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Higher regularity of free boundaries in obstacle problems

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by
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Chapter 1

Introduction

Obstacle problems are canonical examples of the so called *free boundary problems* – a type of non-linear elliptic (or parabolic) PDE, where we are not interested only in the study of the solution to the problem but also in the study of an a priori unknown interphase, called the free boundary. Typically the domain splits into two regions: one where the solution vanishes and the other where the solution solves some kind of PDE. As the local behaviour of solutions to the corresponding PDE is well understood, we are mostly interested in the global regularity of the solution as well as the regularity of the interphase separating the two regions. Obstacle problems appear naturally in many different areas of science such as physics, biology, or finance, see examples described below.

In general it turns out that the free boundary is not always smooth. Still, it can be often shown that it splits into the regular part, where it is infinitely smooth, and the singular part, which is rare in some sense. The regularity near regular points is usually established in two independent steps: one shows that the free boundary is $C^{1,\alpha}$, for some small $\alpha > 0$, and the other shows that if the free boundary is $C^{1,\alpha}$ then it is actually C^∞.

In this thesis we mostly (but not only) study the higher regularity of the free boundary in some versions of the obstacle problem, where only the initial $C^{1,\alpha}$ regularity was known.

1.1 The classical obstacle problem

Let us start with describing the main known results for the classical obstacle problem. It is a minimization problem of the form

$$\min_{v\in\mathcal{D}} \frac{1}{2}\int_\Omega |\nabla v|^2 dx,$$

where $\mathcal{D}$ consists of all functions that are greater or equal than a given function φ called the obstacle (i.e. we impose $u \geq \varphi$ in Ω), and attain given boundary values g. When the domain is two dimensional, we can interpret the solution as an elastic membrane constrained (fastened) over the obstacle, see Figure 1.1.1.

To derive the Euler-Lagrange equation for the minimizer v we compare the minimizer with competitors, which are the minimizer perturbed with bump functions. Since the minimizer and competitors are constrained to be above the obstacle, the perturbations are only allowed to be negative in the set $\{v > \varphi\}$. This results in the following equations

$$\begin{cases} v \geq \varphi & \text{in } \Omega \\ \Delta v \leq 0 & \text{in } \Omega \\ \Delta v = 0 & \text{in } \{v > \varphi\}, \end{cases} \tag{1.1.1}$$

together with the boundary conditions $v|_{\partial\Omega} = g$. As explained above, having the constraint implies that only one inequality holds everywhere, while the other one holds only in the set $\{v > \varphi\}$.

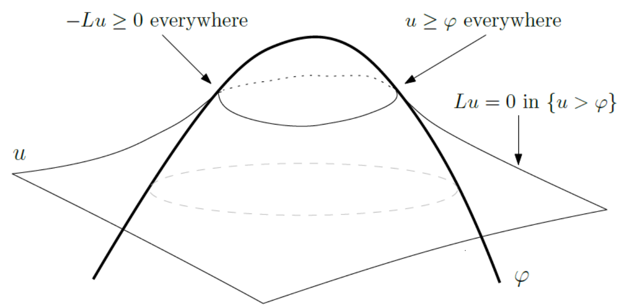

Figure 1.1.1: Example of solution to the obstacle problem.

We can consider the new variable $u = v - \varphi$, sometimes called *the height function*. Then the problem is equivalent to

$$\begin{cases} u \geq 0 & \text{in } \Omega \\ \Delta u \leq f & \text{in } \Omega \\ \Delta u = f & \text{in } \{u > 0\}, \end{cases}$$

where $f = -\Delta\varphi$, together with boundary data $u|_{\partial\Omega} = g - \varphi|_{\partial\Omega}$. The solution can be obtained as the minimizer of the following problem

$$\min_{u \in \mathcal{D}'} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu \right) dx, \tag{1.1.2}$$

where $\mathcal{D}'$ consists of all non-negative functions $u \geq 0$ attaining suitable boundary values. We see that we can trade off the obstacle being zero for adding a right-hand side f to the equations.

The Euler-Lagrange equations can alternatively be written as

$$\min\{-\Delta v, v - \varphi\} = 0 \qquad \text{in } \Omega,$$

which stresses the non-linearity of the problem. Of course the same can be done for the height function, which solves

$$\min\{-\Delta u + f, u\} = 0 \qquad \text{in } \Omega.$$

Additionally we can show that the minimizers of (1.1.2) are the same as the ones of

$$\min_{u \in \mathcal{D}''} \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + fu^+ \right) dx,$$

where $\mathcal{D}''$ now consists of all functions attaining the correct boundary values and $u^+ = \max\{u, 0\}$ stands for the non-negative part of u. This is beneficial because in this formulation we no longer have a constraint, but now the functional is no longer smooth. The Euler-Lagrange equation for this functional is

$$\Delta u = f\chi_{\{u>0\}}, \qquad \text{in } \Omega, \tag{1.1.3}$$

where χ_A stands for the indicator function of the set A. The equation needs to be understood in the weak sense.

We see that in all the formulations the domain Ω splits into a region $\{v > \varphi\} = \{u > 0\}$ called the positivity set and $\{v = \varphi\} = \{u = 0\}$ called the contact set. In the positivity set the solution solves some elliptic PDE, while in the contact set it matches the obstacle. Since we know a lot about solutions to such equations, the challenging part of the obstacle problem is to determine the contact set, or specifically the boundary of the positivity set, called the free boundary. Usually we denote it as $\Gamma = \partial\{u > 0\} \cap \Omega$.

If we want to stress the role of the free boundary in the equations for the solution of the obstacle problem, we can show that the above equations are equivalent to the following

$$\begin{cases} \Delta u &= f \quad \text{in } \{u > 0\} \\ u &= 0 \quad \text{in } \Gamma, \\ \nabla u &= 0 \quad \text{in } \Gamma. \end{cases}$$

That ∇u vanishes on the boundary follows from the fact that $u \in C^1$, thanks to (1.1.3). We see that this formulation consists of both Dirichlet and Neumann conditions, which would in general be an over-determined problem. But since the domain $\{u > 0\}$ is also an unknown, it turns out that the problem has an unique solution.

1.2 Motivations and generalizations

The obstacle problem appears in a wide range of areas. We present some of these here. For more examples and applications we refer to the books [33, 44, 53, 69] and the survey papers [36, 71].

Elasticity

As Figure 1.1.1 hints, the simplest interpretation of the obstacle problem is the deformation of a thin, elastic membrane spanning over an obstacle.

We describe the membrane as a graph of a function $v \colon \Omega \to \mathbb{R}$, so that at a point $(x, y) \in \Omega$ the value $v(x, y)$ represents the vertical displacement of the membrane above that point. We furthermore suppose that the membrane has a fixed boundary which we describe with prescribing the boundary values of v on the boundary by some function $g \colon \partial\Omega \to \mathbb{R}$. The shape of the membrane is determined by the surface tension, which we model so that u needs to be such that the area of its graph is minimized. Namely u minimizes the following functional

$$\int_\Omega \sqrt{1 + |\nabla u|^2} dSy.$$

This yields the classical Plateau problem (with an obstacle). We can simplify the functional by expanding the square root into the Taylor series up to the first order to get

$$\int_\Omega \sqrt{1+|\nabla u|^2}dxdy \approx \int_\Omega 1+\frac{1}{2}|\nabla u|^2 dxdy.$$

This is a good approximation if we assume that values of ∇u are small, which means that the membrane is flat. Hence the minimizers of the Dirichlet energy $\int \frac{1}{2}|\nabla u|^2 dxdy$ are roughly the minimizers of the area.

Hence the solution of the obstacle problem represents an elastic membrane that has fixed boundary and lays above a given obstacle.

Actually, if we compute the Euler-Lagrange equations of the area functional directly, we obtain

$$\begin{cases} v & \geq & \varphi & \text{in } \Omega \\ Lv & \leq & 0 & \text{in } \Omega \\ Lv & = & 0 & \text{in } \{v>\varphi\}, \end{cases}$$

where $Lu(x) = \operatorname{div}(A(x)\nabla u(x))$, for $A(x) = (1+|\nabla u(x)|^2)^{-1/2}$. Once we establish that solutions are $C^{1,1}$, this can be viewed as an obstacle problem with Lipschitz coefficients.

Optimal stopping, finance

Another important appearance of the obstacle problem is raised by the optimal stopping problem, motivated by financial mathematics. In such problem we are given an option, that at any time we can stop a running (stochastic) process in $\mathbb{R}^n$ and receive the pay-off which equals the value of some given function φ at the stopped point. If $(X_t)_{t\in\mathbb{R}}$ is the process from the model, then we are interested in determining

$$u(x) = \max_\tau \mathbb{E}(\varphi(x+X_\tau)),$$

where τ varies over all stopping times, see [31] for more details. Since τ can be chosen as 0, it is clear that $u(x) \geq \varphi(x)$. Furthermore if let the process run for some time $t>0$ and then run the maximum over all stopping times, we get at most as much as running the maximum straight ahead, so

$$u(x) \geq \max_\tau \mathbb{E}(\varphi(x+X_{t+\tau})) = \max_\tau \mathbb{E}(\varphi(x+X_t+X_\tau)) = u(x+X_t),$$

if the process X_t has no memory ($X_{t+\tau} = X_t + X_\tau$). Hence we have

$$Lu(x) := \lim_{t\downarrow 0} \frac{u(x)-\mathbb{E}(u(x+X_t))}{t} \leq 0,$$

if the limit exists. The above operator L is called *the infinitesimal generator* of the process X_t. Furthermore if $x \in \{u>\varphi\}$, then we are not stopping at $\tau = 0$. It turns out that we get the equality in the above expressions

$$Lu(x) = \lim_{t\downarrow 0} \frac{u(x)-\mathbb{E}(u(x+X_t))}{t} = 0, \qquad x\in\{u>\varphi\}.$$

Hence we obtained

$$\begin{cases} v & \geq & \varphi & \text{in } \Omega \\ Lv & \leq & 0 & \text{in } \Omega \\ Lv & = & 0 & \text{in } \{v > \varphi\}, \end{cases}$$

which is exactly the obstacle problem for the infinitesimal generator of the process.

This yields a solution of the optimal stopping problem: we solve the obstacle problem for u, then let the process run as long until we hit the contact set $\{u = \varphi\}$.

A relevant choice of X_t is the class of so called *Lévy* processes – it is any stochastic process starting at the origin with independent, stationary increments (for $s < t$ is $X_t - X_s$ independent of x_s and equally distributed as X_{t-s}), and which is continuous in probability, see [31] for complete definition. By the Lévy-Khintchine theorem the general infinitesimal generator of such process is always of the form

$$Lu(x) = b \cdot \nabla u(x) + \text{div}(A\nabla u(x)) + \int_{\mathbb{R}^n} (u(x+y) - u(x) - \nabla u(x) \cdot y\chi_{B_1}(y))d\mu(y),$$

where $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n^2}$ is a symmetric, positive semidefinite matrix, and μ is a Lévy (jump) measure satisfying

$$\int_{\mathbb{R}^n} \min\{|y|^2, 1\}d\mu(y) < \infty.$$

This gives rise to various different interesting obstacle problems. If we assume that the process has continuous trajectories (no jumps), no drift ($b = 0$), and is rotationally symmetric, then the obtained generator becomes a multiple of the Laplace operator.[1] On the other hand, if we assume that there is no diffusion and that the process is symmetric and scale invariant, we end up with an operator of the form

$$Lu(x) = p.v. \int_{\mathbb{R}^n} (u(x+y) - u(x))K(y)dy, \tag{1.2.1}$$

where

$$K(y) = K(y/|y|)\frac{1}{|y|^{n+2s}},$$

for some $s \in (0,1)$. These are called stable processes, and have been widely studied, both from the point of view of Analysis and Probability. The most canonical example of such non-local operator is the *fractional Laplacian*, denoted $(-\Delta)^s$, whose kernel is a multiple of $|y|^{-n-2s}$. When the process has a drift, we get operators like

$$(-\Delta)^s + b \cdot \nabla,$$

with $b \in \mathbb{R}^n$.

Optimal stopping for Lévy processes has been used in pricing models for American options since the 1970s, see [26], which is a strong motivation for studying such operators.

[1]Such process is called the *Brownian* motion.

Parabolic obstacle problem for non-local operators

The parabolic version of the previously described problem arises for example in American option pricing models. The American option gives its holder the right to buy a stock at a given price before a given time $T > 0$. Thus we get an optimal stopping problem with a "deadline" at time $t = T$. It turns out that the fair price $v(\tau, x)$ at time $\tau < T$ and underlying stock value x, with a pay-off φ, solves the obstacle problem of the following form

$$\min\{-\partial_\tau v - Lv, v - \varphi\} = 0$$
$$v(T, \cdot) = \varphi,$$

where L is either the Laplacian or an integro-differential operator of the form (1.2.1). Making a transformation $t = -\tau$ gives exactly the parabolic version of the obstacle problem for integro-differential operators. For more details we refer to the book [26].

Stefan problem

Dating back to the 19th century, the Stefan problem describes the temperature in a homogeneous medium undergoing a phase change, such as ice melting into water. The problem is named after the Slovenian physicist who introduced the general class of such problems, related to problems of ice formation.

In the most classical formulation the problem is the following: given a bounded domain $\Omega \subset \mathbb{R}^n$, non-negative initial values $\theta_0 \colon \Omega \to \mathbb{R}$ and non-negative boundary condition $g \colon \partial\Omega \times \mathbb{R}^+$, we want to find a non-negative function $\theta(x, t)$ describing the temperature of the medium at the point $x \in \Omega$ at time $t > 0$, so that $\theta(x, 0) = \theta_0(x)$ and $\theta(x, t) = g(x, t)$ for $(x, t) \in \partial\Omega \times (0, \infty)$. The set $\{\theta = 0\}$ represents the ice, while $\{\theta > 0\}$ represents the water. In the water, the temperature function is assumed to solve the heat equation, namely

$$\partial_t \theta + \Delta\theta = 0 \qquad \text{in } \{\theta > 0\},$$

while the evolution of the interface $\partial\{\theta > 0\}$ is dictated by the Stefan condition

$$\partial_t \theta = |\nabla\theta|^2 \qquad \text{on } \partial\{\theta > 0\}. \tag{1.2.2}$$

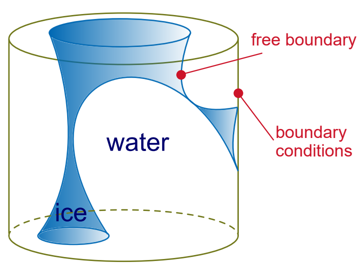

Figure 1.2.2: Example of a contact set at a fixed time.

To study this problem it is convenient to consider a new function

$$u(x, t) = \int_0^t \theta(x, s) ds.$$

Then, $u\colon \Omega \times \mathbb{R}^+$ locally solves the *parabolic obstacle problem*

$$\partial_t u - \Delta u = -\chi_{\{u>0\}}, \qquad u \geq 0, \qquad \partial_t u \geq 0. \tag{1.2.3}$$

Note that knowing u is equivalent to knowing θ, as $\theta(x,t) = \partial_t u(x,t)$, and that sets $\{\theta > 0\}$ and $\{u > 0\}$ agree. Let us present informally, how to obtain the obstacle problem from the Stefan problem. First notice that the ice truly is melting, since the Stefan condition (1.2.2) says that $\partial_t \theta \geq 0$ on $\partial\{\theta > 0\}$. Therefore we can denote $\tau(x)$ the moment when the ice present at x melts into water. In particular

$$\theta(x, \tau(x)) = 0,$$

and differentiating it, we get

$$\nabla\theta(x,\tau(x)) + \nabla\tau(x)\partial_t\theta(x,\tau(x)) = 0. \tag{1.2.4}$$

Now since $\theta(x,t) = 0$ for $t < \tau(x)$, we have

$$u(x,t) = \int_{\tau(x)}^t \theta(x,s)ds.$$

To derive the equation for u, we differentiate the above expression

$$\partial_i u(x,t) = \int_{\tau(x)}^t \partial_i\theta(x,s)ds - \theta(x,\tau(x))\partial_i\tau(x) = \int_{\tau(x)}^t \partial_i\theta(x,s)ds.$$

Differentiating it again leads to

$$\partial_{ii} u(x,t) = \int_{\tau(x)}^t \partial_{ii}\theta(x,s)ds - \partial_i\theta(x,\tau(x))\partial_i\tau(x).$$

Summing over i, we get

$$\Delta u(x,t) = \int_{\tau(x)}^t \Delta\theta(x,s)ds - \nabla\theta(x,\tau(x))\cdot\nabla\tau(x),$$

and recalling (1.2.2) and (1.2.4) we conclude

$$\Delta u(x,t) = \int_{\tau(x)}^t \partial_t\theta(x,s)ds - 1 = \theta(x,t) - 1 = \partial_t u(x,t) - 1,$$

which is exactly the parabolic version of the obstacle problem (1.2.3).

Heat control

Given a domain Ω and a temperature T_0 we have heating devices evenly distributed on Ω which need to ensure that the temperature w is as close to T_0 as possible by injecting flux of magnitude $q > 0$ whenever $w(x) < T_0$. In equilibrium the temperature satisfies

$$\Delta w = -q\chi_{\{w<T_0\}} \qquad \text{in } \Omega.$$

To get a structure of the obstacle problem, we define $u = T_0 - w$, which solves

$$\Delta u = q\chi_{\{u>0\}} \qquad \text{in } \Omega.$$

If additionally $u \geq 0$, this is exactly the obstacle problem. The condition $u \geq 0$ is fulfilled as soon the boundary values of w are at most T_0.

Thin obstacle problem

As the name suggests, the thin obstacle problem is a variation of the obstacle problem where the obstacle provides a constraint only on a surface of co-dimension one.

The obtained equations for the solution u to the thin obstacle problem are then the following

$$\begin{cases} \Delta u & \leq & 0 & \text{in } \Omega \\ u & \geq & \varphi & \text{on } S \\ \Delta u & = & 0 & \text{in } \Omega \backslash (S \cap \{u = \varphi\}), \end{cases}$$

for a surface $S \subset \Omega$ and a given function $\varphi \colon S \to \mathbb{R}$. Normally the solution also needs to attain prescribed values on $\partial\Omega$.

The most classical version of the thin obstacle problem is the case when $S = \{x_1 = 0\}$ and when the domain Ω is symmetric over S. In that case the problem can be reformulated as

$$\begin{cases} \Delta u & = & 0 & \text{in } \Omega \cap \{x_1 > 0\} \\ u & \geq & \varphi & \text{on } \Omega \cap \{x_1 = 0\} \\ \partial_1 u & \leq & 0 & \text{on } \Omega \cap \{x_1 = 0\} \\ \partial_1 u & = & 0 & \text{on } \Omega \cap \{x_1 = 0\} \cap \{u > \varphi\}, \end{cases}$$

as the even reflection gives the solution to the original formulation.

If we study a solutions u of the thin obstacle problem on the full space $\Omega = \mathbb{R}^{n+1}$, which is even in x_1, it turns out that the restriction to the thin space $v(x) = u(0, x)$ gives a solution to the obstacle problem for the fractional Laplacian with parameter $s = \frac{1}{2}$ (also called the *half Laplacian* $\sqrt{-\Delta}$). This follows from the fact that the Dirichlet-to-Neumann map for the Laplace equation corresponds to the square root of the Laplacian. This can be roughly verified as follows. Denote with L the operator that takes a function v and maps it to $\partial_1 u$, where u is the harmonic extension of v to the upper half-space $\{x_1 > 0\}$. To compute the square of the operator L, notice that the harmonic extension of $\partial_1 u|_{\{x_1=0\}}$ is simply $\partial_1 u$, and hence

$$L^2 v = \partial_1 \partial_1 u|_{\{x_1=0\}} = -\Delta u|_{\{x_1=0\}} = -\Delta v. \tag{1.2.5}$$

Interacting particles

Large systems of interacting particles arise in several models in the natural sciences (Physics or Biology for example). In such systems the discrete energy can be well approximated by the continuum interacting energy. We denote μ the (probability) measure representing the distribution of the particles. The interaction energy associated to the interaction potential W is given by

$$E[\mu] := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W(x - y) d\mu(x) d\mu(y).$$

In general, the interaction potential can have very different structures. It is common to assume a repulsive behaviour for particles that are very close (blowing up at zero distance). A typical assumption is to have $W(z) \approx |z|^{-\beta}$ near the origin. One wants to understand their behaviour of the system in presence of some external force field that confines the particles. In that case, the interaction energy associated with the system is given by

$$E[\mu] := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} W(x-y)d\mu(x)d\mu(y) + \int_{\mathbb{R}^n} V(x)d\mu(x).$$

One of the main questions when dealing with these systems is to understand the equilibrium of configurations – minimizers of the energy E. It turns out that for a minimizer μ_0 the associated potential function $u(x) = W * \mu_0(x) = \int_{\mathbb{R}^n} W(x-y)d\mu_0(y)$ (locally) solves the obstacle problem of the form

$$\min\{-Lu, u-\varphi\} =, 0$$

for an operator L depending on W and for some obstacle φ that depends on V. The free boundary corresponds to the boundary of the region in which the particles concentrate.

In the case when the potential is given by $W(z) = |z|^{-n+2s}$, $n \geq 2$ and $s \in (0,1)$ the associated operator becomes the fractional Laplacian $(-\Delta)^s$. Similarly, when $W(z) = |z|^{n-2}$, $n \geq 3$, the operator equals a multiple of the Laplacian.[2] We refer to [5, 24, 80] for a detailed study of these problems.

1.3 Known results

The regularity theory for different variations of obstacle problems goes fairly parallel one to another (even though some of the proofs may be completely different from each other). The main goals in study of the obstacle problems are:

1. determine the optimal regularity of the solution,
2. find the splitting of the free boundary into regular and singular/degenerate points,
3. show that near regular points the free boundary is $C^{1,\alpha}$ for some $\alpha > 0$,
4. show that if the free boundary is $C^{1,\alpha}$ then it is actually C^∞,
5. study the singular points.

Below we present the detailed statements corresponding to these steps for various obstacle problems. The most details we give in the classical case, as it serves as the model case for all the generalizations.

1.3.1 The classical obstacle problem

The development of the regularity theory for the obstacle problem goes back to the 1960s. There were several results regarding the regularity of solutions, see for example [13, 43, 60, 65]. Still, nothing was known about the free boundary until the late 70s.

[2] In general L is the operator whose fundamental solution is the potential W.

Existence and uniqueness

Existence an uniqueness of solutions to the obstacle problem follow from the fact that the Dirichlet energy functional is strictly convex and that the functions vary over a closed, convex set

$$\{v \in H^1(\Omega);\ v \geq \varphi,\ v|_{\partial\Omega} = g\}.$$

We can construct the minimizer as the limit of the realizing sequence of the minimum.

Alternatively we can construct the solution with *Perron's* method: take the least supersolution above the obstacle. Namely we define

$$v = \inf\{w \in C(\bar{\Omega});\ \Delta w \leq 0,\ w \geq 0,\ w|_{\partial\Omega} \geq g\},$$

where the inequality $\Delta w \leq 0$ has to be understood in the viscosity sense. It turns out that v itself is a continuous supersolution satisfying $\Delta v = 0$ in $\{v > \varphi\}$ and attains correct boundary values. For more details see [33].

Optimal regularity and nondegeneracy

As we have seen, obtaining the minimizer is not hard if we consider $H^1(\Omega)$ functions. But then also the minimizer is only an $H^1(\Omega)$ function. Hence our next goal is to see how regular is the solution of the obstacle problem.

Remember that the minimizer solves (1.1.3). This is in particular an elliptic PDE with a bounded right-hand side, and so by Schauder estimates (see e.g. [33, Proposition 2.27]) the solution u is $C^{1,\alpha}$ for any $\alpha \in (0,1)$ in the interior. Moreover from the equation we see that Δu is in general not continuous, so $u \notin C^2$. Exploiting the additional structure of (1.1.3), one can show that the solution is actually $C^{1,1}$, which is therefore is optimal.

Theorem 1.3.1 ([13, 43])**.** *Let u be a solution of* (1.1.1) *with $\varphi \in C^\infty(B_1)$. Then $u \in C^{1,1}(B_{1/2})$ with the estimate*

$$||u||_{C^{1,1}(B_{1/2})} \leq C\left(||u||_{L^\infty(B_1)} + ||\varphi||_{C^{1,1}(B_1)}\right).$$

The constant C depends only on n.

Moreover under additional assumption that $f \geq c_0 > 0$ it holds that the solution actually grows quadratically. Combined with the optimal regularity estimate we get

$$0 < cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2, \qquad \text{for all } r \in (0,1). \tag{1.3.1}$$

Free boundary

As said above, the regularity of solutions can be quickly obtained using some basic tools of elliptic PDE. A much more challenging problem is to understand the geometry of the free boundary. Concretely the main question regarding free boundaries is the following:

Is the the free boundary C^∞ if the obstacle is C^∞?

Note that just from regularity of the solution we can not deduce anything about the regularity of the free boundary. A priori it can be a very irregular set potentially of infinite perimeter.

The answer to the above question is in general negative: adjusting the boundary conditions in a suitable way, we can create a contact set as on Figure 1.3.3, where the free boundary is not a smooth manifold.

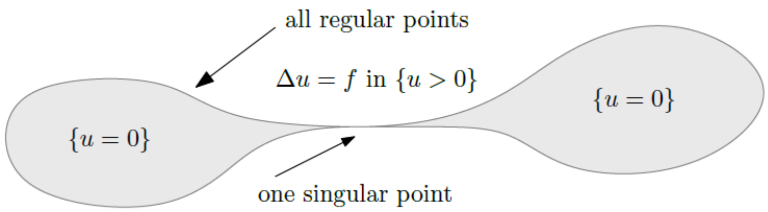

Figure 1.3.3: Example of a contact set with a singular point.

Nevertheless, apart from the one singular point the boundary seems smooth. Therefore our goal is to find a classification of the points at which the free boundary can be irregular, to study the regularity of the free boundary away from those points, and to discover the size of the set of singular points.

Blow-up

The main tool to study the free boundary is the so called *blow-up* method. Intuitively a blow-up is an infinite zoom of a function around some point. We aim to study the properties of blow-ups so as to deduce information about the solution and the free boundary. The blow-up of u at a free boundary point x_0 is

$$u_0(x) := \lim_{k\to\infty} \frac{1}{r_k^2} u(x_0 + r_k x),$$

for some sequence $r_k \downarrow 0$. The rescaling is taken quadratic, since the solution grows quadratically near the free boundary, see (1.3.1). The existence of this limit (along subsequences, locally in $C^{1,1}$) is assured by the optimal regularity estimates, while the non-degeneracy estimate tells us that it is non-trivial. Note that if u was twice differentiable at x_0, the blow-up would converge to its second order Taylor polynomial. Let us stress that translations dilatations of the solution to the obstacle problem are again solutions to an obstacle problem. So as we zoom closer to the point, the right-hand side goes closer to being constant. To avoid treating the error terms, let us from now on assume that the right-hand side f is constantly equal to 1 and that 0 is a free boundary point.

Classification of blow-ups

As Figure 1.3.3 suggests, there are more possibilities for blow-ups. Near regular points we expect the contact set of the blow-up to be a half-space, while at singular points not. Hence to understand which points are regular and which singular, as well as to deduce properties of the free boundary, we want to classify all possible blow-ups. It turns out that the classification is possible and the obtained blow-ups are very simple functions. In 1977 Caffarelli proved the following:

Theorem 1.3.2 ([16]). *Let u be a solution of* (1.1.1) *let* 0 *be a free boundary point and let u_0 be a blow-up of u at* 0. *Then*

(i) either

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2,$$

for some $e \in \mathbb{S}^{n-1}$,

(ii) or

$$u_0(x) = \frac{1}{2}x^T Ax,$$

for some matrix $A \geq 0$ with $\operatorname{tr} A = 1$.

This dichotomy we use to formally define regular and singular points. The points where there is a blow-up as in (i) we call regular and the points where all blow-ups are as in (ii) we call singular. Note that the obtained blow-ups match with our expectation from Figure 1.3.3.

Regularity of the free boundary near regular points

The main regularity theory for free boundaries in the obstacle problem was established by Caffarelli in 1977, see [16]. He introduced the blow-up technique and established the regularity of the free boundaries near regular points.

At regular points there are blow-ups whose contact set is a half-plane, which means that the more we zoom in, the flatter the free boundary is. This can be used to prove the following result.

Theorem 1.3.3 ([16]). *Let u be a solution of* (1.1.1) *with $\varphi \in C^\infty$. The set of regular free boundary points is relatively open and the free boundary is locally a graph of a $C^{1,\alpha}$ function near them, for some $\alpha > 0$.*

In transferring the information from the blow-up to the original solution a key role is played by the "almost positivity" lemma. It claims that if a harmonic function is close to being non-negative in a positivity domain of some solution to an obstacle problem, then it actually has to be non-negative (in some smaller neighbourhood), see [33, Lemma 5.32] for detailed a formulation. We can use this on derivatives of the solution near regular points. If $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$, then by $C^{1,1}$ convergence to the blow-up for every $\varepsilon > 0$ there is $r_0 > 0$, so that

$$|\partial_\tau u_{r_0}(x) - (e \cdot \tau)(x \cdot e)_+| \leq \varepsilon \qquad x \in B_1,$$

where $u_r(x)$ stands for $r^{-2}u(rx)$. If τ is chosen so that $e \cdot \tau > 0$, then $\partial_\tau u$ is the almost non-negative function to which the above applies. We deduce that there is $r_0 > 0$ so that

$$\partial_\tau u \geq 0 \qquad \text{in } B_{r_0}, \tag{1.3.2}$$

for all $\tau \cdot e > \frac{1}{2}$. This readily implies that the free boundary is Lipschitz in B_{r_0}. Deducing that the free boundary is then $C^{1,\alpha}$ is done through the boundary Harnack inequality.

An alternative approach was introduced by Weiss in [85], where he proved that the flatness of the free boundary improves as we zoom in more and more, which yields $C^{1,\alpha}$ regularity near regular points.

Higher regularity

Once the initial regularity of the free boundary is established we can use results for harmonic functions in $C^{k,\alpha}$ domains ($k \geq 1$) to get a bootstrap argument which implies that the free boundary is in fact C^∞.

Theorem 1.3.4 ([52]). *Let u be a solution of* (1.1.1) *with $\varphi \in C^\infty$. Then the free boundary is C^∞ near regular points.*

Originally the higher regularity was proven by Kinderlehrer and Nirenberg in 1977 in [52], where they prove that if the free boundary is C^1 and the solution is C^2 in the positivity set up to the boundary, then the boundary is C^∞. They prove this result using the so called *Hodograph* transform, which is a boundary flattening map depending on the solution itself. This was actually the first result regarding the regularity of the free boundary.

More recently, De Silva and Savin introduced a new (and simpler) way to prove the higher regularity of the free boundary. They exploit the fact that the normal to the level-sets $\{u = t\}$, $t > 0$, can be expressed with quotients of its partial derivatives as follows

$$\nu = \frac{\nabla u}{|\nabla u|} = \left(\frac{\partial_i u / \partial_n u}{\sqrt{1 + \sum_{j=0}^{n-1} (\partial_j u / \partial_n u)^2}} \right)_{i=1,\ldots,n}.$$

After a rotation we can assume that $\partial_n u \geq 0$, see (1.3.2). Note that the derivatives $\partial_i u$ are harmonic in $\{u > 0\}$ and hence we can apply the following result, called the higher order boundary Harnack inequality.

Theorem 1.3.5 ([27]). *Assume that Ω is $C^{k,\alpha}$ for some $k \in \mathbb{N}$ and $\alpha \in (0,1)$, and let u_i, $i = 1, 2$ solve*

$$\begin{cases} \Delta u_i &= 0 \quad \text{in } \Omega \cap B_1 \\ u_i &= 0 \quad \text{on } \partial\Omega \cap B_1. \end{cases}$$

Assume furthermore that $u_2 > 0$ inside $\Omega \cap B_1$.

Then,

$$\left|\left| \frac{u_1}{u_2} \right|\right|_{C^\beta(\Omega \cap B_{1/2})} \leq C ||u_1||_{L^\infty(\Omega \cap B_1)},$$

with C dpending only on n, β, Ω and $||u_2||_{L^\infty(\Omega \cap B_1)}$.

Applying the higher order Harnack inequality gives that the quotients $\partial_i u / \partial_n u$ are $C^{k,\alpha}$ if the free boundary is C^β. But then the normal vector to the free boundary is C^β as well, and so the free boundary is actually $C^{\beta+1}$. Hence the free boundary is C^∞ near regular points:

$$\partial\Omega \in C^{1,\alpha} \overset{BH}{\Longrightarrow} \frac{\partial_i u}{\partial_n u} \in C^{1,\alpha} \Longrightarrow \nu \in C^{1,\alpha} \Longrightarrow \partial\Omega \in C^{2,\alpha} \Longrightarrow \ldots \Longrightarrow \partial\Omega \in C^\infty.$$

Singular points

Let us now briefly discuss the singular set of the free boundary. The singular set is closed and examples show that it is in general not possible to prove that it forms a manifold. But knowing the blow-up we are still able to deduce that at that point we can touch the free boundary from at least two sides by C^1 "parabolas". This roughly says that the singular set has a tangent plane at any singular point. This leads to the following result by Caffarelli:

Theorem 1.3.6 ([14]). *Let u be a solution of* (1.1.1)*, and let $\Sigma \subset B_1$ be the set of all singular free boundary points in B_1. Then $\Sigma \cap B_{1/2}$ is locally contained in a C^1 manifold of dimension $n-1$.*

The result is sharp in the dimension of the covering manifold. It is not hard to construct examples where the singular set is $n-1$ dimensional, see Figure 1.3.4.

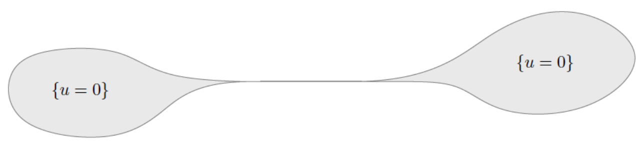

Figure 1.3.4: Example of a singular set of dimension $n-1$.

Nevertheless all the examples of free boundaries containing singular points seem "unlikely" in the sense that if the boundary values were slightly perturbed, it seems that there would be no singular points any more. It has been conjectured by Schaeffer in 1974,[3] that generically, the free boundary has *no* singular points.

The first result in this direction was established by Monneau in 2003, who proved the conjecture in two dimensional case, see [66]. Furthermore, in 2020 Figalli, Ros-Oton and Serra proved the conjecture in dimensions 3 and 4 ([37]). It remains an open problem to decide whether Schaeffer's conjecture holds in dimensions $n \geq 5$ or not.

1.3.2 Parabolic obstacle problem

The results for the parabolic version of the obstacle problem are completely analogous to the ones for the classical one. The main result for the free boundaries states the following:

Theorem 1.3.7 ([14]). *Let u be a solution of* (1.2.3)*. Let $(x_0, t_0) \in \partial\{u > 0\}$ be a free boundary point. Then the limit $u_0 = \lim_{r\downarrow 0} r^{-2}u(x_0 + rx, t_0 + r^2t)$ exists and the following holds true:*

(i) (regular point) either

$$u_0(x,t) = \frac{1}{2}(x \cdot e)_+^2,$$

for some $e \in \mathbb{S}^{n-1}$,

[3] That is even before the first results about the regularity of the free boundaries in 1977.

(ii) (singular point) or

$$u_0(x,t) = \frac{1}{2}x^T Ax,$$

for some matrix $A \geq 0$ with $\operatorname{tr} A = 1$.

Moreover the set of points where (i) holds is relatively open and the free boundary is C^∞ in space and time near them.

The study of the parabolic version of the obstacle problem mostly went hand in hand with the classical one. The optimal regularity of solutions – $C^{1,1}$ in space and C^1 in time – was provided by Brezis and Kinderlehrer in [13] and Caffarelli in [20].

In [52] Kinderlehrer and Nirenberg proved that once the free boundary is C^1 it is C^∞ also for the parabolic setting and moreover Caffarelli developed the regularity theory for the free boundaries in [16] simultaneously.

The study of the singular points was independent. The first results by Blanchet in 2006 proved the analogue of Theorem 1.3.6 for a fixed time – the set of singular points at some fixed time is contained in a $(n-1)$-dimensional manifold of class C^1, see [11]. This was later improved by Lindgren and Monneau in 2015 – [64] – when they showed that the entire singular set can be covered with a $(n-1)$-dimensional manifold of class C^1 in space and $C^{1/2}$ in time. More recently Figalli, Ros-Oton and Serra furthermore improved the result and established a very fine description of the singular set, see [39].

1.3.3 Obstacle problem for fully non-linear operators

The parabolic obstacle problem for fully non-linear operator is of the following form

$$\partial_t u - F(D^2u, x) = f(x)\chi_{\{u>0\}}, \quad \partial_t u \geq 0 \qquad \text{in } \Omega \times (0,T), \tag{1.3.3}$$

where F is a C^∞, convex function, that satisfies the uniform ellipticity condition

$$\lambda||N|| \leq F(M+N, x) - F(M, x) \leq \Lambda||N||, \tag{1.3.4}$$

for any symmetric matrix M, $x \in B_1$ and $N \geq 0$, and for some $0 < \lambda \leq \Lambda$, called the ellipticity constants.

In the stationary version of the problem, the optimal regularity of solutions and $C^{1,\alpha}$ regularity of the free boundary was proved by Lee in his PhD thesis [59]. Moreover the optimal regularity was studied in a more general setting by by Figalli and Shahgholian in [41], where they additionally prove that if the free boundary is Lipschitz, then it is C^1. Furthermore in [40] they extend the results to the parabolic setting, see also [49] by Indrei and Minne and [68] by Petrosyan and Shahgholian. The higher regularity of the free boundary in both elliptic and parabolic setting is provided by Kinderlehrer and Nirenberg [52]. Note that there was still a small gap between the initial regularity and the higher regularity results, since in [52] the solution is assumed to be C^2 in the positivity set up to the boundary, while in the above papers the solutions are only proved to be $C^{1,1}$.

The singular set has been studied in the elliptic case in [12] where Bonorino establishes that the singular set can be locally covered with a Lipschitz $(n-1)$-dimensional manifold, and in [77] where Savin and Yu improve the regularity of the covering manifold to $C^{1,\log^\varepsilon}$, see also [78]. For the parabolic case no results were known for singular points.

1.3.4 Obstacle problem for integro-differential operators

The obstacle problem for integro-differential operators is

$$\min\{Lu, u-\varphi\} = 0 \qquad \text{in} \quad \mathbb{R}^n, \tag{1.3.5}$$

where L is of the form (1.2.1). Due to the non-locality of the operator the behaviour of solutions near the free boundary is different as in the classical case. In this case the following holds true.

Theorem 1.3.8 ([22, 23]). *Let u solve* (1.3.5) *and let $x_0 \in \partial\{u > 0\}$ be a free boundary point. Then*

(i) either

$$0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^{1+s}$$

(ii) or

$$0 \leq \sup_{B_r(x_0)} (u-\varphi) \leq Cr^{1+s+\alpha},$$

for some $\alpha > 0$. The points where (i) *holds are called regular, they form a relatively open set and the free boundary is $C^{1,\alpha}$ near them.*

The study of the obstacle problem for integro-differential operators started in the 2000s. Of particular interest is the fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, that serves as the model example for general integro-differential operators. The obstacle problem for the fractional Laplacian generalises the thin obstacle problem (the case $s = \frac{1}{2}$), and recovers the classical obstacle problem in the limit $s \uparrow 1$.

The first results for the obstacle problem for the fractional Laplacian were obtained by Silvestre in [82], where he proved the almost optimal regularity of solutions $C^{1,s-\varepsilon}$, for all $\varepsilon > 0$. Later on the result was improved by Caffarelli, Salsa and Silvestre in [23] where they established the dichotomy from Theorem 1.3.8 and that near regular points the free boundary is $C^{1,\alpha}$ for some $\alpha > 0$. Moreover they provided an equivalence for all $s \in (0,1)$ between the fractional Laplacian and a Dirichlet-to-Neumann map in $\mathbb{R}^{n+1}_+$ for a local operator with a weight

$$\operatorname{div}(y^{1-2s}\nabla_{x,y} v), \qquad \text{for } (x,y) \in \mathbb{R}^n \times \mathbb{R}_+.$$

Note that in the case when $s = \frac{1}{2}$ it corresponds to the Dirichlet-to-Neumann map described in (1.2.5). This lead to development of new results regarding the higher regularity of the free boundary near regular points as well as the singular set, see [9, 51]. Since these results used the above described Dirichlet-to-Neumann map, the results could not be extended to the general class of integro-differential operators as in (1.2.1). To establish analogous results, completely new techniques were needed. First Caffarelli, Ros-Oton and Serra in [22] extended the results from [3] and later on Abatangelo and Ros-Oton proved the higher regularity of the free boundaries, see [1].

Theorem 1.3.9 ([1, 51]). *Let u solve* (1.3.5) *and let $x_0 \in \partial\{u > 0\}$ be a regular free boundary point. Then the free boundary is C^∞ in a neighbourhood of x_0.*

If in (1.3.5) the operator is $L = (-\Delta)^s + b \cdot \nabla$, for some $s \in (0,1)$ and $b \in \mathbb{R}^n$, we get the obstacle problem for fractional Laplacian with drift. As seen in Subsection 1.2 it arises as the optimal stopping problem for a specific type of Lévy processes.

Note that in this problem there is a strong dependence of the nature of the problem on the value of the parameter s. When $s > \frac{1}{2}$ the leading term is the fractional Laplacian, and we expect the problem to behave as in the drift-free case, while in the case $s < \frac{1}{2}$ the leading term is the drift term and we can not expect any regularity for the free boundary.

Not much is known for the regularity of solutions and free boundaries in this setting. In [67] Petrosyan and Pop show the optimal $C^{1,s}$ regularity of solutions in the sub-critical regime $s > \frac{1}{2}$ and in [47] Garofalo, Petrosyan, Pop and Smit Vega Garcia extend their result and prove that the following theorem.

Theorem 1.3.10 ([47]). *Let u solve* (1.3.5) *and let $x_0 \in \partial\{u > 0\}$ be a free boundary point. Then*

(i) either

$$0 < cr^{1+s} \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s}$$

(ii) or

$$0 \leq \sup_{B_r(x_0)} (u - \varphi) \leq Cr^{1+s+\alpha},$$

for some $\alpha > 0$. The points where (i) holds are called regular, they form a relatively open set and the free boundary is $C^{1,\alpha}$.

The C^∞ regularity of the free boundary in this case was an open problem. Finally, the fractional Laplacian with drift in the critical regime $s = \frac{1}{2}$ is studied in [35], where Fernández-Real and Ros-Oton establish optimal regularity of solutions[4] and the $C^{1,\alpha}$ regularity of the free boundary.

1.3.5 Parabolic obstacle problem for integro-differential operators

First of all notice that the nature of the parabolic version of the obstacle problem for integro-differential operators strongly depends on the value of the parameter s, as in the drift case. For $s > \frac{1}{2}$ the fractional Laplacian is the leading term (the subcritical regime), while for $s < \frac{1}{2}$ the time derivative is (the supercritical regime). The regularity of solutions was first addressed by Caffarelli and Figalli in [19], where they prove that the solutions are C_x^{1+s} in space and $C_t^{\min(\frac{1+s}{2s},2)-\varepsilon}$ in time, for all $s \in (0,1)$. Later on in [9] Barrios, Figalli and Ros-Oton show the following:

Theorem 1.3.11 ([9]). *Let $s \in (\frac{1}{2}, 1)$ and let u solve*

$$\min\{\partial_t u + (-\Delta)^s u, u - \varphi\} = 0 \qquad \text{in} \quad \mathbb{R}^n \times (0,\infty),$$

for some $\varphi \in C_c^4(\mathbb{R}^n)$.

Then for each free boundary point $(x_0, t_0) \in \partial\{u > 0\}$ we have

[4] They establish a result analogous to one above. It turns out that the growth near regular points depends on the angle between the drift direction b and the normal vector to the free boundary.

(i) either

$$0 < cr^{1+s} \leq \sup_{Q_r(x_0)} (u - \varphi) \leq Cr^{1+s}$$

(ii) or

$$0 \leq \sup_{Q_r(x_0)} (u - \varphi) \leq Cr^{1+s+\alpha},$$

for some $\alpha > 0$. The points where (i) holds are called regular, they form a relatively open and the free boundary is $C^{1,\alpha}$ in space and time near them.[5]

Furthermore, very recently in [75] Ros-Oton and Torres-Latorre improve the regularity of solutions in the supercritical regime $s < \frac{1}{2}$ to $C^{1,1}$ in space and time, which is optimal. Finally, in a forthcoming paper [38] Figalli, Ros-Oton and Serra extend the results from [9] to a more general class of operators as well as to the case $s = \frac{1}{2}$.

1.4 New results and structure of the thesis

In this thesis we mainly study the higher regularity of the free boundaries in some of the above presented obstacle problems.

This introduction is followed by four chapters, each corresponding to a paper or a preprint as follows.

- T. Kukuljan, *The fractional obstacle problem with drift: higher regularity of free boundaries*, J. Funct. Anal. **281** (2021), 109114, 60pp.
- T. Kukuljan, *$C^{2,\alpha}$ regularity of free boundaries in parabolic non-local obstacle problems*, preprint arXiv (2022), 40pp.
- T. Kukuljan, *Higher order parabolic boundary Harnack inequality*, Disc. Cont. Dyn. Syst. **42** (2022), 2667–2698.
- A. Audrito, T. Kukuljan, *Obstacle problem for fully nonlinear parabolic operators*, preprint (2022), 30pp.

As explained above, the key to proving the higher regularity of the free boundaries is the higher order boundary Harnack inequalities similar to the one in Theorem 1.3.5. Our strategies in Chapters 2, 3 and 4 rely on developing such kind of results. Notice that any solution to the obstacle problem is C^1 and the derivatives $\partial_i v$ solve the linear equation

$$\begin{cases} Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \Omega^c, \end{cases} \tag{1.4.1}$$

for $\Omega = \{v > 0\}$. We will use this observation in Chapters 2, 3 and 4.

[5] $Q_r(x_0, t_0) = \{|x - x_0| < r \text{ and } |t - t_0| < r^{2s}\}$.

1.4.1 Obstacle problem for fractional Laplacian with drift

In Chapter 2 we study the fractional obstacle problem with drift in the sub-critical regime $s > \frac{1}{2}$. Our main result establishes for the first time the higher regularity of the free boundary, from $C^{1,\alpha}$ to C^∞, provided that s is *irrational*.

Theorem 1.4.1. *Let $s \in (\frac{1}{2}, 1)\backslash\mathbb{Q}$, φ be any $C_c^\infty(\mathbb{R}^n)$ obstacle and $b \in \mathbb{R}^n$. Assume v solves*

$$\min\{(-\Delta)^s v + b \cdot \nabla v, v - \varphi\} = 0 \qquad \text{in} \quad \mathbb{R}^n.$$

Then the free boundary $\partial\{v > \varphi\}$ is C^∞ in a neighbourhood of any regular point.

The assumption $s \notin \mathbb{Q}$ might seem quite surprising, as it is very uncommon to have this kind of assumption in a regularity theorem. However we will see below how this assumption arises in a natural way here.

As said before, we prove the higher regularity of the free boundary in the fractional obstacle problem with drift through some new boundary Harnack type inequalities. In this direction we closely study the boundary behaviour of solutions to

$$\begin{cases} ((-\Delta)^s + b \cdot \nabla)u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \Omega^c \\ \partial\Omega &\in C^\beta, \end{cases}$$

when $s \in (\frac{1}{2}, 1)$. When $b = 0$, first Jhaveri and Neumayer in [51], and later Abatangelo and Ros-Oton in a more general setting in [1], established that the solutions behave as polynomials times the distance to the power s near boundary points up to order β, or equivalently that $u/d^s \in C^{\beta-1}$. If we look at that in the one dimensional case this gives an expansion of the form

$$u(x) = c_0(x_+)^s + c_1(x_+)^{1+s} + c_2(x_+)^{2+s} + \ldots + O(|x|^{\beta-1+s}), \qquad \text{for } b = 0.$$

Roughly this follows from the fact that $(x_+)^{k+s}$ are in the kernel of the fractional Laplacian, since $(-\Delta)^s(x_+)^s = 0$, see Chapter 2. When we add the drift term $b \cdot \nabla$ to the fractional Laplacian, this is no longer the case: $((-\Delta)^s + \frac{d}{dx})(x_+)^s = s(x_+)^{s-1}$. Analysing evaluations of the fractional Laplacian of powers of x_+ we see that $(-\Delta)^s(x_+)^p = c_{p,s}x^{p-2s}$ for $x > 0$, when $p - 2s \notin \mathbb{N}$, where $c_{p,s}$ is an explicit constant. This implies that the kernel of $(-\Delta)^s + \frac{d}{dx}$ is spanned by the functions

$$(x_+)^{s+k} + c_{1,k}(x_+)^{s+k+(2s-1)} + c_{2,k}(x_+)^{s+k+2(2s-1)} + \ldots,$$

for integers $k \geq 0$ and some constants $c_{l,k}$, as long as s is *irrational*.

In the spirit of this, we are able to derive expansions of solutions near a boundary point z of the form

$$u = p_0 d^s + p_1 d^{s+(2s-1)} + p_2 d^{s+2(2s-1)} + \ldots + O(|x - z|^{\beta-1+s}), \tag{1.4.2}$$

for some polynomials p_i. This gives that in general the solutions divided by the distance to the power s are only $2s - 1$ regular up to the boundary,

$$\frac{u}{d^s} \in C^{2s-1}(\overline{\Omega}),$$

and it is optimal. But as the powers of the distance function are still smooth in the tangential directions, we are able to deduce from the expansion, that the quotient u/d^s is $C^{\beta-1}$ when restricted to the boundary, i.e.

$$\frac{u}{d^s} \in C^{\beta-1}(\partial\Omega).$$

Such phenomena is new and had not been used before in such type of boundary estimates: that higher regularity holds only in tangential directions.

Moreover the same approach can also be used to derive expansions of one solution with respect to the other one. Concretely replacing the term of the lowest order in the above expansion ($p_0(z)d^s$) with a constant multiple of another solution, we improve the expansion by an order $2s-1$:

$$u_1 = c_0 u_2 + q_0 d^s + q_1 d^{s+(2s-1)} + q_2 d^{s+2(2s-1)} + \ldots + O(|x-z|^{\beta-1+s+(2s-1)}),$$

where $q_0(z) = 0$. From this we are able to deduce that the quotient u_1/u_2 is $C^{\beta-1+(2s-1)}$ when restricted to the boundary

$$\frac{u_1}{u_2} \in C^{\beta-1+\varepsilon_0}(\partial\Omega),$$

where $\varepsilon_0 = 2s-1 > 0$.

This provides enough information to give rise to a bootstrap argument similar to the one described in Section 1.4. Namely if the free boundary is C^β, then its normal is $C^{\beta-1+\varepsilon_0}$, and hence the boundary is actually $C^{\beta+\varepsilon_0}$:

$$\partial\Omega \in C^\beta \Rightarrow \frac{u_1}{u_2} \in C^{\beta-1+\varepsilon_0}(\partial\Omega) \Rightarrow \nu \in C^{\beta-1+\varepsilon_0}(\partial\Omega) \Rightarrow \partial\Omega \in C^{\beta+\varepsilon_0} \Rightarrow \ldots \Rightarrow \partial\Omega \in C^\infty.$$

When s is rational, we do not expect (1.4.2) to hold and several logarithmic terms would appear. It is not clear what happens in that case.

