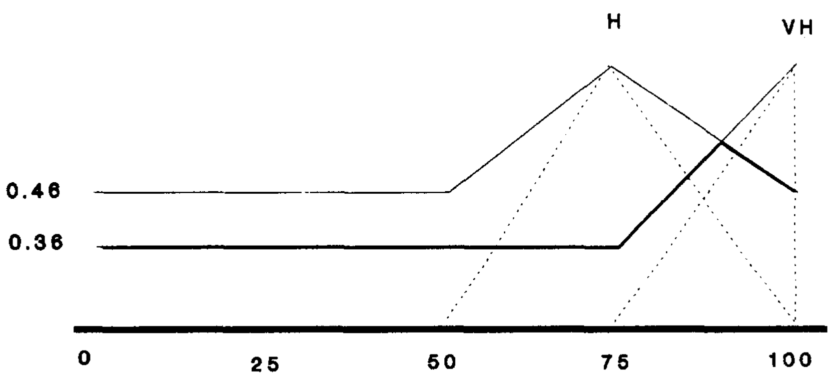
ii) Let the value of X_4 be the input to the third rule. Since $C(B, \neg VH) = 0.2$, the output is: X_5 is O_3, with $\mu_{O_3}(x) = \mu_{VH}(x) \oplus (0.2 \oplus 0.2) = \mu_{VH}(x) \oplus 0.36$.

iii) Since X_5 has two possible values at the end of the process, we can combine them by using the min operator, that is, X_5 is O_4, with $\mu_{O_4}(x) = \min\{\mu_H(x) \oplus 0.46, \mu_{VH}(x) \oplus 0.36\}$.

The membership functions of the outputs of X_3 and X_5 are shown in Figure 4.



(a)



(b)

Figure 4. Outputs for a) X_3, b) X_5.

9. CONCLUDING REMARKS

The inference approach we have proposed:

- is very easy to implement and computationally efficient,
- is able to do inference with different kinds of fuzzy propositions. Moreover, the proposed model has the following characteristics:
- it can be interpreted, in a particular case, as a fuzzy modus ponens,
- it verifies the most usual properties required by a fuzzy inference model, and
- it introduces a very flexible framework within which fuzzy inference can be done, with different freedom degrees: matching process, uncertainty representation, propagation model, integration model, etc.

The restriction on the uncertainty intervals $[\alpha, 1]$ for the rules, that forces us to use only possibilistic uncertainty, will be removed in forthcoming work. This will allow us to manage a more general kind of uncertainty (for example, probabilistic uncertainty). The implementation of this method, together with its application to real problems, will be also the object of further work.

APPENDIX

This appendix contains basic concepts about the Choquet integral and the expression of the propagation model (4) for this particular measure. This result is used in several sections to derive the propagated uncertainty measure on the consequent.

If the lower–upper probability measures (l_x, u_x) are Choquet capacities of order two (see [14]), then the resultant measure $u_Y(B)$ on D_y can be obtained in the following way:

$$u_Y(B) = E_{u_x}(f^B),$$

where f^B is a function on D_x defined by $f^B(x_i) = u(B/x_i)\ \forall x_i \in D_x$, and $E_{u_x}(.)$ is the Choquet integral [12], [13]) with respect to the measure $u_x(.)$. This result is interesting because it is easy to calculate the Choquet integral, and therefore the calculation of $u_Y(.)$ is direct. The measure, defined in (12), which has been propagated in section 4 is a Choquet capacity of order two. Therefore the propagation can use the above result.

Let us briefly show how to calculate the Choquet integral on finite domains: Given a fuzzy measure g on a set D_x and a function f:

$D_x \to \mathbb{R}^+$, the Choquet integral of f with respect to the measure g is

$$E_g(f) = \int_0^{+\infty} g(F_\alpha)\, d\alpha,$$

where $F_\alpha = \{x \in D_x / f(x) \geq \alpha\}$.

If the set D_x is finite ($D_x = \{x_1, \ldots, x_n\}$) and the values of the function f are ordered in the following way: $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n)$, then the Choquet integral can be written as

$$E_g(f) = \sum_{i=1}^{n} f(x_i)(g(A_i) - g(A_{i+1})),$$

where $A_i = \{x_i, x_{i+1}, \ldots, x_n\}$, $A_{n+1} = \emptyset$. ∎

References

1. Dempster, A. P., Upper and lower probabilities induced by a multivalued mapping, *Ann. Math. Statistics* 38, 325–339, 1967.
2. Shafer, G., *A Mathematical Theory of Evidence*, Princeton University Press, Princeton, 1976.
3. Dubois, D., and Prade, H., *Possibility Theory: An Approach to Computerized Processing of Uncertainty*, Plenum Press, New York, 1988.
4. Zadeh, L. A., Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets Syst.* 1, 3–28, 1978.
5. Zadeh, L. A., A theory of approximate reasoning, in *Machine Intelligence* 9 (J. E. Hayes, D. Michie, and L. I. Mikulich, Eds.), Halstead Press, New York, 149–194, 1979.
6. Zadeh, L. A., Outline of a new approach to the analysis of complex systems and decision process, *IEEE Trans. Systems Man Cybernet.* 3, 159–176, 1973.
7. Dubois, D., and Prade, H., Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions, *Fuzzy Sets Syst.* 40, 143–202, 1991.
8. Magrez, P., and Smets, P., Fuzzy modus ponens: A new model suitable for applications in knowledge-based systems, *Int. J. Intell. Syst.* 4, 181–200, 1989.
9. Campos, L. M., de, and Moral, S., Propagating uncertain information forward, *Int. J. Intell. Syst.* 7, 15–24, 1992.
10. Zadeh, L. A., The role of fuzzy logic in the management of uncertainty in expert systems, *Fuzzy Sets Syst.* 11, 199–227, 1983.
11. Campos, L. M. de, and González, A., Aplicación de técnicas de propagación de incertidumbre a la modelización del modus ponens difuso, *Proceedings of Primer Congreso Español sobre Tecnologías y Lógica Fuzzy*, Granada, 97–101, 1991.

12. Campos, L. M. de, and Bolaños, M. J., Representation of fuzzy measures through probabilities, *Fuzzy Sets and Systems* 31, 23–36, 1989.

13. Choquet, G., Theory of capacities, *Ann. Inst. Fourier* 5, 131–295, 1953.

14. Lamata, M. T., and Moral, S., Classification of fuzzy measures, *Fuzzy Sets Syst.* 33, 243–253, 1989.

15. Campos, L. M. de, Lamata, M. T., and Moral, S., Un modelo para el tratamiento de la incertidumbre en programación lógica basado en medidas difusas, *Proceedings of the III Reunión Técnica de la Asociación Española para la Inteligencia Artificial*, Madrid, 377–388, 1989.

16. Delgado, M., Trillas, E., Verdegay, J. L., and Vila, M. A., The generalized "Modus Ponens" with linguistic labels, *Proceedings of the International Conference on Fuzzy Logic and Neutral Networks*, Iizuka, 725–728, 1990.

17. Trillas, E., and Valverde, L., On inference in fuzzy logic, *Proceedings of the 2nd, Inter. Fuzzy Systems Assoc. (IFSA) Congress*, Tokio, 294–297, 1987.