1.4.2 Parabolic obstacle problem for fractional Laplacian

Recall that in the parabolic fractional obstacle problem only the initial $C^{1,\alpha}$ regularity of the free boundary was known, see Theorem 1.3.11 above. Our goal in Chapter 3 is to improve such regularity via some new boundary Harnack estimate. To prove the higher order boundary Harnack inequality for the fractional heat operator we face combined difficulties of the parabolic case and the non-local drift case: the domain under study is moving in time, and the evaluation $(\partial_t + (-\Delta)^s)d^s \approx d^{s-1}$, which ruins the argument from the elliptic non-local case in [1]. Additionally we face two additional struggles. The first one is that the interior estimates for the fractional heat operator require global regularity in time of the solution, and the second one is that there is no suitable way of treating operator evaluation of functions that grow more than $|x|^{2s}$ at infinity. Nevertheless we are able to establish the boundary Harnack inequality of order up to $2s$, when $s > \frac{1}{2}$. Applied to derivatives of the solutions of the obstacle problem, this yields the following:

Theorem 1.4.2. *Let $s \in (\frac{1}{2}, 1)$. Let u be a solution of*

$$\begin{aligned} \min\{(\partial_t + (-\Delta)^s)v, v-\varphi\} &= 0 \qquad \text{in } \mathbb{R}^n \times (0,T) \\ v(\cdot,0) &= \varphi \qquad \text{in } \mathbb{R}^n, \end{aligned}$$

with $\varphi \in C^4(\mathbb{R}^n)$.

Then the free boundary $\partial\{v > \varphi\}$ *is* $C_p^{2+\alpha}$ *near regular free boundary points.*[6]

To establish the corresponding boundary Harnack inequality, we study the boundary behaviour of solutions of the linear equation

$$\begin{cases} \partial_t u + (-\Delta)^s u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \Omega^c. \end{cases} \tag{1.4.3}$$

First we find a regularized distance function, and extend the result about regularity of $L(d^s)$ from [1] to the time variable as well. This yields that the solutions of (1.4.3) behave like d^s near the boundary, from which we deduce the optimal regularity for u and u/d^s:

$$u \in C^s_{x,t}(\overline{\Omega}) \qquad \text{and} \qquad \frac{u}{d^s} \in C_p^{2s-1}(\overline{\Omega}).$$

Refining the argument, comparing one solution to another provides expansions near any boundary point (z, τ) of the following form

$$|u_1(x,t) - p(x,t)u_2(x,t)| \leq C\left(|x-z|^{3s-\varepsilon} + |t-\tau|^{\frac{3s-\varepsilon}{2s}}\right),$$

for a polynomial p and any $\varepsilon > 0$. In combination with interior regularity estimates we obtain that the quotient satisfies

$$\frac{u_1}{u_2} \in C_p^{2s-\varepsilon}(\overline{\Omega}),$$

provided that both solutions are globally $C^{\frac{1}{2}}$ in the time variable.

When applying the result to the partial derivatives of the solution of the fractional parabolic obstacle problem we can only deduce that the normal vector is $C_p^{\gamma+s}$ up to the boundary, where $1-s < \gamma < 2-s$. That is because the time derivative of the solution is globally only $C_t^{\frac{\gamma}{2s}}$, see [9], [19] and [38, Corollary 1.6], and hence the quotient $\partial_t u/\partial_n u$ is only C_p^γ up to the boundary. Therefore we can deduce that the free boundary is $C^{2,\alpha}$ near regular points, but not more using this method. It remains a challenging open problem to decide whether the free boundary is C^∞ or not.

1.4.3 Parabolic obstacle problem

In the local parabolic setting we establish boundary Harnack inequalities in C^1 and $C^{k,\alpha}$ domains. In C^1 domains our results are new, and in $C^{k,\alpha}$ domains we give a new proof of previously known result. This gives a new proof of the higher regularity of the free boundary (C^1 implies C^∞) in the parabolic obstacle problem, that does not rely on the use of the Hodograph transform. The main result states the following:

Theorem 1.4.3. *Let* $\beta \geq 1$. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *be* C_p^β *in* Q_1 *according to Definition 4.1.1. For* $i = 1, 2$ *let* u_i *be a solution to*

$$\begin{cases} \partial_t u_i - \Delta u_i &= f_i \quad \textit{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \textit{on } \partial\Omega \cap Q_1, \end{cases}$$

[6] Parabolic Hölder space C_p^β roughly consists of functions that are C^β in space and $C^{\frac{\beta}{2s}}$ in time. See Definition 3.2.1 for more details.

with $f_i \in C_p^{\beta-1}(\Omega \cap Q_1)$. *Assume that* $|u_2| \geq c_2 d$ *with* $c_2 > 0$ *and* $||f_2||_{C_p^{\beta-1}(\Omega\cap Q_1)} + ||u_2||_{L^\infty(\Omega\cap Q_1)} \leq C_2$.

Then we have

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{\beta}(\overline{\Omega}\cap Q_{1/2})} \leq C\left(||f_1||_{C_p^{\beta-1}(\Omega\cap Q_1)} + ||u_1||_{L^\infty(\Omega\cap Q_1)}\right),$$

with C *depending only on* n, β, c_2, C_2 *and* Ω.[7]

As seen in (1.4.1) in the parabolic obstacle problem the derivatives of the solution solve exactly the equation from the assumption of the above theorem. Even though this is the simplest form of a parabolic equation, the boundary regularity for solutions is quite delicate, as the domain is not cylindric - it is a *moving domain.* Still, around regular points we know that blowing up results in a half-space of the form $\{x \cdot e > 0\}$, for some $e \in \mathbb{S}^{n-1}$. In such domains we are able to prove the boundary Harnack inequality stated above. To establish such result, we show that at every boundary point (z, s) we can approximate any solution with a polynomial times the non-trivial solution up to order $\beta + 1$, concretely

$$|u_1(x,t) - p_{(z,s)}(x,t)u_2(x,t)| \leq C\left(|x-z|^{\beta+1} + |t-s|^{\frac{\beta+1}{2}}\right) \qquad (x,t) \in Q_1(z,s),$$

under the assumption that the solutions are C_p^β up to the boundary. In combination with the interior regularity estimates we can deduce that the quotient has to be as regular as the boundary up to the boundary.

We are also able to show the boundary Schauder type estimates, which provide that the solutions are indeed C_p^β up to the boundary. Even though the result was known in case of $C_p^{k,\alpha} \cap C^{1,\alpha}$ domains, see [6], our result is new in case of C_p^1 and $C_p^{1,\alpha}$ domains and we give a new proof for $C_p^{k,\alpha}$ domains for $k \geq 2$. Our proof bases on the construction of a regularized distance function $d \in C^\beta(\overline{\Omega})$ vanishing on the boundary, comparable to the (parabolic) distance to the boundary, but is infinitely smooth in the interior of the domain, with suitable estimates on growth of the higher derivatives near the boundary. The regularized distance can be used to approximate any solution near a boundary point (z, s) up to order β. We prove that

$$|u(x,t) - p_{(z,s)}(x,t)d(x,t)| \leq C\left(|x-z|^{\beta} + |t-s|^{\frac{\beta}{2}}\right) \qquad (x,t) \in Q_1(z,s),$$

for some polynomial $p_{(z,s)}$, which implies that u is $C_p^\beta(\overline{\Omega})$, thanks to interior regularity estimates. For more details see Chapter 4. As a corollary we find a new proof of the fact that the free boundary in the parabolic obstacle problem is C^∞ near regular points, as explained in the beginning of Section 1.4.

1.4.4 Fully non-linear parabolic obstacle problem

In Chapter 5 we consider a general parabolic obstacle problem (1.3.3), where F satisfies (1.3.4), and has smooth dependence on x. Because of the non-linearity of F, many of the tools used in the classical obstacle problem are no longer available, especially the

[7] When $\beta \in \mathbb{N}$ the regularity of the quotient is $\beta - \varepsilon$ for any $\varepsilon > 0$.

monotonicity formulas that show that blow-ups are homogeneous and to study the singular set. Here we develop the full regularity theory for the free boundaries presented in Section 1.1 to the parabolic obstacle problem for fully non-linear operators (1.3.3).

Concretely, we give a new proof of the optimal $C_x^{1,1}$ regularity of solutions and prove for the first time the continuity of $\partial_t u$. With analysis of global solutions we find a dichotomy between regular and singular points and show that the free boundary is Lipschitz near regular points. Then we improve the regularity in space to C^1 and show that the solution is C_x^2 in the positivity set, up to the free boundary. This allows us to apply either boundary Harnack inequalities from Chapter 4 or the results by Kinderlehrer and Nirenberg from [52], which yield that the free boundary is C^∞ near regular points.

Theorem 1.4.4. *Let u be a solution of* (1.3.3). *For every free boundary point $(x_0, t_0) \in \partial\{u > 0\}$ it holds*

(i) either

$$\lim_{r\downarrow 0} \frac{1}{r^2} u(x_0 + rx, t_0 + r^2 t) = c_0 (x \cdot e_0)_+^2,$$

for some $c_0 > 0$ and $e_0 \in \mathbb{R}^n$,

(ii) or

$$\lim_{r\to 0} \frac{|\{u = 0\} \cap B_r(x_0) \times \{t_0\}|}{|B_r(x_0)|} = 0,$$

and every blow-up at (x_0, t_0) is a quadratic polynomial.

The points where (i) holds are called regular, they form an open subset of $\partial\{u > 0\}$ and the free boundary is C^∞ near them. Points where (ii) holds are called singular.

At singular points we show that every blow-up is a quadratic polynomial. In combination with the new result that the free boundary can be presented as a graph $(x, \tau(x))$ of a Lipschitz function τ, this implies that the singular set can be covered with a Lipschitz $n - 1$ dimensional manifold, which is pointwise C_x^1.

Theorem 1.4.5. *Let u be a solution of* (1.3.3) *and let $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ be the set of all singular points. It holds that for every $\varepsilon > 0$ the singular set Σ can locally be covered by a Lipschitz manifold of dimension $n - 1$ whose Lipschitz norm in space is bounded by ε.*

Once we know that the full free boundary is Lipschitz in time, we exploit a version of the "almost positivity" lemma to deduce that the solution of the obstacle problem is convex near singular points in directions that are close to directions where the blow-up is positive. This furthermore implies that the solution cannot vanish at any point in these directions relative to any singular point nearby. We conclude that the projection of the singular set in time can locally be covered by a $n - 1$ dimensional manifold, that is pointwise C^1, which readily implies the claim of the theorem.

Chapter 2

The fractional obstacle problem with drift: higher regularity of free boundaries

2.1 Introduction

Obstacle problems for integro-differential operators naturally appear in probability and mathematical finance, for example in the optimal stopping problem for Lévy processes with jumps, which has been used in pricing models for American options since 1970, see [26]. More recently, obstacle problems of this kind appeared also in other fields of science, for example biology and material science, see [24, 71, 79] and references therein.

Henceforth, more and more effort has been put into understanding obstacle problems for nonlocal operators. The obstacle problem is a non-linear equation that can be written in the form

$$\min\{Lu, u - \varphi\} = 0 \qquad \text{in} \qquad \mathbb{R}^n,$$

where φ is a given smooth function with compact support, called the obstacle, and L is some kind of nonlocal operator. The main goal of study is to understand the set $\{u > \varphi\}$, concretely to find out the regularity of its boundary $\partial\{u > \varphi\}$, called the free boundary. The most basic and canonical example of a nonlocal operator is the fractional Laplacian, $(-\Delta)^s$, given by

$$(-\Delta)^s u(x) = c_{n,s} p.v. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \quad s \in (0,1),$$

where the constant $c_{n,s}$ is chosen so that the Fourier symbol of the operator is $|\xi|^{2s}$. The main result regarding the regularity of the free boundary in the case of the fractional Laplacian states that the free boundary is C^∞ outside of a set of singular or degenerate points. Results of this type are usually established in three steps:

(a) The free boundary splits into regular and singular/degenerate points,

(b) Near regular points the free boundary is $C^{1,\alpha}$ for some $\alpha > 0$,

(c) If the free boundary is $C^{1,\alpha}$, then it is C^∞.

Parts (a) and (b) were established in [3, 23], and part (c) in [28, 51, 54, 55]. Analogous results have been established for a family of integro-differential operators of the form

$$\begin{aligned} Lu(x) &= p.v. \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)dy \\ &= \frac{1}{2} \int_{\mathbb{R}^n} (2u(x) - u(x+y) - u(x-y))K(y)dy, \end{aligned}$$

where the kernel K satisfies

$$\begin{aligned} &K \text{ is even, homogeneous and} \\ &\frac{\lambda}{|y|^{n+2s}} \le K(y) \le \frac{\Lambda}{|y|^{n+2s}}, \text{ for all } y \in \mathbb{R}^n \text{ with } 0 < \lambda \le \Lambda,\ s \in (0,1). \end{aligned} \tag{2.1.1}$$

Steps (a) and (b) has been established in [22], and step (c) in [1].

Two more cases of operators are of particular interest for study. The parabolic case, when the operator is given with $(-\Delta)^s + \partial_t$, and the drift case or the fractional Laplacian with drift, when the operator equals $(-\Delta)^s + b\cdot\nabla$. In order to be able to apply similar tools as in the case $b \equiv 0$, we must assume that the parameter $s > \frac{1}{2}$ – the subcritical regime. Then the fractional Laplacian is of higher order than the derivative terms, which allows us to treat the drift term as reminders. Still, much less is known in these cases. In the parabolic case, step (a) and (b) were established in [9, 19], and there are no generalisations to a wider class of elliptic operators, and (c) is an open problem. Similarly, for the fractional Laplacian with drift, steps (a) and (b) have been established in [47, 67], but again (c) remained an open problem. On the other hand, the fractional Laplacian with drift in the critical regime $s = \frac{1}{2}$ is studied in [35], where again steps (a) and (b) are established.

The aim of this paper is to continue the study of the free boundary around regular points for the obstacle problem with drift in the subcritical regime $s > \frac{1}{2}$:

$$\min\{Lu + b \cdot \nabla u, u - \varphi\} = 0 \qquad \text{in} \qquad \mathbb{R}^n, \tag{2.1.2}$$

where b is a constant vector in $\mathbb{R}^n$. We show that if $s \notin \mathbb{Q}$, once the free boundary is C^1, it is in fact C^∞, as long as φ is C^∞. Our main result states the following.

Theorem 2.1.1. *Let L be the fractional Laplacian. Let $s \in (\frac{1}{2}, 1)\backslash\mathbb{Q}$, and φ be any $C_c^\infty(\mathbb{R}^n)$ obstacle, and u be any solution to* (2.1.2). *Then the free boundary $\partial\{u > \varphi\}$ is C^∞ in a neighbourhood of any regular point.*[1]

This is the analogue of the step (c) explained above, since the initial regularity is provided in [47]. We explain the exclusion of the rational parameters s in the subsection below. Let us also point out, that our proof from the initial C^1 regularity to C^∞ is done for a family of operators L whose kernels satisfy (2.1.1) and are $C^\infty(\mathbb{S}^{n-1})$. Hence as soon as the initial regularity is provided for this class as well, we immediately obtain the analogue of the above theorem (see Corollary 2.7.2).

[1] We refer to [47] or Section 7 below, for the definition of regular points.

2.1.1 Strategy of the proof

In order to obtain the higher regularity of the free boundary, we exploit the fact that the normal vector can be expressed with the quotients of the partial derivatives of the height function $w := u - \varphi$ (see [27, 28] or [1, Section 5]). Hence, we closely study the quotients w_i/w_n in the domain $\Omega = \{w > 0\}$.

Some ideas are drawn from [1], but we are faced with several difficulties arising from the drift term.

In [1], it is established that the quotients w_i/w_n are as smooth as the boundary, say $C^\beta(\overline{\Omega})$. This implies that the normal vector ν to the boundary is C^β as well, and hence the boundary itself is $C^{\beta+1}$. A key step to show this is to establish that the quotients w_i/d^s are $C^{\beta-1}(\overline{\Omega})$. Here, unfortunately, these results fail due to the presence of the drift term, and the best regularities we can get are $w_i/d^s, w_i/w_n \in C^{2s-1}(\overline{\Omega})$. Still, one may wonder if $w_i/w_n \in C^\beta(\partial\Omega)$, i.e., the regularity in *tangential* directions only. This is indeed what we prove here, but it turns out to be quite delicate, as explained next.

We use the expansion result from [1], stating that

$$\begin{cases} Lu = f & \text{in } \Omega \cap B_1, \\ u = 0 & \text{in } \Omega^c \cap B_1, \end{cases} \implies u(x) = Q(x)d^s + O(|x|^{\beta-1+s}), \tag{2.1.3}$$

for some polynomial Q of degree $\lfloor \beta - 1 \rfloor$, provided that f is smooth enough. For simplicity, let us turn to the one-dimensional case, $\Omega = \{x > 0\} \subset \mathbb{R}$. When there is no drift term, (2.1.3) applies directly on w_i and gives the expansion of the form

$$w_i(x) = c_0 x_+^s + c_1 x_+^{1+s} + c_2 x_+^{2+s} + \ldots + O(|x|^{\beta-1+s}),$$

which yields that w_i/d^s agrees with a polynomial up to an error term of order $\beta - 1$. This provides enough information to deduce the wanted regularity. When the drift term appears, the partial derivatives solve

$$\begin{cases} Lw_i = f_i - b \cdot \nabla w_i & \text{in } \Omega \cap B_1, \\ w_i = 0 & \text{in } \Omega^c \cap B_1. \end{cases}$$

Since a priori we have very little regularity for ∇w_i, the expansion result (2.1.3) gives us only that $w_i = c_0 x_+^s + O(|x|^{s+\alpha})$, for some small α. To continue the expansion, we deduce that than the gradient is of the form $\nabla w_i = c_0' x_+^{s-1} + O(|x|^{s-1+\alpha})$, and then find a suitable constant c_1, such that $L(c_1 x_+^{s+\varepsilon_0}) = c_0' x_+^{s-1}$. Then we expand the function $w_i - c_1 x_+^{s+2s-1}$, which has a bit better right-hand side, to get

$$w_i = c_0 x_+^s + c_1 x_+^{s+2s-1} + O(|x|^{s+2s-1+\alpha}).$$

We proceed in the similar manner, to get the expansion

$$w_i = c_0 x_+^s + c_1 x_+^{s+2s-1} + c_2 x_+^{s+2(2s-1)} + \ldots + O(|x|^{\beta-1+s}),$$

where the powers in the expansion are of the form $s + k(2s-1) + l$, for $k, l \in \mathbb{N}$, but smaller than $\beta - 1 + s$. The procedure is based on the fact that $Lx_+^p = c_p x^{p-2s}$ for some non-zero constant c_p. This equality fails when p is of the form $m + s$ or $m + 2s$ for any integer m. This leads to exclusion of rational parameters s.

In fact, similar happens in the general case. If $\partial\Omega \subset \mathbb{R}^n$ is C^β, we are able to establish the expansion of the form

$$w_i(x) = \sum_{k,l\geq 0}^{k(2s-1)+l\leq\beta-1} Q_{k,l,z}(x) d^{s+k(2s-1)+l}(x) + O(|x-z|^{\beta-1+s}),$$

around every regular free boundary point z, where the zero order term of $Q_{0,0,z}$ is $C^{\beta-1}(\partial\Omega)$ as a function of z. The expansion tells us, that the best regularity of $\frac{w_i}{d^s}$ and $\frac{w_i}{w_n}$ is $C^{2s-1}(\overline{\Omega})$, but moreover also that $\frac{w_i}{d^s} \in C^{\beta-1}(\partial\Omega)$. Furthermore, with some extra steps we are able to deduce also $\frac{w_i}{w_n} \in C^{\beta-1+(2s-1)}(\partial\Omega)$, which provides that the normal ν to the boundary is of the same regularity, and so the boundary has to be $C^{\beta+(2s-1)}$. Since $2s-1>0$, this is enough to bootstrap and deduce that $\partial\Omega$ is actually C^∞.

Let us describe the expansion part a little bit more in details. We improve the accuracy of the expansion of w_i gradually. The improvement is obtained in two steps. First we show how the expansion of w_i translates to the gradient. This presents a rather cumbersome step, which needs some additional interior regularity estimates and Corollary 2.8.11.

Then we need to correct the expansion of w_i, in such a way that its operator evaluation becomes small. Concretely, for every term $Qd^{\tilde{p}}$ in the drift term expansion, we need to find polynomial $\tilde{Q}$ and a suitable power p so that $L(\tilde{Q}d^p) \approx Qd^{\tilde{p}}$. In order to establish it, one needs to show a result of the type

$$\partial\Omega \in C^\beta \quad \Longrightarrow \quad L(\tilde{Q}d^p) = \phi d^{p-2s} + R, \tag{2.1.4}$$

where $\phi \in C^\beta(\overline{\Omega})$ and $R \in C^{\beta-1+p-2s}(\overline{\Omega})$, together with additional information about function ϕ. To get a correspondence between $\tilde{Q}$ and ϕ we perform a blow-up argument with a limiting result, that reduces to the flat case. Together with the explicit computation in the flat case, we are able to argue the existence of $\tilde{Q}$, so that $L(\tilde{Q}d^{p+2s}) \approx Q_i d^p$, up to some error terms. Note that in the computation in the flat case, we need that the power is not in $\mathbb{N}+2s$, otherwise logarithmic terms would appear. Since we apply it on the powers of the form $(2k+1)s-l$, our argument with expansions works only when s is not of the form $\frac{m}{(2k+1)}$, for two integers m,k, and hence we set s to be irrational.

2.1.2 Boundary regularity for linear nonlocal equations with drift

Our results hold for arbitrary solutions to linear equations of the form

$$\left\{ \begin{array}{rcll} Lu + b\cdot\nabla u &=& f & \text{in } \Omega\cap B_1 \\ u &=& 0 & \text{in } \Omega^c\cap B_1. \end{array} \right. \tag{2.1.5}$$

With its aid we are able to establish the following boundary Harnack-type estimate.

Theorem 2.1.2. *Let $s>\frac{1}{2}$, $s\notin\mathbb{Q}$, and assume $\Omega\subset\mathbb{R}^n$ is a C^β domain with $\beta>2-s$, $\beta\notin\mathbb{N}$ and $\beta\pm s\notin\mathbb{N}$. Let L be an operator whose kernel K is $C^{2\beta+1}(\mathbb{S}^{n-1})$ and satisfies conditions* (2.1.1)*. For $i=1,2$, let u_i be two solutions to*

$$\left\{ \begin{array}{rcll} Lu_i + b\cdot\nabla u_i &=& f_i & \textit{in } \Omega\cap B_1 \\ u_i &=& 0 & \textit{in } \Omega^c\cap B_1, \end{array} \right.$$

with $f_i \in C^{\beta-2+s}(\overline{\Omega}\cap B_1)$ *and* $b \in \mathbb{R}^n$. *Assume that* $u_2 \geq c_2 d^s$ *in* B_1, *for some positive* c_2. *Then*

$$\frac{u_1}{u_2} \in C^{2s-1}(\overline{\Omega}\cap B_{1/2}), \tag{2.1.6}$$

and

$$\frac{u_1}{u_2} \in C^{\beta-1+(2s-1)}(\partial\Omega\cap B_{1/2}).$$

An important step towards towards the proof of Theorem 2.1.2 is the following boundary Schauder-type estimate for solutions to (2.1.5).

Theorem 2.1.3. *Let* $s > \frac{1}{2}$, $s \notin \mathbb{Q}$, *and assume* $\Omega \subset \mathbb{R}^n$ *is a* C^β *domain with* $\beta > 1+s$, $\beta \notin \mathbb{N}$, *and* $\beta \pm s \notin \mathbb{N}$. *Let* L *be an operator whose kernel* K *is* $C^{2\beta+1}(\mathbb{S}^{n-1})$ *and satisfies conditions* (2.1.1). *Let* u *be a solution to* (2.1.5) *with* $f \in C^{\beta-1-s}(\overline{\Omega})$ *and* $b \in \mathbb{R}^n$.

Then

$$\frac{u}{d^s} \in C^{2s-1}(\overline{\Omega}\cap B_{1/2}), \tag{2.1.7}$$

and

$$\frac{u}{d^s} \in C^{\beta-1}(\partial\Omega\cap B_{1/2}).$$

We emphasise that (2.1.6) and (2.1.7) are optimal; and therefore the higher regularity of u_1/u_2 and u/d^s holds only in the tangential directions.

2.1.3 Organisation of the paper

In section 2 we present the notation and some tools we use throughout the paper. In Section 3, we prove (2.1.4) and some similar results regarding the evaluation of the operator L on powers of the distance function. Section 4 is devoted to the computation of the flat case and establishing the correspondence between polynomials described above. Furthermore, it provides a boundary regularity estimate needed in our framework. In Section 5, we prove expansion type results which find use in Section 6, where we prove Theorem 2.1.3 and Theorem 2.1.2. In Section 7 we prove Theorem 2.1.1 and related results. At the end there is an appendix, where we prove technical auxiliary results, to lighten the body of the paper.

2.2 Notation and preliminaries

When $\beta \notin \mathbb{N}$, with C^β we mean the Hölder space $C^{\lfloor\beta\rfloor,\langle\beta\rangle}$, where $\lfloor\cdot\rfloor$ denotes the integer part of a number and $\langle x\rangle = x - \lfloor x\rfloor$. Moreover, with $C_0(\mathbb{R}^n)$ we denote the closure of continuous, compactly supported functions with respect to the L^∞ norm.

For $y \in \mathbb{R}^n$ we denote $\langle y\rangle = y/|y|$. Also, we use the multi-index notation $\alpha \in \mathbb{N}^n$, $\alpha = (\alpha_1,\ldots,\alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and accordingly we denote

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

With $\mathbf{P}_k$ we denote the space of polynomials of order k. For a function f, we denote $T_a^k f$ its Taylor polynomial of order k at a point a.

Throughout the paper s is a parameter in $(0,1)$, often also $s > 1/2$, and $\varepsilon_0 = 2s - 1$. Furthermore for real numbers a, b we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

The unit sphere is denoted with $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. Sometimes the following comes in handy

$$\mathbf{1}_{condition} = \begin{cases} 1 & \text{if } condition \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

Finally, C indicates an unspecified constant not depending on any of the relevant quantities, and whose value is allowed to change from line to line. We make use of sub-indices whenever we will want to underline the dependencies of the constant.

2.2.1 Generalised distance function

Throughout the paper, for a domain Ω the distance function to its complement is of great use. Since we need it to be more regular inside Ω, we work with the generalised distance function defined as follows.

Definition 2.2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set with C^β boundary. We denote with $d \in C^\infty(\Omega) \cap C^\beta(\overline{\Omega})$ a function satisfying

$$\frac{1}{C}\operatorname{dist}(\cdot, \Omega^c) \leq d \leq C \operatorname{dist}(\cdot, \Omega^c),$$

$$|D^k d| \leq C_k d^{\beta - k}, \quad \text{for all } k > \beta.$$

The definition follows [1, Definition 2.1] and the precise construction is provided by [1, Lemma A.2].

2.2.2 Nonlocal equations for functions with polynomial growth at infinity

We are often in situation when the function on which we want to evaluate the operator L does not satisfy the growth control. In our case it mostly occurs when doing the limiting arguments and blow-ups. Then we are faced with a function which has polynomial growth. The evaluation is done according the following definition.

Definition 2.2.2. Let $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain, $u \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $f \in L^\infty(\Omega)$. Assume that u satisfies

$$\int_{\mathbb{R}^n} \frac{u(y)}{1 + |y|^{n+k+2s}} dy < \infty.$$

We say that

$$Lu \overset{k}{=} f, \quad \text{in } \Omega, \tag{2.2.1}$$

if there exists a family of polynomials $(p_R)_{R>0} \subset \mathbf{P}_{k-1}$, and a family of functions $(f_R \colon \Omega \to \mathbb{R})_{R>0}$, such that for all $R > \operatorname{diam}(\Omega)$ we have

$$L(u\chi_{B_R}) = f_R + p_R \quad \text{in } \Omega,$$

and

$$f_R \xrightarrow{R\to\infty} f \quad \text{uniformly in } \Omega.$$

In the case when Ω is unbounded, we say that (2.2.1) holds, if it holds in any bounded subdomain.

The definition follows the one in [1, Section 3]. There one can also find some properties of solutions to (2.2.1).

2.3 Non-local operators and the distance function

The goal of this section is to prove the following result:

Theorem 2.3.1. *Let $K \in C^{2\beta+1}(\mathbb{S}^{n-1})$ be a kernel as in (2.1.1). Let $p \in (0, 2s)$ and Ω be a domain in $\mathbb{R}^n$ with C^β boundary, for some $\beta > 1 + 2s - p$. Let $\eta \in C^\infty(\mathbb{R}^n)$. Then we have*

$$L(\eta d^p) = \varphi d^{p-2s} + R,$$

where $\varphi \in C^\beta(\overline{\Omega})$, and $R \in C^{\beta-1-2s+p}(\overline{\Omega})$, with

$$||R||_{C^{\beta-1-2s+p}(\overline{\Omega})} \le C,$$

where C depends only on n, s, η, $||K||_{C^{2\beta-1}(\mathbb{S}^{n-1})}$, p, β and the C^β norm of $\partial\Omega$.

Let us start with the result which estimates $L(d^{s+\varepsilon})$ in a $C^{1,\alpha}$ domain, where $\varepsilon < s$ and $\alpha \le 1$. It is done in a similar manner as [72, Proposition 2.3]. This result already gives all the regularity of $L(d^p)$ we need in this setting and we do not develop it further.

Lemma 2.3.2. *Let Ω be a $C^{1,\alpha}$ domain with $\alpha \le 1$ and K a kernel satisfying (2.1.1). Then for $\varepsilon \in (-s, s)$ we have the following equality*

$$L(d^{s+\varepsilon})(x) = c_{s+\varepsilon}|\nabla d(x)|^{2s} d^{\varepsilon-s}(x) + R(x), \qquad x \in \Omega,$$

where $c_{s+\varepsilon}$ is an explicit constant and R satisfies $|R| \le C d^{(\alpha+\varepsilon-s)\wedge 0}$. Moreover, $c_{s+\varepsilon} = 0$ if and only if $\varepsilon = 0$.

Proof. Let $x_0 \in \Omega$ and $\rho = d(x_0)$. Notice that when $\rho \ge \rho_0 > 0$, then $d^{\varepsilon+s}$ is smooth in the neighbourhood of x_0, and thus $L(d^{\varepsilon+s})(x_0)$ is bounded by a constant depending only on ρ_0. Thus we may assume that $\rho \in (0, \rho_0)$, for some small ρ_0 depending only on Ω.

Let us denote $l(x) = (d(x_0) + \nabla d(x_0) \cdot (x - x_0))_+$. With explicit computation and knowing the one-dimensional case, we get

$$\begin{aligned} L(l^{s+\varepsilon}) &= L\left(|\nabla d(x_0)|^{\varepsilon+s}\left(\frac{d(x_0)}{|\nabla d(x_0)|} + \frac{\nabla d(x_0)}{|\nabla d(x_0)|}(\cdot - x_0)\right)_+^{s+\varepsilon}\right) \\ &= |\nabla d(x_0)|^{\varepsilon+s} c_{s+\varepsilon}\left(\frac{d(x_0)}{|\nabla d(x_0)|} + \frac{\nabla d(x_0)}{|\nabla d(x_0)|}(\cdot - x_0)\right)^{\varepsilon-s}, \quad \text{in } \{l > 0\}, \end{aligned}$$

where $c_{s+\varepsilon} = 0$ if and only if $\varepsilon = 0$ (see [1, Theorem 3.10]), and so we have

$$L(l^{s+\varepsilon})(x_0) = |\nabla d(x_0)|^{2s} c_{s+\varepsilon} d^{\varepsilon-s}(x_0).$$

Then, in the same way as in [72, Proposition 2.3] get the estimates

$$|d(x_0+y)-l(x_0+y)|\leq C|y|^{1+\alpha},\quad y\in\mathbb{R}^n,$$

$$|\nabla d(x_0+y)-\nabla l(x_0+y)|\leq C|y|^{\alpha},\quad y\in B_{\rho/2}.$$

With bounding the derivatives carefully this gives

$$|d^{s+\varepsilon}(x_0+y)-l^{s+\varepsilon}(x_0+y)|\leq\begin{cases}C\rho^{s+\varepsilon+\alpha-2}|y|^2 & y\in B_{\rho/2},\\ C|y|^{1+\alpha}(d^{s+\varepsilon-1}(x_0+y)+l^{s+\varepsilon-1}(x_0+y)) & y\in B_1\backslash B_{\rho/2}\\ C|y|^{s+\varepsilon} & y\in B_1^c.\end{cases}$$

The first one bases on the estimate $||D^2(d^{s+\varepsilon}-l^{s+\varepsilon})||_{L^\infty(B_{\rho/2})}\leq C\rho^{s+\varepsilon+\alpha-2}$, which follows after explicit computation and various application of the estimate $|a^p-b^p|\leq|a-b|(a^{p-1}+b^{p-1})$, which is true for all positive a,b and $p\in(-\infty,2)$. The second one is straight forward application of the latter estimate, and the last is based on the growth of l at infinity.

Now we are in position to estimate

$$\begin{aligned}\big|L(d^{s+\varepsilon})(x_0)-c_{s+\varepsilon}|\nabla d(x_0)|^{2s}d^{\varepsilon-s}(x_0)\big|&=\big|L(d^{s+\varepsilon}-l^{s+\varepsilon})(x_0)\big|\\ &\leq\int_{\mathbb{R}^n}|d^{s+\varepsilon}-l^{s+\varepsilon}|(x_0+y)\frac{\Lambda}{|y|^{n+2s}}dy\\ &\leq\int_{B_{\rho/2}}C\rho^{s+\varepsilon+\alpha-2}|y|^{2-n-2s}dy\\ &\quad+\int_{B_1\backslash B_{\rho/2}}C|y|^{1+\alpha}(d^{s+\varepsilon-1}+l^{s+\varepsilon-1})(x_0+y)\frac{1}{|y|^{n+2s}}dy\\ &\quad+\int_{B_1^c}C|y|^{-n-s+\varepsilon}dy\\ &\leq C\rho^{\alpha+\varepsilon+s}+C(1+\rho^{\varepsilon+\alpha-s})\leq C\rho^{(\alpha+\varepsilon-s)\wedge 0}.\end{aligned}$$

The estimation of the first and the third integrand are computations, and on the middle one we apply [72, Lemma 2.5] twice. □

Now we turn to establishing Theorem 2.3.1. It extends [1, Corollary 2.3] in the sense that we allow wider choice of the power of the distance function. Basically we want to compute $L(d^p)$, for $p\in(0,2s)$. The approach is analogue to the one in [1]. We transform the integral into a suitable form, then we expend the kernel into terms of increasing homogeneities and then treat each of them separately. Due to the change of power some cancellations are not happening and we have to track the additional terms.

We start with the following result, which computes $L(d^p)$ in the case where the domain is a half-space (often referred as the flat case). We strongly use the fact that the distance function is one-dimensional in that case and apply the computation from one dimension.

Lemma 2.3.3. *Let $a\colon\mathbb{S}^{n-1}\to\mathbb{R}$ be an even function, such that $\int_{\mathbb{S}^{n-1}}|\theta_n|^{2s}a(\theta)d\theta<\infty$ and let $p\in(0,2s)$. Then*

$$p.v.\int_{\mathbb{R}^n}(x_n^p-(x_n+y_n)_+^p)\frac{a(\langle y\rangle)}{|y|^{n+2s}}dz=c_px_n^{p-2s}\int_{\mathbb{S}^{n-1}}|\theta_n|^{2s}a(\theta)d\theta,\quad \textit{when } x_n>0.$$

Moreover, $c_p=0$ if and only if $p=s$.

Proof. We use the knowledge that $\frac{1}{4}\int_{\mathbb{R}}(2\xi^p - (\xi+r)_+^p - (\xi-r)_+^p)\frac{1}{|r|^{1+2s}}dr = c_p\xi^{p-2s}$, for positive numbers ξ, where $c_p = 0$ if and only if $p = s$. This follows from the Liouville theorem in a half-space, see for example [1, Theorem 3.10]. Using the symmetry of the kernel, we rewrite the integral in the statement as

$$\frac{1}{2}\int_{\mathbb{R}^n}(2x_n^p - (x_n+y_n)_+^p - (x_n-y_n)_+^p)\frac{a(\langle y\rangle)}{|y|^{n+2s}}dz,$$

and then applying the polar coordinates and using evenness of the integrand, we get

$$\frac{1}{4}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}\left(2x_n^p - (x_n+r\theta_n)_+^p - (x_n-r\theta_n)_+^p\right)\frac{a(\theta)}{|r|^{n+2s}}r^{n-1}drd\theta =$$

$$= \int_{\mathbb{S}^{n-1}}|\theta_n|^p\frac{1}{4}\int_{\mathbb{R}}\left(2\left(\frac{x_n}{|\theta_n|}\right)^p - \left(\frac{x_n}{|\theta_n|}+r\right)_+^p - \left(\frac{x_n}{|\theta_n|}-r\right)_+^p\right)\frac{1}{|r|^{1+2s}}dr\cdot a(\theta)d\theta =$$

$$= \int_{\mathbb{S}^{n-1}}|\theta_n|^p c_p\left(\frac{x_n}{|\theta_n|}\right)^{p-2s}a(\theta)d\theta = c_p x_n^{p-2s}\int_{\mathbb{S}^{n-1}}|\theta_n|^{2s}a(\theta)d\theta.$$

□

We proceed with a result which connects the previous lemma with the terms which appear in computation further on.

Lemma 2.3.4. *Let $a\colon \mathbb{S}^{n-1}\to\mathbb{R}$ be an odd, bounded function, and let $p\in(0,2s)$. Then*

$$p.v.\int_{\mathbb{R}^n}(z_n)_+^{p-1}\frac{a(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz = \tilde{c}_p x_n^{p-2s}, \quad x\in\{x_n>0\}.$$

Moreover, $\tilde{c}_p = 0$ if and only if $p = s$.

Proof. Let us start with the left-hand side. By [1, Lemma 2.4], we have

$$p.v.\int_{\mathbb{R}^n}(z_n)_+^{p-1}\frac{a(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz = p.v.\int_{\mathbb{R}^n}\nabla(z_n)_+^p\cdot(z-x)\frac{a(\langle z-x\rangle)/\langle z-x\rangle_n}{|z-x|^{n+2s}}dz$$

$$= \tilde{L}((x_n)_+^p),$$

where the kernel of $\tilde{L}$ is $\frac{a(\langle z-x\rangle)/\langle z-x\rangle_n}{|z-x|^{n+2s}}$. Since a is bounded, $\frac{a(\theta)}{\theta_n}$ satisfies the assumption of Lemma 2.3.3, and so the above expression equals $\tilde{c}_p x_n^{p-2s}$, where $\tilde{c}_p$ depends only on p and a, and vanishes if and only if $p = s$. □

After splitting the kernel into homogeneities, we will have to deal with the terms of increasing power in the kernel. The next result shows how to deal with them. This is the main step towards Theorem 2.3.8. The extra parameter ζ is introduced because we want to apply the results on functions of the form $P(x-\zeta)d^p(x)$ for some polynomial P. It is going to be crucial to know the regularity in the additional parameter and hence need to track it carefully.

Lemma 2.3.5. *Let $q \in \{x \in \mathbb{R}^n : x_n > 0\}$ and $r > 0$ be such that $\overline{B_r(q)} \subset B_1$. Let $k \in \mathbb{N}$, non-integer $\beta > 1$, $p \in (0, 2s)$, $k > \beta - 2s + p$. Assume that for every ζ varying over some bounded C^β surface we have $\psi_\zeta \in C^{\beta-1}(\overline{B_1}) \cap C^k(\overline{B_r(q)})$, so that $\left|\left|D^j\psi_\zeta\right|\right|_{C^\beta_\zeta} \le C_0$, for $j \ge 0$. For $j \in \mathbb{N}$, let $a_j \in C^{j+k-1}(\mathbb{S}^{n-1})$ satisfy*

$$a_j(-\theta) = (-1)^{j+1}a_j(\theta), \quad \theta \in \mathbb{S}^{n-1}.$$

Then the function defined as

$$I_j(x) := \text{p. v.} \int_{B_1} (z_n)_+^{p-1} \frac{a_j(\langle z - x\rangle)}{|z - x|^{n+2s-j-1}} \psi_\zeta(z)dz$$

satisfies

$$I_j(x) = P_\zeta(x)x_n^{p-2s} + R_\zeta(x),$$

where P_ζ is a polynomial whose coefficients are bounded with $C||\psi_\zeta||_{C^{\beta-1}(B_1)}||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}$, and are C^β smooth as functions of ζ with bounded C^β_ζ norm, and R_ζ is of class C^{k+j-1} in $B_{r/2}(q)$ with

$$|D^{k+j-1}R_\zeta(x)| \le C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}\times$$

$$\times \left(\sum_{|\gamma|=\lfloor\beta\rfloor}^{k-1} ||\psi_\zeta||_{C^{|\gamma|}(B_r(q))} r^{p-2s-k+1+|\gamma|} + ||\psi_\zeta||_{C^k(B_r(q))} r^{1+p-2s} + ||\psi_\zeta||_{C^{\beta-1}(B_1)} r^{\beta-k+p-2s} \right).$$

The constant C depends only on n, s, β and p.

Proof. Fix some point x_0 in $B_{r/2}(q)$ and write

$$\begin{aligned}\psi_\zeta(z) &= \sum_{|\gamma|\le k-1} \partial^\gamma\psi_\zeta(x_0)(z - x_0)^\gamma + P_k(x_0, z)\\ &= \sum_{|\gamma|\le k-1} \partial^\gamma\psi_\zeta(x_0) \sum_{\alpha\le\gamma} \binom{\gamma}{\alpha} (z - x)^\alpha(x - x_0)^{\gamma-\alpha} + P_k\\ &= \sum_{|\alpha|\le k-1} \Psi^k_\alpha(x_0, x)(z - x)^\alpha + P_k(x_0, z), \qquad z \in B_r(q),\end{aligned}$$

where $\Psi^k_\alpha(x_0, x) = \sum_{\gamma\ge\alpha}^{|\gamma|\le k-1} \binom{\gamma}{\alpha}\partial^\gamma\psi_\zeta(x_0)(x - x_0)^{\gamma-\alpha}$ and $|P_k(x_0, z)| \le ||\psi_\zeta||_{C^k(B_r(q))}|z - x_0|^k$. Note that all the dependence in ζ comes through $\partial^\gamma\psi_\zeta$, which is by assumption of bounded C^β_ζ norm. Moreover, Ψ^k_α does not depend on z and hence it exits all the integrals over the variable z. Similarly, we expand ψ_ζ in $B_1\backslash B_r(q)$ using $C^{\beta-1}$ regularity

$$\begin{aligned}\psi_\zeta(z) &= \sum_{|\gamma|\le\beta-1} \partial^\gamma\psi_\zeta(x_0)(z - x_0)^\gamma + P_{\beta-1}(x_0, z)\\ &= \sum_{|\alpha|\le\beta-1} \Psi^{\beta-1}_\alpha(x_0, x)(z - x)^\alpha + P_{\beta-1}(x_0, z), \qquad z \in B_1\backslash B_r(q),\end{aligned}$$

where $\Psi_\alpha^{\beta-1}(x_0,x)=\sum_{\gamma\geq\alpha}^{|\gamma|\leq\beta-1}\binom{\gamma}{\alpha}\partial^\gamma\psi_\zeta(x_0)(x-x_0)^{\gamma-\alpha}$ and $|P_{\beta-1}(x_0,z)|\leq||\psi_\zeta||_{C^{\beta-1}(B_1)}|z-x_0|^{\beta-1}$. We plug these expansions into the definition of I_j so that

$$
\begin{aligned}
I_j(x) &= \int_{B_r(p)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}\psi_\zeta(z)dz+\int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}\psi_\zeta(z)dz\\
&= \sum_{|\alpha|\leq\beta-1}\Psi_\alpha^{\beta-1}(x_0,x)\int_{B_1}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz \qquad (2.3.1)\\
&\quad+\sum_{\lfloor\beta\rfloor\leq|\alpha|\leq k-1}\Psi_\alpha^{k}(x_0,x)\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz \qquad (2.3.2)\\
&\quad+\sum_{|\alpha|\leq\lfloor\beta\rfloor-1}\Big(\Psi_\alpha^{k}(x_0,x)-\Psi_\alpha^{\beta-1}(x_0,x)\Big)\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz\\
&\qquad (2.3.3)\\
&\quad+\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_k(x_0,z)dz\\
&\quad+\int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_{\beta-1}(x_0,z)dz. \qquad (2.3.4)
\end{aligned}
$$

Consider first the term (2.3.1). Note that $\Psi_\alpha^{\beta-1}(x_0,x)$ are polynomials in x, whose coefficients are derivatives of ψ_ζ. To analyse the integral part, choose a multi-index δ with $|\delta|=j$ and calculate

$$
\begin{aligned}
\partial_x^{\alpha+\delta}\int_{B_1}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz &= \int_{B_1}(z_n)_+^{p-1}\frac{\tilde a_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz\\
&= \tilde L((x_n)_+^p)(x)-\int_{\mathbb{R}^n\setminus B_1}(z_n)_+^{p-1}\frac{\tilde a_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz,
\end{aligned}
$$

where we used that $\tilde a_j$ represents a kernel of some operator $\tilde L$. But due to Lemma 2.3.4, $\tilde L((x_n)_+^p)(x)=c_p x_n^{p-2s}$ for $x_n>0$, where c_p depends only on p and a_j, and the remaining integrals is as smooth as $\tilde a_j$, which is $C^{j+k-1-|\alpha|-j}=C^{k-1-|\alpha|}$. Hence the original function equals $Q(x)x_n^{p-2s}+S(x)$ where Q is a polynomial and S is of class $C^{k-1+j}(B_{1/2})$. Because $\Psi_\alpha^{\beta-1}$ are polynomials in x with coefficients bounded with $||\psi_\zeta||_{C^{\beta-1}(B_1)}$ and C_ζ^β smooth, the term (2.3.1) contributes $P(x)x_n^{p-2s}+R_1(x)$, where P is as stated in the claim, and R_1 satisfies $|D^{j+k-1}R_1(x)|\leq C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}||\psi_\zeta||_{C^{\beta-1}(B_1)}$, for $x\in B_{1/2}\cap\{x_n>0\}$.

Proceed now to term (2.3.2). Let us rewrite it for convenience:

$$
\sum_{\lfloor\beta\rfloor\leq|\alpha|\leq k-1}\sum_{\gamma\geq\alpha}\binom{\gamma}{\alpha}\partial^\gamma\psi_\zeta(x_0)(x-x_0)^{\gamma-\alpha}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz.
$$

Choose such multi-indices α,γ, and another one with $|\delta|=j+k-1$. Then

$$
\partial_x^\delta\left((x-x_0)^{\gamma-\alpha}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz\right)=
$$

$$= \sum_{\eta \le \min(\gamma-\alpha,\delta)} \binom{\delta}{\eta} c_{\gamma-\alpha,\eta}(x-x_0)^{\gamma-\alpha-\eta}\partial^{\delta-\eta}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz.$$

Note that since $|\eta| \le |\gamma-\alpha| \le k-1-|\alpha|$ we have that $|\delta-\eta| \ge k-1+j-(k-1-|\alpha|) = j+|\alpha|$. Let us analyse the integral part. Choose any $\epsilon \le \delta-\eta$ of order $|\epsilon| = j+|\alpha|$, and compute first

$$\partial^{\epsilon}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz = \int_{B_r(q)}(z_n)^{p-1}\frac{\tilde{a}_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz \tag{2.3.5}$$

$$= \tilde{L}((x_n)_+^p) - \int_{\mathbb{R}^n\setminus B_1}(z_n)^{p-1}\frac{\tilde{a}_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz - \int_{B_1\setminus B_r(q)}(z_n)^{p-1}\frac{\tilde{a}_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz$$

$$= c_p x_n^{p-2s} + g(x) - \int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{\tilde{a}_j(\langle z-x\rangle)}{|z-x|^{n+2s-1}}dz,$$

where c_p is an explicit constant depending on a_j, obtained in Lemma 2.3.4, and g is or class $C^{k-1-|\alpha|}$. Differentiating the above equation $\delta-\eta-\epsilon$ more, we get

$$\partial^{\delta-\eta}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz =$$

$$= \tilde{c}_p x_n^{p-2s-|\delta-\eta-\epsilon|} + \partial^{\delta-\eta-\epsilon}g(x) + \int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{\hat{a}_j(\langle z-x\rangle)}{|z-x|^{n+2s-1+|\delta-\eta-\epsilon|}}dz.$$

Since $x \in B_{r/2}(q)$, the first term is bounded with

$$C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}r^{p-2s-|\delta-\eta-\epsilon|} = C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}r^{p-2s-k+1+|\eta|+|\alpha|},$$

and the second one just with $||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}$. For the third one we use Lemma 2.8.1 to bound it with $||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}r^{p-2s-|\delta-\eta-\epsilon|}$. To apply the lemma, we need $2s-1+|\delta-\eta-\epsilon| \ge 0$. The only time, this does not happen is, when $|\delta-\eta-\epsilon| = 0$, and $s < \frac{1}{2}$. But then, $\delta-\eta = \epsilon$, and the pole we get in the right-hand side in (2.3.5) is integrable, since $2s-1 < 0$. Therefore we can directly apply Lemma 2.8.1, to end up with the same estimate. Putting it all together, we got the following estimate for the norm of D^{k-1+j} of the term (2.3.2):

$$\sum_{\lfloor\beta\rfloor\le|\alpha|\le k-1}\sum_{\gamma\ge\alpha}\binom{\gamma}{\alpha}\partial^{\gamma}\psi_{\zeta}(x_0)\sum_{\eta\le\min(\gamma-\alpha,\delta)}\binom{\delta}{\eta}c_{\gamma-\alpha,\eta}(x-x_0)^{\gamma-\alpha-\eta}\times$$

$$||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}r^{p-2s-|\delta-\eta-\epsilon|}$$

$$\le C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}\sum_{\gamma,\alpha,\eta}\partial^{\gamma}\psi_{\zeta}(x_0)r^{|\gamma-\alpha-\eta|}r^{p-2s-k+1+|\eta|+|\alpha|}$$

$$\le C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}\sum_{\lfloor\beta\rfloor\le|\gamma|\le k-1}||\psi_{\zeta}||_{C^{|\gamma|}(B_r(q))}r^{p-2s-k+1+|\gamma|}.$$

With (2.3.3) we proceed similarly. Note first, that

$$\Psi_{\alpha}^{k}(x_0,x) - \Psi_{\alpha}^{\beta-1}(x_0,x) = \sum_{\lfloor\beta\rfloor\le|\gamma|\le k-1}^{\gamma\ge\alpha}\binom{\gamma}{\alpha}\partial^{\gamma}\psi_{\zeta}(x_0)(x-x_0)^{\gamma-\alpha},$$

and so

$$(2.3.3) = \sum_{|\alpha|\leq\lfloor\beta\rfloor-1}\ \sum_{\lfloor\beta\rfloor\leq|\gamma|\leq k-1}^{\gamma\geq\alpha} \binom{\gamma}{\alpha}\partial^\gamma\psi_\zeta(x_0)(x-x_0)^{\gamma-\alpha}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle}{|z-x|^{n+2s-j-1-|\alpha|}}dz.$$

When we differentiate the expression $j+k-1$ times (choose multi-index δ), perform the Leibnitz rule (with multi-index η), we end up with the same terms as when we treated (2.3.2), just that it also happens that $|\delta-\eta|<j+|\alpha|$. Then we just estimate the integral we get with Lemma 2.8.1, to obtain the bound with the desired power of r. Therefore we get the same estimate as before, just the range of the index α is different

$$|D^{j+k-1}(2.3.3)|\leq C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}\sum_{\lfloor\beta\rfloor\leq|\gamma|\leq k-1}||\psi_\zeta||_{C^{|\gamma|}(B_r(q))}r^{p-2s-k+1+|\gamma|}.$$

Finally, to estimate (2.3.4), use the argumentation as in [1, Lemma 2.9] to get

$$\begin{aligned}&\left|D^{k-1+j}|_{x_0}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_k(x_0,z)dz\right|\leq\\ \leq& C||a_j||_{C^{k+j-1}}||\psi_\zeta||_{C^k(B_r(q))}\int_{B_r(q)}(z_n)^{p-1}\frac{1}{|z-x|^{n+2s-2}}dz\\ \leq& C||a_j||_{C^{k+j-1}}||\psi_\zeta||_{C^k(B_r(q))}r^{1-2s+p}\end{aligned}$$

in view of Lemma 2.8.1. Similarly,

$$\begin{aligned}&\left|D^{k-1+j}|_{x_0}\int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_{\beta-1}(x_0,z)dz\right|\leq\\ \leq& C||a_j||_{C^{k+j-1}}||\psi_\zeta||_{C^{\beta-1}(B_1)}\int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{1}{|z-x|^{n+2s-1+k-\beta}}dz\\ \leq& C||a_j||_{C^{k+j-1}}||\psi_\zeta||_{C^{\beta-1}(B_1)}r^{\beta-k-2s+p},\end{aligned}$$

and the result follows. □

Next, we show how to connect the obtained estimates on the derivatives with the regularity up to the boundary. Of course we need to have suitable assumptions on the function ψ.

Corollary 2.3.6. *Let $p\in(0,2s)$ and $\beta>1+2s-p$. Suppose for every ζ varying over some bounded C^β surface, we have $\psi_\zeta\in C^{\beta-1}(B_1)\cap C^\infty(B_1\cap\{x_n>0\})$ satisfying*

$$|D^k\psi_\zeta|\leq C_k d^{\beta-1-k},\qquad k>\beta-1,$$

$$D^k\psi_\zeta \text{ is } C^\beta \text{ in variable } \zeta\quad k\in\mathbb{N}.$$

For some $j\in\mathbb{N}$, let $a_j\in C^{j+\lfloor\beta-2s+p\rfloor}(\mathbb{S}^{n-1})$ satisfy

$$a_j(\theta)=(-1)^{j+1}a_j(\theta),\quad \theta\in\mathbb{S}^{n-1}.$$

Then the function defined as

$$I_j(x) := p.v. \int_{B_1} (z_n)_+^{p-1} \frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}} \psi_\zeta(z) dz$$

satisfies

$$I_j(x) = P_\zeta(x) x_n^{p-2s} + R_\zeta(x),$$

where P_ζ is a polynomial whose coefficients are bounded with $C||\psi_\zeta||_{C^{\beta-1}(B_1)}||a_j||_{C^{j+\lfloor\beta-2s+p\rfloor}(\mathbb{S}^{n-1})}$ and C^β in variable ζ. Furthermore, the function R_ζ is of class $C^{j+\beta-1-2s+p}(B_{1/2}\cap\{x_n \geq 0\})$, with

$$||R_\zeta||_{C^{j+\beta-1-2s+p}} \leq C||a_j||_{C^{j+\beta}(\mathbb{S}^{n-1})}||\psi_\zeta||_{C^{\beta-1}(B_1)},$$

where the constant C depends only on n, s, β and p.

Proof. Choose $k = \lfloor \beta - 2s + p\rfloor + 1$. From the above lemma, whenever $x \in B_r(x_0)$ and $d(x_0) = 2r$, we get

$$|D^{k+j-1}R_\zeta(x)| \leq C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}\times$$

$$\times\left(\sum_{|\gamma|=\lfloor\beta\rfloor}^{k-1} ||\psi_\zeta||_{C^{|\gamma|}(B_r(q))} r^{p-2s-k+1+|\gamma|} + ||\psi_\zeta||_{C^k(B_r(q))} r^{1+p-2s} + ||\psi_\zeta||_{C^{\beta-1}(B_1)} r^{\beta-k+p-2s}\right),$$

which we furthermore estimate with

$$\leq C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})} r^{\beta-k+p-2s},$$

where we used the assumption on ψ_ζ. This gives, that every derivative of order $k+j-2$ is a $C^{\langle\beta-2s+p\rangle}$ function in $B_{1/2}\cap\{x_n \geq 0\}$. □

The following develops analogous result as Lemma 2.3.5, in the setting when the function ψ has a zero of order one at origin. Then, the regularity estimate improves roughly by one power as well. We establish it in the similar way as before, just the cases we treat change a little. Note also, that this step is not done completely correct in [1], and deserves more attention.

Lemma 2.3.7. *Let $q \in \{x \in \mathbb{R}^n : x_n > 0\}$ and $r > 0$ be such that $\overline{B_r(q)} \subset B_1$. Take $k \in \mathbb{N}$, non-integer $\beta > 1$, $p \in (0, 2s)$, $k > \beta + 1 - 2s + p$. Assume that for every ζ varying over some bounded C^β surface we have a function $\psi_\zeta \in C^{\beta-1}(\overline{B_1}, \mathbb{R}^n) \cap C^k(\overline{B_r(q)}, \mathbb{R}^n)$, satisfying $\left|\left|D^j\psi_\zeta\right|\right|_{C^\beta_\zeta} \leq C_0$, for $j \geq 0$. For $j \in \mathbb{N}$, let $a_j \in C^{j+k}(\mathbb{S}^{n-1})$ satisfy*

$$a_j(-\theta) = (-1)^{j+1}a_j(\theta), \quad \theta \in \mathbb{S}^{n-1}.$$

Then the function defined as

$$I_j(x) := \text{p. v.} \int_{B_1} (z_n)^{p-1} \frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}} z\cdot\psi_\zeta(z) dz$$

satisfies

$$I_j(x) = P_\zeta(x) x_n^{p-2s} + R_\zeta(x),$$

where P_ζ is a polynomial whose coefficients are bounded with $C||\psi_\zeta||_{C^{\beta-1}(B_1)}||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}$ and C^β functions of variable ζ with bounded C^β_ζ norm, and R_ζ is of class C^{k+j-1} in $B_{r/2}(q)$ with

$$|D^{k+j-1}R_\zeta(x)| \leq C||a_j||_{C^{k+j-1}(\mathbb{S}^{n-1})}|q|\times$$

$$\times\left(\sum_{|\gamma|=\lfloor\beta\rfloor}^{k-1}||\psi_\zeta||_{C^{|\gamma|}(B_r(q))}r^{p-2s-k+1+|\gamma|}+||\psi_\zeta||_{C^k(B_r(q))}r^{1+p-2s}+||\psi_\zeta||_{C^{\beta-1}(B_1)}r^{\beta-k+p-2s}\right).$$

Proof. The beginning is the same as in the lemma above, but now we deal with integrals of the form

$$\int_{B_1} z_i(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz, \tag{2.3.6}$$

$$(x-x_0)^{\gamma-\alpha}\int_{B_r(q)} z_i(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz, \tag{2.3.7}$$

$$\int_{B_r(q)} z_i(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_k(x_0,z)dz, \tag{2.3.8}$$

$$\int_{B_1\setminus B_r(q)} z_i(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)}{|z-x|^{n+2s-j-1}}P_{\beta-1}(x_0,z)dz. \tag{2.3.9}$$

First we split $z_i = x_i + (z_i - x_i)$, and treat both terms analogously as in the previous lemma. Let us start with (2.3.6):

$$(2.3.6) = x_i\int_{B_1}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz+\int_{B_1}(z_n)_+^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^{\alpha+e_i}}{|z-x|^{n+2s-j-1-|\alpha+e_i|}}dz.$$

Performing the same as in previous lemma, we see, that it equals $P_\zeta(x)x_n^{p-2s}+R_1(x)$, where P_ζ is a polynomial with the suitable bound and regularity in ζ, and R_1 is as smooth as the kernel a_j. Since the power of the pole in the second integral is for one bigger than usual we need to assume, that a_j is of one class smoother.

The term (2.3.7) splits into

$$(x-x_0)^{\gamma-\alpha}x_i\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^\alpha}{|z-x|^{n+2s-j-1-|\alpha|}}dz+$$

$$+(x-x_0)^{\gamma-\alpha}\int_{B_r(q)}(z_n)^{p-1}\frac{a_j(\langle z-x\rangle)\,\langle z-x\rangle^{\alpha+e_i}}{|z-x|^{n+2s-j-1-|\alpha+e_i|}}dz,$$

which we analyse just like (2.3.2) and (2.3.3). The first integrand brings $|x_i|$ more to the estimate, while the second brings r, so together the estimate improves for $C|q|$.

The reminder terms, (2.3.8) and (2.3.9), both still work. In analysis of the first one we get an integrable pole, so we use the other case of Lemma 2.8.1 which improves the estimate for $C|q|$, and for the other one we get a new restriction on k: $\beta+1-2s+p<k$, so that we can use Lemma 2.8.1, and we get one extra power of r, which is majorised with $C|q|$. □

In the following result we show how to connect the computation of $L(\psi d^p)$ with the previous results, to get the desired statement. As in the model case [1, Section 2], we split the kernel into homogeneities, we perform the flattening of the boundary, and carefully treat the obtained terms.

Theorem 2.3.8. *Let Ω be a domain in $\mathbb{R}^n$, such that $0 \in \partial\Omega$ and $\partial\Omega \cap B_1 \in C^\beta$, for some $\beta > 1$, and $p \in (0, 2s)$. Let an integer $k > \beta - 2s + p - 1$ and the kernel K be a C^{2k+1} function satisfying (2.1.1). Let for every $\zeta \in \partial\Omega \cap B_1$ the function $\psi_\zeta \in C^{\beta-1}(\overline{B}_1, \mathbb{R}^n) \cap C^\infty(\Omega \cap B_1, \mathbb{R}^n)$ satisfies*

$$\begin{aligned} \left|\left|D^j \psi_\zeta(x)\right|\right|_{C^\beta_\zeta(\partial\Omega\cap B_1)} &\leq C_0, \quad \forall x \in \Omega \cap B_1, j \geq 0, \text{ and} \\ |D^j \psi_\zeta(x)| &\leq C_j r^{\beta-1-j}, \qquad j > \beta - 1,\ x \in B_r(x_0), \end{aligned} \tag{2.3.10}$$

whenever $B_{2r}(x_0) \subset \Omega \cap B_1$. Then

$$\begin{aligned} L_\psi(d^p)(x) :=& p.v. \int_{B_1} d^{p-1}(y) K(y-x) \psi_\zeta(y) \cdot (y-x) dy \\ =& \varphi_\zeta(x) d^{p-2s}(x) + R_\zeta(x), \end{aligned}$$

where φ_ζ is composition of the boundary flattening map with a polynomial whose coefficients' C^β norm in variable ζ is bounded with a constant depending on C_0. We have $||\varphi_\zeta||_{C^\beta(B_1)} \leq C||\psi_\zeta||_{C^{\beta-1}(B_1)}$, and R_ζ satisfies the following estimate:

$$|D^k R_\zeta(x)| \leq C d^{\beta-1-2s+p-k}(x),$$

for $x \in B_{1/2}$.

Moreover, if $\psi_\zeta = f F_\zeta$, where f is a C^β function satisfying $|D^j f(x)| \leq C_j r^{\beta-j}$, $j > \beta$, $x \in B_r(x_0)$, $B_{2r}(x_0) \subset \Omega$, and vanishing at 0, and F_ζ a vector function satisfying (2.3.10), then R_ζ satisfies

$$|D^k R_\zeta(x)| \leq C|x| d^{\beta-1-2s+p-k}(x),$$

for $x \in B_{1/2}$. The constant C depends only on k, s, $||K||_{C^{2k+1}(\mathbb{S}^{n-1})}$, β and $||\partial\Omega||_{C^\beta}$.

Proof. Flattening the boundary (with ϕ) just like in [1, Section 2.2], and using the same notation as therein, we get

$$p.v. \int_{B_1} d^{p-1}(y) K(y-x) \psi_\zeta(y) \cdot (y-x) dy = p.v. \int_{B_1} (z_n)_+^{p-1} J(\phi(z) - \phi(\hat{x})) \cdot \rho_\zeta(z) dz =: I(\hat{x}).$$

Remember that $\rho_\zeta(z) = \psi_\zeta(\phi(z))|\det D\phi(z)|$. Condition (2.3.10) is closed under multiplication of functions and composites. Since all ϕ, ψ_ζ and $D\phi$ satisfy it, so does ρ_ζ. For simplicity we omit the hat script. We want to prove that $I(x) = P_\zeta(x) x_n^{p-2s} + \hat{R}_\zeta(x)$, where P_ζ is a polynomial and $\hat{R}_\zeta$ satisfies

$$|D^k R_\zeta(x)| \leq C d^{\beta-1-2s+p-k}(x).$$

We start with fixing a multi-index α of order $|\alpha| = k$, and a point $q \in \{x_n > 0\}$. We denote $r = d(q)/2$. We split $I(x) = I_1(x) + I_r(x, x)$, and proceed as in [1, proof of

Theorem 2.2] to write

$$\partial_x^\alpha I_r(x,x) = \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\sum_{i=0}^{|\gamma|}\partial_x^\gamma\partial_\xi^{\alpha-\gamma}\int_{B_r(q)}(z_n)_+^{p-1}\frac{b_i(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-i-1}}\cdot\rho_\zeta(z)dz$$
$$+\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\partial_x^\gamma\partial_\xi^{\alpha-\gamma}\int_{B_r(q)}(z_n)_+^{p-1}\frac{R_{|\gamma|+1}(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-1}}\cdot\rho_\zeta(z)dz.$$

To do expansions as in [1], we need that the kernel is $|\alpha|+|\gamma|+1$ times differentiable, so since $\gamma\le\alpha$, we need $2k+1$ regularity of the kernel.

We estimate the error term in the same way as in [1], but we use $|\rho_\zeta(z)|\le C$:

$$\left|\partial_x^\gamma\partial_\xi^{\alpha-\gamma}\int_{B_r(q)}(z_n)_+^{p-1}\frac{R_{|\gamma|+1}(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-1}}\cdot\rho_\zeta(z)dz\right|\le$$
$$\le Cr^{\beta-|\alpha|-2}\int_{B_r(q)}(z_n)^{p-1}|z-x|^{-n-2s+2}dz\le Cr^{\beta-2s+p-1-k}, \tag{2.3.11}$$

in view of Lemma 2.8.1.

When $|\gamma|>i+\beta-1-2s+p$, we write

$$\partial_x^\gamma\int_{B_r(q)}(z_n)_+^{p-1}\frac{\partial_\xi^{\alpha-\gamma}b_i(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-i-1}}\cdot\rho_\zeta(z)dz=$$
$$=-\partial_x^\gamma\int_{B_1\setminus B_r(q)}(z_n)_+^{p-1}\frac{\partial_\xi^{\alpha-\gamma}b_i(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-i-1}}\cdot\rho_\zeta(z)dz+\Gamma(x)+\partial^\gamma(P_1(x)x_n^{p-2s}),$$

where we denoted

$$\Gamma(x):=\partial^\gamma\int_{B_1}(z_n)_+^{p-1}\frac{\partial_\xi^{\alpha-\gamma}b_i(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-i-1}}\cdot\rho(z)dz-\partial^\gamma(P_1(x)x_n^{p-2s}),$$

for suitable polynomial P_1 obtained in Lemma 2.3.5. Note that the lemma gives that the coefficients of obtained polynomial are C^β in ζ with bounded C^β_ζ norm. Due to condition on $|\gamma|$ above, we can apply Lemma 2.3.5 (we use $k'=|\gamma|-i+1$ in the assumption ot the lemma) to every summand of Γ, to say

$$|\Gamma(x)|\le C||\partial_\xi^{\alpha-\gamma}b_i(\xi,\cdot)||_{C^{|\gamma|}(\mathbb{S}^{n-1})}r^{\beta-|\gamma|+i-1-2s+p},$$

where we already used the estimates on ρ_ζ. Using this, together with the estimates on $||\partial_\xi^{\alpha-\gamma}b_i(\xi,\cdot)||_{C^{|\gamma|}(\mathbb{S}^{n-1})}$ in [1, (26)], we end up with

$$|\Gamma(x)|\le Cr^{\beta-|\gamma|+i-1-2s+p}\left\{\begin{array}{ll}1 & \text{if } |\alpha|-|\gamma|+i+1<\beta,\\ r^{\beta-i-1-|\alpha|+|\gamma|} & \text{otherwise,}\end{array}\right\}\le Cr^{\beta-1-2s+p-k}, \tag{2.3.12}$$

since $|\gamma|\le k$ and $i\ge 0$, and $\beta>1$. We postpone the analysis of the "annular" integral.

When $|\gamma|<i+\beta-1-2s+p$ write

$$\int_{B_r(q)}(z_n)_+^{p-1}\frac{b_i(\xi,\langle z-x\rangle)}{|z-x|^{n+2s-i-1}}\cdot\rho_\zeta(z)dz=$$

$$= \int_{B_1} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz - P_2(x) x_n^{p-2s} +$$

$$+ P_2(x) x_n^{p-2s} - \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz,$$

for some suitable polynomial P_2 obtained in Lemma 2.3.5. Notice, that the first term (the integral minus the polynomial) is smoother in $B_{1/2}$ than the derivation order, due to Corollary 2.3.6, so it brings a bounded term.

Let us now deal with the "annular" integrals. In the case $i > \beta - |\alpha| + |\gamma| - 1$ we do the same as in [1], to get

$$\begin{aligned} &\left| \partial_x^\gamma \partial_\xi^{\alpha-\gamma} \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz \right| \leq \\ &\leq C r^{\beta - i - 1 - |\alpha| + |\gamma|} \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} |z-x|^{-n-2s+i+1-|\gamma|} |\rho_\zeta(z)| dz \\ &\leq C r^{\beta - 2s + p - 1 - k}, \end{aligned}$$

in view of Lemma 2.8.1 (note that $i \leq |\gamma|$).

So we are left with

$$\partial_x^\alpha|_p I_r(x,x) - \partial^\alpha (P_1(x) + P_2(x)) x_n^{p-2s} =$$

$$= \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \sum_{i=0}^{\lfloor \beta \rfloor - |\alpha| + |\gamma| - 1} \partial_x^\gamma \partial_\xi^{\alpha-\gamma} \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz + \Theta_\alpha(r,q),$$

where $|\Theta_\alpha(r,q)| \leq C r^{\beta - 1 - 2s + p - k}$.

Finally, the same steps as in [1, Estimate (27)] give

$$\left| \partial^\alpha I_1(q) - \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \sum_{i=0}^{\lfloor \beta \rfloor - |\alpha| + |\gamma| - 1} \partial_x^\gamma|_q \partial_\xi^{\alpha-\gamma}|_q \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz \right| \leq$$

$$\leq C r^{\beta - |\alpha|} \int_{B_1 \setminus B_r(q)} (z_n)_+^{p-1} |z-q|^{-n-2s} |\rho_\zeta(z)| dz \leq C r^{\beta - 1 - 2s + p - k},$$

where we estimated $|\rho_\zeta(z)| \leq C$ and applied Lemma 2.8.1.

Therefore we got that $L_\psi(\phi(x)) = P(x) x_n^{p-2s} + \hat{R}(x)$, with $\hat{R}$ satisfying $|D^k \hat{R}| \leq C d^{\beta - 1 - 2s + p - k}$ for $k > \beta - 1 - 2s + p$. But since ϕ^{-1} is a diffeomorphism satisfying $|D^j(\phi^{-1})| \leq C d^{\beta - j}$, when $j > \beta$, we have that

$$L_\psi(x) = P(\phi^{-1}(x)) d^{p-2s}(x) + \hat{R}(\phi^{-1}(x)) = \varphi_\zeta(x) d^{p-2s}(x) + R_\zeta(x),$$

where $|D^k R_\zeta| \leq C d^{\beta - 1 - 2s + p - k}$, which we get by explicit calculation of the composite:

$$\partial^\alpha (R \circ \phi) = \sum_{j=1}^{|\alpha|} c_j D^j R(\partial^{\alpha_1} \phi, \ldots, \partial^{\alpha_j} \phi), \qquad \text{where } |\alpha_i| \geq 1, \text{ and } \alpha_1 + \ldots + \alpha_j = \alpha.$$

The expression $D^j R(v_1, \ldots, v_j)$ means evaluation of j-linear form on j vectors.

Let us now turn to the "moreover" case; when $\psi = fF_\zeta$ with $f \in C^\beta(B_1)$ vanishing at 0. Then ρ_ζ is also of such form, and since ϕ also satisfies regularity condition (2.3.10), we have that $\rho_\zeta(z) = g(z)G_\zeta(z)$, where g vanishes at 0, and g and G_ζ satisfy the same conditions on growth of the derivatives as f and F_ζ. Next, we apply Lemma 2.8.4 to g, to write $g(z) = z \cdot h(z)$ for a $C^{\beta-1}(B_1, \mathbb{R}^n)$ vector function h, which by regularity condition of g satisfies (2.3.10). We proceed with the estimation. Note that now, we have additional z in the function ρ_ζ, which improves all estimates with $|q|$. We do the same steps as above with some minor differences.

In the error term, we change the estimate $|\rho_\zeta(z)| \leq C$ to $|\rho_\zeta(z)| \leq C|z|$, which leads to the estimate

$$Cr^{\beta-|\alpha|-2} \int_{B_r(q)} (z_n)^{p-1}|z-x|^{-n-2s+2}|z|dz \leq C|q|r^{\beta-1-2s+p-k},$$

in view of Lemma 2.8.1.

Then we want to use Lemma 2.3.7 instead of Lemma 2.3.5, which leads to splitting cases on $|\gamma| > i + \beta - 2s + p$ and $|\gamma| < i + \beta - 1 - 2s + p$. In the estimate of $|\Gamma|$, we get additional $|q|$ from application of Lemma 2.3.7. Note that the case $|\gamma| = i + \lfloor \beta - 2s + p \rfloor$ is problematic (note that this is "never" also $i = |\gamma|$, since $\beta > 1 + 2s - p$ in practice). In this case, we act as usual. First we split the integral on the ball B_1 minus on the "annular" region (we deal with this one later). Now on B_1, we use the fact, that $\rho_\zeta(z) = z \cdot h(z)G_\zeta(z)$, write it in components, to end up with the integrals of the form

$$\int_{B_1} (z_n)_+^{p-1} \frac{\tilde{b}_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \tilde{\rho}_\zeta(z) z_l dz,$$

where $\tilde{\rho}_\zeta$ still satisfies condition (2.3.10). We proceed with splitting $z_l = (z_l - x_l) + x_l$, and treat the two cases separately. The one with $z_l - x_l$ decreases the power of the pole, which is the same as increasing i by one, which we already treated. We differentiate the term with x_l. When we derive x_l, the derivative on the integral term decreases by one, and the integral is smoother than the derivation, so it gives a bounded term. The last term which remains is

$$x_l \partial^\gamma \int_{B_1} (z_n)_+^{p-1} \frac{\tilde{b}_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \tilde{\rho}_\zeta(z) dz, \tag{2.3.13}$$

which we treat just like the one that lead to (2.3.12). Note that this time we have additional x_l in the front, which improves the estimate with $|q|$.

The "annular" integrals are treated similarly. We split the cases in the same way as before. When $i > \beta - |\alpha| + |\gamma| - 1$ we now estimate $|\rho_\zeta(z)| \leq C|z|$ and apply Lemma 2.8.1 to get the improved estimate with one power of r. Notice that we can not use that lemma only in the case $i = |\gamma|$. Then we have to return to the integral on the region $B_r(q)$. We expand

$$b_i(\xi, \langle z-x \rangle) \cdot \rho_\zeta(z) = \sum_j \sum_l b_i^j(\xi, \langle z-x \rangle) G^j(z) h^l(z) z_l =$$

$$= \sum_j \sum_l b_i^j(\xi, \langle z-x \rangle) G^j(z) h^l(z)(x_l + (z_l - x_l)),$$

and plug it into the integral

$$\int_{B_r(q)} (z_n)_+^{p-1} \frac{b_i(\xi, \langle z-x \rangle)}{|z-x|^{n+2s-i-1}} \cdot \rho_\zeta(z) dz.$$

We treat the cases as follows: when the pole, obtained after the estimate [1, (26)], is integrable we directly apply Lemma 2.8.1 to the integral in the region $B_r(q)$ and obtain additional power or r. When the pole is not integrable, we first make the transformation to $B_1 \backslash B_r(q)$ and then apply Lemma 2.8.1. We get additional x_l in front, which brings $|q|$ to the result. □

With one last step we obtain a more concrete statement. Notice, that it is stated in a broader setting than Theorem 2.3.8, due to its applications in other sections.

Corollary 2.3.9. *Let $p \in (0, 2s)$ and $\beta > 1 + 2s - p$. Let $\partial\Omega \cap B_1$ be C^β, and let K be a $C^{2k+1}(\mathbb{S}^{n-1})$ kernel satisfying (2.1.1), for some integer $k > \beta - 1 - 2s + p$. Let for every $\zeta \in \partial\Omega$ the function $\eta_\zeta \in C^\beta(\overline{\Omega} \cap B_1) \cap C^\infty(\Omega \cap B_1)$ satisfy*

$$|D^j \eta_\zeta(x)| \le C_j d^{\beta - j}, \qquad j > \beta \quad \text{and}$$

$$\left|\left|D^j \eta_\zeta\right|\right|_{C^\beta_\zeta} \le C_0, \quad j \ge 0.$$

Then, $L(\eta_\zeta d^p) = \varphi_\zeta d^{p-2s} + R_\zeta$, where φ_ζ is composition of the boundary flattening map with a polynomial whose coefficients' C^β_ζ norm is bounded with a constant depending on C_0. We have $\varphi_\zeta \in C^\beta(\overline{\Omega} \cap B_{1/2})$ with $||\varphi_\zeta||_{C^\beta(\overline{\Omega} \cap B_{1/2})} \le ||\eta_\zeta||_{C^\beta(B_1)}$, and $R_\zeta \in C^{\beta-1-2s+p}(\overline{\Omega} \cap B_{1/2})$, with

$$|D^k R_\zeta(x)| \le C d^{\beta - 1 - 2s + p - k}(x), \qquad \text{in } \overline{\Omega} \cap B_{1/2}.$$

Moreover, if at some boundary point z we have $\eta_\zeta(z) = 0$, the estimate on R_ζ improves to

$$|D^k R_\zeta(x)| \le C|x - z| d^{\beta - 1 - 2s + p - k}(x), \qquad \text{in } \overline{\Omega} \cap B_{1/2}.$$

The constant C depends only on k, s, $||K||_{C^{2k+1}(\mathbb{S}^{n-1})}$, β and $||\partial\Omega||_{C^\beta}$.

Proof. We rewrite $L(\eta_\zeta d^p)$ in the same way as in [1, Proof of Corollary 2.3]:

$$\begin{aligned} L(\eta_\zeta d^p)(x) = &-\frac{1}{2s} p.v. \int_{\mathbb{R}^n} \eta_\zeta(y) \nabla(d^p)(y) \cdot (y-x) K(y-x) dy \\ &- \frac{1}{2s} p.v. \int_{\mathbb{R}^n} d^p(y) \nabla \eta_\zeta(y) \cdot (y-x) K(y-x) dy \\ =& p.v. \int_{\mathbb{R}^n} d^{p-1}(y) K(y-x) \psi_\zeta(y) \cdot (y-x) dy, \end{aligned}$$

where we denoted

$$\psi_\zeta = -\frac{p}{2s} \eta_\zeta \nabla d - \frac{1}{2s} d \nabla \eta_\zeta.$$

By assumption on η_ζ, the vector function $\psi_\zeta \in C^{\beta-1}(\overline{\Omega} \cap B_1)$, smooth in the interior, and satisfies $|D^j \psi(x)| \le C_j d^{\beta-1}(x)$, for $j > \beta - 1$, as well as β regularity of all the derivatives with respect to ζ.

Therefore splitting the integral above in the regions B_1 and $\mathbb{R}^n \backslash B_1$, we deduce the result from the above theorem.

Note that the moreover case in the theorem gives the moreover part of the corollary, since both d and η_ζ are C^β and vanish at z. □

With not much work, we now prove Theorem 2.3.1.

Proof of Theorem 2.3.1. It is a special case of the above corollary. □

To complete the result, we show how to generalise the statement to the powers of distance greater than $2s$. Thanks to the clear representation of the function φ and the improvement of the estimate in the "moreover" cases above, we are able to deduce the corollary below. In short words, we treat the surplus $d^{\lfloor p \rfloor}$ as a part of the function η. That is the reason, why we allow the function η to be less smooth than C^∞. Note that the condition $\langle p \rangle < 2s$ does not play any role in the set-up $s > 1/2$.

Corollary 2.3.10. *Let $\partial\Omega \cap B_1$ be C^β, $\zeta \in \partial\Omega \cap B_1$, η be a polynomial, and $p \in \mathbb{R}$ with $p > 2s$, $p - 2s \notin \mathbb{N}$, $p \leq \lfloor \beta \rfloor + 2s$. Additionally, if $s \leq 1/2$, then we need $p \notin \mathbb{N}$ and $\langle p \rangle < 2s$. Assume $K \in C^{2\beta+1}(\mathbb{S}^{n-1})$ is the kernel of L satisfying* (2.1.1)*.*

Then

$$L(\eta(\cdot - \zeta)d^p)(x) = \varphi_\zeta(x)d^{p-2s}(x) + R_\zeta(x),$$

where $\varphi_\zeta \in C^\beta(\overline{\Omega})$ whose all derivatives in variable ζ are of bounded C^β_ζ norm. Furthermore, we have $||\varphi_\zeta||_{C^\beta(\overline{\Omega}\cap B_{1/2})} \leq C||\eta(\cdot - \zeta)||_{C^\beta(\overline{\Omega})}$, and $R_\zeta \in C^{\beta-1+\langle p-2s\rangle}(\overline{\Omega} \cap B_{1/2})$.

Proof. We apply the above corollary on $\tilde{\eta}_\zeta = \eta(\cdot - \zeta)d^{\lfloor p \rfloor}$. We get $L(\eta(\cdot - \zeta)d^p) = L(\tilde{\eta}_\zeta d^{\langle p \rangle}) = \tilde{\varphi}_\zeta d^{\langle p \rangle - 2s} + R_\zeta$. Note that $\lfloor p \rfloor \geq 1$, and so $\tilde{\eta}_\zeta$ vanishes at every boundary point, so we can use the moreover case of the above corollary everywhere. Let us also stress that $\tilde{\eta}_\zeta$ satisfies the condition (2.3.10), since η is a polynomial, ζ varies over the C^β boundary, and d satisfies the same condition. Hence $\tilde{\varphi}_\zeta$ is a composition of a polynomial whose coefficients are C^β_ζ, with a boundary flattening map. We now argue that $R_\zeta \in C^{\beta-2s+\langle p \rangle}$. We choose $|x_0 - z| = 2r$, $x, y \in B_r(x_0)$ and $|\gamma| = \lfloor \beta - 2s + \langle p \rangle \rfloor$. Compute

$$\begin{aligned}
|\partial^\gamma R_\zeta(x) - \partial^\gamma R_\zeta(y)| &\leq ||D^{|\gamma|+1}R_\zeta||_{L^\infty(B_r(x_0))}|x - y| \\
&\leq C|x - z|r^{\beta-2s+\langle p \rangle-1-|\gamma|-1}r^{1-\langle \beta-2s+\langle p \rangle\rangle}|x - y|^{\langle \beta-2s+\langle p \rangle\rangle} \\
&\leq C|x - y|^{\langle \beta-2s+\langle p \rangle\rangle},
\end{aligned}$$

where we used the moreover case of Corollary 2.3.9.

We can write both regularities as $R_\zeta \in C^{\beta-1+\langle p-2s\rangle}(\overline{\Omega} \cap B_{1/2})$.

To extract the correct power of d on the right-side, proceed as follows. For simplicity we work with 0 as the boundary point. We use [1, Lemma 3.5] to do blow-ups on every compact ball inside Ω. We start with defining $u_r(x) = \frac{1}{r^{\langle p \rangle}}\eta(rx - \zeta)d^p(rx)$, for $r < 1$. We have that u_r converge to 0 in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. We can also estimate $|\eta(rx - \zeta)d^p(rx)| \leq C|x|^p$, which implies the convergence of $\int_{\mathbb{R}^n} \frac{|u_r(x)|}{1+|x|^{n+2s+k}}dx$ to 0, if we choose $k = \lceil p - 2s \rceil$. On every compact ball B inside Ω we have that

$$L(u_r)(x) - P_r(x) = \tilde{\varphi}_\zeta(rx)\frac{1}{r^{\langle p \rangle-2s}}d^{\langle p \rangle-2s}(rx) + \frac{1}{r^{\langle p \rangle-2s}}(R(rx) - \tilde{P}_r) \longrightarrow \tilde{\varphi}_\zeta(0)(x \cdot \nu)_+^{\langle p \rangle-2s}$$

in $L^\infty(B)$, for $\tilde{P}_r$ being the suitable Taylor polynomials of R, of order $k-1$, and ν being the unit normal of Ω at zero. Therefore, the lemma gives us that

$$0 = L(0) \overset{k}{=} \tilde{\varphi}_\zeta(0)(x\cdot\nu)_+^{\langle p\rangle - 2s} \qquad \text{in } B,$$

which implies that $\tilde{\varphi}_\zeta(0) = 0$. Now we change the order of blow-up to $1+\langle p\rangle$ to conclude also that $\nabla\tilde{\varphi}_\zeta(0) = 0$. We can go on with the procedure as long as the order of blow-up is strictly lower than p, so that we have the local uniform convergence of u_r to zero. In the last step the order of blow up has to be taken $p-\varepsilon$, so that $p-\varepsilon < p$, and $p-\varepsilon > \lfloor p\rfloor$, to get that the Taylor polynomial of $\tilde{\varphi}_\zeta$ of order $\lfloor p\rfloor$ vanishes. We also need $p-2s-\varepsilon < \beta - 1 + \langle p-2s\rangle$ (blow-up order has to be smaller than the regularity of R_ζ), so that the blow up of R_ζ vanishes, after subtracting suitable Taylor polynomials. Hence, when $p\notin\mathbb{N}$, we conclude that $\tilde{\varphi}_\zeta$ has zero of order $\lfloor p\rfloor$ at every boundary point. This means that before the pre-composition with the boundary-flattening map, $\tilde{\varphi}_\zeta$ was divisible by $x_n^{\lfloor p\rfloor}$, and all the coefficients were bounded with $||\eta(\cdot-\zeta)||_{C^\beta}$, and C^β in variable ζ. Therefore, if we set $\varphi_\zeta = \frac{\tilde{\varphi}_\zeta}{d^{\lfloor p\rfloor}}$ it is still a $C^\beta(\overline{\Omega}\cap B_{1/2})$ function which satisfies $||\varphi_\zeta||_{C^\beta} \le C||\eta(\cdot-\zeta)||_{C^\beta}$.

Notice also, that in the case $p\in\mathbb{N}$ and $s > 1/2$, we can do the above procedure with taking one power of d less into η. So then we work with $L((\eta(\cdot-\zeta)d^{p-1})d^1)$, and we obtain the same result. □

Remark 2.3.11. Note that in the case when $p\in\mathbb{N}+s$, the polynomial part falls off, so we have that $L(\eta d^p)\in C^{\beta-s}(\overline{\Omega})$, together with the suitable estimate.

It seems that the regularity of the reminder could be improved to $\beta-1+p-2s$. To establish it, we would need to improve the estimates in Lemma 2.3.5, assuming ψ has a zero of some higher order at 0. Since the proof is already very cumbersome, and what we have proven is enough for our purposes, we do not dig into the improvement.

2.4 Equations for non-local operators

2.4.1 Existence and computation

In this subsection we aim to answer the question of solvability of $L(Pd^p) = qd^{p-2s}$, where q is a known polynomial. Briefly, the answer consists of two steps. First we perform a blow-up argument using [1, Lemma 3.5], which reduces the problem to the flat case. Then we investigate the flat case, which is explicitly computable.

We start with computation in the flat case. We need a preliminary result, which says when can we evaluate L on a function with polynomial growth in the generalised sense, recall Definition 2.2.1.

Lemma 2.4.1. *Let L be an operator whose kernel K satisfy* (2.1.1) *and is $C^{k+1}(\mathbb{S}^{n-1})$. Let for some $\varepsilon > 0$ a function $u\colon \mathbb{R}^n\to\mathbb{R}$ be $C^{2s+\varepsilon}(B_2)$ with*

$$\left|\left|\frac{u}{1+|x|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} < \infty,$$

for some $\alpha < s$. Then there exists a function $f\colon B_1 \to \mathbb{R}$ and polynomials $P_R \in \boldsymbol{P}_{k-1}$, so that $L(u\chi_R) - P_R \to f$ in $L^\infty(B_1)$, with

$$||f||_{L^\infty(B_1)} \leq C\left(\left|\left|\frac{u}{1+|x|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} + ||u||_{C^{2s+\varepsilon}(B_2)}\right).$$

Proof. Denote $\tilde{u} = u\chi_2$. Then we can compute $L\tilde{u} = \tilde{f}$, and we have $||\tilde{f}||_{L^\infty(B_1)} \leq C(||u||_{C^{2s+\varepsilon}(B_2)})$. For $R > 2$ we have $L(u\chi_R) = L(u\chi_2) + L(u\chi_{B_R\setminus B_2}) = \tilde{f} + g_R$. Choose now $\gamma \in \mathbb{N}_0^n$ with $|\gamma| = k$. Then

$$\begin{aligned}|\partial^\gamma g_R(x)| &= |\partial^\gamma \int_{B_R\setminus B_2} u(z)K(z-x)dz| = |\int_{B_R\setminus B_2} u(z)\partial^\gamma K(z-x)dz| \\ &\leq C\int_{\mathbb{R}^n\setminus B_2} \frac{u(z)}{|z-x|^{n+k+2s}}dz \leq C\left|\left|\frac{u}{1+|x|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} < \infty.\end{aligned}$$

Also, the same estimate with the dominated convergence theorem gives that $\partial^\gamma g_R(x) \to f_\gamma$ as $R \to \infty$, where we denoted $f_\gamma(x) = \int_{\mathbb{R}^n\setminus B_2} u(z)\partial^\gamma K(z-x)dz$. (This convergence is uniform for $x \in B_1$, which we can check with an ε, R_0 calculus, and some estimates of the kernel.) With another dominated convergence argument we see, that whenever $\gamma + e_i = \gamma' + e_j$ then $\partial_i f_\gamma = \partial_j f_{\gamma'}$. Since B_1 is simply connected, we can integrate k times to get a function f_0 such that $\partial^\gamma f_0 = f_\gamma$. Denote with $T^j\phi$ the j-th Taylor polynomial of ϕ centred at 0. Then

$$||g_R + \tilde{f} - T^{k-1}g_R - f_0 + T^{k-1}f_0 - \tilde{f}||_{L^\infty(B_1)} \leq ||D^k(\ldots)||_{L^\infty(B_1)} \to 0,$$

and so f can be taken as $f_0+\tilde{f}$. Since $||f_0||_{L^\infty(B_1)} \leq C\left|\left|\frac{u}{1+|x|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)}$ by construction, we have

$$||f||_{L^\infty(B_1)} \leq C\left(\left|\left|\frac{u}{1+|x|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} + ||u||_{C^{2s+\varepsilon}(B_2)}\right)$$

as wanted. □

Remark 2.4.2. Note that if u is smooth as the growth (in B_2), the polynomials P_R can be taken as the Taylor polynomial of $Lu\chi_R$ of the correct order centred at 0. If we take a smooth cut-off χ_2 above, then $\tilde{f}$ becomes smooth enough. Note also, that if additionally for some positive ε we have $u \in C^{k+1+2s+\varepsilon}(B_2)$, the convergence is actually happening in $C^{k+1}(B_1)$.

Next, we compute $L(Pd^p)$ in the flat case, for any polynomial P. All the equations should be understood in the sense of Definition 2.2.2, when the function inside the operator grows too much at infinity. Let $\Omega = \{x_n > 0\}$. Then $d(x) = \max\{x_n, 0\} = (x_n)_+$.

Because L is linear, it is enough to compute $L(M(x_n)_+^p)$ for every monomial M.

Proposition 2.4.3. *Suppose $p - 2s \notin \mathbb{N}$. Let $\gamma \in \mathbb{N}^{n-1} \times 0$ and $k > |\gamma| + p - 2s$. If L is an operator whose kernel is $C^k(\mathbb{S}^{n-1})$ and satisfies* (2.1.1)*, we have that*

$$L\left(x^\gamma (x_n)_+^p\right) \overset{k}{=} \sum_{\alpha\leq\gamma} c_{\gamma,\alpha,s,p}x^\alpha x_n^{p-2s+|\gamma-\alpha|} \quad \text{in } \{x_n > 0\}.$$

Furthermore, if and only if $p - s \notin \mathbb{N}$, then the coefficient $c_{\gamma,\gamma,s,p} \neq 0$.

Proof. We prove it with induction on $|\gamma|$. When $|\gamma| = 0$, we have that $L((x_n)_+^p)$ is a $p-2s$-homogeneous function dependent only on x_n. Hence it equals $c_{s,p}x_n^{p-2s}$. Additionally, when $p - s \notin \mathbb{N}$, then $c_{s,p} \neq 0$, which we conclude from the Liouville theorem [1, Theorem 3.10].

Take now $|\gamma| > 0$. Let $\gamma_i > 0$. Differentiating on x_i and using the induction we get

$$\partial_i L\left(x^\gamma (x_n)_+^p\right) = \gamma_i L\left(x^{\gamma - e_i}(x_n)_+^p\right) = \gamma_i \sum_{\alpha \le \gamma - e_i} c_{\gamma - e_i, \alpha, s, p} x^\alpha x_n^{p-2s+|\gamma - e_i - \alpha|}.$$

Integrating back we get

$$\begin{aligned} L\left(x^\gamma (x_n)_+^p\right) &= \sum_{\alpha \le \gamma - e_i} \frac{\gamma_i c_{\gamma - e_i, \alpha, s, p}}{\alpha_i + 1} x^{\alpha + e_i} x_n^{p-2s+|\gamma - e_i - \alpha|} + f(x_1, \dots, \hat{x_i}, \dots, x_n) \\ &= \sum_{e_i \le \alpha \le \gamma} \frac{\gamma_i c_{\gamma - e_i, \alpha - e_i, s, p}}{\alpha_i} x^\alpha x_n^{p-2s+|\gamma - \alpha|} + f(x_1, \dots, \hat{x_i}, \dots, x_n), \end{aligned}$$

for some function f independent of x_i. Differentiating on other x_j, for $j < n$ and comparing the terms, we get

$$L\left(x^\gamma (x_n)_+^p\right) = \sum_{0 < \alpha \le \gamma} c_{\gamma, \alpha, s, p} x^\alpha x_n^{p-2s+|\gamma - \alpha|} + f(x_n) =: h(x) + f(x_n).$$

This means that for some polynomials P_R of degree $\lfloor |\gamma| + p - 2s \rfloor$, we have

$$L(x^\gamma (x_n)_+^p \chi_{B_R})(x) - P_R(x) \longrightarrow h(x) + f(x_n).$$

If $|\gamma| + p - 2s$ is a natural number, the polynomials are of one degree bigger. Let us now use the homogeneity of $L(x^\gamma (x_n)_+^p \chi_{B_R})(x)$ in (x, R) and of h. Concretely, evaluate the above formula at λR and λx. We get

$$f(\lambda x_n) + h(\lambda x) = \lim_{R \to \infty} L(x^\gamma (x_n)_+^p \chi_{B_{\lambda R}})(\lambda x) - P_{\lambda R}(\lambda x)$$

$$= \lambda^{|\gamma| + p - 2s} \lim_{R \to \infty} L(x^\gamma (x_n)_+^p \chi_{B_R})(x) - P_R(x) + \lim_{R \to \infty} \lambda^{|\gamma| + p - 2s} P_R(x) - P_{\lambda R}(\lambda x)$$

$$= \lambda^{|\gamma| + p - 2s}(f(x_n) + h(x)) + P_\lambda(x),$$

for some polynomial P_λ of the same degree. But since h is homogeneous, this implies

$$f(\lambda x_n) = \lambda^{|\gamma| + p - 2s} f(x_n) + P_\lambda(x),$$

and hence P_λ depends only on x_n. Differentiating it $\lceil |\gamma| + p - 2s \rceil$, we conclude that f equals a sum of a polynomial and a $(|\gamma| + p - 2s)$-homogeneous function. This proves the claim, because the equalities in the generalised sense hold up to polynomials of this order. Here we need, that $|\gamma| + p - 2s$ is not a natural number, since otherwise we could get logarithmic terms.

Finally, differentiating the result γ-times, we get the desired inequality (See [1, Theorem 3.10]). $\square$

Observing the outcome, we deduce the following.

Corollary 2.4.4. *When $p - 2s \notin \mathbb{N}$, for every polynomial P we have $L\left(P \cdot (x_n)_+^p\right) = qx_n^{p-2s}$ for some polynomial q of the same degree. Furthermore, if P is homogeneous, so is q, and if $p - s \notin \mathbb{N}$, then if P is non-zero, also q is.*

Proof. The corollary is a direct consequence of the above proposition. □

Remark 2.4.5. The mapping $P \mapsto q$ is a linear map, let us denote it Φ_{e_n}. When $p - s \notin \mathbb{N}$, it is also bijective. If we order the monomials first by order and then lexicographically, we get a lower triangular matrix with non-zero diagonals. [2]

Let us connect the above computation with the computation of $L(P \cdot (xe)_+^p)$ in $\{xe > 0\}$, for a polynomial P and a unit vector e. For a polynomial P we denote $\Phi_e(P) := q$, when $L(P(xe)_+^p) = q(xe)^{p-2s}$ in $\{xe > 0\}$. Similarly, for a linear map Q, we denote $\Phi_e^Q(P) := q$, when $L^Q(P(xe)_+^p) = q(xe)^{p-2s}$ in $\{xe > 0\}$, where the kernel of L^Q is the kernel of L pre-composed with Q.

Lemma 2.4.6. *Let $p - 2s \notin \mathbb{N}$, let L be an operator whose kernel K satisfies the condition* (2.1.1) *and let $e \in \mathbb{S}^{n-1}$. Let Q be an orthogonal matrix which maps e_n into e. Then the following formula holds:*

$$\Phi_e^I(P) = \Phi_{e_n}^Q(P \circ Q) \circ Q^T,$$

where Φ_e^I and $\Phi_{e_n}^Q$ are as described above.

Proof. In general we have

$$\begin{aligned} Lu(x) &= \int_{\mathbb{R}^n} (u(x) - u(x+y))K(y)dy = \int_{\mathbb{R}^n} \big(u(QQ^Tx) - u(Q(Q^Tx + Q^Ty)\big)K(QQ^Ty)dy \\ &= \int_{\mathbb{R}^n} \big(u \circ Q(Q^Tx) u \circ Q(Q^Tx + y)\big)K \circ Q(y)dy = L^Q(u \circ Q)(Q^Tx), \end{aligned}$$

where the kernel of the operator L^Q is $K \circ Q$. To get the same equality for the equation for function with polynomial growth, we just use the above equality on every ball B_R.

Let now Q be an orthogonal matrix whose last column equals e, meaning that we find an orthonormal basis of $\mathbb{R}^n$ so that the last vector is e. Then $Qxe = xQ^Te = xe_n$. So if we denote $u(x) = P(x)(xe)_+^p$ then $u \circ Q(x) = P \circ Q(x)(xe_n)_+^p$. Since Q is linear, L^Q has the same properties as L needed for computation of $L(P(x_n)_+^p)$, we conclude

$$L(P(xe)_+^p)(x) = L^Q(P \circ Q(x_n)_+^p)(Q^Tx) = \Phi_{e_n}^Q(P \circ Q)(Q^Tx) \cdot (Q^Txe_n)^{p-2s} = q(x)(xe)^{p-2s}$$

where q is still polynomial, because pre-composition with linear map preserves polynomials. Therefore we have

$$\Phi_e^I(P) = \Phi_{e_n}^Q(P \circ Q) \circ Q^T,$$

as wanted. □

[2] Note also, that if $p - 2s = m \in \mathbb{N}$, we can get terms of the form $f(x) = x^m \log x$, since it satisfies the condition

$$f(\lambda x) = (\lambda x)^m \log(\lambda x) = \lambda^m x^m \log x + \lambda^m \log \lambda \cdot x^m = \lambda^n f(x) + P_\lambda(x).$$

This is why we need $p - 2s \notin \mathbb{N}$ in the assumptions.

We are now in position to state the result, which answers the question of the beginning of this subsection. The result is of great use when proving the main statement of the paper, Theorem 2.1.1.

Theorem 2.4.7. *Assume L is an operator with kernel K satisfying* (2.1.1). *Let $\Omega \subset \mathbb{R}^n$ be a domain with $\partial\Omega \cap B_1 \in C^\beta$ and $p > \varepsilon_0$, $p - 2s \notin \mathbb{N}$, $p - s \notin \mathbb{N}$. Let for every $z \in \partial\Omega \cap B_1$ we have a polynomial $Q_z \in \boldsymbol{P}_{\lfloor\alpha\rfloor}$, for some $\alpha \leq \beta - 1$, such that the coefficient in front of x^γ is $C^{\alpha-|\gamma|}$ as a function in z with $\left|\left|(Q_z)^{(\gamma)}\right|\right|_{C_z^{\alpha-|\gamma|}(\partial\Omega\cap B_1)} \leq C_0$. Let also $\lfloor\alpha\rfloor + \lfloor p - 2s\rfloor \vee 0 < \beta - 1$.*

Then there exists a polynomial $\tilde{Q}_z \in \boldsymbol{P}_{\lfloor\alpha\rfloor}$, with $\left|\left|(\tilde{Q}_z)^{(\gamma)}\right|\right|_{C_z^{\alpha-|\gamma|}(\partial\Omega\cap B_1)} \leq CC_0$, so that

$$L(\tilde{Q}_z(\cdot - z)d^p) = Q_z(\cdot - z)d^{p-2s} + R_z + \eta_z,$$

where the function $R_z \in C^{\beta-1+\langle p-2s\rangle - \mathbf{1}_{(p<2s)}}(\overline{\Omega}\cap B_{1/2})$ and $\eta_z = \phi_z d^{p-2s}$, for some function $\phi_z \in C^\beta(\overline{\Omega} \cap B_{1/2})$ with $|\phi_z(x)| \leq C|x - z|^{\lfloor\alpha\rfloor+1}$.

Proof. We analyse the map $\Phi_z \colon \mathbf{P}_{\lfloor\alpha\rfloor} \to \mathbf{P}_{\lfloor\alpha\rfloor}$, defined as $\Phi_z(P) = q$, when $L(P(\cdot - z)d^p) = \varphi_z d^{p-2s} + R_z$, as before (see Corollary 2.3.10) and $q = T_z^{\lfloor\alpha\rfloor}(\varphi_z)$. The map Φ_z is linear.

Let us show that Φ_z is surjective. Therefore we choose a multi-index γ of order less or equal than $\lfloor\alpha\rfloor$. With straight forward computation we see that

$$L((\cdot - z)^\gamma d^p)(x) = L((\cdot)^\gamma d_z^p)(x - z) = \varphi_\gamma(x - z)d_z^{p-2s}(x - z) + R_\gamma(x - z),$$

where $d_z(x) = d(x + z)$. We want to perform a blow-up of the function $(x)^\gamma d_z^p(x)$, to get the $|\gamma|-$th Taylor polynomial of φ_γ. We use [1, Lemma 3.5].

Define $u_r(x) = \frac{1}{r^{|\gamma|+p}}u(rx)$, where $u(x) = x^\gamma d_z^p(x)$. We have the $C^p_{\text{loc}}(\mathbb{R}^n)$ convergence of $u_r \to u_0$, for $u_0 = x^\gamma(x\nu_z)^p$. Together with the estimate $|u_r(x)| \leq C|x|^{|\gamma|}|x|^p$, we get all the assumptions of the lemma, with $k' = \lceil|\gamma| + p\rceil$. True: $u_r \to u_0$ in L^∞_{loc} follows from the convergence above and $\int_{\mathbb{R}^n} \frac{|u_r|}{1+|x|^{n+2s+k}} < C$ independently or r, due to the growth estimate. Consequently, $\int_{\mathbb{R}^n} \frac{|u_r-u_0|}{1+|x|^{n+2s+k}} \to 0$ follows from both the growth estimate and the convergence; we obtain it with splitting the integral on B_R and the complement. On B_R we have uniform convergence, on the complement the integral is small. Taking into account the right-hand side as well, we get

$$L(u_r)(x) = \frac{1}{r^{|\gamma|}}\varphi_\gamma(rx)\frac{1}{r^{p-2s}}d^{p-2s}(rx) + \frac{1}{r^{|\gamma|+p}}R_\gamma(rx).$$

So for suitable rescaled Taylor polynomials of R_γ of order $\lceil|\gamma| + p - 2s\rceil$, we get that $L(u_r) - P_r$ converges in $L^\infty_{\text{loc}}(\mathbb{R}^n)$, similarly as in the proof of Corollary 2.3.10.

The lemma gives that $L(u_0) \stackrel{k}{=} T_0^{|\gamma|}(\varphi_\gamma)(x\nu_z)_+^{p-2s}$, where ν_z is the unit normal to $\partial\Omega$ at z. Now we use Corollary 2.4.4 and Lemma 2.4.6, to get that

$$T_0^{|\gamma|}(\varphi_\gamma) = \phi^I_{\nu_z}(x^\gamma) = \Phi^Q_{e_n}(\varphi_\gamma \circ Q) \circ Q^T,$$

for any orthogonal matrix Q mapping e_n into ν_z. Since $T_0^{|\gamma|}(\phi_\gamma)$ is the projection of $\Phi_z(x^\gamma)$ on $\mathbf{P}_{|\gamma|}$, we deduce, that Φ_z is in some basis a lower triangular operator (with ordering the monomials first by homogeneity and then lexicographically), and surjective (due to the assumption $p - s \notin \mathbb{N}$) - and hence bijective, since the dimensions of the domain and codomain agree.

Let us now turn to the regularity of entries of Φ_z. Since $P(x-z)$ is a polynomial in x, all its derivative are C^β smooth in z. Therefore, Corollary 2.3.10 renders that when $L(P(\cdot - z)d^p) = \varphi_z d^{p-2s} + R_z$, the derivatives of φ_z at any point are C^β in z as well. Since φ_z is a C^β function, then $\partial^\gamma \varphi_z(z)$ is $C^{\beta-|\gamma|}$ regular in z. Therefore the coefficient of Φ_z in the γ-th row is $C_z^{\beta-|\gamma|}$. Now we define the inverse map,

$$\Psi_z := \Phi_z^{-1},$$

which has the same properties, since Φ_z is a lower triangular and surjective. Note that the entry of inverse of the lower diagonal matrix depends only on the entries which lie in the rows of lower or equal index in the original matrix.

Finally, take Q_z as in the assumptions. We define

$$\tilde{Q}_z := \Psi_z(Q_z) \quad \text{and} \quad \eta_z := \left(\varphi_z - T_z^{\lfloor \alpha \rfloor}(\varphi_z)\right) d^{p-2s},$$

where

$$L(\tilde{Q}_z(\cdot - z)d^p) = \varphi_z d^{p-2s} + R_z,$$

which proves the claim. □

2.4.2 Regularity estimates

In the second part of this section, we establish some regularity results for equations with polynomial growth, where the right-hand side explodes at the boundary. The ideas are taken from the regularity results in [72, 1, 4].

The strategy for treating functions with polynomial growth is often through cut-off. Then we have a term which is compactly supported and another one which grows at infinity, but vanishes around the point of evaluation. We start with the cut-off lemma, which is a generalisation of [1, Lemma 3.6].

Lemma 2.4.8. *Assume $U \subset B_1$ is a C^β domain with $\beta > 1$. For some $k \in \mathbb{N}$ let the kernel K of operator L be $C^{k+1}(S^n)$ and satisfy condition* (2.1.1). *Let u be the solution of*

$$\begin{cases} Lu \overset{k}{=} f & \text{in } U, \\ \left|\left|\dfrac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} \leq C & \text{with } \alpha < s, \end{cases}$$

with $|f| \leq K_0 d^{\varepsilon-2s}$, for some $\varepsilon \in (0,s)$.

Then the function defined as

$$\tilde{u} := u\chi_{B_2},$$

satisfies $L\tilde{u} = \tilde{f}$, with

$$||\tilde{f} d^{2s-\varepsilon}||_{L^\infty(U)} \leq C\left(||f d^{2s-\varepsilon}||_{L^\infty(U)} + \left|\left|\frac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)}\right).$$

Proof. We compute
$$L\tilde{u} \stackrel{k}{=} Lu - L\hat{u},$$
for $\hat{u} := u\chi_{B_2^c}$. From Lemma 2.4.1, and Remark 2.4.2, we know that $L\hat{u} \stackrel{k}{=} \hat{f}$, for some $\hat{f} \in C^{k+1}(U)$, satisfying
$$||\hat{f}||_{L^\infty(U)} \leq C \left|\left|\frac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)}.$$
Therefore,
$$L\tilde{u} \stackrel{k}{=} f - \hat{f}.$$
But since $\tilde{u}$ is compactly supported it follows from the definition of evaluation of L in the generalised sense, that
$$L\tilde{u} = f - \hat{f} - P,$$
for some polynomial P of degree k.

Now, we split $\tilde{u} = u_1 + u_2 + u_3$, so that
$$\begin{cases} L(u_1) = & f - \hat{f} \quad \text{in } U, \\ u_1 = & 0 \quad \text{in } U^c, \end{cases} \quad \begin{cases} L(u_2) = & 0 \quad \text{in } U, \\ u_2 = & \tilde{u} \quad \text{in } U^c, \end{cases} \quad \text{and} \quad \begin{cases} L(u_3) = & P \quad \text{in } U, \\ u_3 = & 0 \quad \text{in } U^c. \end{cases}$$
The existence of u_2 and u_3 is provided, and then we define $u_1 = \tilde{u} - u_2 - u_3$. By the above proposition we have
$$||u_1||_{L^\infty(U)} \leq C_U(||fd^{2s-\varepsilon}||_{L^\infty(U)} + ||\hat{f}||_{L^\infty(U)}).$$
By standard elliptic estimates,
$$||u_2||_{L^\infty(U)} \leq ||\tilde{u}||_{L^\infty(\mathbb{R}^n)},$$
and so applying [1, Lemma 3.7], we get
$$\begin{aligned}||P||_{L^\infty(U)} \leq C||u_1||_{L^\infty(U)} &\leq ||\tilde{u}||_{L^\infty(U)} + ||u_1||_{L^\infty(U)} + ||u_2||_{L^\infty(U)} \\ &\leq C\left(||fd^{2s-\varepsilon}||_{L^\infty(U)} + \left|\left|\frac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)}\right).\end{aligned}$$
Hence defining $\tilde{f} := f - \hat{f} - P$, the lemma is proven. □

This immediately gives the boundary regularity result with which we conclude this section.

Lemma 2.4.9. *Let L be an operator whose kernel K satisfies* (2.1.1). *Let Ω be a domain of class C^β, $\beta > 1$. Let for some $k \in \mathbb{N}$ a function u be the solution to*
$$\begin{cases} Lu \stackrel{k}{=} f & \text{in } \Omega \cap B_1, \\ u = 0 & \text{in } B_1 \backslash \Omega, \\ \left|\left|\frac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)} \leq C & \text{with } \alpha < s, \end{cases}$$

with $|f| \leq Cd^{\varepsilon-2s}$ *and* $\varepsilon \in (0,s)$.

Then we have

$$||u||_{C^{\varepsilon}(\overline{B}_{1/2})} \leq C\left(||fd^{2s-\varepsilon}||_{L^\infty(U)} + \left|\left|\frac{u}{1+|\cdot|^{k+s+\alpha}}\right|\right|_{L^\infty(\mathbb{R}^n)}\right),$$

where C *does not depend on* u.

Proof. We define $\tilde{u} := u\chi_{B_2}$. It suffices to prove the claim for $\tilde{u}$, since functions agree on $B_{1/2}$. By [4, Theorem 1.1],[3] we get

$$||u||_{C^{\varepsilon}(B_{1/2})} \leq \left(||\tilde{f}d^{2s-\varepsilon}||_{L^\infty(\Omega\cap B_1)} + ||\tilde{u}||_{L^\infty(\mathbb{R}^n)}\right),$$

and by Lemma 2.4.8, we get the result. □

2.5 Expansion results

In this section we investigate how well can functions be approximated with the generalised distance function, depending on the regularity of the right-hand side and the regularity of the boundary of the domain. We sort the results according to the increasing regularity of the boundary.

The proofs are all done in the same way. We perform a blow-up with compactness argument. First proofs are done with all the details, and in the later ones we just explain what are the differences from the previous cases.

We start with a result, similar to [72, Proposition 3.2].

Lemma 2.5.1. *Let* $\alpha = \varepsilon_0 + \varepsilon \in (0,s)$, *and* Ω *be a* $C^{1,\alpha}$ *domain. Let* $0 \in \partial\Omega$, *and let* $\partial\Omega \cap B_1$ *be a graph of some* $C^{1,\alpha}$ *function of* $C^{1,\alpha}$ *norm less than* 1.

Let L *be an operator with kernel* K *satisfying* (2.1.1). *Suppose that for every* $z \in \partial\Omega \cap B_{3/4}$ *there is a function* g_z, *so that*

$$\begin{cases} L(u-g_z) &= f_z, \quad \text{in } \Omega\cap B_1 \\ u-g_z &= 0, \quad \text{in } B_1\backslash\Omega, \end{cases}$$

whit f_z *satisfying* $|f_z(x)| \leq C_0 d^{\varepsilon_0 - s}|x-z|^{\varepsilon}$ *in* $\Omega \cap B_1$. *We set* $K_0 = C_0 + \sup_z ||u - g_z||_{L^\infty(\mathbb{R}^n)} < \infty$.

Then, for every $z \in \partial\Omega \cap B_{1/2}$ *there exists a constant* Q_z *satisfying* $|Q_z| \leq CK_0$ *so that*

$$|u(x) - g_z(x) - Q_z d^s(x)| \leq CK_0|x-z|^{\alpha+s}.$$

Moreover, whenever $d(x_0) = |x_0 - z| = 2r$, *we have*

$$[u - g_z - Q_z d^s]_{C^{2s}(B_r(x_0))} \leq CK_0 r^{\alpha - s}$$

and if $d(x_1) = 2r_1 = |x_1 - z_1|$, *we have*

$$[u - g_z - Q_z d^s]_{C^{\alpha+s}(B_{r_1}(x_1))} \leq CK_0\left(\frac{|x_1 - z|}{r_1}\right)^{\alpha+s}.$$

The constant C *depends only on* n, s, α *and ellipticity constants.*

[3] Even though the result is stated with the right-hand side equal to zero, the proof is done under the assumptions we have here.

Proof. First, we prove the existence claim. Due to the assumption on the norm of the boundary defining function, we can prove the claim for $z = 0$. For simplicity we denote $u - g_0 = \tilde{u}$. We proceed in the same way as in [72, Proposition 3.2]. We can assume that $K_0 = 1$ and that the normal vector of $\partial\Omega$ at 0 is e_n. Let us assume by contradiction there exist Ω_k, L_k, u_k, f_k satisfying the assumptions, so that for every Q_k we have

$$\sup_{r>0} \frac{1}{r^{s+\alpha}} ||u_k - Q_k d_k^s||_{L^\infty(B_r)} > k.$$

Now we define the constants $Q_{k,r} = \frac{\int_{B_r} u_k d^s}{\int_{B_r} d^{2s}}$, and the monotone function

$$\theta(r) = \sup_k \sup_{\rho>r} \frac{1}{\rho^{\alpha+s}} ||u_k - Q_{k,r} d_k^s||_{L^\infty(B_\rho)},$$

which by [72, Lemma 3.3] converges to ∞ as $r \downarrow 0$. Hence there are sequences r_m, k_m, such that

$$\theta(r_m) \geq \frac{1}{r_m^{\alpha+s}} ||u_{k_m} - Q_{k_m,r_m} d_{k_m}^s||_{L^\infty(B_{r_m})} \geq \frac{1}{2}\theta(r_m).$$

With them, we define the blow-up sequence

$$v_m(x) := \frac{1}{r_m^{s+\alpha}\theta(r_m)} (u_{k_m}(r_m x) - Q_{r_m,k_m} d_{k_m}^s(r_m x)),$$

which by the definition of the constants $Q_{r,k}$ satisfies

$$\int_{B_1} v_m(x) d^s(r_m x) dx = 0,$$

and by the definition of θ also $||v_m||_{L^\infty(B_1)} \geq \frac{1}{2}$.

With the same arguments as in [72, Proposition 3.2], we get the estimates $|Q_{k,r} - Q_{k,2r}| \leq C r^\alpha \theta(r)$ and $|Q_{k,r} - Q_{k,Rr}| \leq C(Rr)^\alpha \theta(r)$, where both C are independent of k, which furthermore give that $||v_m||_{L^\infty(B_R)} \leq C R^{s+\alpha}$.

We proceed to computing $L(v_m)(x) = \frac{r_m^{s-\alpha}}{\theta(r_m)} \big(f(r_m x) + Q_{r_m,k_m} L(d_m^s)(x)\big)$, which we estimate using [72, Proposition 2.3] with

$$\begin{aligned} |L(v_m)(x)| &\leq \frac{r_m^{s-\alpha}}{\theta(r_m)} d_m^{\varepsilon_0 - s}(r_m x) |r_m x|^\varepsilon + \frac{Q_{r_m,k_m}}{\theta(r_m)} r_m^{s-\alpha} d_m^{\alpha-s}(r_m x) \\ &\leq \frac{C}{\theta(r_m)} d_m^{\varepsilon_0 - s}(x)|x|^\varepsilon + \frac{Q_{k_m,r_m}}{\theta(r_m)} d_m^{\alpha-s}(x), \end{aligned}$$

which converges to 0 as $m \to \infty$ in every compact set in $\{x_n > 0\}$, since $\theta(r_m)^{-1} \to 0$ and $\frac{Q_{r_m,k_m}}{\theta(r_m)} \to 0$. This implies that $|L(v_m)(x)| \leq C_M d^{\varepsilon_0 - s}(x)$ on B_M, which is needed to get the uniform bound (in m) on $||v_m||_{C^s(B_M)}$, using [72, Proposition 3.1].

The rest of the proof is analogous to the original one. We conclude that v_m converge to $\kappa(x_n)_+^s$, for some $\kappa \in \mathbb{R}$, which together with passing the integral quantity to the limit gives $\kappa = 0$. But this is a contradiction with the non-triviality of $||v||_{L^\infty(B_1)}$.

Let us now show, that the growth control proven above gives the interior regularity we want. For this, we use [74, Corollary 3.6] and [72, Proposition 2.3]. Concretely, take

$2r = d(x_0) = |x_0 - z|$. Then the mentioned corollary, used on $(u - g_z - Q_z d^s)(x_0 + r\cdot)$, gives

$$r^{2s}\left[u - g_z - Q_z d^s\right]_{C^{2s}(B_r(x_0))} \leq C(K_0 r^{\alpha+s} + K_0 r^{2s} r^{\alpha-s}) \leq C K_0 r^{\alpha+s}.$$

We bound the first term using Lemma 2.8.3 while for the other one we use the assumption on $L(u - g_z)$ and Lemma 2.3.2.

Let now $d(x_1) = 2r_1 = |x_1 - z_1|$. Then denoting $u_r(x) := (u - g_z - Q_z d^s)(x_1 + 2r_1 x)$, and applying the same two results, we get

$$r_1^{\alpha+s}\left[u - g_z - Q_z d^s\right]_{C^{\alpha+s}(B_{r_1}(x_1))} \leq C(K_0|x_1 - z|^{\alpha+s} + K_0 r_1^{\varepsilon_0+s}|x_1 - z|^{\varepsilon}) \leq C K_0 |x_1 - z|^{\alpha+s},$$

again with aid of Lemma 2.8.3 and Lemma 2.3.2. □

This result, together with the generalisation of [1, Proposition 4.1] cover all cases when doing expansions with d^s varying the regularity of Ω. The result states as follows:

Lemma 2.5.2. *Let $\beta > 1 + s$, $\beta \notin \mathbb{N}$, and let $\Omega \subset \mathbb{R}^n$ be C^β domain. Assume $0 \in \partial\Omega$, and let $\partial\Omega \cap B_1$ be a graph of some C^β function of C^β norm less than 1. Let L be an operator whose kernel K is $C^{2\beta+1}(\mathbb{S}^{n-1})$ and satisfies (2.1.1). Let $u\colon \mathbb{R}^n \to \mathbb{R}$ and assume for every $z \in \partial\Omega \cap B_{3/4}$ we have a function g_z so that*

$$\begin{cases} L(u - g_z) &= f_z \quad \text{in } \Omega \\ u - g_z &= 0 \quad \text{in } \Omega^c \cap B_1. \end{cases}$$

For some $\varepsilon > 0$, let us have $|f_z(x) - P_z(x)| \leq C|x - z|^{\beta-1+s-\varepsilon} d^{\varepsilon-2s}$, for $x \in B_1 \cap \Omega$, where C does not depend on the boundary point z, for some $P_z \in \boldsymbol{P}_{\lfloor\beta-1\rfloor}$. Furthermore, let also

$$[f_z]_{C^{\beta-1-s}(B_{\frac{3r}{2}}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{\beta-s},$$

whenever $d(x_1) = 2r$, independently of z, r, x_1. Then for every $z \in \partial\Omega \cap B_{1/2}$ there exists a polynomial $Q_z \in \boldsymbol{P}_{\lfloor\beta-1\rfloor}$, so that

$$|u(x) - g_z(x) - Q_z(x) d^s(x)| \leq C|x - z|^{\beta-1+s} \quad \text{for every } x \in B_1,$$

and C is independent of z. Also for if $x_1 \in \Omega \cap B_1$, $d(x_1) = 2r_1$, then

$$[u - g_z - Q_z d^s]_{C^{\beta-1+s}(B_r(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{(\beta-1+s)\vee(\beta-s)}$$

with C depending only on n, s, β and $||K||_{C^{2\beta+1}(\mathbb{S}^{n-1})}$.

Proof. First we argue that without loss of generality we work at $z = 0$, because of the assumption on the norm of the domain. Then the proof goes exactly the same as in [1, Proposition 4.1] denoting $\tilde{u} = u - g_0$. Notice that in the cited result, we needed that $Lu \in C^{\beta-1-s}$ to find a polynomial P such that $Lu - P = O(|x|^{\beta-1-s})$. Now by assumption we have $Lu - P = O(|x|^{\beta-1+s-\varepsilon} d^{\varepsilon-2s})$, which gives that the blow up sequence satisfies

$$|L_m v_m(x) - P_m(x)| \leq \frac{C}{\theta(r_m)}|x|^{\beta-1+s-\varepsilon} d_m^{\varepsilon-2s}(x),$$

which still converges locally uniformly to 0 in the half-space. Instead of [1, Proposition 3.8] we now use Lemma 2.4.9, which allows us to apply Arzela-Ascoli Theorem, so that v_m converges locally uniformly in $\mathbb{R}^n$ to some function v. Hence by [1, Lemma 3.5] in every compact set in $\{x_n > 0\}$ the limit function satisfies $L_\star v \overset{k}{=} 0$. The conclusion of the proof is then the same.

Second part, where we needed the regularity of f is for the interior regularity estimate. Using [1, Proposition 3.9], similarly as at the end of the previous lemma, we obtain

$$r_1^{\beta-1+s}\left[u-g_z-Q_zd^s\right]_{C^{\beta-1+s}(B_{r_1}(x_1))} \leq C\left(|x_1-z|^{\beta-1+s}+r_1^{\beta-1+s}\left(\frac{|x_1-z|}{r_1}\right)^{\beta-s}\right)$$
$$\leq C\left(\frac{|x_1-z|}{r_1}\right)^{(\beta-1+s)\vee(\beta-s)},$$

as desired. □

In the similar manner we also want to give the complete answer to expansions of one solution with respect to another, non-trivial one. Replacing the distance function with a solution basically increases the order of approximation by one. Now the cases split in three categories with respect to the smoothness of the right-hand side. First one is when the right-hand side f is only bounded by $|f| \leq Cd^{\varepsilon-s}$, for some $\varepsilon \in (0,s)$, the second one is $f \in C^{\varepsilon-s}$, $\varepsilon \in (s,1)$, and the third one is $\varepsilon > 1$. In the first two cases we only need that the boundary $\partial\Omega$ is C^1, while in the third case we need to assume that the boundary is C^β, with $\beta = \varepsilon + s$. The model result here is [1, Proposition 4.4], which deals with the third case.

First we prove the basic claims in all settings. Let us begin with the case $\varepsilon < s$.

Lemma 2.5.3. *Let L be an operator with kernel K satisfying* (2.1.1). *Let Ω be a domain with $0 \in \partial\Omega$, so that $\partial\Omega \cap B_1$ is a graph of a function whose C^1 norm is smaller than* 1. *For $i = 1, 2$, let u_i be the solution to*

$$\begin{cases} Lu_i &= f_i \quad \text{in } \Omega\cap B_1 \\ u_i &= 0 \quad \text{in } B_1\backslash\Omega, \end{cases}$$

for some f_i satisfying $|f_i| \leq C_0 d^{\varepsilon-s}$, $\varepsilon \in (0,s)$. Furthermore assume the existence of $C_2, c_2 > 0$, such that $C_2 d^s \geq u_2 \geq c_2 d^s$ in B_1. Denote $K_0 = C_0 + ||u_1||_{L^\infty(\mathbb{R}^n)} + ||u_2||_{L^\infty(\mathbb{R}^n)}$.

Then for every $z \in \partial\Omega \cap B_{1/2}$ there exist a constant Q_z, such that

$$|u_1(x) - Q_z u_2(x)| \leq CK_0|x-z|^{\varepsilon+s}, \quad x \in B_{1/2}(z)$$

and

$$[u_1 - Q_z u_2]_{C^{s+\varepsilon}(B_r(x_0))} \leq CK_0,$$

whenever $d(x_0) = |x_0 - z| = 2r$. The constant C depends only on $n, s, c_2, C_2, \varepsilon$ and ellipticity constants.

Proof. We start with the expansion estimate part. Without loss of generality assume $K_0 = 1$, $z = 0$ and the normal vector $\nu_0 = e_n$. Assume by contradiction that there exist $L_j, \Omega_j, u_{i,j}, f_{i,j}$ satisfying the assumptions of the lemma, but for every Q_j we have

$$\sup_{r>0}\frac{1}{r^{s+\varepsilon}}||u_{1,j} - Q_j u_{2,j}||_{L^\infty(B_r)} > j.$$

We define

$$Q_{r,j} := \frac{\int_{B_r} u_{1,j} u_{2,j}}{\int_{B_r} u_{2,j}^2},$$

so that $\int_{B_r} (u_{1,j} - Q_{r,j} u_{2,j}) u_{2,j} dx = 0$. Define now the monotone quantity

$$\theta(r) := \sup_j \sup_{\rho > r} \frac{1}{\rho^{\varepsilon+s}} ||u_{1,j} - Q_{\rho,j} u_{2,j}||_{L^\infty(B_\rho)},$$

which by contradiction assumption and [1, Lemma 4.5] converges to infinity as r goes to zero. We find the realising sequences r_m, j_m so that

$$m \leq \frac{\theta(r_m)}{2} \leq \frac{1}{r_m^{\varepsilon+s}} ||u_{1,j_m} - Q_{r_m,j_m} u_{2,j_m}||_{L^\infty(B_{r_m})} \leq \theta(r_m).$$

The blow-up sequence is defined with

$$v_m(x) := \frac{1}{\theta(r_m) r_m^{\varepsilon+s}} \big(u_{1,j_m}(r_m x) - Q_{j_m,r_m} u_{2,j_m}(r_m x) \big).$$

By construction, we have both $||v_m||_{L^\infty(B_1)} \geq \frac{1}{2}$ and $\int_{B_1} v_m(x) u_{2,j_m}(r_m x) dx = 0$. In the limit of some subsequence of v_m, these quantities will contradict each other.

We proceed with some estimations

$$\begin{aligned} |Q_{r,j} - Q_{2r,j}| &\leq C \frac{2^s}{r^s} ||Q_{r,j} u_{2,j} - Q_{2r,j} u_{2,j}||_{L^\infty(B_r \cap \{d > r/2\})} \\ &\leq \frac{C}{r^s} \big(||u_{1,j} - Q_{r,j} u_{2,j}||_{L^\infty(B_r)} + ||u_{1,j} - Q_{2r,j} u_{2,j}||_{L^\infty(B_{2r})} \big) \\ &\leq C r^{-s} \big(\theta(r) r^{s+\varepsilon} + \theta(2r)(2r)^{s+\varepsilon} \big) \leq C \theta(r) r^\varepsilon, \end{aligned}$$

where we used the monotonicity of θ. Summing the geometric series, this gives

$$\begin{aligned} |Q_{r,j} - Q_{2^k r,j}| \leq \sum_{i=0}^{k-1} |Q_{2^{i+1}r,j} - Q_{2^i r,j}| &\leq \sum_{i=0}^{k-1} C\theta(2^i r)(2^i r)^\varepsilon \\ &\leq C\theta(r) r^\varepsilon C_\varepsilon 2^{\varepsilon k} \leq C\theta(r)(2^k r)^\varepsilon, \end{aligned}$$

which furthermore gives

$$|Q_{r,j} - Q_{Rr,j}| \leq C_\varepsilon \theta(r)(Rr)^\varepsilon.$$

The latter estimate implies the growth of v_m at infinity, as follows

$$\begin{aligned} ||v_m||_{L^\infty(B_R)} =& \left|\left| \frac{1}{\theta(r_m) r_m^{\varepsilon+s}} (u_{1,j_m} - Q_{j_m,r_m} u_{2,j_m}) \right|\right|_{L^\infty(B_{Rr_m})} \\ \leq& \frac{1}{\theta(r_m) r_m^{\varepsilon+s}} ||u_{1,j_m} - Q_{j_m,Rr_m} u_{2,j_m}||_{L^\infty(B_{Rr_m})} + \\ &+ \frac{1}{\theta(r_m) r_m^{\varepsilon+s}} |Q_{r_m,j_m} - Q_{Rr_m,j_m}| \cdot ||u_{2,j_m}||_{L^\infty(B_{Rr_m})} \\ \leq& \frac{\theta(Rr_m)(Rr_m)^{\varepsilon+s}}{\theta(r_m) r_m^{\varepsilon+s}} + \frac{C_\varepsilon \theta(r_m)(Rr_m)^\varepsilon C_2 (Rr_m)^s}{\theta(r_m) r_m^{\varepsilon+s}} \leq CR^{s+\varepsilon}. \end{aligned}$$

We proceed to computing the L_{j_m} on the blow-up sequence:

$$L_{j_m}v_m(x) = \frac{1}{\theta(r_m)r_m^{\varepsilon+s}} r_m^{2s}(f_{1,j_m} - Q_{j_m,r_m}f_{2,j_m})(r_m x) = \frac{r_m^{s-\varepsilon}}{\theta(r_m)}(f_{1,j_m} - Q_{j_m,r_m}f_{2,j_m})(r_m x),$$

so when estimating the modulus we get

$$|L_{j_m}v_m(x)| \le \frac{C}{\theta(r_m)} r_m^{s-\varepsilon}(1+Q_{j_m,r_m})d_{j_m}^{\varepsilon-s}(r_m x) \le \frac{C}{\theta(r_m)}(1+Q_{j_m,r_m})d_{j_m}^{\varepsilon-s}(x).$$

Let us now show, that $\frac{Q_{j,r}}{\theta(r)} \to 0$ uniformly in j, as $r \to 0$. Let $2^{-k-1} \le r \le 2^{-k}$. Then

$$\begin{aligned} |Q_{j,r}| &\le |Q_{j,2^k r}| + \sum_{i=0}^{k-1} |Q_{j,2^i r} - Q_{j,2^{i+1}r}| \le C + \sum_{i=0}^{k-1} C\theta(2^i r)(2^i r)^\varepsilon \\ &\le C(1 + \sum_{i=0}^{k-1} \theta(2^{i-k})(2^{i-k})^\varepsilon) \le C\left(1 + \sum_{i=0}^{k-1} \theta(2^{-i})(2^{-i})^\varepsilon\right). \end{aligned}$$

Now we divide with $\theta(r)$, which is better then dividing with $\theta(2^{-k})$, we get a series $\sum_{i=0}^{k-1} \frac{\theta(2^{-i})}{\theta(2^{-k})}(2^{-i})^\varepsilon$, which is of the form "convergent series times coefficients which all go to 0", and so we can show (with an ϵ, r_0 calculus) that it goes to 0 indeed. Above we also used the uniform bound $|Q_{r,j}| \le C$, $r \in (1/2, 1)$ which follows by the definition of $Q_{r,j}$, uniform boundedness of $u_{1,j}$, $u_{2,j}$ and the uniform boundedness from below on $u_{2,j}$. As a consequence, we get that $L_{j_m}v_m \to 0$ uniformly in every compact set in $\{x_n > 0\}$ as well as $|L_{j_m}v_m| \le Cd_{j_m}^{\varepsilon-s}$. This bound, together with previously obtained growth control of v_m, allows us to apply [72, Lemma 5.2], to get a uniform bound $[v_m]_{C^\alpha(B_M)} \le C(M)$, for some $\alpha > 0$.

The same argumentation as in [72, Proposition 3.2] we get that a subsequence still denoted v_m converges locally uniformly to v in $\mathbb{R}^n$, and v solves $L_* v = 0$ in $\{x_n > 0\}$, and $v = 0$ in $\{x_n \le 0\}$, which implies that $v(x) = \kappa(x_n)_+^s$ (see [73, Proposition 5.1].) But passing the integral quantity to the limit, gives $\kappa = 0$, which gives contradiction with $||v||_{L^\infty(B_1)} \ge \frac{1}{2}$.

To get the interior regularity from the claim, we use the growth estimate just proven together with [74, Corollary 3.6], as follows. Take $d(x_0) = |x_0 - z| = 2r$, and denote $v_r(x) = (u_1 - Q_z u_2)(x_0 + 2rx)$. The growth estimate implies $v_r(x) \le CK_0|2rx + x_0 - z|^{\varepsilon+s} \le Cr^{\varepsilon+s}(1+|x|)^{\varepsilon+s}$. We apply the corollary, to get

$$[v_r]_{C^{2s}(B_{1/2})} \le C(CK_0 r^{\varepsilon+s} + r^{2s}CK_0 r^{\varepsilon-s}),$$

which after passing to smaller exponent and the rescaling gives

$$[u_1 - Q_z u_2]_{C^{\varepsilon+s}(B_r(x_0))} \le CK_0,$$

as wanted. □

Notice, that we need the boundary to be at least C^1, so that the blow-up of the domain converges to the half space.

We now state the analogous result, when $\varepsilon \in (s, 1)$.

Lemma 2.5.4. *Let L be an operator with kernel K satisfying* (2.1.1)*. Let Ω be a domain with $0 \in \partial\Omega$, so that $\partial\Omega \cap B_1$ is a graph of a function whose C^1 norm is smaller than 1. For $i = 1, 2$, let u_i be the solution to*

$$\begin{cases} Lu_i = f_i & \text{in } \Omega \cap B_1 \\ u_i = 0 & \text{in } B_1 \backslash \Omega, \end{cases}$$

for some $f_i \in C^{\varepsilon - s}(\Omega \cap B_1)$, $\varepsilon \in (s, 1)$. Furthermore assume the existence of $C_2, c_2 > 0$, such that $C_2 d^s \geq u_2 \geq c_2 d^s$ in B_1. Denote $K_0 = ||f_1||_{C^{\varepsilon-s}} + ||f_2||_{C^{\varepsilon-s}} + ||u_1||_{L^\infty(\mathbb{R}^n)} + ||u_2||_{L^\infty(\mathbb{R}^n)}$.

Then for every $z \in \partial\Omega \cap B_{1/2}$ there exist a constant Q_z, such that

$$|u_1(x) - Q_z u_2(x)| \leq CK_0 |x - z|^{\varepsilon + s}, \quad x \in B_{1/2}(z)$$

and

$$[u_1 - Q_z u_2]_{C^{s+\varepsilon}(B_r(x_0))} \leq CK_0,$$

whenever $d(x_0) = |x_0 - z| = 2r$. The constant C depends only on $n, s, c_2, C_2, \varepsilon$ and ellipticity constants.

Proof. The proof goes along the same lines of the proof of the lemma above, just the converging arguments changes a little. To obtain the uniform C^s bound on the blow-up sequence v_m, we use [1, Proposition 3.8] instead, and since the growth of the limit function is too large, we use Liouville theorem [1, Theorem 3.10], to get that the limit $v(x) = p(x)(x_n)_+^s$, for some polynomial p of degree one. But then the growth control of v ensures that the polynomial is indeed of degree zero.

To obtain the interior regularity estimate we now apply [1, Proposition 3.9]. □

The third case, $\varepsilon > 1$ is done in [1, Proposition 4.4].

Proceed now to the generalisations. First we state the result with the least regularity assumptions.

Lemma 2.5.5. *Let L be an operator whose kernel K satisfies* (2.1.1)*. Let Ω be a domain with $0 \in \partial\Omega$, so that $\partial\Omega \cap B_1$ is a graph of a function whose C^1 norm is smaller than 1. For $i = 1, 2$ and $z \in \partial\Omega \cap B_{3/4}$, let $u_i - g_{i,z}$ be solutions to*

$$\begin{cases} L(u_i - g_{i,z}) = f_{i,z} & \text{in } \Omega \cap B_1, \\ u_i - g_{i,z} = 0 & \text{in } B_1 \backslash \Omega, \end{cases}$$

for some $f_{i,z}$ satisfying $|f_{i,z}(x)| \leq C_0 d^{\varepsilon_0 - s}|x - z|^{\varepsilon}$, $\varepsilon_0 + \varepsilon \in (0, s)$. Furthermore assume the existence of $C_2, c_2 > 0$, such that $C_2 d^s \geq u_2 - g_{2,z} \geq c_2 d^s$ in B_1. Denote

$$K_0 = C_0 + \sup_z ||u_1 - g_{1,z}||_{L^\infty(\mathbb{R}^n)} + \sup_z ||u_2 - g_{2,z}||_{L^\infty(\mathbb{R}^n)}.$$

Then for every $z \in \partial\Omega \cap B_{1/2}$ there exist a constant Q_z, such that

$$|u_1(x) - g_{1,z}(x) - Q_z(u_2(x) - g_{2,z}(x))| \leq CK_0 |x - z|^{\varepsilon_0 + \varepsilon + s}, \quad x \in B_{1/2}(z)$$

and

$$[u_1 - g_{1,z} - Q_z(u_2 - g_{2,z})]_{C^{s+\varepsilon_0+\varepsilon}(B_r(x_0))} \leq CK_0,$$

whenever $d(x_0) = |x_0 - z| = 2r$. The constant C depends only on $n, s, c_2, C_2, \varepsilon$ and ellipticity constants.

Proof. The proof works in the same way as the the one in Lemma 2.5.1, but following the proof of Lemma 2.5.3, just that the exponent changes to $\varepsilon_0 + \varepsilon$ instead of ε. □

Similarly happens with Lemma 2.5.4.

Lemma 2.5.6. *Let L be an operator whose kernel K satisfies condition* (2.1.1)*. Let Ω be a domain with $0 \in \partial\Omega$, so that $\partial\Omega \cap B_1$ is a graph of a function whose C^1 norm is smaller than 1. For $i = 1, 2$ and $z \in \partial\Omega \cap B_{3/4}$, let $u_i - g_{i,z}$ be solutions to*

$$\begin{cases} L(u_i - g_{i,z}) &= f_{i,z} \quad \text{in } \Omega \cap B_1, \\ u_i - g_{i,z} &= 0 \quad \text{in } B_1 \backslash \Omega. \end{cases}$$

Let there exist constants P_z, so that $|f_{i,z}(x) - P_z| \leq C_0 d^{\varepsilon_0 - s}|x - z|^\varepsilon$, $\varepsilon, \varepsilon_0 > 0$, $\varepsilon_0 < s$, $\varepsilon_0 + \varepsilon \in (s, 1)$. Assume also that whenever $d(x_0) = 2r = |x_0 - z|$, we have the bound

$$[f_z]_{C^{\varepsilon_0+\varepsilon-s}(B_{\frac{3r}{2}})} \leq C_0.$$

Furthermore assume the existence of $C_2, c_2 > 0$, such that $C_2 d^s \geq u_2 - g_{2,z} \geq c_2 d^s$ in B_1. Denote $K_0 = C_0 + \sup_z ||u_1 - g_{1,z}||_{L^\infty(\mathbb{R}^n)} + \sup_z ||u_2 - g_{2,z}||_{L^\infty(\mathbb{R}^n)}$.

Then for every $z \in \partial\Omega \cap B_{1/2}$ there exist a constant Q_z, such that

$$|u_1(x) - g_{1,z}(x) - Q_z(u_2(x) - g_{2,z}(x))| \leq CK_0|x - z|^{\varepsilon+\varepsilon_0+s}, \quad x \in B_{1/2}(z)$$

and

$$[u_1 - g_{1,z} - Q_z(u_2 - g_{2,z})]_{C^{s+\varepsilon+\varepsilon_0}(B_r(x_0))} \leq CK_0,$$

whenever $d(x_0) = |x_0 - z| = 2r$. The constant C depends only on $n, s, c_2, C_2, \varepsilon$ and ellipticity constants.

Proof. The proof is again done with the blow up argument as in the proof of Lemma 2.5.4, just that the assumption $f \in C^\varepsilon$ is replaced with approximation estimate ($|f_{i,z}(x) - P_z| \leq C_0|x - z|^\varepsilon d^{\varepsilon_0 - s}$) together with the interior regularity ($[f_z]_{C^{\varepsilon+\varepsilon_0-s}(B_{\frac{3r}{2}})} \leq C_0$). The convergence argument is done in the same way as in Lemma 2.5.5, just that for the equation of the limit function we use [1, Lemma 3.5] on every compact set in $\{x_n > 0\}$.

For the interior regularity estimate we again use [1, Proposition 3.9]. □

We conclude the section with the last generalised result.

Lemma 2.5.7. *Let $\beta > 1$, such that $\beta \notin \mathbb{N}$ and $\beta \pm s \notin \mathbb{N}$. Let L be an operator whose kernel K satisfies the condition* (2.1.1) *and is $C^{2\beta+1}(\mathbb{S}^{n-1})$. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^β, with $0 \in \partial\Omega$. Let $z \in \partial\Omega \cap B_{3/4}$ and $u_1, u_2 \in L^\infty(\mathbb{R}^n)$ solutions of*

$$\begin{cases} L(u_i - g_{i,z}) &= f_{i,z} \quad \text{in } \Omega \cap B_1 \\ u_i - g_{i,z} &= 0 \quad \text{in } \Omega^c \cap B_1, \end{cases}$$

for some functions $g_{i,z}$. Suppose that $\partial\Omega \cap B_1$ is a graph of some C^β function whose C^β norm is bounded by one. Assume that for some $c_1 > 0$,

$$u_1(x) - g_{1,z}(x) \geq c_1 d^s(x), \quad \text{for any } x \in B_1(z).$$

Let there exist two polynomials $P_{i,z} \in \boldsymbol{P}_{[\beta-s]}$*, so that for some* $\varepsilon > 0$*, we have* $|f_{i,z}(x) - P_{i,z}(x)| \le C_1|x-z|^{\beta+s-\varepsilon}d^{\varepsilon-2s}, \quad x \in B_1(z)$*, with* C_1 *independent of* z*. Assume also*

$$[f_{i,z}]_{C^{\beta-s}(B_{\frac{3r}{2}}(x_0))} \le C_2,$$

whenever $d(x_0) = 2r = |x_0 - z| > 0$*, independently of* z, r, x_0*. Then for every* $z \in \partial\Omega \cap B_{\frac{1}{2}}$ *there exists a polynomial* $Q_z \in \boldsymbol{P}_{\lfloor\beta\rfloor}$*, such that*

$$|u_2(x) - g_{2,z}(x) - Q_z(x)(u_1(x) - g_{1,z}(x))| \le C|x-z|^{\beta+s}, \quad \textit{for any } x \in B_1(z).$$

Moreover, when $d(x_0) = 2r = |x_0 - z| > 0$*,*

$$[u_2 - g_{2,z} - Q_z(u_1 - g_{1,z})]_{C^{\beta+s}(B_r(x_0))} \le C.$$

The constant C *depends only on* n, s, c_1, C_1, C_2 *and* $||K||_{C^{2\beta+1}(\mathbb{S}^{n-1})}$*.*

Proof. Again we can assume $z = 0$ without loss of generality. Denote $\tilde{u}_1 = u_1 - g_{1,0}$ and $\tilde{u}_2 = u_2 - g_{2,0}$. Applying Lemma 2.5.2 (or Lemma 2.5.1 when β is to small) to $\tilde{u}_1$, we see with the same argument as in [1, Proposition 4.4], that it is equivalent finding $\tilde{Q} = \tilde{q}^{(0)} + \tilde{Q}^{(1)}(x)$ such that

$$|\tilde{u}_2(x) - \tilde{q}^{(0)}\tilde{u}_1(x) - \tilde{Q}^{(1)}(x)d^s(x)| \le C|x|^{\beta+s}, \quad \text{for any } x \in B_1.$$

Just like in Lemma 2.5.2 we assumed all the necessary interior regularity, which replaces the assumption $f_i \in C^{\beta-s}(\overline{\Omega})$. Again it is needed to find a polynomial so that when subtracting it from the operator on the blow-up sequence we get something small, and to get a uniform bound on the C^ε norm of the blow-up sequence (where we potentially use Lemma 2.4.9), and the interior regularity estimate in the claim. Note also that the equation of the limit function is obtained through [1, Lemma 3.5] on every compact subset of $\{x_n > 0\}$, just like in the proof of Lemma 2.5.2. □

2.6 Regularity estimates for solutions to linear equations

In this section, we study the behaviour near the boundary of solutions to the linear equation (2.1.5) when $s > \frac{1}{2}$. We denote

$$\varepsilon_0 = 2s - 1.$$

In the first part we establish the optimal C^s regularity of the solutions up to the boundary and in the second we study quotients of solutions with d^s and quotients of two solutions.

We deal with the weak solutions according to the following definition.

Definition 2.6.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $b \in \mathbb{R}^n$, $f \in L^2(\Omega)$ and let $u\colon \mathbb{R}^n \to \mathbb{R}$ satisfy the growth control $|u(x)| \le C(1 + |x|^{2s-\delta})$, for some $\delta > 0$. Assume L is an operator whose kernel K satisfies (2.1.1). We say that u is a weak solution of

$$Lu + b\cdot\nabla u = f \quad \text{in } \Omega,$$

whenever

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)dydx - \int_\Omega u(x)b\cdot\nabla\varphi(x)dx = \int_\Omega f(x)\varphi(x)dx,$$

holds for every $\varphi \in C_c^\infty(\Omega)$.

Remark 2.6.2. Due to the fact that $K(x-y) = K((x-t)-(y-t))$ the following fact holds. If $\eta_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ is a mollifier and u a solution to $Lu + b\cdot\nabla u = f$ in Ω in the weak sense, then $u_\varepsilon := u * \eta_\varepsilon$ satisfies $Lu_\varepsilon + b\cdot\nabla u_\varepsilon = f * \eta_\varepsilon$ in Ω_ε in the weak sense, for $\Omega_\varepsilon = \{x\in\Omega;\ \operatorname{dist}(x,\partial\Omega) > \varepsilon\}$.

2.6.1 Interior and boundary regularity

We present a version of interior regularity estimate that we need in our work. Even though it follows from already known results we provide a short proof.

Lemma 2.6.3. *Let $s\in(1/2,1)$, and let L be an operator whose kernel satisfies condition* (2.1.1)*. Let u be any solution of*

$$Lu + b\cdot\nabla u = f \qquad \text{in } B_1,$$

with $f\in L^\infty(B_1)$ and $b\in\mathbb{R}^n$. Then for every $\delta>0$,

$$||u||_{C^{2s}(B_{1/2})} \le C\left(||f||_{L^\infty(B_1)} + \sup_{R\ge 1} R^{\delta-2s}||u||_{L^\infty(B_R)}\right).$$

The constant C depends only on n, s, δ and ellipticity constants.

Proof. Let us first assume that u is a smooth solution. Then, by interior regularity estimates for the equation

$$Lu = f - b\cdot\nabla u \quad \text{in } B_1,$$

see [74, Corollary 3.6], we get

$$||u||_{C^{2s}(B_{1/2})} \le C\left(||f||_{L^\infty(B_1)} + \sup_{R\ge 1} R^{\delta-2s}||u||_{L^\infty(B_R)} + ||\nabla u||_{L^\infty(B_1)}\right).$$

Since $s > \frac{1}{2}$, the interpolation inequality gives that for every $\varepsilon > 0$ there is a constant C so that

$$||u||_{L^\infty(B_1)} \le \varepsilon\,[u]_{C^{2s}(B_1)} + C_\varepsilon ||u||_{L^\infty(B_1)}.$$

Combining the two inequalities, we get that for every $\varepsilon > 0$,

$$||u||_{C^{2s}(B_{1/2})} \le \varepsilon\,[u]_{C^{2s}(B_1)} + C_\varepsilon\left(||f||_{L^\infty(B_1)} + \sup_{R\ge 1} R^{\delta-2s}||u||_{L^\infty(B_R)}\right),$$

which thanks to [33, Lemma 2.26] gives

$$||u||_{C^{2s}(B_{1/2})} \le C\left(||f||_{L^\infty(B_1)} + \sup_{R\ge 1} R^{\delta-2s}||u||_{L^\infty(B_R)}\right).$$

Using a mollifier and taking the limit proves the general case, see Remark 2.6.2. □

We proceed with the comparison principle for equation (2.1.5).

Lemma 2.6.4. *Let $s \in (1/2, 1)$, and let L be an operator whose kernel satisfies condition (2.1.1) and let Ω be a bounded domain. Let $u \in C(\mathbb{R}^n)$ be a solution to*

$$\begin{cases} Lu + b \cdot \nabla u &= f \quad \text{in } \Omega \\ u &\geq 0 \quad \text{in } \Omega^c, \end{cases}$$

with $f \in L^\infty(\Omega)$, $f \geq 0$.

Then $u \geq 0$ in $\mathbb{R}^n$.

Proof. Since the weak formulation of equations is preserved under mollification, the proof is the same as the proof of [30, Lemma 4.1]. □

In order to achieve the optimal boundary regularity C^s, we need a suitable barrier – a supersolution which grows as d^s near the boundary. The construction is the same as in [72, Lemma 2.8], we only need to estimate the gradient terms, which behave nicely thanks to the assumption $2s > 1$.

Lemma 2.6.5. *Let $s \in (1/2, 1)$, and let L be an operator whose kernel satisfies condition (2.1.1) and let Ω be a $C^{1,\alpha}$ domain. Then for a given $\varepsilon \in (0, \alpha \wedge \varepsilon_0)$ there exists a $\rho_0 > 0$ and a function ϕ satisfying*

$$\begin{cases} L\phi + b \cdot \nabla\phi &\leq -d^{\varepsilon - s} \quad \text{in } \Omega \cap \{d \leq \rho_0\} \\ C^{-1}d^s \leq \phi &\leq Cd^s \quad \text{in } \Omega \\ \phi &= 0 \quad \text{in } \Omega^c. \end{cases}$$

The constant C depends only on $n, s, \varepsilon, \Omega$ and ellipticity constants.

Proof. By Lemma 2.3.2, we have that

$$|Ld^s| \leq Cd^{(\alpha - s)\vee 0},$$

and by [72, Lemma 2.7], we have

$$Ld^{s+\varepsilon} \geq c_0 d^{\varepsilon - s} - C_0,$$

for some $c_0 > 0$. Consider the function

$$\phi_1 = d^s - cd^{s+\varepsilon},$$

with $c > 0$ small enough. Then we have

$$\begin{aligned} L\phi_1 + b\nabla\phi_1 &\leq Cd^{(\alpha-s)\vee 0} + C|\nabla d^s| - cc_0 d^{\varepsilon - s} + C|\nabla d^{s+\varepsilon}| \leq \\ &\leq -cc_0 d^{\varepsilon - s} + C\left(d^{(\alpha - s)\vee 0} + Cd^{s-1} + Cd^{s+\varepsilon-1}\right) \leq \frac{cc_1}{2} d^{\varepsilon - s}, \end{aligned}$$

in $\Omega \cap \{d \leq \rho_0\}$ provided that $\varepsilon < \varepsilon_0 \wedge \alpha$. By construction we have

$$\phi_1 = 0 \quad \text{in } \Omega^c,$$

as well as

$$C^{-1}d^s \leq \phi_1 \leq Cd^s.$$

Hence a suitable rescaling of ϕ_1 is the searched function. □

Having all the above ingredients, we can now prove the following.

Proposition 2.6.6. *Let $s \in (1/2, 1)$, and let L be an operator whose kernel satisfies condition* (2.1.1) *and let Ω be a $C^{1,\alpha}$ domain. Let u be a solution to* (2.1.5)*, with*

$$|f| \le C_0 d^{\varepsilon - s} \quad in\ \Omega \cap B_1,$$

for some positive ε.

Then

$$||u||_{C^s(B_{1/2})} \le C \left(C_0 + \sup_{R \ge 1} R^{\delta - 2s} ||u||_{L^\infty(B_R)} \right).$$

The constant C depends only on $n, s, \varepsilon, \delta, \Omega$ and ellipticity constants.

Proof. The wanted regularity follows from Lemma 2.6.3, Lemma 2.6.5 and Lemma 2.6.4 exactly as in [72, Proposition 3.1]. □

Before proceeding to the finer description of the boundary behaviour of solutions, let us derive the following estimates.

Lemma 2.6.7. *Let $s \in (1/2, 1)$, and let L be an operator whose kernel satisfies condition* (2.1.1) *and let Ω be a C^1 domain. Let u be a solution to* (2.1.5)*, with $|u| \le C_0 d^s$ in $\Omega \cap B_1$, and $f \in C^\theta(\overline{\Omega})$, for some $\theta > 0$.*

Then we have

$$[u]_{C^\gamma(B_r(x_0))} \le C_\gamma r^{s-\gamma}, \tag{2.6.1}$$

whenever $d(x_0) = 2r$, as long as $\gamma \le \theta + 2s - 1$. Furthermore,

$$|\nabla u| \le C d^{s-1}. \tag{2.6.2}$$

The constants C and C_γ depend only on $n, s, C_0, ||f||_{C^\theta(\overline{\Omega})}$, and ellipticity constants.

Proof. Choose a point $x_0 \in \Omega \cap B_{1/2}$, denote $2r = d(x_0)$ and apply interior estimates (Lemma 2.6.3) on $u_r(x) := u(x_0 + rx)$. With aid of Lemma 2.8.3 and the bound on growth of u near the boundary, we get

$$r^{2s} [u]_{C^{2s}(B_r(x_0))} \le C(r^{2s} + r^s),$$

which tells

$$[u]_{C^{2s}(B_r(x_0))} \le C r^{-s}.$$

Rewriting it for the gradient we get

$$[\nabla u]_{C^{2s-1}(B_r(x_0))} \le C r^{-s}.$$

Using Lemma 2.8.5, we deduce (2.6.2).

To get higher order interior estimate (2.6.1), we successively apply higher order interior estimates [1, Proposition 3.9]. Concretely assume we already have estimate

$$[u]_{C^{\gamma'}(B_r(x_0))} \le C r^{s-\gamma'},$$

for some $\gamma' > 1$. Then the gradient satisfies $[\nabla u]_{C^{\gamma'-1}(B_r(x_0))} \leq Cr^{s-\gamma'}$, and so applying [1, Proposition 3.9],[4] we get

$$r^{\gamma'+\varepsilon_0}[u]_{C^{\gamma'+\varepsilon_0}(B_r(x_0))} \leq C\left(\left|\left|\frac{u(x_0+r\cdot)}{1+|\cdot|^{\gamma'+\varepsilon_0}}\right|\right|_{L^\infty(\mathbb{R}^n)} + r^{\gamma'+\varepsilon_0}\left([f-b\cdot\nabla u]_{C^{\gamma'-1}(B_r(x_0))}\right)\right)$$
$$\leq C(r^s + r^{s-\gamma'}),$$

in view of Lemma 2.8.3, provided $\gamma' - 1 \leq \theta$, and so

$$[u]_{C^{\gamma'+\varepsilon_0}(B_r(x_0))} \leq Cr^{s-\gamma'-\varepsilon_0}.$$

With similar argumentation as in Lemma 2.8.5, we get the wanted estimate for all parameters between γ' and $\gamma' + \varepsilon_0$ as well. □

Remark 2.6.8. Note that the assumption $|u| \leq Cd^s$ can be omitted if we have that the domain is $C^{1,\alpha}$. Then the growth control follows from the boundary regularity.

2.6.2 Boundary Schauder and boundary Harnack estimates

We continue to study of the behaviour near the boundary of solutions to (2.1.5).

The strategy is to gradually expand the solution in terms of the powers of the distance function with use of the expansion results established in the previous section. We improve the accuracy of the expansion of the solution in three steps. First we show how the already established expansion translates to the gradient, which we treat as the right-hand side. Then we need to correct the expansion of the solution, in such a way that its operator evaluation becomes small. In this step, the key ingredient is Theorem 2.4.7. Finally, we apply one of the expansion results from previous section.

Since the procedure differs in three different cases of the regularity of the data we split the result according to the relevant settings. Let us point out, that in the proofs we are doing expansions with functions of the form $Qd^{s+k\varepsilon_0+l}$, where Q is a polynomial. We use Theorem 2.4.7, where the power of the distance has to satisfy $s + k\varepsilon_0 + l - 2s \notin \mathbb{N}$ and $s + k\varepsilon_0 + l - s \notin \mathbb{N}$. This is equivalent to $s \notin \mathbb{Q}$. This is the reason we need to assume that $s \in (\frac{1}{2}, 1)\backslash\mathbb{Q}$.

Let us start with the following:

Proposition 2.6.9. *Let $s > \frac{1}{2}$, $s \notin \mathbb{Q}$, and assume $\Omega \subset \mathbb{R}^n$ is a C^β domain with $1 < \beta < 1 + s$, $\beta + s \notin \mathbb{N}$. Let L be an operator whose kernel K satisfies conditions* (2.1.1). *Let $u \in L^\infty(\mathbb{R}^n)$ be a solution to* (2.1.5) *with $|f| \leq C_1 d^{\beta-1-s}$ and $b \in \mathbb{R}^n$.*

Then
$$\left|\left|\frac{u}{d^s}\right|\right|_{C^{\varepsilon_0\wedge(\beta-1)}(\overline{\Omega}\cap B_{1/2})} \leq C\left(C_1 + ||u||_{L^\infty(\mathbb{R}^n)}\right),$$
and
$$\left|\left|\frac{u}{d^s}\right|\right|_{C^{\beta-1}(\partial\Omega\cap B_{1/2})} \leq C\left(C_1 + ||u||_{L^\infty(\mathbb{R}^n)}\right).$$
The constant C depends only on n, s, β, Ω and ellipticity constants.

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[4]We have to apply the interior estimates on smaller and smaller balls, but for simplicity let us always write the estimates with $B_r(x_0)$.

Moreover, there exist constants Q_z^j*, for* $j = 0, \ldots, k$*, where* $k = \lceil \frac{\beta-1}{\varepsilon_0} \rceil - 1$*, with* $Q_z^j \in C_z^{\beta-1-j\varepsilon_0}(\partial\Omega \cap B_{1/2})$*, so that for*

$$\tilde{u}_z := u - Q_z^k d^{s+k\varepsilon_0} + \ldots + Q_z^1 d^{s+\varepsilon_0},$$

we have $|L\tilde{u}_z| \leq |x-z|^{\beta-2s} d^{s-1}$*,*

$$\left|\tilde{u}_z - Q_z^0 d^s\right| \leq C|x-z|^{\beta-1+s},$$

and

$$\left[\tilde{u}_z - Q_z^0 d^s\right]_{C^{\beta-1+s}(B_{r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{\beta-1+s},$$

whenever $d(x_1) = 2r_1$*. The constant* C *depends only on* n, s, β, Ω *and ellipticity constants.*

Proof. Because of the homogeneity of the equation, we can assume that $C_1 + ||u||_{L^\infty(\mathbb{R}^n)} = 1$. Notice, that u solves

$$\begin{cases} Lu &= f - b \cdot \nabla u & \text{in } \Omega \cap B_1 \\ u &= 0 & \text{in } \Omega^c \cap B_1, \end{cases}$$

where $|f - b \cdot \nabla u| \leq C d^{s-1}$, thanks to (2.6.2). Therefore we are in position to apply Lemma 2.5.1, to get that for every $z \in \partial\Omega \cap B_{3/4}$ there is a constant Q_z, such that

$$|u - Q_z d^s| \leq C|x-z|^{(s+\varepsilon_0)\wedge(s+\beta-1)}.$$

The interior regularity result [74, Corollary 3.6] together with Lemma 2.8.3 render

$$[u - Q_z d^s]_{C^{2s}(B_r(x_0))} \leq C r^{\varepsilon_0 \wedge (\beta-1) - s},$$

whenever $d(x_0) = |x_0 - z| = 2r$. The corollary can be applied due to Lemma 2.3.2, which assures that $|L(d^s)| \leq C d^{\beta-1-s}$. In particular, this implies that

$$\left|\left|\frac{u}{d^s}\right|\right|_{C^{\varepsilon_0\wedge(\beta-1)}(\overline{\Omega}\cap B_{1/2})} \leq C,$$

thanks to Lemma 2.8.6. Moreover, Lemma 2.8.8 gives that $|Q_z - Q_{z'}| \leq C|z-z'|^{\varepsilon_0\wedge(\beta-1)}$. Note that $Q_z = \frac{u}{d^s}(z)$, and so the proposition is proven if $\beta - 1 < \varepsilon_0$. Therefore we continue assuming $\varepsilon_0 < \beta - 1$.

Reading the interior regularity estimate for the gradient, we get

$$\left[\nabla w_i - s Q_z^i \nabla d d^{s-1}\right]_{C^{2s-1}(B_r(x_0))} \leq C r^{\varepsilon_0 - s},$$

which by Lemma 2.8.5 gives ($2\varepsilon_0 - s < 0$)

$$|\nabla w_i(x) - s Q_z^i \nabla d(x) d^{s-1}(x)| \leq C d^{2\varepsilon_0 - s}(x)$$

on the cone $\mathcal{C}_z := \bigcup \left\{B_r(x_0);\ \text{where } d(x_0) = 2r = |x-z|\right\}$. This furthermore implies

$$|\nabla u(x) - s Q_z \nabla d(z) d^{s-1}(x)| \leq C|x-z|^{\varepsilon_0} d^{s-1}(x),$$

since $\nabla d \in C^{\varepsilon_0}(\overline{\Omega})$, and on such cone $|z-x|$ is comparable to $d(x)$. To derive the same inequality on the neighbourhood of the boundary point, we do the following

$$\begin{aligned}|\nabla u(x) - sQ_z \nabla d(z) d^{s-1}(x)| \leq & \, |\nabla u(x) - sQ_{z'} \nabla d(z') d^{s-1}(x)| + \\ & + s|Q_{z'} \nabla d(z') - Q_z^i \nabla d(z)| d^{s-1}(x) \\ \leq & \, d^{2\varepsilon_0 - s}(x) + C|z-z'|^{\varepsilon_0} d^{s-1}(x) \\ \leq & \, C|x-z|^{\varepsilon_0} d^{s-1}(x),\end{aligned}$$

and so

$$|b \cdot \nabla u - sQ_z b \cdot \nabla d(z) d^{s-1}(x)| \leq C|x-z|^{\varepsilon_0} d^{s-1}(x).$$

Then we use Lemma 2.3.2, to find a constant $\tilde{Q}_z = -sQ_z b \cdot \nabla d(z) (c_{s+\varepsilon_0} |\nabla d(z)|^{2s})^{-1}$, which is $C_z^{\varepsilon_0}$, so that

$$L(u - \tilde{Q}_z d^{s+\varepsilon_0}) = f - b \cdot \nabla u + \tilde{Q}_z c_{s+\varepsilon_0} |\nabla d(x)|^{2s} d^{s-1}(x) + R(x),$$

where $|R| \leq d^{2\varepsilon_0 - s}$. Combining it with the expansion of the gradient of u and the regularity of ∇d, we get

$$\left| L(u - \tilde{Q}_z d^{s+\varepsilon_0}) \right| \leq C|x-z|^{\varepsilon_0} d^{s-1}(x).$$

We apply Lemma 2.5.1, to get a constant Q_z, such that

$$\left| u - \tilde{Q}_z d^{s+\varepsilon_0} - Q d^s \right| \leq C|x-z|^{2\varepsilon_0 \wedge (\beta-1)+s},$$

and Lemma 2.8.8 gives that Q_z is $C^{2\varepsilon_0 \wedge (\beta-1)}$ in variable z.

We proceed in a similar manner to get the expansion

$$\tilde{u}_z := u - Q_z^k d^{s+k\varepsilon_0} + \ldots + Q_z^1 d^{s+\varepsilon_0},$$

so that $|L\tilde{u}_z| \leq C|x-z|^{\beta-2s} d^{s-1}$, and $Q_z^j \in C_z^{(k-j+1)\varepsilon_0}(\partial\Omega \cap B_{1/2})$. Note that this is not less regularity than stated. It remains to apply Lemma 2.5.1 one last time, to get a constant Q_z^0, so that

$$|\tilde{u}_z - Q_z^0 d^s| \leq C|x-z|^{\beta-1+s},$$

and

$$\left[\tilde{u}_z - Q_z^0 d^s \right]_{C^{\beta-1+s}(B_{r_1}(x_1))} \leq C \left(\frac{|x_1 - z|}{r_1} \right)^{\beta-1+s},$$

whenever $d(x_1) = 2r_1$. Finally, Lemma 2.8.8 gives that $||\frac{u}{d^s}||_{C^{\beta-1}(\partial\Omega \cap B_{1/2})} \leq C$, as wanted. $\square$

Let us now show, how the arguments change, when $(k+1)\varepsilon_0$ just exceeds s. Basically, the difference is that we can not use Lemma 2.3.2, since it does not provide enough regularity. Instead we apply Theorem 2.4.7, while the rest remains the same.

Proposition 2.6.10. *Let $s > \frac{1}{2}$, $s \notin \mathbb{Q}$, and assume $\Omega \subset \mathbb{R}^n$ is a C^β domain with $1 + s < \beta \leq 1 + k_0\varepsilon_0$, where k_0 is the least integer such that $k_0\varepsilon_0 > 1$. Let L be an operator* *whose kernel K is $C^{2\beta+1}(\mathbb{S}^{n-1})$ and satisfies conditions* (2.1.1). *Let $u \in L^\infty(\mathbb{R}^n)$ solve* (2.1.5) *with $f \in C^{\beta-1-s}(\overline{\Omega})$ and $b \in \mathbb{R}^n$. Then*

$$\Big|\Big|\frac{u}{d^s}\Big|\Big|_{C^{\beta-1}(\partial\Omega\cap B_{1/2})} \le C\left(||f||_{C^{\beta-1-s}(\Omega\cap B_1)} + ||u||_{L^\infty(\mathbb{R}^n)}\right)$$

The constant C depends only on n, s, β, Ω and $||K||_{C^{2\beta+1}(\mathbb{S}^{n-1})}$.

Moreover, there exist polynomials $Q_z^j \in \boldsymbol{P}_{\lfloor \beta-1-j\varepsilon_0\rfloor}$,[5] *for $j = 0, \ldots, k$, where $k = \lceil \frac{\beta-1}{\varepsilon_0}\rceil - 1$, with the γ-th coefficient $(Q_z^j)^{(\gamma)} \in C_z^{\beta-1-j\varepsilon_0-|\gamma|}(\partial\Omega \cap B_{1/2})$, so that for*

$$\tilde{u}_z := u - Q_z^k d^{s+k\varepsilon_0} + \ldots + Q_z^1 d^{s+\varepsilon_0},$$

there is a polynomial P_z so that

$$\begin{aligned} |L\tilde{u}_z - P_z| \le& C|x-z|^{\beta-1-\varepsilon_0} d^{s-1}, \\ [L\tilde{u}_z]_{C^{\beta-1-s}(B_{\frac{3}{2}r_1}(x_1))} \le& C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-s}, \\ \left|\tilde{u}_z - Q_z^0 d^s\right| \le& C|x-z|^{\beta-1+s}, \quad \textit{and} \\ \left[\tilde{u}_z - Q_z^0 d^s\right]_{C^{\beta-1+s}(B_{\frac{3}{2}r_1}(x_1))} \le& C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-1+s}, \end{aligned}$$

whenever $d(x_1) = 2r_1$. The constant C depends only on n, s, β, Ω and ellipticity constants.

Proof. We want to continue with the expansion of u, where we stopped in the previous proposition. Note that u admits (2.6.1) and (2.6.2). Hence, for k such that $k\varepsilon_0 < s < (k+1)\varepsilon_0$, the previous proposition indeed gives existence of constants $Q_z^j \in C_z^{(k-j)\varepsilon_0}(\partial\Omega\cap B_{3/4})$, for $j = 0, \ldots, k-1$, so that

$$\left|u - Q_z^{k-1} d^{s+(k-1)\varepsilon_0} - \ldots - Q_z^0 d^s\right| \le C|x-z|^{k\varepsilon_0+s}$$

and

$$\left[u - Q_z^{k-1} d^{s+(k-1)\varepsilon_0} - \ldots - Q_z^0 d^s\right]_{C^{k\varepsilon_0+s}(B_{\frac{3}{2}r_1}(x_1))} \le C\left(\frac{|x_1-z|}{r_1}\right)^{k\varepsilon_0+s},$$

whenever $d(x_1) = r_1$. Notice that $k\varepsilon_0 + s > 1$, since $(k+1)\varepsilon_0 > s$. With aid of Lemma 2.8.2 we conclude that in the cone $\mathcal{C}_z$ we have

$$\left|\nabla\big(u - Q_z^{k-1} d^{s+(k-1)\varepsilon_0} - \ldots - Q_z^0 d^s\big)\right| \le C|x-z|^{k\varepsilon_0+s-1},$$

as well as

$$\left[\nabla\left(u - Q_z^{k-1} d^{s+(k-1)\varepsilon_0} - \ldots - Q_z^0 d^s\right)\right]_{C^{k\varepsilon_0+s-1}(B_{\frac{3}{2}r_1}(x_1))} \le C\left(\frac{|x_1-z|}{r_1}\right)^{k\varepsilon_0+s},$$

whenever $d(x_1) = 2r_1$. Taking into account that $\nabla d \in C^{k\varepsilon_0}(\overline{\Omega})$ and $|\nabla d(x) - \nabla d(z)| \le C|x-z|^{k\varepsilon_0}$, the above implies that for $x \in \mathcal{C}_z$, we have

$$\left|\nabla u - ((k-1)\varepsilon_0 + s)Q_z^{k-1}\nabla d(z) d^{(k-1)\varepsilon_0+s-1} - \ldots - sQ_z^0 \nabla d(z) d^{s-1}\right| \le C|x-z|^{k\varepsilon_0+s-1},$$

[5]The only case when polynomial is non-constant is when $j = 1$ and $\beta > 2$.

as well as

$$\left[\nabla u - ((k-1)\varepsilon_0 + s)Q_z^{k-1}\nabla d(z) d^{(k-1)\varepsilon_0+s-1} - \ldots - sQ_z^0 \nabla d(z) d^{s-1}\right]_{C^{k\varepsilon_0+s-1}} \leq$$
$$\leq C\left(\frac{|x_1 - z|}{r_1}\right)^{k\varepsilon_0+s},$$

in $B_{\frac{3}{2}r_1}(x_1)$, thanks to Lemma 2.8.9. As in the previous proposition we derive the expansion of the gradient in the neighbourhood, namely for $x \in \Omega \cap B_1$,

$$\left|\nabla u - ((k-1)\varepsilon_0 + s)Q_z^{k-1}\nabla d(z) d^{(k-1)\varepsilon_0+s-1} - \ldots - sQ_z^0 \nabla d(z) d^{s-1}\right| \leq C|x-z|^{k\varepsilon_0} d^{s-1}(x).$$

Using Theorem 2.4.7 we now find constants $\tilde{Q}_z^j \in C^{(k+1-j)\varepsilon_0}(\partial\Omega)$, $j = 1, \ldots, k$, so that

$$L(\tilde{Q}_z^j d^{j\varepsilon_0+s}) = ((j-1)\varepsilon_0 + s)Q_z^{j-1} \cdot b \cdot \nabla d(z) d^{(j-1)\varepsilon_0+s-1} + R_j + \eta_j,$$

where $R_j \in C^{k\varepsilon_0-2s+j\varepsilon_0+s}$ and $\eta_j = \phi_{j\varepsilon_0+s} d^{(j-1)\varepsilon_0+s-1}$. The result gives that $\phi_{j\varepsilon_0+s} \in C^{1+k\varepsilon_0}(\overline{\Omega})$, and by construction $\phi_{j\varepsilon_0+s}(z) = 0$. Hence we have $|\eta_j| \leq C|x-z| d^{(j-1)\varepsilon_0+s-1} \leq |x-z|^{k\varepsilon_0} d^{s-1}(x)$, and by Lemma 2.8.9 also $[\eta_j]_{C^{k\varepsilon_0+s-1}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1-z|}{r_1}\right)^{k\varepsilon_0}$. So if we define

$$\tilde{u}_z := u - \tilde{Q}_z^k d^{k\varepsilon_0+s} - \ldots - \tilde{Q}_z^1 d^{\varepsilon_0+s},$$

we have

$$|L\tilde{u}_z - P_z| \leq C|x-z|^{k\varepsilon_0} d^{s-1}(x),$$

for a suitable Taylor polynomial P_z and

$$[L\tilde{u}_z]_{C^{k\varepsilon_0+s-1}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{k\varepsilon_0+s}.$$

Applying Lemma 2.5.2, we get a polynomial Q_z of degree $\lfloor (k+1)\varepsilon_0 \wedge (\beta - 1)\rfloor$, so that

$$|\tilde{u}_z - Q_z d^s| \leq C|x-z|^{(k+1)\varepsilon_0 \wedge (\beta-1)+s},$$

and

$$[\tilde{u}_z - Q_z d^s]_{C^{(k+1)\varepsilon_0\wedge(\beta-1)+s}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{(k+1)\varepsilon_0\wedge(\beta-1)+s}.$$

With aid of Lemma 2.8.8, we get the wanted regularity of the coefficients, in particular the zero order satisfies

$$||(Q_z)^{(0)}||_{C^{(k+1)\varepsilon_0\wedge(\beta-1)}(\partial\Omega\cap B_{1/2})} = \left|\left|\frac{u}{d^s}\right|\right|_{C^{(k+1)\varepsilon_0\wedge(\beta-1)}(\partial\Omega\cap B_{1/2})} \leq C.$$

If $\beta - 1 \leq (k+1)\varepsilon_0$ we are done, otherwise we continue following the same steps. □

When regularity becomes greater than 2, it becomes harder to translate the expansion of the function to the expansion of its gradient, since the polynomials in the expansions start getting non-constant terms. This is where Lemma 2.8.10 and Corollary 2.8.11 become useful. Note also, that these results give a slightly worse estimate, which is the reason for establishing the boundary regularity result - Lemma 2.4.9 which plays the key role when establishing Lemma 2.5.2 and Lemma 2.5.7.

Proposition 2.6.11. *Let $s > \frac{1}{2}$, $s \notin \mathbb{Q}$, and assume $\Omega \subset \mathbb{R}^n$ is a C^β domain with $\beta > k\varepsilon_0$, for $k\varepsilon_0 > 1 > (k-1)\varepsilon_0$, and $\beta \pm s \notin \mathbb{N}$. Let L be an operator whose kernel K is $C^{2\beta+1}(\mathbb{S}^{n-1})$ and satisfies conditions (2.1.1). Let $u \in L^\infty(\mathbb{R}^n)$ be a solution to (2.1.5) with $f \in C^{\beta-1-s}(\overline{\Omega})$ and $b \in \mathbb{R}^n$. Then*

$$\left|\left|\frac{u}{d^s}\right|\right|_{C^{\beta-1}(\partial\Omega\cap B_{1/2})} \leq C\left(||f||_{C^{\beta-1-s}(\Omega\cap B_1)} + ||u||_{L^\infty(\mathbb{R}^n)}\right)$$

The constant C depends only on n, s, β, Ω and $||K||_{C^{2\beta+1}(\mathbb{S}^{n-1})}$.

Moreover, there exist polynomials $Q_z^{j,l} \in \boldsymbol{P}_{\lfloor\beta-1-j\varepsilon_0-l\rfloor}$, for $j, l \geq 0$, such that $j\varepsilon_0 + l < \beta - 1$, with the γ-th coefficient $(Q_z^{j,l})^{(\gamma)} \in C_z^{\beta-1-j\varepsilon_0-l-|\gamma|}(\partial\Omega \cap B_{1/2})$, so that for

$$\tilde{u}_z := u - \sum_{j\geq 1, l\geq 0} Q_z^{j,l} d^{j\varepsilon_0+l+s}$$

there is a polynomial P_z so that

$$\begin{aligned}
|L\tilde{u}_z - P_z| &\leq C|x-z|^{\beta-1-\varepsilon_0+s}d^{-1}, \\
[L\tilde{u}_z]_{C^{\beta-1-s}(B_{\frac{3}{2}r_1}(x_1))} &\leq C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-s}, \\
|\tilde{u}_z - Q_z^{0,0}d^s| &\leq C|x-z|^{\beta-1+s}, \quad \text{and} \\
\left[\tilde{u}_z - Q_z^{0,0}d^s\right]_{C^{\beta-1+s}(B_{\frac{3}{2}r_1}(x_1))} &\leq C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-1+s},
\end{aligned}$$

whenever $d(x_1) = 2r_1$. The constant C depends only on n, s, β, Ω and ellipticity constants.

Proof. We apply the previous proposition, to get polynomials $Q_z^j \in \mathbf{P}_{\lfloor(k-j)\varepsilon_0\rfloor}$ with the γ-th coefficient $(Q_z^j)^{(\gamma)} \in C^{(k-j)\varepsilon_0-|\gamma|}$, for $j = 0, \ldots, k-1$ so that

$$|u - Q_z^{k-1}d^{(k-1)\varepsilon_0+s} - \ldots - Q_z^0 d^s| \leq C|x-z|^{k\varepsilon_0+s},$$

and

$$\left[u - Q_z^{k-1}d^{(k-1)\varepsilon_0+s} - \ldots - Q_z^0 d^s\right]_{C^{k\varepsilon_0+s}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1-z|}{r_1}\right)^{k\varepsilon_0+s},$$

whenever $d(x_1) = 2r_1$.

We proceed with applying Corollary 2.8.11, we get

$$\left|\nabla\left(u - Q_z^{k-1}d^{(k-1)\varepsilon_0+s} - \ldots - Q_z^0 d^s\right)\right| \leq C|x-z|^{k\varepsilon_0+s}d^{-1}(x),$$

while

$$\left[\nabla\left(u - Q_z^{k-1}d^{(k-1)\varepsilon_0+s} - \ldots - Q_z^0 d^s\right)\right]_{C^{k\varepsilon_0+s-1}(B_{r_1}(x_1))} \leq C\left(\frac{|x_1-z|}{r_1}\right)^{k\varepsilon_0+s}$$

follows straightforward from the interior estimate above. Taking into account that $\nabla d - T_z^1(\nabla d)$ is $C^{k\varepsilon_0} \cap O(|x-z|^{k\varepsilon_0})$, we get the refined expansion of the form

$$\left|\nabla u - \sum_{j,l\geq 0}^{j\varepsilon_0+l<k\varepsilon_0} P_z^{j,l}d^{j\varepsilon_0+l+s-1}\right| \leq C|x-z|^{k\varepsilon_0+s}d^{-1}(x),$$

where $P_z^{j,l}$ is a polynomial of degree $\lfloor k\varepsilon_0 - j\varepsilon_0 - l \rfloor$, with the $\alpha-$th coefficient being $C_z^{k\varepsilon_0 - j\varepsilon_0 - l - |\alpha|}$ smooth (multiplying a polynomial with this property with the Taylor polynomial of ∇d preserves this property). Using Lemma 2.8.9 we also get the interior part of the estimate

$$\left[\nabla u - \sum_{j,l\geq 0}^{j\varepsilon_0 + l < k\varepsilon_0} P_z^{j,l} d^{j\varepsilon_0 + l + s - 1}\right]_{C^{k\varepsilon_0 + s - 1}(B_{r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{k\varepsilon_0 + s}.$$

Using Theorem 2.4.7 we find polynomials $\tilde{Q}_z^{j,l}$, $j \geq 1, l \geq 0$ of degree $\lfloor (k+1)\varepsilon_0 - j\varepsilon_0 - l \rfloor$, with the $\alpha-$th coefficient being $C_z^{(k+1)\varepsilon_0 - j\varepsilon_0 - l - |\alpha|}$ smooth, so that

$$L(\tilde{Q}_z^{j,l} d^{j\varepsilon_0 + l + s}) = P_z^{j-1,l} d^{(j-1)\varepsilon_0 + l + s - 1} + R_{j,l} + \eta_{j,l},$$

with $R_{j,l} \in C^{k\varepsilon_0 - s + j\varepsilon_0 + l}(\overline{\Omega})$ (since $j \geq 1$, the power is greater than $k\varepsilon_0 + s - 1$) and $\eta_{j,l} = \phi_{j,l} d^{(j-1)\varepsilon_0 + l + s - 1}$, and $\phi_{j,l} \in C^{1 + k\varepsilon_0} \cap O(|x - z|^{\lceil (k+1)\varepsilon_0 - j\varepsilon_0 - l \rceil})$ so the worst is when $j = 1, l = 0$, when we get $|\eta_{j,l}| \leq C|x - z|^{k\varepsilon_0} d^{s-1}$, which gives $[\eta_{j,l}]_{C^{k\varepsilon_0 + s - 1}(B_{r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{k\varepsilon_0}$. Hence, defining

$$\tilde{u}_z := u - \sum_{j\geq 1, l\geq 0} \tilde{Q}_z^{j,l} d^{j\varepsilon_0 + l + s}$$

and refining the estimates with Lemma 2.8.9 we have

$$|L\tilde{u}_z - P_z| \leq C|x - z|^{k\varepsilon_0 + s} d^{-1}(x),$$

for a suitable Taylor polynomial P_z and

$$[L\tilde{u}_z]_{C^{k\varepsilon_0 + s - 1}(B_{r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{k\varepsilon_0 + s},$$

whenever $d(x_1) = 2r_1$. We apply Lemma 2.5.2, we get a polynomial Q_z of degree $\lfloor (k+1)\varepsilon_0 \wedge (\beta - 1) \rfloor$, so that

$$|\tilde{u}_z - Q_z d^s| \leq C|x - z|^{(k+1)\varepsilon_0 \wedge (\beta - 1) + s},$$

and

$$[\tilde{u}_z - Q_z d^s]_{C^{(k+1)\varepsilon_0 \wedge (\beta - 1) + s}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1 - z|}{r_1}\right)^{(k+1)\varepsilon_0 \wedge (\beta - 1) + s}.$$

With aid of Lemma 2.8.8, we get the wanted regularity of the coefficients of Q_z, in particular the zero order satisfies

$$\left|\left|(Q_z)^{(0)}\right|\right|_{C^{(k+1)\varepsilon_0 \wedge (\beta - 1)}(\partial\Omega \cap B_{1/2})} = \left|\left|\frac{u}{d^s}\right|\right|_{C^{(k+1)\varepsilon_0 \wedge (\beta - 1)}(\partial\Omega \cap B_{1/2})} \leq C.$$

If $\beta - 1 \leq (k+1)\varepsilon_0$ we are done, otherwise we continue following the same steps. □

We can now also provide the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. It is a special case of Proposition 2.6.11. □

Now we turn to the ratio of two solutions to (2.1.5). In the following result we establish the optimal regularity of the quotient up to the boundary.

Proposition 2.6.12. *Let $s > \frac{1}{2}$, and assume $\Omega \subset \mathbb{R}^n$ is a C^1 domain. Let L be an operator whose kernel K satisfies conditions* (2.1.1)*. For $i = 1, 2$ let $u_i \in L^\infty(\mathbb{R}^n)$ satisfy*

$$|u_i| \le C_1 d^s,$$

and solve

$$\begin{cases} Lu_i + b \cdot \nabla u_i &= f_i \quad \text{in } \Omega \cap B_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap B_1, \end{cases}$$

with $|f_i| \le K_0 d^{s-1}$ and $b \in \mathbb{R}^n$. Assume $C_2 d^s \ge u_2 \ge c_2 d^s$ for some positive C_2, c_2. Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C^{\varepsilon_0}(\overline{\Omega} \cap B_{1/2})} \le C\left(K_0 + ||u||_{L^\infty(\mathbb{R}^n)}\right)$$

The constant C depends only on $n, s, \beta, \Omega, C_1, C_2, c_2$, and ellipticity constants.

Proof. In particular, u_i solve

$$\begin{cases} Lu_i &= f_i - b \cdot \nabla u_i \quad \text{in } \Omega \cap B_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap B_1, \end{cases}$$

where $|f_i - b \cdot \nabla u_i| \le C d^{s-1}$, thanks to (2.6.2). Hence Lemma 2.5.3 assures that for every $z \in \partial\Omega \cap B_{1/2}$ we get the existence of Q_z, so that

$$|u_1(x) - Q_z u_2(x)| \le C|x - z|^{\varepsilon_0 + s}, \quad x \in B_{1/2}(z)$$

and

$$[u_1 - Q_z u_2]_{C^{s+\varepsilon_0}(B_r(x_0))} \le C,$$

whenever $d(x_0) = |x_0 - z| = 2r$. The latter, together with assumptions on u_2 allow us to apply Lemma 2.8.6 to get

$$[u_1/u_2]_{C^{\varepsilon_0}(B_r(x_0))} \le C,$$

which gives that the quotient is $C^{\varepsilon_0}(\overline{\Omega} \cap B_{1/2})$ □

Similarly as in the case of the quotient with the distance function, the higher order expansions provide the higher regularity only on the boundary. But in the case of two solutions, we can make one more step of expansions, and then apply Lemma 2.5.7. This yields an improvement of the regularity of size ε_0 with respect to the case of the quotient with the distance function.

Proposition 2.6.13. *Let $s > \frac{1}{2}$, $s \notin \mathbb{Q}$, and assume $\Omega \subset \mathbb{R}^n$ is a C^β domain with $\beta > 1$ and $\beta \pm s \notin \mathbb{N}$. Let L be an operator whose kernel K is $C^{2\beta+1}(\mathbb{S}^{n-1})$ and satisfies conditions* (2.1.1)*. For $i = 1, 2$, let $u_i \in L^\infty(\mathbb{R}^n)$ be two solutions to*

$$\begin{cases} Lu_i + b \cdot \nabla u &= f_i \quad \text{in } \Omega \cap B_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap B_1, \end{cases}$$

with $K_i := ||f_i||_{C^{\beta-1-s+\varepsilon_0}(\overline{\Omega}\cap B_1)} < \infty$ *and* $b \in \mathbb{R}^n$.[6] *When* $\beta < 1+s-\varepsilon_0$ *we assume* $K_i := ||fd^{1+s-\varepsilon_0-\beta}||_{L^\infty(\Omega\cap B_1)} < \infty$ *instead. Assume that* $C_2 d^s \geq u_2 \geq c_2 d^s$*, for some positive* C_2, c_2*. Then*

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C^{\beta-1+\varepsilon_0}(\partial\Omega\cap B_{1/2})} \leq C\left(K_1 + ||u_1||_{L^\infty(\mathbb{R}^n)}\right).$$

The constant C *depends only on* $n, s, \beta, \Omega, C_2, c_2$ *and* $||K||_{C^{2\beta+1}(\mathbb{S}^{n-1})}$.

Proof. We can assume that $K_1 + ||u_1||_{L^\infty(\mathbb{R}^n)} = 1$. Depending on β, we apply the relevant result amongst Proposition 2.6.9, Proposition 2.6.10, or Proposition 2.6.11, to get polynomials $Q^{j,l}_{i,z} \in \mathbf{P}_{\lfloor\beta-1-j\varepsilon_0-l\rfloor}$, for $j, l \geq 0$, such that $j\varepsilon_0 + l < \beta - 1$, with the γ-th coefficient $(Q^{j,l}_{i,z})^{(\gamma)} \in C_z^{\beta-1-j\varepsilon_0-l-|\gamma|}(\partial\Omega\cap B_{1/2})$, so that

$$\left|u_i - \sum_{j\geq 0, l\geq 0} Q^{j,l}_{i,z} d^{j\varepsilon_0+l+s}\right| \leq C|x-z|^{\beta-1+s},$$

and

$$\left[u_i - \sum_{j\geq 0, l\geq 0} Q^{j,l}_{i,z} d^{j\varepsilon_0+l+s}\right]_{C^{\beta-1+s}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-1+s},$$

whenever $d(x_1) = 2r_1$. Since the regularity of f_i is $C^{\beta-1-s+\varepsilon_0}(\overline{\Omega})$, we are able to make one more step when doing the expansions, to get polynomials $\tilde{Q}^{j,l}_{i,z} \in \mathbf{P}_{\lfloor\beta-1+\varepsilon_0-j\varepsilon_0-l\rfloor}$, for $j \geq 1, l \geq 0$, such that $j\varepsilon_0 + l < \beta - 1 + \varepsilon_0$, with the γ-th coefficient $(\tilde{Q}^{j,l}_{i,z})^{(\gamma)} \in C_z^{\beta-1+\varepsilon_0-j\varepsilon_0-l-|\gamma|}(\partial\Omega\cap B_{1/2})$, so that for

$$\tilde{u}_{i,z} := u_i - \sum_{j\geq 1, l\geq 0} \tilde{Q}^{j,l}_{i,z} d^{j\varepsilon_0+l+s}$$

there is a polynomial P_z so that

$$|L\tilde{u}_{i,z} - P_z| \leq \begin{cases} Cd^{\beta-2+s} & \text{if } \beta < 1+s-\varepsilon_0 \\ C|x-z|^{\beta-1}d^{s-1} & \text{if } 1+s-\varepsilon_0 < \beta < 2 \\ C|x-z|^{\beta-1+s}d^{-1} & \text{if } 2 < \beta, \end{cases}$$

and when $\beta > 1+s-\varepsilon_0$ also

$$[L\tilde{u}_{i,z}]_{C^{\beta-1-s+\varepsilon_0}(B_{\frac{3}{2}r_1}(x_1))} \leq C\left(\frac{|x_1-z|}{r_1}\right)^{\beta-s+\varepsilon_0},$$

whenever $d(x_1) = r_1$. Applying the relevant expansion result, Lemma 2.5.5, Lemma 2.5.6, or Lemma 2.5.7, gives the existence of Q_z of degree $\lfloor\beta-1+\varepsilon_0\rfloor$, so that

$$|\tilde{u}_{1,z} - Q_z\tilde{u}_{2,z}| \leq C|x-z|^{\beta-1+\varepsilon_0+s} \quad \text{and} \quad [\tilde{u}_{1,z} - Q_z\tilde{u}_{2,z}]_{C^{\beta-1+\varepsilon_0+s}(B_r(x_0))} \leq C.$$

[6] We can even allow them to be different for $i = 1, 2$.

Finally, Lemma 2.8.6 and Lemma 2.8.7 assure that

$$\left\|\frac{u_1}{u_2}\right\|_{C^{\beta-1+\varepsilon_0}(\partial\Omega\cap B_{1/2})} \le C,$$

as wanted. ☐

We conclude this section with noticing that this also provides the proof of Theorem 2.1.2.

Proof of Theorem 2.1.2. It is a special case of Proposition 2.6.13. ☐

2.7 Smoothness of the free boundary

In this section we use the developed tools on the height function

$$w := u - \varphi,$$

for solution u of problem (2.1.2). Note that in particular, w solves

$$\begin{cases} Lw = f - b\cdot\nabla w & \text{in } \Omega\cap B_1 \\ w = 0 & \text{in } B_1\cap\Omega^c, \end{cases} \tag{2.7.1}$$

where $f := -(L + b\cdot\nabla)\varphi$ and $\Omega := \{w > 0\}$. The main goal of this section is to prove Theorem 2.1.1. Its proof, as well as this section is divided in two parts. In the first one we establish a general result, stating that if the free boundary $\partial\Omega$ and the height function w satisfy

$$\begin{aligned} 0 &\in \partial\Omega, \\ \partial\Omega\cap B_1 &\in C^1, \\ w &\in C^1(B_1), \\ |Dw| &\le Cd^s, \quad x\in B_1, \\ \partial_\nu w \ge cd^s, \quad c &> 0, \quad x\in B_1, \end{aligned} \tag{2.7.2}$$

where ν is the normal vector to $\partial\Omega$ at 0, then the free boundary $\partial\Omega\cap B_{1/2}$ is roughly as smooth as the obstacle. If the above conditions hold true for some point $x_0\in\partial\Omega$, we say that x_0 is *a regular free boundary point.* Then in the second part we show how to obtain (2.7.2) near the regular free boundary points in the case when the operator is the fractional Laplacian. Let us stress, that in the first part the operator can be very general, its kernel only has to satisfy (2.1.1), and some regularity condition. Therefore as soon as the conditions (2.7.2) are established for solutions of (2.1.2) in this setting, we get the same regularity of the free boundary. Notice also, that if $\partial\Omega\cap B_1$ is $C^{1,\alpha}$, then the fourth assumption in (2.7.2) follows from the fact that $w\in C^1(B_1)$ and Proposition 2.6.6.

For the height function w, we denote

$$w_i := \partial_i w, \qquad i = 1,\dots,n.$$

2.7.1 General stable operators

Suppose that 0 is a free boundary point, at which the height function w satisfies (2.7.2). Without loss of generality we also assume that the normal vector to the free boundary $\nu(0) = e_n$. In order to obtain the higher regularity of the free boundary, we apply the results established in Section 2.6.2 to the quotients w_i/w_n, for $i = 1, \dots, n-1$. Since we can express the normal vector of the free boundary with these quotients (see [1, Section 5]), we get the regularity of the free boundary.

Proposition 2.7.1. *Let φ be an $C^{\theta+s+\varepsilon_0}(\mathbb{R}^n)$ obstacle for some $\theta > 1+s-\varepsilon_0$ with $\theta \notin \mathbb{N}$, $\theta+\varepsilon_0 \pm s \notin \mathbb{N}$. Let L be an operator whose kernel K is $C^{2\theta+1}(\mathbb{S}^{n-1})$ and satisfies* (2.1.1)*. Assume that 0 is a free boundary point at which the conditions* (2.7.2) *hold true. Then the free boundary is C^θ around 0.*

Proof. Note that the partial derivatives w_i solve

$$\begin{cases} Lw_i + b\cdot\nabla w_i &= f_i \quad \text{in } \Omega\cap B_1,\\ \qquad\qquad w_i &= 0 \quad \text{in } B_1\cap\Omega^c, \end{cases}$$

where $f_i = \partial^i f \in C^{\theta-1-s+\varepsilon_0}$. Thanks to Lemma 2.6.7 we can apply Proposition 2.6.12 on w_i and w_n, to get that the quotient $\frac{w_i}{w_n}$ is $C^{\varepsilon_0}(\overline{\Omega}\cap B_{1/2})$. Therefore the normal vector $\nu \in C^{\varepsilon_0}(\overline{\Omega}\cap B_{1/2})$ and hence the boundary is $C^{1+\varepsilon_0}$. We proceed with induction. Assume that the free boundary is C^β around 0. Applying Proposition 2.6.13 to w_i and w_n, gives that the quotient w_i/w_n is $C^{\beta-1+\varepsilon_0}(\partial\Omega\cap B_{1/2})$. Therefore the normal vector $\nu \in C^{\beta-1+\varepsilon_0}(\partial\Omega\cap B_{1/2})$ and hence the free boundary is in fact $C^{\beta+\varepsilon_0}$. We can proceed with the induction as long as the regularity of f_i is better than required, $\theta-1-s+\varepsilon_0 \geq \beta-1-s+\varepsilon_0$, which indeed renders the wanted regularity. □

Corollary 2.7.2. *Let L be an operator whose kernel K satisfies* (2.1.1) *and is $C^\infty(\mathbb{S}^{n-1})$. Let $0 \in \partial\Omega$ be a free boundary point, and assume that* (2.7.2) *holds. If $\varphi \in C^\infty$, then the free boundary is C^∞ around 0.*

2.7.2 Fractional Laplacian

We conclude with establishing conditions (2.7.2) for the height function in the case when the operator is the fractional Laplacian. All the work has already been done in [47, 67], we just show how to translate their results to our setting.

Lemma 2.7.3. *Let L be the fractional Laplacian and let φ be an obstacle in the space $C^{3s}(\mathbb{R}^n)\cap C_0(\mathbb{R}^n)$, which satisfies*

$$((-\Delta)^s\varphi + b\cdot\nabla\varphi)^+ \in L^\infty(\mathbb{R}^n).$$

Let u be the solution to problem (2.1.2)*. Let Let 0 be a regular free boundary point with e_n as the normal vector. Then the height function $w := u-\varphi \in C^{1+s}(\mathbb{R}^n)$, and there exists $r_0 > 0$ and $\alpha > 0$, so that the free boundary is $C^{1,\alpha}$ in $B_{r_0}(0)$. Furthermore, for all $1 \leq i \leq n$ the functions w_i satisfy*

$$|w_i(x)| \leq Cd^s(x), \quad x \in B_{r_0}(0).$$

and,

$$w_n(x) \geq cd^s(x), \quad x \in B_{r_0}(0).$$

Proof. In [67], they establish that under these assumptions, the height function $w \in C^{1+s}(\mathbb{R}^n)$. In [47] the furthermore show that the homogeneous rescaling of the height function $v_0(x, y)$, for a regular free boundary point 0 with the normal vector e_n, converges in $C^{1,\gamma}(\overline{B}^+_{1/8})$ to

$$a(x_n + \sqrt{x_n^2 + y^2})^s (x_n - s\sqrt{x_n^2 + y^2}).$$

where $a > 0$ (See [47, results 3.7, 3.8, 5.2, 5.3, 5.5].) Reading this convergence at $y = 0$, we get

$$\frac{1}{r^{1+s}}(u - \varphi)(0 + rx) \longrightarrow 2^s(1-s)a(x_n)_+^{1+s} \qquad \text{in } C^{1,\gamma}(\overline{B}_{1/8}),$$

for some positive γ and a. This implies that the partial derivatives satisfy

$$\frac{1}{r^s}(u - \varphi)_i(0 + rx) \longrightarrow 2^s(1-s^2)a(x_n)_+^s \delta_{i,n}, \qquad \text{in } L^\infty(\overline{B}_{1/8}).$$

From this it is not hard to get bounds

$$||(u - \varphi)_i||_{L^\infty(B_r)} \le Cr^s, \quad i = 1, \dots, n \quad \text{and}$$

$$||(u - \varphi)_n||_{L^\infty(B_r)} \ge cr^s, \quad c > 0,$$

for all $r \le r_0$.

Let us now show, that we also have $|(u - \varphi)_i| \le Cd^s$ and $|(u - \varphi)_n| \ge cd^s$, for some $c > 0$, where d is the generalised distance function to the free boundary. The first one follows from optimal regularity of solutions for the obstacle problem, namely u is $C^{1+s}(\mathbb{R}^n)$ (see [67, Theorem 1.1]), and so $w := u - \varphi$ is also C^{1+s} and vanishes outside $\Omega := \{w > 0\}$. So in Ω^c we have that also $w_i = 0$ for every $i = 1, \dots, n$. Therefore, since $w_i \in C^s(\mathbb{R}^n)$, we have that $|w_i(x)| = |w_i(x) - w_i(z)| \le ||w_i||_{C^s}|x - z|^s \le Cd^s(x)$, for the closest point z in the free boundary.

Let now 0 be a regular free boundary point. In [47, Theorem 1.3] they show that around 0, the free boundary is $C^{1,\alpha}$ for some positive α. Therefore in a ball $B_{r_2}(0))$, we can apply [72, Proposition 3.3], to get that for every $z \in \Gamma \cap B_{r_2/2}(0)$ there exists $Q(z)$, such that

$$|w_i(x) - Q(z)d^s(x)| \le C|x - z|^{s+\alpha}, \tag{2.7.3}$$

provided that $|\nabla w_i| \le Cd^{s-1}$, which we show later. We already established that around 0 we have $||w_n||_{L^\infty(b_r(0))} \ge cr^s$ for some positive c. This implies, that $Q(0) \ne 0$. Since the function w is non-negative, $Q(0)$ must be positive. Estimate (2.7.3) together with Lemma 2.8.8 give that $z \mapsto Q(z)$ is a C^α map, and so continuous, which gives that in a perhaps smaller neighbourhood $B_{\tilde{r}_0}0$ we have $Q(z) \ge c' > 0$. Choose now a point $x \in B_{r_0'(0)}$, where r_0' will be specified later. Let z be the closest boundary point, which falls into $B_{\tilde{r}_0}(0)$, so that $Q(z) \ge c'$. Denote $x = z + t\nu_z$, so $d(x) = t$. Then (2.7.3), with a triangle inequality gives

$$w_n(x) \ge Q(z)d^s(x) - C|x - z|^{s+\alpha} \ge c't^s - Ct^{s+\alpha} \ge \frac{c'}{2}t^s = \frac{c'}{2}d^s(x), \tag{2.7.4}$$

if $t \le t_0$ suitable chosen ($t_0 \le \left(\frac{c'}{2C}\right)^{1/\alpha}$). Hence we got such r_0 (the minimum of above constraints), that $w_n \ge c_1 d^s$ in $B_{r_0}(0)$, and $\Gamma \cap B_{r_0}(0)$ is $C^{1,\alpha}$. □

This provides the last ingredients for proving Theorem 2.1.1.

Proof of Theorem 2.1.1. The claim follows straight-forward from Lemma 2.7.3 and Corollary 2.7.2. □

2.8 Appendix: Technical tools and lemmas

Lemma 2.8.1. *Let $\alpha \in (-1,0)$, $\beta \in \mathbb{R}$, $r > 0$ and $x_0 \in B_1$ be such that $(x_0)_n > 2r$. Then there exists a constant C, such that for every $x \in B_{r/2}(x_0)$ it holds*

$$\int_{B_1 \setminus B_r(x)} (z_n)_+^\alpha |z-x|^{-n+\beta} dz \le C r^{\alpha+\beta}, \quad \text{if } \alpha+\beta < 0 \quad \text{and}$$

$$\int_{B_r(x)} (z_n)_+^\alpha |z-x|^{-n+\beta} dz \le C r^{\alpha+\beta}, \quad \text{if } \beta > 0.$$

Proof. The proof is exactly the same as the one of [1, Lemma A.9]. □

Lemma 2.8.2. *Let Ω be a domain in $\mathbb{R}^n$ with $0 \in \partial\Omega$ and d the distance function to the boundary. Assume function f satisfies*

$$|f(x)| \le C|x|^\alpha,$$

and

$$[f]_{C^\alpha(B_r(x_0))} \le C,$$

whenever $d(x_0) = 2r = |x_0|$.

Denote $\mathcal{C} = \cup_{d(x_0)=|x_0|} B_r(x_0)$. Then $f \in C^\alpha(\overline{\mathcal{C}})$ with

$$|D^j f| \le C|x|^{\alpha-j}, \quad j \le \alpha.$$

Proof. Let us first show, that $f \in C^\alpha(\overline{\mathcal{C}})$. Choose therefore a multi-index γ of order $\lfloor \alpha \rfloor$ and points $x, y \in \mathcal{C}$. Denote x', y' the points such that $d(x') = |x'|$ and $d(y') = |y'|$, so that $x \in B_{|x'|/2}(x')$ and $y \in B_{|y'|/2}(y')$, so that $|x'-y'| \le |x-y|$, $|x'-x| \le |x-y|$ and $|y-y'| \le |x-y|$. For this we need to assume that x and y do not lie in any of the balls $B_r(x_0)$. Otherwise the desired property holds true by assumption. For simplicity let us also assume that x' and y' lie on the same line as 0. Assume with out loss of the generality that $|x'| < |y'|$. Denote $x_i = 2^i x$. Then

$$|\partial^\gamma f(x) - \partial^\gamma f(y)| \le |\partial^\gamma f(x) - \partial^\gamma f(x')| + |\partial^\gamma f(x') - \partial^\gamma f(y')| + |\partial^\gamma f(y') - \partial^\gamma f(y)|.$$

The first term is bounded with $C|x-x'|^{\langle\alpha\rangle} \le |x-y|^{\langle\alpha\rangle}$, and similarly the last term. For the second one, we choose $K \in \mathbb{N}$ so that $2^K|x'| \le |y'| \le 2^{K+1}|x'|$, and $2^K|x'| \le 2|x'-y'|$ and compute

$$|\partial^\gamma f(x') - \partial^\gamma f(y')| \le \sum_{i=1}^K |\partial^\gamma f(x_{i-1}) - \partial^\gamma f(x_i)| + |\partial^\gamma f(x_K) - \partial^\gamma f(y')|$$

$$\leq \sum_{i=1}^{K} C|x_i - x_{i-1}|^{\langle\alpha\rangle} + C|x_K - y'|^{\langle\alpha\rangle} \leq \sum_{i=1}^{K} C(2^i|x'|)^{\langle\alpha\rangle} + C|x_K - y'|^{\langle\alpha\rangle}$$

$$\sum_{i=1}^{K} C(2^i 2^{-K}|x' - y'|)^{\langle\alpha\rangle} + C|x' - y'|^{\langle\alpha\rangle} \leq C|x' - y'|^{\langle\alpha\rangle},$$

because the series we get is summable. Noticing that $|x' - y'| \leq |x - y|$ finishes the proof.

The only exception is when for example $x = 0$. Then we do the same procedure, with $x_i = 2^{-i}y$.

Once we know that $f \in C^\alpha(\overline{C})$, the growth control implies that $D^j f(0) = 0$ for all $j \leq \lfloor\alpha\rfloor$. Since $D^{\lfloor\alpha\rfloor} f$ is a $C^{\langle\alpha\rangle}$ function, we get $|D^{\lfloor\alpha\rfloor} f| \leq C|x|^{\langle\alpha\rangle}$. Integrating this iteratively, we get the others. □

Lemma 2.8.3. *Let u be a bounded function, satisfying $|u(x)| \leq C|x|^\alpha$ for some $\alpha > 0$. Let $\beta > \alpha$, and $x_0 \in B_{1/2}$ with $|x| = 2r$. Then the function $u_r(x) := u(x_0 + rx)$ satisfies*

$$\left|\left|\frac{u_r}{1 + |\cdot|^\beta}\right|\right| \leq Cr^\alpha.$$

Proof. We compute

$$|u_r(x)| = |u(x_0 + rx)| \leq C|x_0 + rx|^\alpha \leq Cr^\alpha(1 + |x|)^\alpha \leq Cr^\alpha(1 + |x|^\beta).$$

□

Lemma 2.8.4. *Let $f \in C^\alpha(\overline{B_1})$ with $\alpha > 1$. Then there exists a $C^{\alpha-1}$ map $g\colon \overline{B_1} \to \mathbb{R}^n$, such that*

$$f(x) - f(0) = x \cdot g(x).$$

Moreover, $||g||_{C^{\alpha-1}} \leq C||f||_{C^\alpha}$.

Proof. Write $f(x) - f(0) = \int_0^1 \partial_t f(tx)dt = \int_0^1 x \cdot \nabla f(tx)dt$, and hence we can define $g(x) = \int_0^1 \nabla f(tx)dt$. For any multi-index k with $|\gamma| \leq \alpha - 1$, we have $\partial^\gamma g(x) = \int_0^1 \nabla\partial^\gamma f(tx)t^{|\gamma|}dt$. Hence we get $||D^j g||_{L^\infty} \leq C||D^{j+1} f||_{L^\infty}$, for every $j \leq \alpha - 1$ and for $|\gamma| = [\alpha - 1]$

$$|\partial^\alpha g(x_1) - \partial^\alpha g(x_2)| \leq \int_0^1 |\nabla\partial^\alpha f(tx_1) - \nabla\partial^\alpha f(tx_2)|t^{|\alpha|}dt$$

$$\leq C\,[f]_{C^\alpha} \int_0^1 |tx_1 - tx_2|^{\langle\alpha\rangle} t^{|\alpha|}dt \leq C\,[f]_{C^\alpha}\,|x_1 - x_2|^{\langle\alpha\rangle}.$$

□

Lemma 2.8.5. *Let Ω be C^1 domain and $f\colon \Omega \to \mathbb{R}$ a $C(\Omega)$ function, which for some $\alpha \in (0,1)$, $\beta < -\alpha$ satisfies*

$$[f]_{C^\alpha(B_r(x_0))} \leq C_0 r^\beta,$$

for any x_0, r such that $B_{2r}(x_0) \subset \Omega$ and C_0 independent of x_0, r.

Then we have the following bound

$$|f(x)| \leq CC_0 d^{\alpha+\beta}(x).$$

Proof. Choose $r_0 > 0$, so that for $K = \overline{\{y \in \Omega;\ d(y, \partial\Omega) > r_0\}}$, and every $z \in \partial\Omega$ the intersection $K \cap \{z + t\nu_z;\ t \in \mathbb{R}\}$ is non-empty. Since K is compact and f continuous on K, it is bounded there. Choose now a point x in Ω. Let z be the closest boundary point, and $r = |x - z|$. Denote $x_i = z + 2^i r\nu_z$, so that $x_0 = x$ and $x_k \in K$. Then, since x_i and x_{i+1} are both in the ball $B_{2^{i-1}r}(y_i)$ for $y_i = 1/2(x_i + x_{i+1})$, by assumption we have

$$|f(x_i)-f(x_{i-1})| \leq C_0(2^{i-2}r)^\beta |x_i - x_{i+1}|^\alpha \leq C_0(2^{i-2}r)^\beta (2^{i-1}r)^\alpha = C_0 2^{-2\beta} r^{\alpha+\beta} 2^{(\alpha+\beta)(i-1)}.$$

Hence, summing a geometric series and using that $x_k \in K$, we get

$$|f(x)| \leq \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| + |f(x_k)| \leq C_{\alpha+\beta} C_0 r^{\alpha+\beta} + C \leq C C_0 r^{\alpha+\beta}.$$

□

Lemma 2.8.6. *Let $0 \in \partial\Omega$ and let f, g, Q be functions on Ω which satisfy $|f - Qg| \leq C|x|^{a+b}$, and $[f - Qg]_{C^{a+b}(B_r(x_0))} \leq C$, whenever $d(x_0) = |x_0 - 0| = 2r$. Assume g satisfies $|D^k g^{-1}| \leq C_k d^{-b-k}$, for all $0 \leq k \leq \lfloor a \rfloor + 1$.*

Then we have $|\frac{f}{g} - Q| \leq C|x|^a$ on $\mathcal{C}_0$ and $\left[\frac{f}{g} - Q\right]_{C^a(B_r(x_0))} \leq C$.

Proof. The first estimate is just dividing by g and taking into account its growth. We have to restrict ourselves to the cone, so that $|x|$ and d become comparable.

For the second one, pick $|\gamma| = \lfloor a \rfloor$ and compute

$$|\partial^\gamma((f - Qg)g^{-1})(x) - \partial^\gamma((f - Qg)g^{-1})(y)| \leq$$

$$\leq \sum_{\alpha \leq \gamma} \binom{\gamma}{\alpha} |\partial^\alpha(f - Qg)(x) - \partial^\alpha(f - Qg)(y)| \cdot |\partial^{\gamma-\alpha} g^{-1}(x)| +$$

$$+ \sum_{\alpha \leq \gamma} \binom{\gamma}{\alpha} |\partial^{\gamma-\alpha} g^{-1}(x) - \partial^{\gamma-\alpha} g^{-1}(y)| \cdot |\partial^\alpha(f - Qg)(y)|.$$

First, notice that the assumptions on the growth and regularity $f - Qg$ imply that $f - Qg \in C^{a+b}(\mathcal{C}_0)$, with $D^j(f - Qg)(0) = 0$ for $j \leq \lfloor a + b \rfloor$, and so on $B_r(x_0)$ we have $|D^j(f - Qg)| \leq Cr^{a+b-k}$. Now we estimate the above expression with

$$\sum_{\alpha \leq \gamma} C||D^{|\alpha|+1}(f - Qg)||_{L^\infty(B_r(x_0))} |x - y| ||D^{|\gamma-\alpha|} g^{-1}||_{L^\infty(B_r(x_0))} +$$

$$+ \sum_{\alpha \leq \gamma} ||D^{|\gamma-\alpha|+1} g^{-1}||_{L^\infty(B_r(x_0))} |x - y| ||D^{|\alpha|}(f - Qg)||_{L^\infty(B_r(x_0))} \leq$$

$$\leq \left(Cr^{a+b-|\alpha|-1} r^{1-\langle a \rangle} r^{-b-|\gamma-\alpha|}\right) |x - y|^{\langle a \rangle} + \left(Cr^{-b-|\gamma-\alpha|-1} r^{1-\langle a \rangle} r^{a+b-|\alpha|}\right) |x - y|^{\langle a \rangle}$$

$$\leq C|x - y|^{\langle a \rangle}.$$

Note that in the case when $\lfloor a + b \rfloor = \lfloor a \rfloor$, and when $\alpha = \gamma$ we have to use the regularity from the assumption for $(f - Qg)$, since we do not have the estimate on $D^{|\gamma|+1}(f - Qg)$, but we end up with the same powers of r. □

Lemma 2.8.7. *Let Ω be a domain of class β' and let for every boundary point $z \in \partial\Omega \cap B_1$ the following hold true*

$$\left|\frac{u(x) - \sum_k P_z^k(x-z) d^{s+p_k}(x)}{v(x) - \sum_k R_z^k(x-z) d^{s+p_k}(x)} - Q_z(x-z)\right| \leq C_0 |x-z|^{\beta'}, \quad x \in \mathcal{C}_z$$

and

$$\left[\frac{u - \sum_k P_z^k d^{s+p_k}}{v - \sum_k R_z^k d^{s+p_k}} - Q_z\right]_{C^{\beta'}(B_r(x_0))} \leq C_0,$$

for some $p_k > 0$ and polynomials $P_z^k, R_z^k \in \boldsymbol{P}_{\lfloor \beta' - p_k \rfloor}$ whose coefficients of order α are $C_z^{\beta'-p_k-|\alpha|}(\partial\Omega \cap B_1)$ and a polynomial $Q_z \in \boldsymbol{P}_{\lfloor \beta' \rfloor}$. Assume that $|v| \leq C_1 d^s$.

Then the coefficients of Q_z satisfy

$$\left|\left|Q_z^{(\alpha)}\right|\right|_{C_z^{\beta'-|\alpha|}(\partial\Omega \cap B_{1/2})} \leq C.$$

The constant C depends only on n, s, β', C_0, C_1, $\left|\left|(R_z^k)^{(\alpha)}\right|\right|_{C_z^{\beta'-p_k-|\alpha|}(\partial\Omega \cap B_1)}$ and $\left|\left|(P_z^k)^{(\alpha)}\right|\right|_{C_z^{\beta'-p_k-|\alpha|}(\partial\Omega \cap B_1)}$.

Proof. With similar argument as in the proof of Lemma 2.8.8 we argue that the coefficients of Q_z are uniformly bounded.

Let us denote $\tilde{u} = u(x) - \sum_k P_z^k(x-z) d^{s+p_k}(x)$ and $\tilde{v}$ analogously. The assumptions give that on the cone $\mathcal{C}_z$, the function $\eta_z := \frac{\tilde{u}}{\tilde{v}} - Q_z$ is of class $C^{\beta'}$ with $D^{\lfloor \beta' \rfloor}\eta_z(z) = 0$. Hence for $|\gamma| = \lfloor \beta' \rfloor$, we have

$$Q_z^{(\gamma)} = \partial^\gamma Q_z = \partial^\gamma(\tilde{u}/\tilde{v}) - \partial^\gamma \eta_z.$$

Now choose $N \in \mathbb{N}$ big enough and "take" incremental quotient of the above equation in variable z of increment h. We do it through the parametrisation of the boundary, which can be taken of class $C^{\beta'}$. When we get get the boundary points z_i, so that $\Delta_h^N Q_z^{(\gamma)} = \sum_{i=0}^N (-1)^i \binom{N}{i} Q_{z_i}^{(\gamma)}$. Then choose $x \in \cap_{i=0}^N \mathcal{C}_{z_i}$, so that $d(x) \leq C|h|$. Then we have

$$\Delta_h^N Q_z^{(\gamma)} = \Delta_h^N \partial^\gamma \frac{\tilde{u}}{\tilde{v}} - \Delta_h^N \partial^\gamma \eta_z.$$

First, since for all i we have $|\partial^\gamma \eta_{z_i}(x)| \leq C|x - z_i|^{\beta'-|\gamma|}$ we can bound

$$|\Delta_h^N \partial^\gamma \eta_z| \leq C_N d(x)^{\beta'-|\gamma|} \leq C_N |h|^{\beta'-|\gamma|}. \tag{2.8.1}$$

Hence, let us focus on the first term only. We compute

$$\partial^\gamma \frac{\tilde{u}}{\tilde{v}} = \sum_{\alpha \leq \gamma} \binom{\gamma}{\alpha} \partial^\alpha \tilde{u} \partial^{\gamma-\alpha} \frac{1}{\tilde{v}} = \sum_{\alpha \leq \gamma} \binom{\gamma}{\alpha} \partial^\alpha \tilde{u} \sum_{K=0}^{|\gamma-\alpha|} \frac{1}{\tilde{v}^{K+1}} \sum_{\delta_1 + \ldots + \delta_K = \gamma - \alpha} c(\delta, \alpha, \gamma) \partial^{\delta_1} \tilde{v} \cdot \ldots \cdot \partial^{\delta_K} \tilde{v},$$

where when $K = 0$ the inner sum has to be understood as 1 if also $|\gamma - \alpha| = 0$, otherwise 0. Also, the constant $c(\delta, \alpha, \gamma)$ is non-zero if and only if all δ_i are non-zero. So we need to estimate

$$\Delta_h^N \left(\partial^\alpha \tilde{u} \frac{\partial^{\delta_1} \tilde{v} \ldots \partial^{\delta_K} \tilde{v}}{\tilde{v}^{K+1}}\right)$$

which we split even further with the Leibnitz rule into

$$\sum_{N_1+\ldots N_{K+2}=N} \binom{N}{N_1 \ldots N_{K+2}} \Delta_h^{N_1} \partial^\alpha \tilde{u} \Delta_h^{N_2} \partial^{\delta_1} \tilde{v} \ldots \Delta_h^{N_{K+1}} \partial^{\delta_K} \tilde{v} \Delta_h^{N_{K+2}} \frac{1}{\tilde{v}^{k+1}}$$

and furthermore

$$\Delta_h^{N_{K+2}} \frac{1}{\tilde{v}^{k+1}} = \sum_{M=0}^{N_{K+2}} \frac{1}{\tilde{v}^{(K+1)(M+1)}} \sum_{L_1+\ldots+L_M=N_{K+2}} \Delta_h^{L_1} \tilde{v}^{K+1} \ldots \Delta_h^{L_M} \tilde{v}^{K+1}.$$

With final Leibnitz rule on all of the above powers, we conclude that up to some constants

$$\Delta_h^N \partial^\gamma \frac{\tilde{u}}{\tilde{v}} = \sum_\alpha \sum_K \sum_\delta \sum_{N_l} \sum_M \sum_L \Delta_h^{N_1} (\partial^\alpha \tilde{u}) \Delta_h^{N_2} (\partial^{\delta_1} \tilde{v}) \ldots \Delta_h^{N_{K+1}} (\partial^{\delta_K} \tilde{v}) \frac{1}{\tilde{v}^{(M+1)(K+1)}}$$

$$\times \prod_{i=1}^{M} \sum_{L_1^i+\ldots+L_{K+1}^i=L_i} \prod_{j=1}^{K+1} \Delta_h^{L_j^i} \tilde{v}.$$

So we need to estimate

$$\Delta_h^{N_1} (\partial^\alpha \tilde{u}) \Delta_h^{N_2} (\partial^{\delta_1} \tilde{v}) \ldots \Delta_h^{N_{K+1}} (\partial^{\delta_K} \tilde{v}) \frac{1}{\tilde{v}^{(M+1)(K+1)}} \prod_{i,j}^{M,K+1} \Delta_h^{L_j^i} \tilde{v}. \tag{2.8.2}$$

We treat every of the above factors separately. Starting with the first one, we begin with the following manipulation, where we take out the part, which is important for the finite difference:

$$\partial^\alpha \tilde{u}(x) = \partial^\alpha u(x) - \partial^\alpha \sum_k \sum_\eta c_{k,z}^\eta (x-z)^\eta d^{s+p_k}(x)$$

$$= \partial^\alpha u(x) - \sum_k \sum_\eta \sum_\varepsilon \binom{\eta}{\varepsilon} c_{k,z}^\eta (0-z)^\varepsilon \partial^\alpha (x-0)^{\eta-\varepsilon} d^{s+p_k}(x).$$

Now we take the finite difference, and apply the Leibnitz rule to get (with omitting some constants)

$$\Delta_h^{N_1} (\partial^\alpha \tilde{u}) = -\sum_k \sum_\eta \sum_\varepsilon \sum_{N'} \Delta_h^{N'} c_{k,z}^\eta \Delta_h^{N_1-N} (z-0)^\varepsilon \cdot \partial^\alpha (x-0)^{\eta-\varepsilon} d^{s+p_k}(x).$$

When estimating, this gives

$$\left|\Delta_h^{N_1} (\partial^\alpha \tilde{u})\right| \leq C \sum_k \sum_\eta \sum_\varepsilon \sum_{N'} |h|^{\rho \wedge N'} |h|^{(|\varepsilon| \vee (N_1-N')) \wedge \beta'} |h|^{s+p_k+|\eta-\varepsilon|-|\alpha|},$$

where ρ denotes the regularity of the coefficient, so $\rho = \beta' - p_k - |\eta|$. With treating the cases carefully, we can bound it by

$$\leq C \sum_k \sum_\eta |h|^{(N_1+p_k+s-|\alpha|) \wedge (\rho+p_k+s+|\eta|-|\alpha|)} \leq C |h|^{s-|\alpha|+N_1 \wedge \beta'}.$$

Note also, that this is true also when $N_1 = 0$, due to the assumption on the growth of derivatives of u. For the other factors, we perform the same estimations. We plug it in (2.8.2) to get

$$\left|\Delta_h^N \partial^\gamma \frac{\tilde{u}}{\tilde{v}}\right| \leq C|h|^{(K+1)(M+1)s-|\gamma|}|h|^{N_1\wedge\beta'}|h|^{N_2\wedge\beta'}\dots|h|^{N_{K+1}\wedge\beta'}|h|^{-(K+1)(M+1)s}\prod_{i,j}|h|^{L_{i,j}\wedge\beta'}.$$

If we choose N big enough, so that at least one of the minimums give β' we get the claim, due to (2.8.1).

To establish the regularity of other coefficients, we proceed with the same procedure, only that now we treat higher order coefficients of polynomial Q_z as reminders. Concretely, for $|\gamma| < \lfloor\beta'\rfloor$ we have

$$Q_z^{(\gamma)} = \partial^\gamma Q_z - \sum_{\gamma<\gamma'} c_{\gamma,\gamma'} Q_z^{(\gamma')}(x-z)^{\gamma'-\gamma}.$$

After finite differences, we treat the first term as before to get $|h|^{\beta'-|\gamma|}$, while the second one gives at least the same by already proven regularity (we can treat it as we treated polynomials before).

The claim follows from [2, Theorem 2.1]. $\square$

Lemma 2.8.8. *Let $u \in L^\infty(\mathbb{R}^n)$. Suppose that for every $z \in \partial\Omega \cap B_1$ we have polynomials $P_z^{k,l} \in \mathbb{P}_{\lfloor\beta'-k\varepsilon_0-l\rfloor}$ such that*

$$\left|u(x) - \sum_{k,l\geq 0} P_z^{k,l}(x-z)d^{s+k\varepsilon_0+l}(x)\right| \leq C_0|x-z|^{\beta'+s}, \quad x \in B_1(z),$$

and

$$\left[u - \sum_{k,l\geq 0} P_z^{k,l} d^{s+k\varepsilon_0+l}\right]_{C^{\beta'+s}(B_r(x_0))} \leq C_0, \quad \text{if } d(x_0) = 2r = |x_0 - z|.$$

Suppose that when $(k,l) \neq (0,0)$, the coefficient $(P_z^{k,l})^{(\alpha)}$ are $C_z^{\beta'-k\varepsilon_0-l-|\alpha|}(\partial\Omega\cap B_1)$, with

$$\left|\left|(P_z^{k,l})^{(\alpha)}\right|\right|_{C_z^{\beta'-k\varepsilon_0-l-|\alpha|}(\partial\Omega\cap B_1)} \leq C_1.$$

Then the same holds true also for $P_z^{0,0}$ i.e.

$$\left|\left|(P_z^{0,0})^{(\alpha)}\right|\right|_{C_z^{\beta'-|\alpha|}(\partial\Omega\cap B_{1/2})}. \leq C$$

The constant C depends only on n, s, β', C_0, C_1.

Proof. We start with proving that all the coefficients of $P_z^{0,0}$ are uniformly bounded for every $z \in \partial\Omega \cap B_{1/2}(0)$. In this direction, we stress that

$$\bigcap_{z\in\partial\Omega\cap B_{1/2}(0)} B_1(z) \cap \Omega$$

has non-empty interior, hence we can get a ball B inside, with $d(\partial\Omega, B) \geq c > 0$. Now we can bound

$$||P_z^{0,0}||_{L^\infty(B)} \leq \frac{1}{c^s}||P_z^{0,0}d^s||_{L^\infty(B)} \leq \frac{1}{c^s}||\tilde{u} - P_z^{0,0}d^s||_{L^\infty(B)} + \frac{1}{c^s}||\tilde{u}||_{L^\infty(B)},$$

where we denoted $\tilde{u} = u - \sum_{k\geq 1, l\geq 0} P_z^{k,l}d^{s+k\varepsilon_0+l}$. But since both terms on the right-hand side above are bounded independently of z, we have

$$||P_z^{0,0}||_{L^\infty(B)} \leq C.$$

Now [1, Lemma A.10] applies.

We proceed towards finite differences. First, due to Lemma 2.8.6 we rewrite the assumptions into

$$\left|\frac{u}{d^s} - \sum_{k,l\geq 0} P_z^{k,l}d^{k\varepsilon_0+l}\right| \leq C|x-z|^{\beta'}, \quad x \in \mathcal{C}_z, \quad \text{and}$$

$$\left[\frac{u}{d^s} - \sum_{k,l\geq 0} P_z^{k,l}d^{k\varepsilon_0+l}\right]_{C^{\beta'}(B_r(x_0))} \leq C,$$

which implies that on the cone $\mathcal{C}_z$ we have

$$\left|D^j\left(\frac{u}{d^s} - \sum_{k,l\geq 0} P_z^{k,l}d^{k\varepsilon_0+l}\right)\right| \leq C|x-z|^{\beta'-j}$$

for $0 \leq j < \beta'$. We proceed as in Lemma 2.8.7.

Choose $|\gamma| = \lfloor \beta' \rfloor$. On the cone $\mathcal{C}_z$ we have

$$\partial^\gamma \frac{u}{d^s} - \sum_{(k,l)\neq(0,0)} \sum_{\alpha\leq\gamma} \binom{\gamma}{\alpha} \partial^\alpha P_z^{k,l} \partial^{\gamma-\alpha} d^{k\varepsilon_0+l} - \partial^\gamma P_z^{0,0} = \eta_z,$$

with $|\eta_z| \leq C|x-z|^{\beta'-|\gamma|}$. As in Lemma 2.8.7 we take the finite difference ot he above equation of some order big enough, and choose x in a suitable intersection of cones. Then we can estimate

$$|\Delta_h^N \eta_z| \leq C|h|^{\beta'-|\gamma|},$$
$$\left|\Delta_h^N \frac{u}{d^s}\right| = 0,$$

as well as

$$\left|\Delta_h^N \partial^\alpha P_z^{k,l} \partial^{\gamma-\alpha} d^{k\varepsilon_0+l}\right| = \left|\Delta_h^N \sum_{\alpha'\geq\alpha} c_{\alpha',z}(x-z)^{\alpha'-\alpha}\partial^{\gamma-\alpha}d^{k\varepsilon_0+l}\right|$$

$$= \left|\sum_{\alpha'\geq\alpha}\sum_{\delta\leq\alpha'-\alpha}\binom{\alpha'-\alpha}{\delta}\Delta_h^N\left(c_{\alpha',z}(z-0)^\delta\right)\cdot(x-0)^{\alpha'-\alpha-\delta}\partial^{\gamma-\alpha}d^{k\varepsilon_0+l}\right|.$$

The finite difference we estimate as in the proof of Lemma 2.8.7, (N' comes from the Leibnitz rule) to get

$$\leq C\sum_{\alpha',\delta}\sum_{N'}|h|^{(\beta'-k\varepsilon_0-l-|\alpha'|)\wedge N'}|h|^{|\delta|\vee((N-N')\wedge\beta')}\cdot|h|^{|\alpha'-\alpha-\delta|+k\varepsilon_0+l-|\gamma-\alpha|}\leq C|h|^{N\wedge\beta'-|\gamma|}.$$

Once this is established, we conclude that for $N>\beta'$,

$$\left|\Delta_h^N\left(P_z^{0,0}\right)^{(\gamma)}\right|\leq C|h|^{\beta'-|\gamma|}.$$

We proceed in the same way as in Lemma 2.8.7: when dealing with lower order coefficients of $P_z^{0,0}$ we treat higher order ones as "the reminder".

The claim follows from [2, Theorem 2.1]. □

Lemma 2.8.9. *Let $b>0$ and $p>-b$. Suppose $\phi\colon\Omega\to\mathbb{R}$ is a $C^{b+p\vee b}(\overline{\Omega})$ function, with $|\phi(x)|\leq C|x-z|^b$ for some boundary point z.*

Then for any $a<b+p$, and $d(x_1)=2r$ we have

$$[\phi d^p]_{C^a(B_r(x_1))}\leq C\left(\frac{|x_1-z|}{r}\right)^b.$$

Proof. First we deduce, that $|D^k\phi(x)|\leq C|x-z|^{b-k}$, for $k\leq b$. Now choose a multi-index γ od order $\lfloor a\rfloor$, and compute

$$\left|\partial^\gamma(\phi d^p)(x)-\partial^\gamma(\phi d^p)(y)\right|\leq\sum_{\alpha\leq\gamma}\binom{\gamma}{\alpha}|\partial^\alpha\phi(x)-\partial^\alpha\phi(y)|\cdot|\partial^{\gamma-\alpha}d^p(x)|+$$

$$+\sum_{\alpha\leq\gamma}\binom{\gamma}{\alpha}|\partial^\alpha\phi(y)|\cdot|\partial^{\gamma-\alpha}d^p(x)-\partial^{\gamma-\alpha}d^p(y)|$$

$$\leq C\left(\sum_{|\alpha|\leq b-1}+\sum_{b-1<|\alpha|\leq b+p-1}+\sum_{b+p-1<|\alpha|\leq\gamma}\right)||D^{|\alpha|+1}\phi||_{L^\infty}|x-y|||D^{|\gamma-\alpha|}d^p||_{L^\infty}+$$

$$+C\left(\sum_{|\alpha|\leq b}+\sum_{b<|\alpha|\leq\gamma}\right)||D^{|\gamma-\alpha|+1}d^p||_{L^\infty}|x-y|||D^{|\alpha|}\phi||_{L^\infty}$$

which after taking the worst term (second sum, $|\alpha|=0$) gives

$$\leq C|x_1-z|^b r^{p-a}|x-y|^{\langle a\rangle}.$$

But since $r^{p-a}\leq r^{-b}$, the claim is proven. □

Lemma 2.8.10. *Let $p>1$, and assume f satisfies $|f(x)|\leq C|x-z|^p$ on the cone $\mathcal{C}_z$, as well as the interior regularity estimate*

$$[f]_{C^p(B_r(x_1))}\leq C\left(\frac{|x_1-z|}{r}\right)^p,$$

where $d(x_1)=|x_1-z'|=2r$ (x_1 does not need to be in the cone). Then we have

$$|\partial^\gamma f(x)|\leq C|x-z|^p d^{-|\gamma|}(x),\qquad x\in\Omega\cap B_1(z),1\leq|\gamma|\leq\lfloor p\rfloor.$$

Proof. Let us stress, that inside the cone the claim is true due to the growth and the regularity estimates. First let us prove the case $|\gamma| = \lfloor p \rfloor$. From the assumptions of f, we have $|\partial^\gamma f(x)| \le C|x-z|^{p-|\gamma|}$ on the cone. Choose now $x \in \Omega$ outside the cone. Let z' be the closest boundary point. Denote $x_i := z' + 2^i(x - z')$ and $r_i = |x_i - x_{i-1}| = 2^{i-1}d(x)$. Let N be such that x_N is the first point inside the cone $\mathcal{C}_z$. Then we compute

$$
\begin{aligned}
|\partial^\gamma f(x)| &\le \sum_{i=1}^{N} |\partial^\gamma f(x_i) - \partial^\gamma f(x_{i-1})| + |\partial^\gamma f(x_N)| \\
&\le \sum_{1}^{N} C\left(\frac{|x_i - z|}{r_i}\right)^p r_i^{\langle p\rangle} + C|x_N - z|^{p-|\gamma|} \\
&\le C|z-x|^p d^{-\lfloor p\rfloor}(x) \sum 2^{-\lfloor p\rfloor i} \le C|x-z|^p d^{-\lfloor p\rfloor}(x).
\end{aligned}
$$

Since $x_i \notin \mathcal{C}_z$, we have $|x_i - z| \le C|x-z|$.

For the lower order derivatives, we integrate the obtained bound along the line from x to x_N. Choose $|\alpha| = \lfloor p \rfloor - 1$ and compute

$$
\begin{aligned}
|\partial^\alpha f(x)| \le |\partial^\alpha f(x) - \partial^\alpha f(x_N)| + |\partial^\alpha f(x_N)| \le |\int_x^{x_N} |D^{|\alpha|+1} f(t)| dt \\
\le \int_{d(x)}^{d(x_N)} C|x_N - z|^p t^{-|\alpha|-1} dt \le C|x-z|^p d(x)^{-|\alpha|}.
\end{aligned}
$$

Iterating this, we prove the claim. □

Corollary 2.8.11. *In the same setting as above, we get the estimate*

$$|\nabla f(x)| \le C|x-z|^p d^{-1}(x).$$

Remark 2.8.12. We can integrate once more time, to get the estimate for f in the full neighbourhoods of z:

$$|f(x)| \le C|x-z|^p \log d(x).$$

Chapter 3

$C^{2,\alpha}$ regularity of free boundaries in parabolic non-local obstacle problems

3.1 Introduction

The fractional obstacle problem is the following

$$\min\{(-\Delta)^s u, u-\varphi\} = 0 \quad \text{in } \mathbb{R}^n,$$
$$\lim_{|x|\to\infty} u(x) = 0,$$

where $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is a given function called the obstacle and

$$(-\Delta)^s u(x) = c_{n,s} p.v. \int_{\mathbb{R}^n} (u(x) - u(x+y))|y|^{-n-2s} dy,$$

for some $s \in (0,1)$. The operator $(-\Delta)^s$ is called the fractional Laplacian. This problem (and its parabolic version) arises in the study of the optimal stopping problems for stochastic processes for example when modelling the prices of American options. For more information see [26].

The study of the fractional obstacle problem was initiated by Silvestre in [82] and by Caffarelli, Salsa and Silvestre in [23]. Since then there has been put a lot of effort in studying this problem and is nowadays quite well understood. The interest of study in the obstacle problems is twofold. On one hand we are interested in regularity of the solutions, and on the other one we want to understand the set $\partial\{u>\varphi\}$, called the free boundary. In [23] they show the optimal C^{1+s} regularity of solutions, moreover they prove that at any free boundary point $x_0 \in \partial\{u>\varphi\}$ exactly one of the following two statements holds

$$\begin{array}{llcl} \text{(i)} & 0 < cr^{1+s} & \leq \sup_{B_r(x_0)}(u-\varphi) \leq & Cr^{1+s} \\ \text{(ii)} & 0 & \leq \sup_{B_r(x_0)}(u-\varphi) \leq & Cr^2 \end{array} \qquad \forall r \in (0, r_0).$$

The points satisfying (i) are called regular points, they form an open subset of the free boundary, and the free boundary is $C^{1,\alpha}$ there for some $\alpha > 0$. Later on it has been established that near regular points the free boundary is in fact C^∞ if the obstacle is C^∞ (see [51, 55]). For results about the degenerate/singular points - the ones satisfying (ii) - we refer to [34, 42, 45] and references therein.

Many methods used for establishing these results strongly depend on the tools only available in the case of the fractional Laplacian. Therefore it was also challenging to extend the above mentioned results to a more general class of integro-differential operators. The dichotomy of the regular and degenerate points, as well as the $C^{1,\alpha}$ regularity of the free boundary near regular points was proved in [22], and the higher order regularity of the free boundary in [1].

Much less is known in the parabolic version of the problem:

$$\min\{\partial_t + (-\Delta)^s u, u - \varphi\} = 0 \quad \text{in } \mathbb{R}^n \times (0,T),$$
$$u(\cdot, 0) = \varphi.$$

Notice that the nature of the problem strongly depends on the value of the parameter s. For $s > \frac{1}{2}$ the fractional Laplacian is the leading term (the subcritical regime), while for $s < \frac{1}{2}$ the time derivative is (the supercritical regime). The regularity of solutions was first addressed by Caffarelli and Figalli in [19], where they prove that the solutions are C_x^{1+s} in space and $C_t^{\min(\frac{1+s}{2s}, 2) - \varepsilon}$ in time. Later on in [9] Barrios, Figalli and Ros-Oton show that when $s > \frac{1}{2}$ we have the analogous dichotomy as in the elliptic case, and that the free boundary is $C^{1,\alpha}$ in space and time near regular free boundary points. Furthermore, very recently in [75] Ros-Oton and Torres-Latorre improve the regularity of solutions in the supercritical regime $s < \frac{1}{2}$ to $C^{1,1}$ in space and time, which is optimal. Finally in a forthcoming paper [38] Figalli, Ros-Oton and Serra extend the results from [9] to a more general class of operators as well as to the case $s = \frac{1}{2}$.

Despite all these results nothing is known though about the higher regularity of the free boundary. An open question that seems quite challenging is the following:

Is it true that the free boundary is C^∞ (at least in space) near regular points?

The goal of this paper is to study this question for $s > \frac{1}{2}$ and establish that near regular points the free boundary is $C^{2,\alpha}$. We consider the class of non-local operators of the form

$$Lu(x,t) = p.v. \int_{\mathbb{R}^n} (u(x,t) - u(y,t)) K(x-y) dy$$
$$\lambda |y|^{-n-2s} \le K(y) \le \Lambda |y|^{-n-2s}, \; y \in \mathbb{R}^n, \quad K \text{ is even and homogeneous,} \tag{3.1.1}$$

for $0 < \lambda \le \Lambda$, called the ellipticity constants. Our main result reads as follows. (We refer to Section 3.2 for definition of the parabolic Hölder spaces C_p^β.)

Theorem 3.1.1. *Let $s \in (\frac{1}{2}, 1)$. Let L be as in* (3.1.1)*, with its kernel $K \in C^5(\mathbb{S}^{n-1})$. Let u be a solution of*

$$\begin{aligned} \min\{(\partial_t + L)u, u - \varphi\} &= 0 \quad && \text{in } \mathbb{R}^n \times (0,T) \\ u(\cdot, 0) &= \varphi && \text{in } \mathbb{R}^n, \end{aligned}$$

with $\varphi \in C^4(\mathbb{R}^n)$.

Then the free boundary $\partial\{u > 0\}$ is $C_p^{2+\alpha}$ near regular free boundary points for some $\alpha > 0$.

It remains an open problem to improve the regularity from $C^{2,\alpha}$ to C^{∞}; see Remark 3.1.3 below.

To prove this result, we exploit the fact that the normal to the free boundary can be expressed with the quotients of partial derivatives of $u-\varphi$, see [1, 27, 28]. Hence we closely study the boundary behaviour of solutions to

$$\begin{cases} (\partial_t + L)w &= f \quad \text{in } \Omega\cap Q_1 \\ w &= 0 \quad \text{in } \Omega^c\cap Q_1, \end{cases} \tag{3.1.2}$$

for some open set $\Omega\subset\mathbb{R}^{n+1}$, as the mentioned derivatives solve such equation. There was not much attention put into the studying of the boundary regularity of solutions in *moving* domains, the only known results to the bests of our knowledge ([10, 32, 48, 76]) consider cylindric types of domains. We provide that the solutions grow like distance to the boundary to the power s, which in combination with interior regularity estimates gives C^s_p regularity up to the boundary. Refining the argument, and comparing solutions directly amongst each other, we are able to establish the boundary Harnack inequality of the following type.

Theorem 3.1.2. *Let $s\in(\frac{1}{2},1)$. Let $\Omega\subset\mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\gamma,\varepsilon>0$. Let L be an operator of the form (3.1.1), with kernel $K\in C^{1-s}(\mathbb{S}^{n-1})$. Assume $u_i\in C^{\gamma}_p(Q_1)\cap L^{\infty}(\mathbb{R}^n\times(-1,1))$, $i\in\{1,2\}$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega\cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c\cap Q_1, \end{cases}$$

with $[f_i]_{C^{1-s}_p(\Omega\cap Q_1)}\le 1$, and $[u_i]_{C_t^{\frac{1-s}{2s}}(\mathbb{R}^n\times(-1,1))}\le 1$. Assume also that $u_2\ge c_0 d^s$, for some $c_0>0$. Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C^{1-\varepsilon}_p(\overline{\Omega}\cap Q_{1/2})}\le C,$$

where $C>0$ depends only on n,s,ε,c_0,G_0 and ellipticity constants.

If additionally Ω is C^{β}_p in Q_1 for some $\beta>1+s$, $K\in C^{2\beta+1}(\mathbb{S}^{n-1})$, $[f_i]_{C^s_p(\Omega\cap Q_1)}\le 1$, and $[u_i]_{C_t^{\frac{1}{2}}(\mathbb{R}^n\times(-1,1))}\le 1$, then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C^{2s-\varepsilon}_p(\overline{\Omega}\cap Q_{1/2})}\le C.$$

To obtain the boundary Harnack inequality, we develop expansions of the form u_1-Qu_2 at boundary points of orders up to $3s-\varepsilon$, where Q is a polynomial in space variables of degree 1. In combination with interior regularity results this yields estimates for the quotient as well.

Remark 3.1.3. Note that in the parabolic non-local setting the interior regularity estimates require global time regularity of solutions (see [32, Lemma 7.1]). This is also one of the reasons why we only get $C^{2,\alpha}_p$ regularity of the free boundary in Theorem 3.1.1. Namely, the time derivative of the solution to the obstacle problem is globally $C_t^{\frac{\gamma}{2s}}$, where $1-s<\gamma<2-s$ (see [9], [19] and [38, Corollary 1.6]). Hence we can only use the interior estimates for orders up to $\gamma+2s$. Therefore with the method we use, we can get the boundary Harnack estimate of order at most $\gamma+s$, which is smaller than 2, and hence we can not deduce that the free boundary is better than $C^{2,\alpha}$.

It remains an open problem to decide whether the free boundary is C^∞ near regular points or not.

As a consequence of our result we also obtain the optimal Hölder regularity of u and of u/d^s, thus extending the results of [76] to the case of moving domains. More precisely, we prove that any solution of (3.1.2) satisfies

$$u \in C^s_{x,t} \qquad \text{and} \qquad \frac{u}{d^s} \in C^{2s-1}_p,$$

see Corollary 3.3.14.

3.1.1 Organisation of the paper

We start the body with presenting the notation in Section 3.2. In Section 3.3 we provide the results regarding the boundary regularity of solutions to (3.1.2), and we prove Theorem 3.1.2. In Section 3.4 we prove Theorem 3.1.1. In the last section of the body, Section 3.5, we establish results about operator evaluation of the distance function to the power s, which is required in some proofs from Section 3.3. At the end there is an appendix where we prove technical auxiliary results, to lighten the body of the paper.

3.2 Notation and preliminary definitions

The ambient space is $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where the first n coordinates we denote with $x = (x_1, \ldots, x_n)$ and the last one with t. Sometimes we furthermore split $x = (x', x_n)$, for $x' = (x_1, \ldots, x_{n-1})$. Accordingly we use the multi-index notation $\alpha \in \mathbb{N}_0^{n+1}$, $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_t)$, $|\alpha|_p = \alpha_1 + \ldots + \alpha_n + 2s\alpha_t$, and furthermore

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \circ \left(\frac{\partial}{\partial t}\right)^{\alpha_t}.$$

The gradient operator ∇, differential operator D and Laplace operator Δ are taken only in x variables.

For $\Omega \subset \mathbb{R}^{n+1}$ we denote the time slits with $\Omega_t = \{x \in \mathbb{R}^n;\ (x,t) \in \Omega\}$ and the distance function to the boundary in space directions only with $d_t(x) = d_x(x,t) = \inf_{z \in \partial\Omega_t} |z - x|$. We also write $|(x,t) - (x',t')|_p = |x - x'| + |t - t'|^{\frac{1}{2s}}$.

We denote with $Q_r(x_0, t_0)$ the following cylinder of radius r centred at (x_0, t_0),

$$Q_r(x_0,t_0) = B_r(x_0) \times (t_0 - r^{2s}, t_0 + r^{2s}).$$

When $(x_0, t_0) = (0,0)$, we denote it simply Q_r. We denote $Q'_r = \{|x'| < r\} \times (t_0 - r^{2s}, t_0 + r^{2s})$.

Finally, C indicates an unspecified constant not depending on any of the relevant quantities, and whose value is allowed to change from line to line. We make use of sub-indices whenever we will want to underline the dependencies of the constant.

We define the parabolic Hölder seminorms of order $\alpha > 0$ in the following way

Definition 3.2.1. Let Ω be an open subset of $\mathbb{R}^{n+1}$ and let $s \in (\frac{1}{2}, 1)$. For $\alpha \in (0,1]$ we define the parabolic Hölder seminorm of order α as follows

$$[u]_{C_p^\alpha(\Omega)} = \sup_{(x,t),(x',t')\in\Omega} \frac{|u(x,t) - u(x',t')|}{|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2s}}},$$

and

$$[u]_{C_t^\alpha(\Omega)} = \sup_{(x,t),(x,t')\in\Omega} \frac{|u(x,t) - u(x,t')|}{|t-t'|^\alpha}.$$

If $\alpha \in (1, 2s]$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [u]_{C_t^{\frac{\alpha}{2s}}(\Omega)}.$$

For bigger exponents $\alpha > 2s$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [\partial_t u]_{C_p^{\alpha-2s}(\Omega)}.$$

We furthermore define the parabolic Hölder space of order α

$$C_p^\alpha(\Omega) = \{f;\ [f]_{C_p^\alpha(\Omega)} < \infty,\ D^k\partial_t^l f \in C^0(\Omega) \text{ whenever } k + 2sl \leq \alpha\}.$$

We are now able to define what means that a set is C_p^α.

Definition 3.2.2. Let $\Omega \subset \mathbb{R}^{n+1}$ with $0 \in \partial\Omega$, $r > 0$, and $\alpha > 0$. We say that Ω is C_p^α in Q_r, if $\partial\Omega \cap Q_r$ is a graph over the n-th coordinate of some function $g \in C_p^\alpha(Q_r')$, with $[g]_{C_p^\alpha(Q_r')} \leq G_0$, for some $G_0 > 0$.

In the paper we work with a distance function to the boundary. Since we need it to have more regularity than just the euclidean distance, we give precise statement in the following definition.

Definition 3.2.3. Let Ω be C_p^β in Q_1, for some $\beta > 1$. We denote with d a function satisfying

$$d \in C_p^\beta(\mathbb{R}^{n+1}) \cap C_x^\infty(\{d > 0\}), \quad C^{-1}\operatorname{dist}(\cdot, \partial\Omega \cap Q_1) \leq d \leq C\operatorname{dist}(\cdot, \partial\Omega \cap Q_1) \text{ in } \Omega \cap Q_1,$$

$$|D^k d| \leq Cd^{\beta-k}, \quad \text{in } \{d > 0\}, \text{ for all } k > \beta.$$

Any such function is called a *regularized distance.*

Furthermore we denote with ψ a diffeomorphism $\psi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, such that it holds $\psi(\Omega \cap Q_1) = Q_1 \cap \{x_n > 0\}$, and that $\psi \in C_p^\beta(\mathbb{R}^{n+1}) \cap C_x^2(\{d > 0\})$, with

$$|D^k\psi| \leq Cd^{\beta-k},$$

for all $k > \beta$, in $\{d > 0\}$.

The construction of d and ψ is provided in Lemma 3.6.1.

3.3 Boundary regularity in moving domains

In this section we prove the results regarding boundary regularity of solutions to (3.1.2). We follow the ideas from [32, 76], but in our setting the domain does not need to be cylindric. The main tool is using contradiction arguments in combination with blow-up techniques. This is why we work with domains that are at least C^1_p near the origin, since they give a half-space after blowing up. Let us stress that for all the estimates we need to assume that the solutions are already Hölder continuous with some (small) positive exponent. As we use these estimates on the derivatives of the solution to the obstacle problem, which is $C^{1,\alpha}$, this does not cause issues.

3.3.1 Hölder estimates

We begin with establishing a-priori boundary estimates for orders smaller than s.

Proposition 3.3.1. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\gamma \in (0, s)$ and $\varepsilon > 0$. Let L be an operator of the form (3.1.1). Assume $u \in C^\gamma_p(Q_1)$ is a solution of*

$$\begin{cases} (\partial_t + L)u = f & \text{in } \Omega \cap Q_1 \\ u = 0 & \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $fd^s \in L^\infty(\Omega \cap Q_1)$. Then

$$[u]_{C^\gamma_p(Q_{1/2})} \leq C \left(||fd^s||_{L^\infty(\Omega\cap Q_1)} + \sup_{R>1} R^{\varepsilon - 2s} ||u||_{L^\infty(B_R \times (-1,1))} \right).$$

The constant C depends only on $n, s, \gamma, \varepsilon, G_0$ and ellipticity constants.

We prove it in two steps. First we establish a weaker estimate of similar type in the following lemma and then prove that it in fact implies the wanted inequality.

Lemma 3.3.2. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\varepsilon > 0$. Let L be an operator of the form (3.1.1). Assume $u \in C^{s-\varepsilon}_p(\mathbb{R}^n \times (-1, 1))$ is a solution of*

$$\begin{cases} (\partial_t + L)u = f & \text{in } \Omega \cap Q_1 \\ u = 0 & \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $fd^s \in L^\infty(\Omega \cap Q_1)$. Then for every $\delta > 0$ there exists $C > 0$ so that

$$[u]_{C^{s-\varepsilon}_p(\Omega\cap Q_{1/2})} \leq \delta\, [u]_{C^{s-\varepsilon}_p(\mathbb{R}^n \times(-1,1))} + C \left(||fd^s||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\mathbb{R}^n\times(-1,1))} \right).$$

The constant C depends only on $n, s, \varepsilon, \delta, G_0$ and ellipticity constants.

Proof. We can assume that $||fd^s||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\mathbb{R}^n\times(-1,1))} \leq 1$, since otherwise we would divide the equation with a suitable constant. We argue with contradiction. Assume there exists $\delta > 0$, so that for every $k \in \mathbb{N}$ there exist domains Ω_k that are C^1_p in Q_1, u_k, f_k and L_k operators satisfying (3.1.1) so that the suitable equation holds, but

$$[u_k]_{C^{s-\varepsilon}_p(\Omega_k\cap Q_{1/2})} > \delta\, [u_k]_{C^{s-\varepsilon}_p(\mathbb{R}^n\times(-1,1))} + k. \tag{3.3.1}$$

Pick $(x_k,t_k),(y_k,s_k)\in\Omega_k\cap Q_{1/2}$ so that

$$\frac{1}{2}\,[u_k]_{C_p^{s-\varepsilon}(\Omega_k\cap Q_{1/2})}\le\frac{|u_k(x_k,t_k)-u_k(y_k,s_k)|}{|x_k-y_k|^{s-\varepsilon}+|t_k-s_k|^{\frac{s-\varepsilon}{2s}}}.$$

We define $\rho_k:=|x_k-y_k|+|t_k-s_k|^{\frac{1}{2s}}$. Using (3.3.1) we see that

$$\rho_k^{s-\varepsilon}\,[u_k]_{C_p^{s-\varepsilon}(\Omega_k\cap Q_{1/2})}\le C|u(x_k,t_k)-u(y_k,s_k)|\le 2C\le\frac{C}{k}\,[u_k]_{C_p^{s-\varepsilon}(\Omega_k\cap Q_{1/2})},$$

which yields that $\rho_k\to 0$ as $k\to\infty$.

Now we have the following dichotomy

$$\text{Case 1)}\limsup_{k\to\infty}\frac{d_x(x_k,t_k)}{\rho_k}=\infty,$$
$$\text{Case 2)}\limsup_{k\to\infty}\frac{d_x(x_k,t_k)}{\rho_k}=:\gamma<\infty.$$

Let us first treat the case 1). We define

$$v_k(x,t)=\frac{1}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n\times(-1,1))}\,\rho_k^{s-\varepsilon}}\left(u_k(x_k+\rho_k x,t_k+\rho_k^{2s}t)-u_k(x_k,t_k)\right).$$

We have $v_k(0,0)=0$, $[v_k]_K\le 1$ for any compact set $K\subset R^{n+1}$, provided that k is big enough, and moreover

$$\begin{aligned}||v_k||_{L^\infty(Q_1)}&\ge|v_k(\rho_k^{-1}(y_k-x_k),\rho_k^{-2s}(s_k-t_k))|\\&=\frac{1}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n\times(-1,1))}\,\rho_k^{s-\varepsilon}}|u_k(y_k,s_k)-u_k(x_k,t_k)|\\&\ge\frac{1}{C}\frac{[u_k]_{C_p^{s-\varepsilon}(\Omega\cap Q_{1/2})}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n\times(-1,1))}}\ge\frac{\delta}{C}\end{aligned}$$

where in the last step we used (3.3.1). Moreover v_k satisfy

$$(\partial_t+L_k)v_k=\frac{\rho_k^{s+\varepsilon}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n\times(-1,1))}}f_{k,\rho_k}\quad\text{in }U_k\cap Q_{1/\rho_k},\tag{3.3.2}$$

where $U_k=\{(x,t);\ (x_k+\rho_k x,t_k+\rho_k^{2s}t)\in\Omega_k\}$ and $f_{k,\rho_k}(x,t)=f_k(x_k+\rho_k x,t_k+\rho_k^{2s}t)$. Note that U_k converge to $\mathbb{R}^{n+1}$, while

$$\left|\frac{\rho_k^{s+\varepsilon}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n\times(-1,1))}}f_{k,\rho_k}\right|\le\frac{\rho_k^{s+\varepsilon}d_{k,\rho_k}^{-s}}{k}\longrightarrow 0,$$

locally uniformly in R^{n+1}, thanks to (3.3.1). Moreover, L_k converge (up to a subsequence) to an operator L_0 satisfying (3.1.1).

Passing to subsequence we get that v_k converge to a function v locally uniformly in $\mathbb{R}^{n+1}$. The function v satisfies $v(0)=0$, $[v]_{\mathbb{R}^{n+1}}\le 1$, which implies

$$||v||_{L^\infty(Q_R)}\le R^s,\qquad R>0.$$

Thanks to [32, Lemma 3.1], the function v solves

$$(\partial_t + L_0)v = 0 \quad \text{in } \mathbb{R}^{n+1},$$

and hence by [32, Theorem 2.1] v is a constant function. Hence $v(0,0) = 0$ and $||v||_{L^\infty(Q_1)} > 0$ contradict each other.

In the case 2) we proceed similarly. We choose $(z_k, t_k) \in \partial\Omega \cap Q_{1/2}$ so that $d_x(x_k, t_k) = |z_k - x_k|$. Then we define

$$v_k(x,t) = \frac{1}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n \times (-1,1))}\,\rho_k^{s-\varepsilon}} u_k(z_k + \rho_k x, t_k + \rho_k^{2s} t).$$

We have $v_k(0,0) = 0$, $[v_k]_{C_p^{s-\varepsilon}(K)} \leq 1$ for every compact set $K \subset \mathbb{R}^{n+1}$ and k big enough, but moreover

$$|v_k(\xi_k, 0) - v_k(\eta_k, \tau_k)| = \frac{|u_k(x_k,t_k) - u_k(y_k,s_k)|}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n \times (-1,1))}\,\rho_k^{s-\varepsilon}} \geq \frac{1}{C}\frac{[u_k]_{C_p^{s-\varepsilon}(\Omega_k \cap Q_{1/2})}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n \times (-1,1))}} \geq c_0\delta > 0,$$

thanks to (3.3.1), where $\xi_k = \rho_k^{-1}(x_k - z_k)$, $(\eta_k, \tau_k) = (\rho_k^{-1}(y_k - z_k), \rho_k^{-2s}(s_k - t_k))$. All points $(\xi_k, 0), (\eta_k, \tau_k)$ are contained in $Q_{\gamma+1}$.

In this case v_k solves

$$\begin{cases} (\partial_t + L_k)v_k &= \dfrac{\rho_k^{s+\varepsilon}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n \times (-1,1))}} f_{k,\rho_k} & \text{in } U_k \cap Q_{1/\rho_k} \\ v_k &= 0 & \text{in } U_k^c \cap Q_{1/\rho_k}. \end{cases}$$

Due to (3.3.1), it holds $\left|\frac{\rho_k^{s+\varepsilon}}{[u_k]_{C_p^{s-\varepsilon}(\mathbb{R}^n \times (-1,1))}} f_{k,\rho_k}\right| \leq \frac{\rho_k^{s+\varepsilon} d_{k,\rho_k}^{-s}}{k}$. Choosing a suitable subsequence $\{k_l\}_{l \in \mathbb{N}}$, so that $v := \lim_{l\to\infty} v_{k_l}$, $\xi = \lim_{l\to\infty} \xi_{k_l}$, $(\eta, \tau) = \lim_{l\to\infty} (\eta_{k_l}, \tau_{k_l})$ exist, we deduce from [32, Lemma 3.1] that

$$\begin{cases} (\partial_t + L_0)v &= 0 \quad \text{in } \{x_n > 0\} \\ v &= 0 \quad \text{in } \{x_n \leq 0\}, \end{cases}$$

for some homogeneous operator L_0 satisfying (3.1.1). Moreover v inherits the following properties: $v(0,0) = 0$, $[v]_{C_p^{s-\varepsilon}(\mathbb{R}^{n+1})} \leq 1$ and $|v(\xi,0) - v(\eta,\tau)| \geq c_0\delta > 0$. It follows from [32, Theorem 2.1], that v is a constant function, and hence $v = 0$. But this contradicts the fact that v has different values at $(\xi, 0)$ and (η, τ). □

We show next how to conclude the a priori boundary regularity estimate using this lemma.

Proof of Proposition 3.3.1. We choose a smooth cut off function $\eta \in C_c^\infty(Q_1)$, so that $\eta \equiv 1$ in $Q_{3/4}$. We apply Lemma 3.3.2 on the truncated function ηu to obtain that for every $\delta > 0$ there is $C > 0$ so that

$$[u]_{C_p^\gamma(Q_{1/4})} \leq \delta\,[\eta u]_{C_p^\gamma(\mathbb{R}^n \times (-1,1))} + C\left(||gd^s||_{L^\infty(\Omega \cap Q_{1/2})} + ||\eta u||_{L^\infty(\mathbb{R}^n \times (1-,1))}\right),$$

where we denoted $g = (\partial_t + L)(\eta u)$. Since $\eta \equiv 1$ in $Q_{1/2}$ we have $g = f - Lu(1-\eta)$ in $\Omega \cap Q_{1/2}$. We can furthermore estimate

$$[\eta u]_{C_p^\gamma(\mathbb{R}^n \times (-1,1))} \le [u]_{C_p^\gamma(Q_1)} + [(1-\eta)u]_{C_p^\gamma(Q_1)} \le C\,[u]_{C_p^\gamma(Q_1)}\,,$$

and

$$\begin{aligned}|L(u(1-\eta))(x,t)| &\le \int_{B_{3/4}^c} |u(y,t)(1-\eta)(y,t)K(x,y,t)|\,dy \\ &\le \Lambda \sup_{R>1} R^{\varepsilon-2s}||u||_{L^\infty(B_R\times(-1,1))} \int_{B_{3/4}^c} |y|^{2s-\varepsilon}|x-y|^{-n-2s}dy \\ &\le C \sup_{R>1} R^{\varepsilon-2s}||u||_{L^\infty(B_R\times(-1,1))}.\end{aligned}$$

This gives

$$[u]_{C_p^\gamma(Q_{1/4})} \le \delta\,[u]_{C_p^\gamma(Q_1)} + C\left(||fd^s||_{L^\infty(\Omega\cap Q_1)} + \sup_{R>1} R^{\varepsilon-2s}||u||_{L^\infty(B_R\times(1-,1))}\right),$$

which by [33, Lemma 2.23] and the covering argument proves the result. □

In the following result we establish a version of the estimate that is used later on.

Corollary 3.3.3. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^1 in Q_1. Let $\gamma \in (0,s)$ and $\varepsilon > 0$, $\gamma_2 \in (0,1)$. Let L be an operator of the form (3.1.1). Assume $u \in C_p^\gamma(Q_1)$ is a solution of*

$$\begin{cases} (\partial_t + L)u &= f_1 + f_2 \quad \text{in } \Omega \cap Q_1 \\ \qquad u &= 0 \qquad\quad\;\; \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $f_1 d^s \in L^\infty(\Omega \cap Q_1)$, and $f_2 \in C_p^{\gamma_2}(\Omega \cap Q_1)$. Then

$$[u]_{C_p^\gamma(Q_{1/2})} \le C\left(||f_1 d^s||_{L^\infty(\Omega\cap Q_1)} + [f_2]_{C_p^{\gamma_2}} + \sup_{R>1} R^{-2s+\varepsilon}||u||_{L^\infty(B_R\times(-1,1))}\right).$$

The constant C depends only on $n, s, \gamma, \varepsilon, G_0$ and ellipticity constants.

Proof. To pass from L^∞ norm to the Hölder seminorm in the right hand side, we find function v solving

$$\begin{cases} (\partial_t + L)v &= f_2(0,0) \quad \text{in } \Omega \cap Q_1 \\ \qquad v &= 0 \qquad\quad\;\; \text{in } (\Omega \cap Q_1)^c. \end{cases}$$

Then we have $|f_2(0,0)| \le C||v||_{L^\infty(\Omega\cap Q_1)}$ and by comparison principle

$$\begin{aligned}||u - v||_{L^\infty(\Omega\cap Q_1)} &\le C(||f_1 d^s|| + ||f_2 - f_2(0,0)|| + ||u||_{L^\infty(Q_1^c)}) \\ &\le C(||f_1 d^s|| + [f_2]_{C_p^{\gamma_2}} + ||u||_{L^\infty(Q_1^c)}).\end{aligned}$$

Furthermore it holds

$$\begin{aligned}||(f_1+f_2)d^s||_{L^\infty(Q_1\cap\Omega)} &\le ||f_1 d^s||_{L^\infty} + ||f_2 - f_2(0,0)|| + |f_2(0,0)| \\ &\le ||f_1 d^s||_{L^\infty} + [f_2]_{C_p^{\gamma_2}} + C||v||_{L^\infty(Q_1)} \\ &\le C\left(||f_1 d^s||_{L^\infty} + [f_2]_{C_p^{\gamma_2}} + ||u||_{L^\infty(Q_1)} + ||u-v||_{L^\infty(Q_1)}\right) \\ &\le C\left(||f_1 d^s||_{L^\infty} + [f_2]_{C_p^{\gamma_2}} + ||u||_{L^\infty(R^n\times(-1,1))}\right),\end{aligned}$$

and hence by Proposition 3.3.1 we conclude

$$[u]_{C_p^\gamma(Q_{1/2})} \leq C\left(||f_1 d^s||_{L^\infty(\Omega\cap Q_1)} + [f_2]_{C_p^{\gamma_2}} + ||u||_{L^\infty(R^n\times(-1,1))}\right).$$

It remains to replace the L^∞ norm of u at infinity with the term as in the statement, which is done with the same cut-off procedure as in the proof of Proposition 3.3.1. □

3.3.2 Boundary Harnack in C_p^1 domains

We now turn our attention to the deriving the boundary Harnack inequalities. To prove that the quotient of two solutions is regular of some order α up to the boundary, it is heuristically enough to prove that the solution in the numerator can be approximated by the solution from the denominator multiplied with a polynomial of order α up to order $\alpha+s$. The regularity of the quotient can then be deduced from interior regularity estimates. We start with establishing such expansions of order $2s$.

Proposition 3.3.4. *Let $s\in(\frac{1}{2},1)$. Let $\Omega\subset\mathbb{R}^{n+1}$ be C_p^1 in Q_1. Let $\varepsilon>0$ and $\rho\in(0,s)$. Let L be an operator of the form* (3.1.1). *Assume $u_i\in C_p^\gamma(Q_1)$, $i\in\{1,2\}$, $\gamma>0$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega\cap Q_1\\ u_i &= 0 \quad \text{in } \Omega^c\cap Q_1, \end{cases}$$

with $||d^\rho f_i||_{L^\infty(\Omega\cap Q_1)} \leq 1$, $||u_i||_{L^\infty(B_R\times(-1,1))} \leq R^{2s-\rho}$ for $R>1$. Assume also that $u_2\geq c_0 d^s$, for some $c_0>0$.

Then for every $(z,t_0)\in\partial\Omega\cap Q_{1/2}$ there exists a constant $q_{(z,t_0)}$, so that

$$|u_1(x,t) - q_{(z,t_0)}u_2(x,t)| \leq C\left(|x-z| + |t-t_0|^{\frac{1}{2s}}\right)^{2s-\rho}.$$

The constant C depends only on $n,s,\rho,\varepsilon,c_0,G_0$ and ellipticity constants.

Moreover for every $(x_0,t_0)\in\Omega\cap Q_{1/2}$, such that $d_x(x_0)=|x_0-z|=c_\Omega r$, we have

$$\left[u_1 - q_{(z,t_0)}u_2\right]_{C_p^{2s-\rho}(Q_r(x_0,t_0))} \leq C.$$

Proof. Thanks to the translation and rescaling invariance of the statement, we only need to prove the case when $(z,t_0)=(0,0)$. We prove the claim with contradiction. Assume that for every $k\in\mathbb{N}$, there exist domains Ω_k that are C_p^1 in Q_1, $u_{i,k}, f_{i,k}$, with $||d^\rho f_i||_{L^\infty(\Omega\cap Q_1)}, \sup_{R>1} R^{\varepsilon-2s}||u_i||_{L^\infty(B_R\times(-1,1))} \leq 1$, and L_k operators satisfying (3.1.1) so that the suitable equations hold, but for any choice of q_k we have

$$\sup_{r>0}\frac{1}{r^{2s-\rho}}||u_{1,k} - q_k u_{2,k}||_{L^\infty(Q_r)} > k.$$

Next we define

$$q_{k,r} = \frac{\int_{Q_r} u_{1,k}u_{2,k}}{\int_{Q_r} u_{2,k}^2},$$

so that $\int_{Q_r}(u_{1,k} - q_{k,r}u_{2,k})u_{2,k} = 0$, and set

$$\theta(r) = \sup_k \sup_{\rho>r}\frac{1}{\rho^{2s-\rho}}||u_{1,k} - q_{k,\rho}u_{2,k}||_{L^\infty(Q_\rho)}.$$

Notice that θ is monotone decreasing and thanks to [57, Lemma B.7] and the contradiction assumption it holds $\lim_{r\downarrow 0}\theta(r)=\infty$. Now choose sequences r_m and k_m, so that

$$\frac{m}{4}\leq\frac{1}{2}\theta(r_m)\leq\frac{1}{r_m^{2s-\rho}}||u_{1,k_m}-q_{k_m,r_m}u_{2,k_m}||_{L^\infty(Q_{r_m})}\leq\theta(r_m),$$

and define

$$v_m(x,t)=\frac{1}{\theta(r_m)r_m^{2s-\rho}}(u_{1,k_m}-q_{k_m,r_m}u_{2,k_m})(r_mx,r_m^{2s}t).$$

Next we estimate

$$\begin{aligned}|q_{k,r}-q_{k,2r}|&\leq Cr^{-s}||q_{k,r}u_{2,k}-q_{k,2r}u_{2,k}||_{L^\infty(Q_r\cap\{d_k>r/2\})}\\&\leq Cr^{-s}(||u_{1,k}-q_{k,r}u_{2,k}||_{L^\infty(Q_r)}+||u_{1,k}-q_{k,2r}u_{2,k}||_{L^\infty(Q_{2r})})\\&\leq C\theta(r)r^{s-\rho}.\end{aligned}$$

This implies in the same way as in [1, Proposition 4.4], that $\frac{q_{k,r}}{\theta(r)}\to 0$, as $r\downarrow 0$, and $||v_m||_{L^\infty(Q_R)}\leq CR^{2s-\rho}$. Moreover by the definition of θ it holds $||v_m||_{L^\infty(Q_1)}\geq\frac{1}{2}$. Let us turn to the equation that v_m satisfies

$$\begin{cases}(\partial_t+L_{k_m})v_m&=&\frac{r_m^\rho}{\theta(r_m)}(f_{1,k_m}-q_{k_m,r_m}f_{2,k_m})&\text{in }U_m\cap Q_{1/r_m}\\ u&=&0&\text{in }U_m^c\cap Q_{1/r_m},\end{cases}$$

with notation as in (3.3.2). Rescaled Proposition 3.3.1 say, that for every $M>1$ we get that

$$[v_m]_{C_p^\gamma(Q_{M/2})}\leq C(M),$$

provided that m is big enough. This along with the convergence result [32, Lemma 3.1] allows us to pass to the limit (up to a subsequence) to get that $v:=\lim_{m\to\infty}v_m$ solves

$$\begin{cases}(\partial_t+L_0)v&=&0&\text{in }\{x_n>0\}\\ u&=&0&\text{in }\{x_n\leq 0\},\end{cases}$$

for some operator L_0 satisfying (3.1.1). Moreover it holds $||v||_{L^\infty(Q_1)}\geq\frac{1}{2}$, $||v||_{L^\infty(Q_R)}\leq CR^{2s-\rho}$ and $\int_{Q_1}v\cdot(x_n)_+^s=0$. Then the Liouville theorem from [32, Theorem 4.11] implies that $v(x)=q(x_n)_+^s$, which gives rise to the contradiction.

To prove the moreover part, we first notice that $q_{(z,t_0)}$ are bounded independently of the point (z,t_0), thanks to [57, Lemma B.5]. Then we apply interior estimates (Lemma 3.6.3) on rescaled function $u_1-q_{(z,t_0)}u_2$ together with the above established estimate with [58, Lemma A.3], to get the wanted estimate. □

Note that the argument works only for orders smaller than $2s$, since the growth of the blow-up function at infinity inherits this order, which is the largest admissible growth when dealing with non-local operators of order $2s$.

As mentioned before, the established expansion imply the boundary regularity estimate for the quotient of two solutions, for orders smaller than s.

Corollary 3.3.5. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\rho \in (0, s)$. Let L be an operator of the form* (3.1.1). *Assume $u_i \in C^\gamma_p(Q_1)$, $i \in \{1, 2\}$, $\gamma > 0$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $||d^\rho f_2||_{L^\infty(\Omega\cap Q_1)} \le 1$ and $||u_2||_{L^\infty(B_R\times(-1,1))} \le R^{2s-\rho}$ for $R > 1$. Assume also that $u_2 \ge c_0 d^s$, for some $c_0 > 0$.

Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{s-\rho}(\overline{\Omega}\cap Q_{1/2})} \le C\left(||d^\rho f_1||_{L^\infty(\Omega\cap Q_1)} + \sup_{R>1} R^{-2s+\rho}||u_1||_{L^\infty(B_R\times(-1,1))}\right),$$

with $C > 0$ depending only on n, s, ρ, c_0, G_0 and ellipticity constants.

Proof. Dividing u_1 with $||d^\rho f_1||_{L^\infty(\Omega\cap Q_1)} + \sup_{R>1} R^{-2s+\rho}||u_1||_{L^\infty(B_R\times(-1,1))}$ and thanks to [57, Lemma B.2] it suffices to prove

$$\left[\frac{u_1}{u_2}\right]_{C_p^{s-\rho}(Q_r(x_0,t_0))} \le C,$$

independently of (x_0, t_0) and r whenever $Q_{2r}(x_0, t_0) \subset \Omega \cap Q_{1/2}$ and $d_x(x_0, t_0) \le C_1 r$, with C_1 depending only on Ω. Denote $(z, t_0) \in \partial\Omega$ the closest point to (x_0, t_0), and $C_1 r = |x_0 - z|$. Proposition 3.3.4 gives

$$[u_1 - qu_2]_{C_p^{2s-\rho}(Q_r(x_0,t_0))} \le C, \qquad ||u_1 - qu_2||_{L^\infty(Q_r(z,t_0))} \le Cr^{2s-\rho},$$

for a suitable constant q. We now apply Lemma 3.6.2 to get

$$\left[\frac{u_1}{u_2}\right]_{C_p^{s-\rho}(Q_r(x_0,t_0))} \le C,$$

where the interior regularity for u_2 is provided by Lemma 3.6.3. The claim is proven. □

We prove next an estimate for the regularity of the quotient of two solutions in the form needed later on.

Corollary 3.3.6. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\varepsilon, \varepsilon' > 0$ and $\gamma \in (0, 1)$. Let L be an operator of the form* (3.1.1). *Assume $u_i \in C^{\varepsilon'}_p(\mathbb{R}^n \times (-1, 1))$, $i \in \{1, 2\}$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $||f_2||_{L^\infty(\Omega\cap Q_1)} \le 1$, $||u_2||_{L^\infty(B_R\times(-1,1))} \le R^{2s-\varepsilon}$, for $R > 1$. Assume also that $u_2 \ge c_0 d^s$, for some $c_0 > 0$.

Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{s-\varepsilon}(\overline{\Omega}\cap Q_{1/2})} \le C\left([f_1]_{C_p^\gamma(\Omega\cap Q_1)} + ||u_1||_{L^\infty(\mathbb{R}^n\times(-1,1))}\right),$$

with $C > 0$ depending only on $n, s, \varepsilon, c_0, G_0$ and ellipticity constants.

Moreover

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{s-\varepsilon}(\overline{\Omega}\cap Q_{1/2})} \le C\left([f_1]_{C_p^\gamma(\Omega\cap Q_1)} + \sup_{R>1} R^{-2s+\varepsilon}\,[u_1]_{C_p^\gamma(B_R\times(-1,1))} + ||u_1||_{L^\infty(Q_1)}\right),$$

and if the kernel K of the operator L satisfies $[K]_{C^\gamma(B_r^c)} \le Cr^{-n-2s-\gamma}$, we have

$$\begin{aligned}\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{s-\varepsilon}(\overline{\Omega}\cap Q_{1/2})} \le& C\left([f_1]_{C_p^\gamma(\Omega\cap Q_1)} + \sup_{R>1} R^{-2s+\varepsilon}\,[u_1]_{C_t^{\frac{\gamma}{2s}}(B_R\times(-1,1))}\right.\\ &\left.+\sup_{R>1} R^{-2s-\gamma-\varepsilon}||u_1||_{L^\infty(B_R\times(-1,1))}\right).\end{aligned}$$

Proof. We split $u_1 = u + v$, where $(\partial_t + L)v = f_1(0,0)$ in Ω, and $v = 0$ in Ω^c. Then we have that $|f_1(0,0)| \le C||v||_{L^\infty(\Omega)}$ and $||u||_{L^\infty(\Omega\cap Q_1)} \le C(||f_1 - f_1(0,0)||_{L^\infty(\Omega\cap Q_1)}) \le C\,[f_1]_{C_p^\gamma(\Omega\cap Q_1)}$. Hence

$$\begin{aligned}||f_1||_{L^\infty(\Omega\cap Q_1)} &\le |f_1(0,0)| + C\,[f_1]_{C_p^\gamma(\Omega\cap Q_1)} \le C\left(||v||_{L^\infty(\Omega\cap Q_1)} + [f_1]_{C_p^\gamma(\Omega\cap Q_1)}\right)\\ &\le C\left(||u_1||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)} + [f_1]_{C_p^\gamma(\Omega\cap Q_1)}\right)\\ &\le C\left(||u_1||_{L^\infty(\Omega\cap Q_1)} + [f_1]_{C_p^\gamma(\Omega\cap Q_1)}\right).\end{aligned}$$

The first estimate now follows from Corollary 3.3.5.

To prove the moreover case, we use the already proven result on the function $\bar{u} = u_1\chi$ instead of u_1, where $\chi \in C_c^\infty(B_1)$, with $\chi \equiv 1$ in $B_{4/5}$. Then $\bar{u}$ solves $(\partial_t + L)\bar{u} = (\partial_t + L)u_1 + L(u_1(\chi - 1)) = f_1 + \bar{f}$ in $Q_{3/4}\cap\Omega$. We estimate $\left[\bar{f}\right]_{C_p^\gamma(\Omega\cap Q_{3/4})}$ as follows

$$\begin{aligned}|\bar{f}(x,t) - \bar{f}(x',t')| =& \left|\int_{B_{4/5}^c(-x)} u_1(1-\chi)(x+y,t)K(y)dy\right.\\ &\left.-\int_{B_{4/5}^c(-x')} u_1(1-\chi)(x'+y,t')K(y)dy\right|\\ \le& \int_{B_{1/20}^c} |u_1(x+y,t) - u_1(x'+y,t')|K(y)dy\\ &+ ||u_1||_{L^\infty(Q_1)}\int_{B_{1/20}^c} |\chi(x+y) - \chi(x'+y)|K(y)dy\\ \le& (|x-x'|^\gamma + |t-t'|^{\frac{\gamma}{2s}})\left(\int_{B_{1/20}^c} [u_1]_{C_p^\gamma(B_{|y|+1})}\,K(y)dy + C_\chi||u_1||_{L^\infty(Q_1)}\right)\\ \le& (|x-x'|^\gamma + |t-t'|^{\frac{\gamma}{2s}})\left(\Lambda C_0\int_{B_{1/2}^c}\frac{(|y|+1)^{2s-\varepsilon}}{|y|^{n+2s}}dy + C||u_1||_{L^\infty(Q_1)}\right)\\ \le& (|x-x'|^\gamma + |t-t'|^{\frac{\gamma}{2s}})\left(CC_0 + C||u_1||_{L^\infty(Q_1)}\right),\end{aligned}$$

where C_0 stands for $\sup_{R>1} R^{-2s+\varepsilon}\,[u_1]_{C_p^\gamma(B_R\times(-1,1))}$. Applying the already proven inequality for $Q_{3/4}$ and $Q_{1/2}$ (which follows from the covering argument) gives

$$\left[\frac{u_1}{u_2}\right]_{C_p^{s-2\varepsilon}(\overline{\Omega}\cap Q_{1/2})} \le C\left([f_1]_{C_p^\gamma(\Omega\cap Q_1)} + \sup_{R>1} R^{-2s+\varepsilon}\,[u_1]_{C_p^\gamma(B_R\times(-1,1))} + ||u_1||_{L^\infty(Q_1)}\right).$$

To pass from the growth control with the parabolic Hölder seminorms to seminorms in time with L^∞ growth, we proceed in the same way as in the moreover case of Lemma 3.6.3. The claim is proven. □

In the following proposition we establish the expansion for two solutions for orders between $2s$ and $1+s$. In order to exceed order $2s$, we work with Hölder seminorms of the quotient directly. We follow the idea of [76, Proposition 3.3].

Proposition 3.3.7. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\varepsilon > 0$ and $\gamma > 0$. Let L be an operator of the form (3.1.1), with kernel $K \in C^{1-s}(\mathbb{S}^{n-1})$. Assume $u_i \in C^\gamma_p(Q_1)$, $i \in \{1,2\}$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $[f_i]_{C^{1-s}_p(\Omega\cap Q_1)} \leq 1$, $[u_i]_{C_t^{\frac{1-s}{2s}}(B_R\times(-1,1))} \leq R^{2s-1}$, and $||u_2||_{L^\infty(B_R\times(-1,1))} \leq R^s$ and $||u_1||_{L^\infty(B_R\times(-1,1))} \leq R^{3s-1-\varepsilon}$. Assume also that $u_2 \geq c_0 d^s$, for some $c_0 > 0$.

Then for every $(z,t) \in \partial\Omega \cap Q_{1/2}$ we have

$$\left[\frac{u_1}{u_2} - q_r\right]_{C_p^{s-2\varepsilon}(\Omega\cap Q_r(z,t))} \leq C r^{1-s},$$

where q_r equals $\arg\min_{q\in\mathbb{R}} \int_{Q_r\cap\Omega}(\frac{u_1}{u_2} - q)^2$ and $C > 0$ depends only on $n, s, \varepsilon, c_0, G_0$ and ellipticity constants.

Proof. Without loss of generality we can assume that $(z,t) = (0,0)$. Moreover, if we cut off u_1 with a smooth function $\chi \in C^\infty_c(B_1)$ that is 1 in $B_{3/4}$, we can assume that $u_1 = 0$ outside Q_1, since this does not ruin the estimate for C^{1-s}_p seminorm of the right-hand side of u_1 (see the proof of Lemma 3.6.3). We argue with contradiction. Suppose that for each $k \in \mathbb{N}$ there exist $\Omega_k, u_{i,k}, f_{i,k}, L_k$ as in the statement, so that

$$\sup_k \sup_{r>0} r^{s-1}\left[\frac{u_{1,k}}{u_{2,k}} - q_{r,k}\right]_{C_p^{s-2\varepsilon}(\Omega_k\cap Q_r)} = \infty,$$

with $q_{r,k} = \arg\min_{q\in\mathbb{R}} \int_{Q_r\cap\Omega_k}(\frac{u_{1,k}}{u_{2,k}} - q)^2$.

Then we define

$$\theta(r) = \sup_k \sup_{\rho>r} \rho^{s-1}\left[\frac{u_{1,k}}{u_{2,k}} - q_{\rho,k}\right]_{C_p^{s-2\varepsilon}(\Omega_k\cap Q_\rho)},$$

which is monotone in r, finite for $r > 0$ by Corollary 3.3.6, and goes to ∞ as $r \downarrow 0$, thanks to the contradiction assumption. Hence for every $m \in \mathbb{N}$ we can find k_m and r_m, so that

$$\frac{m}{4} \leq \frac{\theta(r_m)}{2} \leq r_m^{s-1}\left[\frac{u_{1,k_m}}{u_{2,k_m}} - q_{r_m,k_m}\right]_{C_p^{s-2\varepsilon}(\Omega_{k_m}\cap Q_{r_m})} \leq \theta(r_m).$$

In particular $r_m \to 0$ as $m \to \infty$. We define the blow up sequence

$$v_m(x,t) = \frac{1}{\theta(r_m) r_m^{1-2\varepsilon}}\left(\frac{u_{1,k_m}}{u_{2,k_m}} - q_{r_m,k_m}\right)(r_m x, r_m^{2s} t).$$

Notice that by definition of r_m, k_m and $q_{r,m}$ we have

$$[v_m]_{C_p^{s-2\varepsilon}(\Omega_m\cap Q_1)} \geq \frac{1}{2} \quad \text{and} \quad \int_{\Omega_m\cap Q_1} v_m = 0, \tag{3.3.3}$$

where $\Omega_m = \{(x,t);\ (r_m x, r_m^{2s}t) \in \Omega_{k_m}\}$. Moreover, the way θ is defined gives a control on the growth of v_m in the following manner

$$\begin{aligned}
[v_m]_{C_p^{s-2\varepsilon}(\Omega_m\cap Q_R)} =& \frac{r_m^{s-2\varepsilon}}{\theta(r_m)r_m^{1-2\varepsilon}}\left[\frac{u_{1,k_m}}{u_{2,k_m}} - q_{r_m,k_m}\right]_{C_p^{s-2\varepsilon}(\Omega_m\cap Q_{Rr_m})} \\
\leq& \frac{1}{\theta(r_m)r_m^{1-s}}\left[\frac{u_{1,k_m}}{u_{2,k_m}} - q_{Rr_m,k_m}\right]_{C_p^{s-2\varepsilon}(\Omega_m\cap Q_{Rr_m})} \\
\leq& \frac{\theta(Rr_m)(Rr_m)^{1-s}}{\theta(r_m)r_m^{1-s}} \leq R^{1-s},
\end{aligned} \tag{3.3.4}$$

for every $R \geq 1$. In combination with $\int_{\Omega_m\cap Q_1} v_m = 0$ we deduce $||v_m||_{L^\infty(Q_1\cap\Omega_m)} \leq C$ uniformly in m. Combining it with the growth of the seminorms, we deduce $||v_m||_{L^\infty(Q_R\cap\Omega_m)} \leq CR^{1-2\varepsilon}$. Notice that these estimates hold true for general k, r, since we only used the definition of θ and not of r_m, k_m. Therefore we have

$$\begin{aligned}
\frac{|q_{r,k} - q_{2r,k}|}{\theta(r)} &= \frac{||q_{r,k} - q_{2r,k}||_{L^\infty(Q_r\cap\Omega_k)}}{\theta(r)} \\
&\leq \frac{||u_{1,k}/u_{2,k-q_{r,k}}||_{L^\infty(Q_r\cap\Omega_k)}}{\theta(r)} + \frac{||u_{1,k}/u_{2,k-q_{2r,k}}||_{L^\infty(Q_r\cap\Omega_k)}}{\theta(r)} \\
&\leq r^{1-2\varepsilon}\left|\left|\frac{1}{\theta(r)r^{1-2\varepsilon}}\left(\frac{u_{1,k}}{u_{2,k}} - q_{r,k}\right)\right|\right|_{L^\infty(\Omega_k\cap Q_r)} \\
&\quad + (2r)^{1-2\varepsilon}\left|\left|\frac{1}{\theta(2r)(2r)^{1-2\varepsilon}}\left(\frac{u_{1,k}}{u_{2,k}} - q_{r,k}\right)\right|\right|_{L^\infty(\Omega_k\cap Q_{2r})} \\
&\leq Cr^{1-2\varepsilon}.
\end{aligned}$$

Hence as in the proof of [1, Proposition 4.1], we conclude that $\frac{q_{r,k}}{\theta(r)} \to 0$ as $r \downarrow 0$, uniformly in k.

We now define

$$v_{1,m}(x,t) = \frac{1}{\theta(r_m)r_m^{1+s-2\varepsilon}}\left(u_{1,k_m} - q_{r_m,k_m}u_{2,k_m}\right)(r_m x, r_m^{2s}t),$$

and

$$v_{2,m}(x,t) = \frac{1}{r_m^s}u_{2,k_m}(r_m x, r_m^{2s}t),$$

so that $v_m = \frac{v_{1,m}}{v_{2,m}}$. Thanks to assumptions on $u_{2,k}$ we have that $[v_{2,m}]_{C_p^s(\mathbb{R}^n\times(-r_{k_m}^{-2s}, r_{k_m}^{-2s}))} \leq 1$, $||(\partial_t + L_{k_m})v_{2,m}||_{L^\infty(Q_1\cap\Omega_m)} \leq 1$ and $v_{2,m} \geq cc_0 d_m^s$, were d_m denotes the distance function in Ω_m. Therefore we can apply Corollary 3.3.6 to $u_{1,m}(M\cdot, M^{2s}\cdot)$ and $v_{2,m}(M\cdot, M^{2s}\cdot)$, to get

$$[v_m]_{C_p^{s-\varepsilon}(\Omega_m\cap Q_{M/2})} \le C(M)\Big([(\partial_t+L_{k_m})v_{1,m}]_{C_p^{1-s}(Q_M\cap\Omega_m)} + \sup_{R>M} R^{\varepsilon-2s}[v_{1,m}]_{C_t^{\frac{1-s}{2s}}(B_R\times(-M^{2s},M^{2s}))} + \sup_{R>1} R^{-s-1+\varepsilon}||v_{1,m}||_{L^\infty(B_R\times(-M^{2s},M^{2s}))}\Big),$$

for every $M\in\mathbb{N}$, for m big enough. Let us now show that all the quantities in the right-hand side are bounded uniformly in m. First,

$$(\partial_t+L_{k_m})v_{1,m}(x,t) = \frac{1}{\theta(r_m)r_m^{1-s-2\varepsilon}}(f_{1,k_m}-q_{r_m,k_m}f_{2,k_m})(r_mx, r_m^{2s}t),$$

and hence

$$\begin{aligned}[(\partial_t+L_{k_m})v_{1,m}]_{C_p^{1-s}(Q_M\cap\Omega_m)} &= \frac{r_m^{2\varepsilon}}{\theta(r_m)}[f_{1,k_m}-q_{r_m,k_m}f_{2,k_m}]_{C_p^{1-s}(Q_{Mr_m}\cap\Omega_{k_m})}\\ &\le \frac{C(1+q_{r_m,k_m})}{\theta(r_m)},\end{aligned}$$

which is bounded by assumption on the seminorms of $f_{i,k}$ and the fact that $\frac{q_{r,k}}{\theta(r)}\to 0$ as $r\downarrow 0$ uniformly in k. The growth term we estimate as follows

$$\begin{aligned}[v_{1,m}]_{C_t^{\frac{1-s}{2s}}(B_R\times(-M^{2s},M^{2s}))} &\le [v_mv_{2,m}]_{C_t^{\frac{1-s}{2s}}}\\ &\le [v_m]_{C_p^{s-2\varepsilon}(Q_R)}R^{s-2\varepsilon-1+s}||v_{2,m}||_{L^\infty} + [v_{2,m}]_{C_t^{\frac{1-s}{2s}}}||v_m||_{L^\infty(Q_R)}\\ &\le R^{1-s}R^{2s-1-2\varepsilon}C(M)R^s + C(M)R^{2s-1}R^{1-2\varepsilon}\\ &\le C(M)R^{2s-2\varepsilon},\end{aligned}$$

where we used the growth control of v_m from (3.3.4) and $[u_{2,k}]_{C_p^{1-s}(B_R\times(-1,1))}\le R^{2s-1}$, which holds by assumption. The last term is estimated similarly

$$||v_{1,m}||_{L^\infty(B_R\times(-M^{2s},M^{2s}))} \le ||v_m||_{L^\infty}||v_{2,m}||_{L^\infty} \le C(M)R^{1-2\varepsilon}R^s.$$

Putting it all together, we get that $||v_m||_{L^\infty(\Omega_m\cap Q_M)}$ and $[v_m]_{C_p^{s-\varepsilon}(\Omega_m\cap Q_M)}$ are uniformly bounded, which implies that v_m converge to some function v in $C_p^{s-2\varepsilon}$ in any compact subset of $\{x_n\ge 0\}$. Moreover $v_{2,m}$ converges to $c(x_n)_+^s$ in $C_p^{s-2\varepsilon}$ locally in $\mathbb{R}^{n+1}$. Hence also $v_{1,m}$ converges to $w := v\cdot(x_n)_+^s$ in $C_p^{s-2\varepsilon}$ locally in $\mathbb{R}^{n+1}$.

We claim that w satisfies the hypothesis of the Liouville theorem [76, Theorem 2.1]. To see this, note that for fixed $h\in\mathbb{R}^n$, $h_n\ge 0$, and $\tau\in\mathbb{R}$ we have

$$(\partial_t+L_{k_m})(v_{1,m}(x+h,t+\tau)-v_{1,m}(x,t)) = \frac{1}{\theta(r_m)r_m^{1-s}}\left(\hat f_{1,k_m}+q_{r_m,k_m}\hat f_{2,k_m}\right),$$

where $\hat f_{i,k_m} = f_{i,k_m}(r_m(x+h), r_m^{2s}(t+\tau)) - f_{i,k_m}(r_mx, r_m^{2s}t)$. But

$$|\hat f_{i,k_m}| \le C(h,\tau)r_m^{1-s}[f_{i,k_m}]_{C_p^{1-s}(Q_1\cap\Omega_{k_m})},$$

which is bounded by assumption. Hence

$$|(\partial_t + L_{k_m})(v_{1,m}(x+h,t+\tau) - v_{1,m}(x,t))| \le \frac{C(1+q_{r_m,k_m})}{\theta(r_m)},$$

which goes to 0 as $m\to\infty$. Hence we can pass to the limit with [76, Lemma 3.1], to get

$$(\partial_t + L_0)(w(x+h,t+\tau) - w(x,t)) = 0, \qquad \text{whenever } x_n > 0.$$

Passing the growth control of v_m to the limit gives $\left[\frac{w}{(x_n)_+^s}\right]_{C^{s-2\varepsilon}(Q_R\cap x_n>0)} \le R^{1-s}$, and hence [76, Theorem 2.1] implies that $w = q(x_n)_+^s$, and hence $v = q$. But passing to the limit quantities in (3.3.3), we get that $[v]_{C_p^{s-2\varepsilon}(Q_1\cap\{x_n>0\})} > \frac{1}{2}$ but also $\int_{Q_1\cap\{x_n>0\}} v = 0$. These contradict each other, since v is a constant function. □

Remark 3.3.8. We actually need not work with the decay of the seminorms of the quotient in order to establish expansions of order $1+s$. We nevertheless presented the result, since we believe they might be important for establishing expansions of higher orders and moreover we correct some imprecisions from [76, Proposition 3.3].

As a consequence we get $C^{1-\varepsilon}$ regularity of the quotient of two solutions up to the boundary.

Corollary 3.3.9. *Let $s\in(\frac{1}{2},1)$. Let $\Omega\subset\mathbb{R}^{n+1}$ be C^1_p in Q_1. Let $\gamma,\varepsilon>0$. Let L be an operator of the form (3.1.1), with kernel $K\in C^{1-s}(\mathbb{S}^{n-1})$. Assume $u_i\in C^\gamma_p(Q_1)$, $i\in\{1,2\}$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega\cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c\cap Q_1, \end{cases}$$

with $[f_i]_{C_p^{1-s}(\Omega\cap Q_1)}\le 1$, $[u_i]_{C_t^{\frac{1-s}{2s}}(B_R\times(-1,1))}\le R^{2s-1}$, and $||u_2||_{L^\infty(B_R\times(-1,1))}\le R^s$ and $||u_1||_{L^\infty(B_R\times(-1,1))}\le R^{3s-1-\varepsilon}$. Assume also that $u_2\ge c_0 d^s$, for some $c_0>0$.

Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{1-\varepsilon}(\Omega\cap Q_{1/2})} \le C,$$

where $C>0$ depends only on n,s,ε,c_0,G_0 and ellipticity constants.

Proof. Proposition 3.3.7 in combination with [76, Proposition 3.4] give that for every $(z,\tau)\in\partial\Omega\cap Q_{1/2}$ there is a constant $q_{(z,\tau)}$ so that

$$|u_1(x,t) - q_{(z,\tau)}u_2(x,t)| \le C(|x-z|^{1+s-\varepsilon} + |t-\tau|^{\frac{1+s-\varepsilon}{2s}}).$$

Analogously as in the proof of Corollary 3.3.10, in combination with interior and boundary regularity estimates (Proposition 3.3.1 and Lemma 3.6.3) we conclude for any (x_0,t_0), such that $d_p(x_0,t_0) = |(x_0,t_0)-(z,\tau)|_p \le c_\Omega r$, and that $Q_{2r}(x_0,t_0)\subset\Omega\cap Q_1$ we have

$$\left[u_1 - q_{(z,\tau)}u_2\right]_{C_p^{1+s-\varepsilon}(Q_r(x_0,t_0))} \le C.$$

We can apply Lemma 3.6.2 to get that

$$\left[\frac{u_1}{u_2}\right]_{C_p^{1-\varepsilon}(Q_r(x_0,t_0))} = \left[\frac{u_1}{u_2} - q_{(z,\tau)}\right]_{C_p^{1-\varepsilon}(Q_r(x_0,t_0))} \le C,$$

where Lemma 3.6.3 provides the needed interior estimates for u_2. The result now follows from [57, Lemma B.2]. □

3.3.3 Boundary Harnack in C_p^β domains

We now present a result that is required for establishing boundary Harnack inequalities of higher orders. It shows how well can we approximate solutions with d^s near the boundary and in the interior.

Corollary 3.3.10. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1, for some $\beta > 2s$. Let $\varepsilon > 0$. Let L be an operator of the form (3.1.1). Assume that $u \in C_p^\gamma(\mathbb{R}^n \times (-1,1))$, $\gamma > 0$, solves*

$$\begin{cases} (\partial_t + L)u &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $||d^{1-s}f||_{L^\infty(\Omega\cap Q_1)} \leq 1$, $||u||_{L^\infty(B_R\times(-1,1))} \leq R^s$.

Then for every $(z,t_0) \in \partial\Omega \cap Q_{1/2}$ there exists a constant $q_{(z,t_0)}$, so that

$$\left[u(x,t) - q_{(z,t_0)}d^s(x,t)\right]_{C_p^{s-\varepsilon}(Q_r(z,t_0))} \leq Cr^{2s-1+\varepsilon}, \qquad r < \frac{1}{2}.$$

The constant C depends only on $\beta, n, s, \varepsilon, G_0$ and ellipticity constants.

Moreover, if additionally $\beta > 1+s$ and $[f]_{C_p^\alpha(\Omega\cap Q_1)} \leq 1$, and $[u]_{C_t^{\frac{\alpha}{2s}}(B_R\times(-1,1))} \leq R^{3s-1-\alpha}$, for some $\alpha \in (0, s-\varepsilon]$, then for every $(x_0,t_0) \in \Omega \cap Q_{1/2}$, such that $d_x(x_0) = |x_0 - z| = c_\Omega r$, we have

$$\left[u - q_{(z,t_0)}d^s\right]_{C_p^{\alpha+2s}(Q_r(x_0,t_0))} \leq Cr^{-1+s-\alpha}.$$

Proof. From Corollary 3.3.5, and the fact that $(\partial_t + L)d^s$ is bounded by d^{s-1} (see [58, Lemma 3.2]) we get

$$\left[\frac{u}{d^s}\right]_{C_p^{2s-1}(\Omega\cap Q_{3/4})} \leq C, \qquad |u(x,t) - q_{(z,t_0)}d^s(x,t)| \leq C\left(|x-z| + |t-t_0|^{\frac{1}{2s}}\right)^{3s-1}.$$

Moreover Corollary 3.3.14 gives that $u \in C_p^s(Q_{3/4})$. Since u and d^s grow at most as power s at infinity, the second estimate implies that $||u - q_{(z,t_0)}d^s||_{L^\infty(Q_r(z,t_0))} \leq Cr^{3s-1}$ for all $r > 0$. Using rescaled Proposition 3.3.1 gives

$$\begin{aligned}\left[u(x,t) - q_{(z,t_0)}d^s(x,t)\right]_{C_p^{s-\varepsilon}(Q_r(z,t_0))} \leq& Cr^{-s+\varepsilon}\Big(r^{2s}||d^s(f + (\partial_t + L)d^s)||_{L^\infty(Q_{2r}(z,t_0))} \\ &+ \sup_{R>1} R^{-2s+\varepsilon}||u - q_{(z,t_0)}d^s||_{L^\infty(Q_{Rr}(z,t_0))}\Big) \\ \leq& Cr^{2s-1+\varepsilon},\end{aligned}$$

whenever $r < \frac{1}{2}$.

To prove the moreover case, we use Lemma 3.6.3, which gives

$$\begin{aligned}\left[u - q_{(z,t_0)}d^s\right]_{C_p^{\alpha+2s}(Q_r(x_0,t_0))} \leq& Cr^{-\alpha-2s}\Big(r^{\alpha+2s}\left[f - q_{(z,t_0)}(\partial_t + L)d^s\right]_{C_p^\alpha(Q_{2r}(x_0,t_0))} \\ &+ \sup_{R>1} R^{-2s-\alpha+\varepsilon}||u - q_{(z,t_0)}d^s||_{L^\infty(Q_{2(R+2)r}(x_0,t_0))} \\ &+ \sup_{R>1} R^{-2s+\varepsilon}r^\alpha\left[u - q_{(z,t_0)}d^s\right]_{C_p^\alpha(Q_{2r(R+2)})}\Big).\end{aligned}$$

Lemma 3.6.1 and Proposition 3.5.1 assure that

$$[(\partial_t + L)d^s]_{C_p^{s-\varepsilon}(Q_{2r}(x_0,t_0))} \leq Cr^{-1+\varepsilon},$$

while above we established that

$$||u - q_{(z,t_0)}d^s||_{L^\infty(Q_{2(R+2)r}(x_0,t_0))} \leq C((R+2)r)^{3s-1}.$$

Finally, combining the already proven estimate with the assumptions on u, we get

$$\left[u(x,t) - q_{(z,t_0)}d^s(x,t)\right]_{C_t^{\frac{\alpha}{2s}}(Q_r(z,t_0))} \leq Cr^{3s-1-\alpha},$$

for all $r > 0$, which implies

$$\left[u - q_{(z,t_0)}d^s\right]_{C_t^{\frac{\alpha}{2s}}(Q_{2r(R+2)})} \leq C(r(R+2))^{3s-1-\alpha}.$$

We conclude that

$$\left[u - q_{(z,t_0)}d^s\right]_{C_p^{\alpha+2s}(Q_r(x_0,t_0))} \leq Cr^{-1+s-\alpha},$$

which proves the claim. □

In order to get the expansions for two solutions of orders exceeding $1+s$, we prove the decay of seminorms of the expansion. Since the decay rate is transformed in the growth of the blow up sequence, it can not be taken larger than $2s$. In combination with the C^s nature of solutions, with this strategy we can only achieve expansions of order $3s$. The precise version of the claim is stated below.

Proposition 3.3.11. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1, for some $\beta > 1+s$. Let $\varepsilon > 0$ and $\gamma > 0$. Let L be an operator of the form (3.1.1), with kernel $K \in C^{2\beta+1}(\mathbb{S}^{n-1})$. Assume $u_i \in C_p^\gamma(Q_1)$, $i \in \{1,2\}$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i \quad \text{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $[f_i]_{C_p^\alpha(\Omega\cap Q_1)} \leq 1$, $||u_i||_{L^\infty(B_R\times(-1,1))} \leq R^{2s+\alpha-\varepsilon}$ and $[u_i]_{C_t^{\frac{\alpha}{2s}}(B_R\times(-1,1))} \leq R^{2s-\varepsilon}$, for some $\alpha \in (1-s, s-\varepsilon]$. Assume also that $u_2 \geq c_0 d^s$, for some $c_0 > 0$.

Then for every $(z,\tau) \in \partial\Omega \cap Q_{1/2}$ there are $q_{(z,\tau)} \in \mathbb{R}$ and $Q_{(z,\tau)}$ a polynomial of degree 1 in $\mathbb{R}^n$ with $Q_{(z,\tau)}(z,\tau) = 0$, so that

$$\left[u_1 - q_{(z,\tau)}u_2 - Q_{(z,\tau)}d^s\right]_{C_p^\alpha(Q_r(z,\tau))} \leq Cr^{2s-\varepsilon}.$$

The constant $C > 0$ depends only on $n, s, \varepsilon, c_0, G_0$ and ellipticity constants.

Proof. Thanks to the assumption on the domain, we can assume that $(z,\tau) = (0,0)$. We can also assume $u_i = 0$ outside of Q_1, $u \in C_p^s(\mathbb{R}^n \times (-1,1))$,[1] thanks to the growth control, global regularity in time and regularity of the kernel (see Lemma 3.6.3). We argue with

[1] We achieve it with multiplying it with a smooth cut off, which does not change the right-hand side too much.

contradiction. Assume that there exist $\Omega_k, L_k, ui, k, f_{i,k}$, $i = 1, 2$, as in the statement, so that

$$\sup_k \sup_{r>0} r^{-2s+\varepsilon} \left[u_{1,k} - q_k u_{2,k} - Q_k d_k^s\right]_{C_p^\alpha(Q_r)} = \infty,$$

for any $q_k \in \mathbb{R}$ and Q_k 1−homogeneous polynomial. We define $q_{r,k}, Q_{r,k}$ as the coefficients of $L^2(Q_r)$ projection of $u_{1,k}$ to $\mathbb{R}u_{2,k} + \mathbf{P}d_k^s$, where $\mathbf{P}$ stands for all 1-homogeneous polynomials. Therefore we have

$$\int_{Q_r} (u_{1,k} - q_{r,k}u_{2,k} - Q_{r,k}d_k^s)(qu_{2,k} + Qd_k^s) = 0, \qquad \text{for all } q \in \mathbb{R},\ Q \in \mathbf{P}.$$

Furthermore we define a monotone quantity

$$\theta(r) = \sup_k \sup_{\rho>r} \rho^{-2s+\varepsilon} \left[u_{1,k} - q_{\rho,k}u_{2,k} - Q_{\rho,k}d_k^s\right]_{C_p^\alpha(Q_\rho)}.$$

From Lemma 3.6.4 we deduce that $\theta(r) \to \infty$ as $r \downarrow 0$. Hence we can get a sequences $r_m \downarrow 0$ and k_m, for $m \in \mathbb{N}$, so that

$$\frac{m}{4} \leq \frac{\theta(r_m)}{2} \leq r_m^{-2s+\varepsilon} \left[u_{1,k_m} - q_{r_m,k_m}u_{2,k_m} - Q_{r_m,k_m}d_{k_m}^s\right]_{C_p^\alpha(Q_{r_m})} \leq \theta(r_m).$$

We define the blow-up sequence

$$v_m(x,t) = \frac{1}{\theta(r_m)r_m^{\alpha+2s-\varepsilon}} \left(u_{1,k_m} - q_{r_m,k_m}u_{2,k_m} - Q_{r_m,k_m}d_{k_m}^s\right)(r_m x, r_m^{2s}t). \tag{3.3.5}$$

By definition of $q_{r,k}$ and choice of r_m, k_m we have

$$\int_{Q_1} v_m(qu_{2,m} - Qd_m^s) = 0, \qquad \frac{1}{2} \leq [v_m]_{C_p^\alpha(Q_1)} \leq 1,$$

for any $q \in \mathbb{R}$ and $Q \in \mathbf{P}$.

Next we want to obtain growth control near infinity for the blow-up sequence. In this direction we estimate the decay of coefficients of $Q_{r,k}$ and $q_{r,k}$ at zero. We denote $Q_{r,k}(x) = Q_{r,k} \cdot x$, and proceed in the following way. First, by rescaled [1, Lemma A.10]

$$\begin{aligned}
|Q_{r,k} - Q_{2r,k}| \leq & Cr^{-1}||(Q_{r,k} - Q_{2r,k})||_{L^\infty(Q_r \cap \{d_k > r/2\})} \\
\leq & Cr^{-1-s}||Q_{r,k}d_k^s - Q_{2r,k}d_k^s||_{L^\infty(Q_r \cap \{d_k > r/2\})} \\
\leq & Cr^{-1-s+\alpha}\left[Q_{r,k}d_k^s - Q_{2r,k}d_k^s\right]_{C_p^\alpha(Q_r)} \\
\leq & Cr^{-1-s+\alpha}\Big(\left[u_{1,k} - q_{r,k}u_{2,k} - Q_{r,k}d_k^s\right]_{C_p^\alpha(Q_r)} \\
& \quad + \left[u_{1,k} - q_{2r,k}u_{2,k} - Q_{2r,k}d_k^s\right]_{C_p^\alpha(Q_{2r})} \\
& \quad + \left[q_{r,k}u_{2,k} - q_{2r,k}u_{2,k}\right]_{C_p^\alpha(Q_r)}\Big) \\
\leq & Cr^{-1-s+\alpha}\left(\theta(r)r^{2s-\varepsilon} + \theta(2r)(2r)^{2s-\varepsilon} + |q_{r,k} - q_{2r,k}|r^{s-\alpha}\right) \\
\leq & C\left(\theta(r)r^{s+\alpha-1-\varepsilon} + |q_{r,k} - q_{2r,k}|r^{-1}\right),
\end{aligned}$$

where we used that $[u_{2,k}]_{C^s_p}(Q_1) \le C$, see Corollary 3.3.14. Moreover from the definition of θ and the triangle inequality we get

$$[(q_{r,k} - q_{2r,k})u_{2,k} + (Q_{r,k} - Q_{2r,k}) \cdot x d_k^s]_{C^\alpha_p(Q_r)} \le C\theta(r) r^{2s-\varepsilon},$$

which furthermore implies

$$\begin{aligned} ||(q_{r,k} - q_{2r,k})u_{2,k} + (Q_{r,k} - Q_{2r,k}) \cdot x d_k^s||_{L^\infty(Q_r \cap \{d_k < r/2\})} \le \\ &\le ||(q_{r,k} - q_{2r,k})u_{2,k} + (Q_{r,k} - Q_{2r,k}) \cdot x d_k^s||_{L^\infty(Q_r)} \\ &\le r^\alpha \, [(q_{r,k} - q_{2r,k})u_{2,k} + (Q_{r,k} - Q_{2r,k}) \cdot x d_k^s]_{C^\alpha_p(Q_r)} \\ &\le C\theta(r) r^{2s+\alpha-\varepsilon}, \end{aligned}$$

and hence

$$\left|\left|(q_{r,k} - q_{2r,k})\frac{u_{2,k}}{d_k^s} + (Q_{r,k} - Q_{2r,k}) \cdot x\right|\right|_{L^\infty(Q_r \cap \{d_k < r/2\})} \le C\theta(r) r^{s+\alpha-\varepsilon}.$$

It follows from [1, Lemma A.11] that

$$|q_{r,k} - q_{2r,k}| \le C\theta(r) r^{s+\alpha-\varepsilon},$$

which implies

$$|Q_{r,k} - Q_{2r,k}| \le C\theta(r) r^{s+\alpha-1-\varepsilon}.$$

In the same way as in [1, Proposition 4.4] we deduce that

$$\begin{aligned} |q_{r,k} - q_{Rr,k}| &\le C\theta(r)(Rr)^{s+\alpha-\varepsilon} \\ |Q_{r,k} - Q_{Rr,k}| &\le C\theta(r)(Rr)^{s+\alpha-1-\varepsilon} \\ \frac{|q_{r,k}| + |Q_{r,k}|}{\theta(r)} &\downarrow 0, \quad \text{uniformly in } k. \end{aligned}$$

This implies the growth control of the blow-up sequence

$$\begin{aligned} [v_m]_{C^\alpha_p(Q_R)} \le & \frac{r_m^\alpha}{\theta(r_m) r_m^{\alpha+2s-\varepsilon}} \Big(\theta(Rr_m)(Rr_m)^{2s-\varepsilon} + |q_{r_m,k_m} - q_{Rr_m,k_m}| \, [u_{2,k_m}]_{C^\alpha(Q_{Rr_m})} \\ & + |Q_{r_m,k_m} - Q_{Rr_m,k_m}| \left[x d_{k_m}^s\right]_{C^\alpha(Q_{Rr_m})}\Big) \\ \le & \frac{C r_m^\alpha}{\theta(r_m) r_m^{\alpha+2s-\varepsilon}} \left(\theta(r_m)(Rr_m)^{2s-\varepsilon} + \theta(r_m)(Rr_m)^{\alpha+s-\varepsilon}(Rr_m)^{s-\alpha}\right. \\ & \left. + \theta(r_m)(Rr_m)^{s+\alpha-1-\varepsilon}(Rr_m)^{1+s-\alpha}\right) \\ \le & C R^{2s-\varepsilon}. \end{aligned} \tag{3.3.6}$$

Note that the estimate is valid for all $R > 1$,[2] and that we used $[u_{2,k_m}]_{C^s_p(\mathbb{R}^n \times (-1,1))} \le 1$.

[2]When $R > r_m^{-1}$, then Q_R has to be intersected with $\mathbb{R}^n \cap (-1,1)$.

Denoting $\Omega_m = \{(x,t);\ (r_m x, r_m^{2s} t) \in \Omega_{k_m}\}$, we have that v_m solves

$$(\partial_t + L_{k_m})v_m(x,t) = \frac{1}{\theta(r_m) r_m^{\alpha-\varepsilon}}\big(f_{1,k_m} - q_{r_m,k_m} f_{2,k_m} - sQ_{r_m,k_m} d_{k_m}^{s-1}\partial_t d_{k_m} \\ - L_{k_m}(Q_{r_m,k_m} d_{k_m}^s)\big)(r_m x, r_m^{2s} t)$$

in $\Omega_m \cap Q_{r_m^{-1}}$, as well as

$$v_m = 0, \qquad \text{in } \Omega_m^c \cap Q_{r_m^{-1}}.$$

By assumption we can bound

$$\begin{aligned}\frac{1}{\theta(r_m) r_m^{\alpha-\varepsilon}}\left[f_{1,k_m}(r_m\cdot, r_m^{2s}\cdot)\right]_{C_p^{\beta-1-s}(\Omega_m\cap Q_2)} &\leq C\frac{r_m^{\beta-1-s}}{\theta(r_{k_m}) r_m^{s-\varepsilon}}\left[f_{1,k_m}\right]_{C_p^{\beta-1-s}(\Omega_{k_m}\cap Q_{2r_m})} \\ &\leq C\frac{1}{\theta(r_{k_m})}\left[f_{1,k_m}\right]_{C_p^{\alpha-\varepsilon}(\Omega_{k_m}\cap Q_{2r_m})} \\ &\leq C\frac{1}{\theta(r_{k_m})}\end{aligned}$$

which goes to zero and in particular it is bounded for all m. Bounding the term with f_{2,k_m}, we obtain $C\frac{|q_{r_m,k_m}|}{\theta(r_m)}$ which also converges to zero. We proceed with estimating

$$\begin{aligned}\left|\frac{1}{\theta(r_m) r_m^{\alpha-\varepsilon}} Q_{r_m,k_m} d_{k_m}^{s-1}\partial_t d_{k_m}(r_m x, r_m^{2s} t)\right| &\leq \frac{C|Q_{r_m,k_m}|}{\theta(r_m) r_m^{\alpha-\varepsilon}}|r_m x| r_m^{s-1} d_m^{s-1}(x,t) \\ &\leq C\frac{|Q_{r_m,k_m}|}{\theta(r_m)} d_m^{s-1}(x,t),\end{aligned}$$

where d_m stands for the distance function in Ω_m. Finally by Proposition 3.5.1 we also get that

$$\begin{aligned}&\frac{1}{\theta(r_m) r_m^{\alpha-\varepsilon}}\left[L_{k_m}(Q_{r_m,k_m} d_{k_m}^s)(r_m\cdot, r_m^{2s}\cdot)\right]_{C_p^{\beta-1-s}(\Omega_m\cap Q_2)} \leq \\ &\qquad\leq \frac{r_m^{\beta-1-s}}{\theta(r_m) r_m^{\alpha-\varepsilon}}\left[L_{k_m}(Q_{r_m,k_m} d_{k_m}^s)\right]_{C_p^{\beta-1-s}(\Omega_{k_m}\cap Q_{2r_m})} \\ &\qquad\leq C\frac{|Q_{r_m,k_m}| r_m^{s-\varepsilon}}{\theta(r_m) r_m^{\alpha-\varepsilon}} \\ &\qquad\leq C\frac{|Q_{r_m,k_m}|}{\theta(r_m)},\end{aligned}$$

which also converges to 0 as $m \to \infty$. Hence Corollary 3.3.3 gives that

$$[v_m]_{C_p^{\alpha+\varepsilon/2}(Q_1)} \leq C,$$

independently of m. Note that boundedness of $||v_m||_{L^\infty(Q_2)}$ follows from $v_m(0,0) = 0$ and the uniform control on the seminorm $C_p^\alpha(Q_2)$, see (3.3.6). Moreover the growth control (3.3.6) and the newly obtained estimate together with Arzela-Ascoli theorem give that up to passing to a subsequence v_m converges locally uniformly in $\mathbb{R}^{n+1}$ to some function v, and the convergence is C_p^α in Q_1. Choosing $h \in \mathbb{R}^n$ with $h_n \geq 0$ and $\tau \in \mathbb{R}$, and denoting $v_h(x,t) = v(x+h, t+\tau)$ we conclude from [76, Proposition 3.1], that

$$\begin{cases} (\partial_t + L)(v_h - v) &= 0 & \text{in } \{x_n > 0\} \\ v &= 0 & \text{in } \{x_n \leq 0\} \\ [v]_{C_p^\alpha(Q_R)} &\leq CR^{2s-\varepsilon} & \text{for all } R > 1. \end{cases}$$

Hence the Liouville theorem (Proposition 3.3.12) yields that $v(x,t) = (x_n)_+^s(q + Q \cdot x)$, for some $Q \in \mathbf{P}$. But passing the quantities in (3.3.5) to the limit, we get that $q = 0$ and $Q = 0$, which contradicts $[v]_{C^\alpha(Q_1)} \geq \frac{1}{2}$. □

The Liouville type result used above reads as follows.

Proposition 3.3.12. *Let $s \in (0,1)$, $\alpha \in (0,s)$, and $\beta \in (0, 2s)$. Assume that w satisfies*

$$\begin{cases} (\partial_t + L)(w(\cdot + h, \cdot + \tau) - w) &= 0 & \text{in } \{x_n > 0\} \times (-\infty, 0) \\ w &= 0 & \text{in } \{x_n \leq 0\} \times (-\infty, 0), \end{cases}$$

where L is an operator of the form (3.1.1), $h \in \mathbb{R}^n$ with $h_n \geq 0$ and $\tau < 0$. Assume that w satisfies the growth condition

$$[w]_{C_p^\alpha(Q_R \cap \{t<0\})} \leq R^\beta,$$

for $R > 1$. Then

$$w(x,t) = (x_n)_+^s(p \cdot x + q),$$

for some $p \in \mathbb{R}^n$ and $q \in \mathbb{R}$.

Proof. Let first $h_n = 0$. Denote $v = w(\cdot + h, \cdot + \tau) - w$. We have

$$\begin{cases} (\partial_t + L)v &= 0 & \text{in } \{x_n > 0\} \times (-\infty, 0) \\ v &= 0 & \text{in } \{x_n \leq 0\} \times (-\infty, 0), \end{cases}$$

and $||v||_{L^\infty(Q_R \cap \{t<0\})} \leq CR^\beta$. Applying [32, Theorem 4.11], we conclude that $v(x) = K(x_n)_+^s$ for some constant K. The rest follows in the same way as in [76, Theorem 2.1]. □

We are now well equipped to prove $C^{2s-\varepsilon}$ regularity of the quotient two solutions.

Corollary 3.3.13. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1, for some $\beta > 1 + s$. Let $\varepsilon > 0$. Let L be an operator of the form (3.1.1), with kernel $K \in C^{2\beta+1}(\mathbb{S}^{n-1})$. Assume $u_i \in C_p^\gamma(Q_1)$, $i \in \{1,2\}$, $\gamma > 0$, solve*

$$\begin{cases} (\partial_t + L)u_i &= f_i & \text{in } \Omega \cap Q_1 \\ u_i &= 0 & \text{in } \Omega^c \cap Q_1, \end{cases}$$

with $[f_i]_{C_p^\alpha(\Omega \cap Q_1)} \leq 1$, $||u_i||_{L^\infty(B_R \times (-1,1))} \leq R^{2s+\alpha-\varepsilon}$, $[u_1]_{C_t^{\frac{\alpha}{2s}}(B_R \times (-1,1))} \leq R^{2s-\varepsilon}$ and $[u_2]_{C_t^{\frac{\alpha}{2s}}(B_R \times (-1,1))} \leq R^{3s-1-\alpha}$, for some $\alpha \in (1-s, s-\varepsilon]$. Assume also that $u_2 \geq c_0 d^s$, for some $c_0 > 0$.

Then

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{s+\alpha-\varepsilon}(\Omega \cap Q_{1/2})} \leq C.$$

The constant $C > 0$ depends only on $n, s, \alpha, \varepsilon, c_0, G_0$ and ellipticity constants.

Proof. Let $(x_0,t_0)\in\Omega\cap Q_{1/2}$ be such that $d_x(x_0,t_0)\le C_0r_0$ and that $Q_{2r_0}(x_0,t_0)\subset\Omega$. Let $(z,t_0)\in\partial\Omega$ be the closest point to (x_0,t_0) in the boundary at time t_0. We want to show that $\left[\frac{u_1}{u_2}\right]_{C_p^{\alpha+s}(Q_{r_0}(x_0,t_0))}\le C$, with C not depending on x_0,t_0,r_0. From Proposition 3.3.11 we get that for (z,t_0) we have an expansion of the form

$$\left[u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s\right]_{C_p^{\alpha}(Q_r(z,t_0))}\le Cr^{2s-\varepsilon}.$$

In combination with $(u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s)(z,t_0)=0$, we conclude

$$||u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s||_{L^\infty(Q_r(z,t_0))}\le Cr^{\alpha+2s-\varepsilon}.$$

Moreover since by assumption on the growth of $||u_i||_{L^\infty(B_R\times(-1,1))}$ and $[u_i]_{C_t^{\frac{\alpha}{2s}}(B_R\times(-1,1))}$ we have

$$\left[u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s\right]_{C_t^{\frac{\alpha}{2s}}(Q_r(z,t_0))}\le Cr^{2s-\varepsilon},$$

and

$$||u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s||_{L^\infty(Q_r(z,t_0))}\le Cr^{\alpha+2s-\varepsilon}$$

for all $r>0$.

We now apply Lemma 3.6.3 on the function $v_{r_0}(x,t)=(u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s)(x_0+2r_0x,t_0+(2r_0)^{2s}t)$, to deduce that

$$\begin{aligned}
[u_1-q_zu_2-Q_z\ d^s]_{C_p^{\alpha+2s-\varepsilon}(Q_{r_0}(x_0,t_0))}&\le Cr_0^{\varepsilon}\left[u_1-q_zu_2-Q_zd^s\right]_{C_p^{\alpha+2s}(Q_{r_0}(x_0,t_0))}\\
&\le Cr_0^{-\alpha-2s+\varepsilon}\Big(r_0^{\alpha+2s}\left[f_1+q_{(z,t_0)}f_2\right]_{C_p^{\alpha}(Q_{2r_0}(x_0,t_0))}\\
&\quad+r_0^{\alpha+2s}\left[(\partial_t+L)(Q_{(z,t_0)}d^s)\right]_{C_p^{\alpha}(Q_{2r_0}(x_0,t_0))}\\
&\quad+\sup_{R>1}R^{-2s-\alpha+\varepsilon}||u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s||_{L^\infty(Q_{2(R+2)r_0}(x_0,t_0))}\\
&\quad+\sup_{R>1}R^{-2s+\varepsilon}r_0^{\alpha}\left[u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s\right]_{C_t^{\frac{\alpha}{2s}}(Q_{2(R+2)r_0}(z,t_0))}\Big),
\end{aligned}$$

where we denoted $q_z,Q_z=q_{(z,t_0)},Q_{(z,t_0)}$ in the first line for transparency. The seminorms of f_i are bounded by assumption, $q_{(z,t_0)}$ and $Q_{(z,t_0)}$ are bounded thanks to [58, Lemma B.5]. The term with $\partial_t(Q_{(z,t_0)}d^s)$ we treat in the following way

$$\begin{aligned}
\left[Q_{(z,t_0)}\partial_tdd^{s-1}\right]_{C_p^{s-\varepsilon}(Q_{2r_0}(z,t_0))}&\le\left[Q_{(z,t_0)}\right]||\partial_tdd^{s-1}||+[\partial_td]\,||Q_{(z,t_0)}d^{s-1}||+\left[d^{s-1}\right]||Q\partial_td||\\
&\le C(r_0^{1-s+\varepsilon}r_0^{s-1}+r_0^{(\beta-2s+\varepsilon)-s}r_0^s+r_0^{-1+\varepsilon}r_0)\\
&\le C,
\end{aligned}$$

thanks to Lemma 3.6.1. The estimate

$$\left[L(Q_{(z,t_0)}d^s)\right]_{C_p^{s-\varepsilon}(Q_{2r_0}(x_0,t_0))}\le C$$

is provided by Proposition 3.5.1, while

$$||u_1-q_{(z,t_0)}u_2-Q_{(z,t_0)}d^s||_{L^\infty(Q_{2Rr_0}(x_0,t_0))}\le C((R+2)r_0)^{\alpha+2s-\varepsilon}$$

and

$$\sup_{R>1} R^{-2s+\varepsilon} r_0^{\alpha} \left[u_1 - q_{(z,t_0)}u_2 - Q_{(z,t_0)}d^s\right]_{C_t^{\frac{\alpha}{2s}}(Q_{2(R+2)r_0}(z,t_0))} \le C r_0^{\alpha+2s-\varepsilon}$$

are provided above. Hence we have

$$\left[u_1 - q_{(z,t_0)}u_2 - Q_{(z,t_0)}d^s\right]_{C_p^{\alpha+2s-\varepsilon}(Q_{r_0}(x_0,t_0))} \le C. \tag{3.3.7}$$

Moreover by Corollary 3.3.10 we have

$$\left[u_2 - q_{2,(z,t_0)}d^s\right]_{C_p^{\alpha+2s}(Q_{r_0}(x_0,t_0))} \le C r^{-1+s-\alpha}, \qquad \left[u_2 - q_{2,(z,t_0)}d^s\right]_{C_p^{3s-1}(Q_{r_0}(x_0,t_0))} \le C.$$

Note that the first inequality also implies that

$$||\partial_t(u_2 - q_{2,(z,t_0)}d^s)||_{L^\infty(Q_{r_0}(x_0,t_0))} \le C r^{s-1},$$

see [58, Lemma A.5]. Hence for any polynomial Q of degree 1 with $Q(z) = 0$ we have

$$\begin{aligned}
[Q(u_2 - d^s)]_{C_p^{\alpha+2s}(Q_{r_0}(x_0,t_0))} \le & ||Q||_{L^\infty} \left[u_2 - q_{2,(z,t_0)}d^s\right]_{C_p^{\alpha+2s}(Q_{r_0}(x_0,t_0))} \\
& + |\nabla Q| \left[u_2 - q_{2,(z,t_0)}d^s\right]_{C_p^{\alpha+2s-1}(Q_{r_0}(x_0,t_0))} \\
& + [Q]_{C_p^{\alpha}(Q_{r_0}(x_0,t_0))} ||\partial_t(u_2 - q_{2,(z,t_0)}d^s)||_{L^\infty(Q_{r_0}(x_0,t_0))} \\
\le & C r^1 r^{-1+s-\alpha} + C r^{s-\alpha} + C r^{1-\alpha} r^{s-1} \\
\le & C.
\end{aligned}$$

Combining it with (3.3.7), we get

$$\left[u_1 - \tilde{Q}_{(z,t_0)}u_2\right]_{C_p^{\alpha+2s-\varepsilon}(Q_{r_0}(x_0,t_0))} \le C,$$

for $\tilde{Q}_{(z,t_0)}(x) = Q_{(z,t_0)}(0) + q_{2,(z,t_0)}^{-1} \nabla Q_{(z,t_0)} \cdot (x - z)$. Using Lemma 3.6.2 we conclude

$$\left[\frac{u_1}{u_2} - \tilde{Q}_{(z,t_0)}\right]_{C_p^{\alpha+s-\varepsilon}(Q_{r_0}(x_0,t_0))} \le C,$$

which proves the claim in view of [57, Lemma B.2]. □

To conclude this section we prove Theorem 3.1.2.

Proof of Theorem 3.1.2. It is a direct consequence of Corollaries 3.3.9 and 3.3.13. □

3.3.4 Optimal Hölder estimates

Using the expansion result (Proposition 3.3.4) with d^s in place of one of solutions yields, that solutions grow at most as d^s near the boundary. In combination with interior regularity we can prove that then the solutions must be C^s (in space and time, which is better than C_p^s) up to the boundary.

Corollary 3.3.14. *Let $s \in (\frac{1}{2}, 1)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1, for some $\beta > 2s$. Let $\varepsilon > 0$. Let L be an operator of the form (3.1.1). Assume that $u \in C_p^\gamma(\mathbb{R}^n \times (-1,1))$, $\gamma > 0$, solves*

$$\begin{cases} (\partial_t + L)u &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{in } \Omega^c \cap Q_1, \end{cases}$$

Then

$$||u||_{C^s(Q_{1/2})} \leq C \left(||fd^{1-s}||_{L^\infty(Q_1 \cap \Omega)} + \sup_{R>1} R^{-2s+\varepsilon} ||u||_{L^\infty(B_R \times (-1,1))} \right),$$

and

$$\left|\left| \frac{u}{d^s} \right|\right|_{C_p^{2s-1}(Q_{1/2} \cap \overline{\Omega})} \leq C \left(||fd^{1-s}||_{L^\infty(Q_1 \cap \Omega)} + \sup_{R>1} R^{-2s+\varepsilon} ||u||_{L^\infty(B_R \times (-1,1))} \right),$$

where C depends only on n, s, G_0 and ellipticity constants.

Proof. Dividing u with $\left(||fd^{1-s}||_{L^\infty(Q_1 \cap \Omega)} + \sup_{R>1} R^{-2s+\varepsilon} ||u||_{L^\infty(Q_R \times (-1,1))}\right)$, it suffices to prove that $[u]_{C^s(Q_{1/2})} \leq C$ and $\left[\frac{u}{d^s}\right]_{C_p^{2s-1}(Q_{1/2})} \leq 1$, when it holds $||fd^{1-s}||_{L^\infty(Q_1 \cap \Omega)} + \sup_{R>1} R^{-2s+\varepsilon} ||u||_{L^\infty(Q_R \times (-1,1))} \leq 1$.

Note that $(\partial_t + L)d^s$ is bounded by d^{s-1} (see [58, Lemma 3.2]), so from Corollary 3.3.5 we get

$$\left[\frac{u}{d^s}\right]_{C_p^{2s-1}(\Omega \cap Q_{1/2})} \leq C.$$

Moreover by Proposition 3.3.4

$$\left[u - q_{(z,t_0)} d^s\right]_{C_p^{3s-1}(Q_r(x_0,t_0))} \leq C, \tag{3.3.8}$$

whenever $Q_{2r}(x_0, t_0) \subset \Omega$ and $d(x_0, t_0) \leq 2r$. From this we deduce that

$$[u]_{C_p^s(Q_r(x_0,t_0))} \leq C \, [d^s]_{C_p^s(Q_r(x_0,t_0))} \leq C,$$

in view of [57, Lemma B.5]. From [57, Lemma B.2] we get that

$$[u]_{C_p^s(\Omega \cap Q_{1/2})} \leq C,$$

and hence in particular

$$[u]_{C_x^s(\Omega \cap Q_{1/2})} \leq C.$$

To get the time regularity of solutions we need to work a bit more. Choose $(x,t), (x,t') \in \Omega \cap Q_{1/2}$. First if $|t-t'|^{\frac{1}{2s}} \leq \frac{1}{2} \max\{d(x,t), d(x,t')\}$, then we can find $(x_0, t_0) \in \Omega$, so that $(x,t), (x,t') \in Q_{2r}(x_0,t_0) \subset \Omega$ and $d(x_0,t_0) \leq 2r$. Then from (3.3.8) we deduce that

$$|u(x,t) - u(x,t')| \leq \left(\left[u - q_{(z,t_0)} d^s\right]_{C_t^s(Q_r(x_0,t_0))} + [d^s]_{C_t^s(Q_r(x_0,t_0))} \right) |t-t'|^s \leq C|t-t'|^s,$$

since $\frac{3s-1}{2s} > s$ for $s \in (\frac{1}{2}, 1)$. Let now $|t-t'|^{\frac{1}{2s}} > \frac{1}{2} \max\{d(x,t), d(x,t')\}$. First notice that the regularity of the quotient implies also that

$$\left[u - q_{(z,t_0)} d^s\right]_{C_p^{2s-1}(Q_r(z,t_0))} \leq Cr^s,$$

for all $0 < r < \frac{1}{2}$ and $q_{(z,t_0)}$ from Proposition 3.3.4. Let (x,t) be the point closer to the boundary than (x,t'), (z,t) be its closest boundary point at time t, and let (z',t') be the closest to (x,t'). We can estimate

$$\begin{aligned}|u(x,t)-u(x,t')| \leq & |u(x,t)-q_{(z,t)}d^s(x,t)-u(x,t')+q_{(z',t')}d^s(x,t')| \\ & + |q_{(z,t)}d^s(x,t)-q_{(z',t')}d^s(x,t')| \\ \leq & \mathrm{I}+\mathrm{II}.\end{aligned}$$

To estimate II, we observe that $q_{(z,t)} = \frac{u}{d^s}(z,t)$ which is a C_p^{2s-1} function, and hence $|q_{(z,t)} - q_{(z',t')}| \leq C(|z-z'|^{2s-1} + |t-t'|^{\frac{2s-1}{2s}})$. Moreover by triangle inequality and the assumption that $|t-t'|$ is big, we have that $|z-z'| + |t-t'|^{\frac{1}{2s}} \leq C|t-t'|^{\frac{1}{2s}}$. Hence

$$\begin{aligned}\mathrm{II} \leq & |q_{(z,t)}-q_{(z',t')}||d^s(x,t)| + |q_{(z',t')}||d^s(x,t)-d^s(x,t')| \\ \leq & C|t-t'|^{\frac{2s-1}{2s}}|t-t'|^{\frac{1}{2}} + C|t-t'|^s \\ \leq & C|t-t'|^s.\end{aligned}$$

Let us turn to the term I. We fix x_0 on the line going through z and x, far enough from the boundary (to be specified later) and define $x_k = (1-2^{-k})x + 2^{-k}x_0$, for $k \in \mathbb{N}$. We denote $v_z(\cdot) = u(\xi,\tau) - q_{(z,t)}d^s(\cdot)$ and compute

$$\begin{aligned}\mathrm{II} = & |v_z(x,t)-v_{z'}(x,t')| \\ \leq & |v_z(x_0,t)-v_{z'}(x_o,t')| + \sum_{k=1}^{\infty}|v_z(x_k,t)-v_z(x_{k-1},t)| + \sum_{k=1}^{\infty}|v_{z'}(x_k,t')-v_{z'}(x_{k-1},t')| \\ \leq & |v_z(x_0,t)-v_z(x_0,t')| + |q_{(z,t)}d^s(x_0,t')-q_{(z',t')}d^s(x_0,t')| \\ & + \sum_{k=1}^{\infty}C|x_0-z|^s|x_k-x_{k-1}|^{2s-1} + \sum_{k=1}^{\infty}C|x_0-z'|^s|x_k-x_{k-1}|^{2s-1} \\ \leq & C|t-t'|^{\frac{3s-1}{2s}} + C|t-t|^{\frac{1}{2}}\sum_{k=1}^{\infty}2^{-k(2s-1)}|x-x_0|^{2s-1} \\ \leq & C|t-t'|^s,\end{aligned}$$

if x_0 is chosen so that $|x_0 - z| \leq C_0|t-t'|^{\frac{1}{2s}}$ and that $(x_0,t),(x_0,t') \in Q_r(y_0,t_0) \subset Q_{2r}(y_0,t_0)$ for some $(y_0,t_0) \in \Omega$. Hence also

$$[u]_{C_t^s(\Omega\cap Q_{1/2})} \leq C,$$

and the claim is proven. □

3.4 Regularity of the free boundary in space and time

In this section we connect the established boundary Harnack results to the regularity of the free boundary in the parabolic nonlocal obstacle problem in the subcritical regime.

We say that a function u solves the parabolic obstacle problem for integro-differential operators, if it holds

$$\begin{array}{rcll} \min\{(\partial_t + L)u, u-\varphi\} & = & 0 & \text{in } \mathbb{R}^n \times (-1,1) \\ u(\cdot,-1) & = & \varphi & \text{in } \mathbb{R}^n, \end{array} \tag{3.4.1}$$

for a given function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ called the obstacle and some non-local elliptic operator L of the form (3.1.1). The $C^{1,\alpha}$ regularity of the free boundary near regular points has been established in [9, 38]. Here we prove that the free boundary is $C_p^{2,\alpha}$.

Proposition 3.4.1. *Let $s \in (\frac{1}{2}, 1)$ and set $\alpha = \min\{s, 2-2s\}$. Suppose that u solves (3.4.1) for some operator L of the form (3.1.1), with kernel $K \in C^5(\mathbb{S}^{n-1})$. Assume that the obstacle $\varphi \in C^4(\mathbb{R}^n)$. Suppose that $0 \in \partial\{u > \varphi\}$ is a regular free boundary point.*

Then the normal to the free boundary is $C_p^{s+\alpha-\varepsilon}$ in Q_{r_0}, for some $r_0 > 0$ and every $\varepsilon > 0$.

Proof. In [9] (see also [38]) they prove that there exists $r_0 > 0$ so that in Q_{r_0} the free boundary is $C^{1,\gamma}$ for some $\gamma > 0$ in space-time and that up to a rotation of coordinates $\partial_n u \geq c_1 d^s$, for some $c_1 > 0$. We can assume that $r_0 = 1$, since the problem is scale invariant. Let us denote

$$v = u - \varphi.$$

Since u solves (3.4.1), the partial derivatives of v solve

$$\begin{cases} (\partial_t + L)w & = & \partial_e f & \text{in } \Omega \cap Q_1 \\ w & = & 0 & \text{in } \Omega^c \cap Q_1, \end{cases}$$

for $f = -L\varphi$ and $\Omega = \{u > \varphi\} = \{v > 0\}$.

Remember that f is independent of time and by assumption on φ, we have $f \in C^{1+s}(\mathbb{R}^n)$ and hence $\partial_e f \in C_p^s(\Omega \cap Q_1)$. From [38, Corollary 1.6] (see also [19] for the case $(-\Delta)^s$ and with a smaller α) we deduce, that all the partial derivatives of u are $C_t^{\frac{\alpha-\varepsilon/2}{2s}}(\mathbb{R}^n \times (-1,1))$, and hence the same is true for partial derivatives of v. Therefore we can apply Corollary 3.3.9, which gives that all quotients v_i/v_n and v_t/v_n are $C_p^{1-\varepsilon}(\overline{\Omega} \cap Q_{1/2})$, with bounds on the norms.

Now notice that every component the normal vector $\nu(x,t)$ to the level set $\{v = \tau\}$, $\tau > 0$ can be expressed as

$$\nu^i(x,t) = \frac{\partial_i v}{|\nabla_{(x,t)} v|}(x,t) = \frac{\partial_i v/\partial_n v}{\left(\sum_{j=1}^{n-1}(\partial_j v/\partial_n v)^2 + 1 + (\partial_t v/\partial_n v)^2\right)^{1/2}}.$$

Letting $\tau \downarrow 0$, we get that the normal vector is $C_p^{1-\varepsilon}(\partial\Omega \cap Q_{1/2})$. Hence the free boundary is $C_p^{2-\varepsilon}$ in $Q_{1/2}$.

Now we can apply Corollary 3.3.13, which gives that the quotients v_i/v_n and v_t/v_n are $C_p^{s+\alpha-\varepsilon}(\overline{\Omega} \cap Q_{1/2})$, which furthermore implies that the normal to the free boundary is $C_p^{s+\alpha-\varepsilon}$ in $Q_{1/2}$. □

Note that the above proof works already if the initial regularity of the free boundary is C_p^1.

Theorem 3.1.1 follows.

Proof of Theorem 3.1.1. The claim follows from Proposition 3.4.1, noting that $s+\alpha > 1$. □

3.5 Operator evaluation of d^s

This last section we devote to the analysis of the evaluation of operators satisfying (3.1.1) of the distance function. It is put at the end due to its technical and computational nature. The regularity in space has been studied in [1, 57]. We follow their approach, but we focus on the regularity in time. In fact the regularity in time behaves as expected: the regularity in space is proved to be $C^{\beta-1-s}$, if the domain is of class C^β, while we establish the $C^{\frac{\beta-1-s}{2s}}$ regularity in time. Precisely we prove the following.

Proposition 3.5.1. *Let Ω be C^β_p in Q_1, for some $\beta \in (1+s,2)$. Assume that the operator L satisfies* (3.1.1) *with its kernel $K \in C^{2\beta+1}(\mathbb{S}^{n-1})$. If $Q\colon \mathbb{R}^n \to \mathbb{R}$ is a polynomial of degree one with gradient bounded by 1, we have*

$$\left[L(Qd^s_+)\right]_{C^{\beta-1-s}_p(Q_r\cap\Omega)} \le C(|Q(0)| + r^{s-\varepsilon}),$$

for every $r \le \frac{1}{2}$, and every $\varepsilon \in (0,s)$. The constant C depends only on n, s, G_0 and ellipticity constants.

Furthermore, when $(x_0,t_0) \in \Omega \cap Q_{1/2}$, with $d(x_0,t_0) \le C_0 r$, so that $Q_{2r}(x_0,t_0) \subset \Omega$, we have

$$\left[L(Qd^s_+)\right]_{C^{\beta-1}_p(Q_r(x_0,t_0))} \le C(|Q(0)| + |x_0|)r^{-s}.$$

Moreover if $Q(0) = 0$, whenever $d_x(x_0) = |x_0| = C_0 r$, with a suitable constant C_0 depending only on n, it holds

$$\left[L(Qd^s_+)\right]_{C^{\beta-1-\varepsilon}_p(Q_r(x_0,0))} \le C.$$

Proof. The regularity of $L(Qd^s)$ in space follows from [1, Corollary 2.3], since the generalized distance function satisfies the suitable estimates (see Definition 3.2.3). Let us show the regularity in time.

We start with applying [1, Lemma 2.4], to get

$$L(Qd^s_+)(x,t) = -\frac{1}{2s} p.v. \int_{\mathbb{R}^n} (Q(y)sd^{s-1}_+(y,t)\nabla d(y,t) + \nabla Q d^s(y,t)) \cdot (y-x)K(y-x)dy.$$

We now introduce the new variable $y = \phi_t(\eta)$, where ϕ_t comes as the inverse of the space component of the flattening map ψ: $\psi(x,t) = (\psi_t(x),t)$, $\phi_t = \psi_t^{-1}$. Note that then $d(\phi_t(\eta),t) = \eta_n$ and that $\phi_t \in C^\beta_p \cap C^2_x(\{x_n \ne 0\})$ and satisfies $|D^2\phi_t(x)| \le C|x_n|^{\beta-2}$. We get

$$\begin{aligned} L(Qd^s_+)(x,t) =& -\frac{1}{2s} p.v. \int_{\mathbb{R}^n} (\eta_n)^{s-1}_+ \rho(\eta,t)(\phi_t(\eta)-x)K(\phi_t(\eta)-x)d\eta \\ =& -\frac{1}{2s} p.v. \int_{B_1} (\eta_n)^{s-1}_+ \rho(\eta,t)(\phi_t(\eta)-x)K(\phi_t(\eta)-x)d\eta \\ & -\frac{1}{2s} \int_{B_1^c} (\eta_n)^{s-1}_+ \rho(\eta,t)(\phi_t(\eta)-x)K(\phi_t(\eta)-x)d\eta \end{aligned}$$

where we denoted $\rho(\eta,t) = (Q(\eta)\nabla d(\phi(\eta,t)) + \nabla Q d(\phi(\eta,t))|\nabla\phi_t(\eta)|$. Note that $\phi_t \in C_p^\beta$ and $\rho \in C_p^{\beta-1}$, hence the integral in B_1^c is $C_t^{\frac{\beta-1}{2s}}$ as well. Therefore we only need to study the regularity of the first integral. We denote

$$\hat{I}(x,t) = -\frac{1}{2s} p.v. \int_{B_1} (y_n)_+^{s-1} \rho(y,t)(\phi_t(y) - x) K(\phi_t(y) - x) dy,$$

and

$$I(x,t) = \hat{I}(\psi^{-1}(x,t)) = -\frac{1}{2s} p.v. \int_{B_1} (y_n)_+^{s-1} \rho(y,t)(\phi_t(y) - \phi_t(x)) K(\phi_t(y) - \phi_t(x)) dy.$$

Since ψ^{-1} is C_p^β (and hence Lipschitz), it is sufficient to show that I satisfies the stated estimate. To do so we expand $\phi_t(x) - \phi_t(y) = D\phi_t(x)(x-y) + S(x,y,t)$. Similarly

$$\begin{aligned}
K(\phi_t(x) - \phi_t(y)) &= K(D\phi_t(x)(x-y) + S(x,y,t)) \\
&= |x-y|^{-n-2s} K(D\phi_t(x)\langle x-y\rangle + |x-y|^{-1} S(x,y,t)) \\
&= |x-y|^{-n-2s} \left(K(D\phi_t(x)\langle x-y\rangle) + K_1(x,y,t)\right)
\end{aligned}$$

Before plugging the expansions in the formula above, we estimate the newly obtained terms. Using the fundamental theorem of calculus, we can write

$$\begin{aligned}
S(x,y,t) &= \int_x^y \left(D\phi_t(\xi) - D\phi_t(x)\right)(y-x) d\xi \\
&= \int_x^y \int_x^\xi (\xi - x)^T D^2\phi_t(\eta) d\eta (y-x) d\xi.
\end{aligned}$$

Since $\phi_t \in C_p^\beta$, we can estimate $|S(x,y,t)| \le C|x-y|^\beta$. Analogously,

$$K_1(x,y,t) = \int_0^1 \nabla K \left(D\phi_t(x)\langle x-y\rangle + \xi \frac{S(x,y,t)}{|x-y|}\right) \frac{S(x,y,t)}{|x-y|} d\xi,$$

and hence $|K_1(x,y,t)| \le C|x-y|^{\beta-1}$. Moreover, we want to estimate the incremental differences in time of S and K_1. Since $D\phi_t$ is $C_t^{\frac{\beta-1}{2s}}$, we get

$$|S(x,y,t) - S(x,y,t')| \le C|x-y||t-t'|^{\frac{\beta-1}{2s}}. \tag{3.5.1}$$

Moreover we can extract more using the fine estimate from Lemma 3.6.1 and get

$$|S(x,y,t) - S(x,y,t')| \le Cr^{-1}|x-y|^2|t-t'|^{\frac{\beta-1}{2s}}, \tag{3.5.2}$$

when $(y,t') \in Q_r(x,t)$, and $r = \frac{x_n}{2}$.. Since ∇K is Lipschitz, increments of K_1 inherit the estimates of S, and hence

$$|K_1(x,y,t) - K_1(x,y,t')| \le C|t-t'|^{\frac{\beta-1}{2s}} \tag{3.5.3}$$

and

$$|K_1(x,y,t) - K_1(x,y,t')| \le Cr^{-1}|x-y||t-t'|^{\frac{\beta-1}{2s}}, \tag{3.5.4}$$

whenever $(y,t') \in Q_r(x,t)$, and $r = \frac{x_n}{2}$.

We plug the expansions into the expression for I and get[3]

$$\begin{aligned}-2sI(x,t) =& \int_{B_1} (y_n)_+^{s-1}\rho(y,t)D\phi_t(x)(y-x)K(D\phi_t(x)\langle y-x\rangle)|x-y|^{-n-2s}dy\\ &+\int_{B_1} (y_n)_+^{s-1}\rho(y,t)S(x,y,t)K(D\phi_t(x)\langle y-x\rangle)|x-y|^{-n-2s}dy\\ &+\int_{B_1} (y_n)_+^{s-1}\rho(y,t)D\phi_t(x)(y-x)K_1(x,y,t)|x-y|^{-n-2s}dy\\ &+\int_{B_1} (y_n)_+^{s-1}\rho(y,t)S(x,y,t)K_1(x,y,t)|x-y|^{-n-2s}dy\\ =&I_1+I_2+I_3+I_4.\end{aligned}$$

We start with analysing I_1. We split it furthermore as follows

$$\begin{aligned}I_1 =&\rho(x,t)\int_{B_1} (y_n)_+^{s-1}D\phi_t(x)(y-x)K(D\phi_t(x)\langle y-x\rangle)|x-y|^{-n-2s}dy\\ &+\int_{B_1} (y_n)_+^{s-1}(\rho(y,t)-\rho(x,t))D\phi_t(x)(y-x)K(D\phi_t(x)\langle y-x\rangle)|x-y|^{-n-2s}dy\\ =&I_{11}+I_{12}.\end{aligned}$$

The integral I_{11} is the term with the lowest order of $|x-y|$, but we exploit the fact that it is almost an evaluation of some homogeneous operator of order $2s$ of the function $(x_n)_+^p$. To do so, we decouple the space variables in the following way

$$I_{11} = I_{11}(x,x,t)$$

where

$$I_{11}(\xi,x,t) = \rho(x,t)\int_{B_1} (y_n)_+^{s-1}D\phi_t(\xi)\langle y-x\rangle K(D\phi_t(\xi)\langle y-x\rangle)|x-y|^{-n-2s+1}dy.$$

Then by [58, Lemma 3.4] we conclude that

$$I_{11}(\xi,x,t) = \rho(x,t)\int_{B_1^c} (y_n)_+^{s-1}D\phi_t(\xi)\langle y-x\rangle K(D\phi_t(\xi)\langle y-x\rangle)|x-y|^{-n-2s+1}dy.$$

The obtained integral converges and hence I_{11} is $C_t^{\frac{\beta-1}{2s}}$.

We proceed with analysing I_{12}. We want to bound the time incremental difference of I_{12}. Since we are getting a lot of terms, we simplify the notation in the following way: we denote $\Delta_t f = f(t)-f(t')$, as well as $\Delta_x f = f(x)-f(y)$, furthermore $\rho = (Q\nabla d + d\nabla Q)J$, where J stands for $|\phi_t(y)|$. We furthermore split I_{12}

$$I_{12} = \int_{B_r(x)} \dots dy + \int_{B_1\setminus B_r(x)} \dots dy = B_{12r} + I_{12rc},$$

[3]We omit the principal value symbol in some of the integrals until the end of the proof.

for $r = \frac{x_n}{2}$. In the region near the pole we need to extract both $|t-t'|^{\frac{\beta-1}{2s}}$, as well as $|x-y|$ from the increment of ρ. This is done with adding an subtracting $D\rho(x,t)(y-x)$ from $\rho(y,t)-\rho(x,t)$. We get

$$I_{12r} = \int_{B_r(x)} (y_n)_+^{s-1}(\rho(y,t)-\rho(x,t)-D\rho(x,t)(y-x))D\phi_t(x)(y-x)K(D\phi_t(x)(y-x))dy$$
$$+\int_{B_r(x)} (y_n)_+^{s-1} D\rho(x,t)(x-y)D\phi_t(x)(y-x)K(D\phi_t(x)\langle y-x\rangle)|x-y|^{-n-2s}dy.$$

We can write

$$S_1(x,y,t) = \rho(y,t)-\rho(x,t)-D\rho(x,t)(y-x) = \int_x^y (D\rho(\xi,t)-D\rho(x,t))d\xi\cdot(y-x).$$

Using the simplified notation we compute

$$D\rho = D(J)(Q\nabla d + d\nabla Q) + J(2\nabla Q\nabla d + QD^2 d),$$

and hence

$$\begin{aligned}\Delta_t D\rho =& \Delta_t(D(J))(Q\nabla d + d\nabla Q) + \Delta_t J(2\nabla Q\nabla d + QD^2 d)\\ &+ D(J)(Q\Delta_t\nabla d + \Delta_t d\nabla Q) + J(2\nabla Q\Delta_t\nabla d + Q\Delta_t D^2 d).\end{aligned}$$

Lemma 3.6.1 gives that

$$|\Delta_t D(J)|, |\Delta_t D^2 d| \le Cr^{-1}|t-t'|^{\frac{\beta-1}{2s}}, \qquad |D(J)|, |D^2 d| \le Cr^{\beta-2},$$

and so we can bound

$$|\Delta_t D\rho(\xi,t)| \le Cr^{-1}(Q(0)+|x|+r)|t-t'|^{\frac{\beta-1}{2s}},$$

since $|Q(\xi)| \le |Q(0)| + C|\xi| \le C(|Q(0)+|x|+r)$.

We are now ready to estimate the incremental difference of I_{12r}. Whenever the difference operator does not apply on terms with ρ, we get $|t-t'|^{\frac{\beta-1}{2s}}$ from the other terms, and we are left with

$$Cr^{\beta-2}\int_{B_r(x)} (y_n)_+^{s-1}|y-x|^{-n-2s+2}dy,$$

which we can bound with $Cr^{\beta-s-1}$. When the difference operator lands on the term with ρ, we get the following

$$Cr^{-1}(|Q(0)|+|x|+r)|t-t'|^{\frac{\beta-1}{2s}}\int_{B_r(x)} (y_n)_+^{s-1}|y-x|^{-n-2s+2}dy,$$

which is bounded by $Cr^{-s}(|Q(0)|+|x|+r)|t-t'|^{\frac{\beta-1}{2s}}$.

We proceed with I_{12rc}. We can estimate

$$|\Delta_t I_{12rc}| \le \int_{B_1\setminus B_r(x)} y_{n+}^{s-1}|\Delta_t\Delta_x\rho||x-y|^{-n-2s+1}dy + C|t-t'|^{\frac{\beta-1}{2s}}\int_{B_1} y_{n+}^{s-1}|x-y|^{-n-2s+\beta},$$

where the second term bounds all the other contributions of Δ_t that do not come from $\Delta_x\rho$. We continue with computing the double incremental difference

$$\begin{aligned}\Delta_t\Delta_x\rho \leq&\Delta_t\left(\Delta_x J(Q\nabla d + d\nabla Q) + J(\Delta_x Q\nabla d + Q\Delta_x\nabla d + \Delta_x d\nabla Q)\right)\\ \leq&\Delta_t\Delta_x J(Q\nabla d + d\nabla Q) + \Delta_t J(\Delta_x Q\nabla d + Q\Delta_x\nabla d + \Delta_x d\nabla Q)\\ &+ \Delta_x J(Q\Delta_t\nabla d + \Delta_t d\nabla Q) + J(\Delta_x Q\Delta_t\nabla d + Q\Delta_t\Delta_x\nabla d + \Delta_t\Delta_x d\nabla Q).\end{aligned}$$

Where we have the single increments we do the straight forward estimate. It holds $|\Delta_t\Delta_x J| \leq C|t-t'|^{\frac{\beta-1}{2s}}$. The double increment of the distance function is better, since $d \in C_p^\beta$, so using the fundamental theorem of calculus we can extract $|\Delta_t\Delta_x d| \leq |x-y||t-t'|^{\frac{\beta-1}{2s}}$. Hence we get

$$|\Delta_t\Delta_x\rho| \leq C(|Q(0)|+|x|)|t-t'|^{\frac{\beta-1}{2s}} + C|x-y||t-t'|^{\frac{\beta-1}{2s}} + C|x-y|^{\beta-1}|t-t'|.$$

Plugging it inside the integral we end up with

$$|\Delta_t I_{12rc}| \leq C\left((|Q(0)|+|x|)|t-t'|^{\frac{\beta-1}{2s}}r^{-s} + |t-t'|^{\frac{\beta-1}{2s}}r^{-\varepsilon}\right)$$

in view of [1, Lemma A9] and Lemma 3.6.5.

The other terms I_2, I_3, I_4 are estimated in a similar manner. Let us analyse the term I_2. Taking the incremental difference and estimating we get

$$|\Delta_t I_2| \leq \int_{B_1} y_{n+}^{s-1}|\rho(y)||\Delta_t S||x-y|^{-n-2s}dy + C|t-t'|^{\frac{\beta-1}{2s}}\int_{B_1} y_{n+}^{s-1}|x-y|^{-n-2s+\beta}.$$

The second integral gives $Cx_n^{-\varepsilon}|t-t'|^{\frac{\beta-1}{2s}}$. In the first one we split $|\rho(y)| \leq C(|Q(0)| + |x| + |x-y|)$. In the term with $|x-y|$ we can use estimate (3.5.1) for $|\Delta_t S|$, to get

$$C|t-t'|^{\frac{\beta-1}{2s}}\int_{B_1} y_{n+}^{s-1}|x-y|^{-n-2s+2}dy,$$

which is bounded with $Cx_n^{-\varepsilon}|t-t'|^{\frac{\beta-1}{2s}}$, thanks to Lemma 3.6.5. The remaining term

$$(|Q(0)|+|x|)\int_{B_1} y_{n+}^{s-1}|\Delta_t S||x-y|^{-n-2s}dy$$

we split into the integrals in regions $B_r(x)$ and $B_1\backslash B_r(x)$. Away from the pole we can still use the estimate (3.5.1) and estimate the obtained integral with [1, Lemma A9], to get $C(|Q(0)|+|x|)|t-t'|^{\frac{\beta-1}{2s}}r^{-s}$. In the region near the pole we use (3.5.2), which gives

$$(|Q(0)|+|x|)r^{-1}|t-t'|^{\frac{\beta-1}{2s}}\int_{B_1} y_{n+}^{s-1}|x-y|^{-n-2s+2}dy$$

which is also bounded by $C(|Q(0)|+|x|)|t-t'|^{\frac{\beta-1}{2s}}r^{-s}$.

The analysis of the integral I_3 is the same as the one of I_2, using (3.5.3) and (3.5.4) instead of (3.5.1) and (3.5.2). The term I_4 is the same.

We obtained

$$|I(x,t) - I(x,t')| \leq C\left((|Q(0)|+|x|)r^{-s} + r^{-\varepsilon}\right)|t-t'|^{\frac{\beta-1}{2s}},$$

whenever $(x,t),(x,t') \in Q_r(x_0,t_0)$ with $(x_0)_n = 2r$. This implies the second and the third estimate in the claim, as well as

$$\begin{aligned}|I(x,t)-I(x,t')| &\le C\left(|Q(0)|+|x|+r^{s-\varepsilon}\right)|t-t'|^{\frac{\beta-1-s}{2s}}\\ &\le C(|Q(0)|+r_0^{s-\varepsilon})|t-t'|^{\frac{\beta-1-s}{2s}},\end{aligned}$$

if additionally $|x_0| \le r_0$. It follows from [57, Lemma B.2], that $[I]_{C_p^{\beta-1-s}(Q_{r_0})} \le C(|Q(0)|+r_0^{s-\varepsilon})$, since the space part of the estimate is provided by [1, Corollary 2.3]. Since the flattening map ϕ is Lipschitz the claim is proved. □

3.6 Appendix: Technical tools and lemmas

Lemma 3.6.1. *Let Ω be C_p^β in Q_1, for some $\beta \in (2s,2)$. Then there exists a function $d\colon \mathbb{R}^{n+1} \to \mathbb{R}$ satisfying*

$$d \in C_p^\beta(\mathbb{R}^{n+1}) \cap C_x^\infty(\{d>0\}), \quad C^{-1}\operatorname{dist}(\cdot,\partial\Omega\cap Q_1) \le d \le C\operatorname{dist}(\cdot,\partial\Omega\cap Q_1) \text{ in } \Omega\cap Q_1,$$

$$|(\partial_t)^l D^k d| \le C_{k,l} d^{\beta-k-2sl}, \quad \text{in } \{d>0\}, \text{ whenever } 2sl+k>\beta,$$

$$\left[D^2 d\right]_{C_t^{\frac{\beta-1}{2s}}(Q_r(x_0,t_0))} \le Cr^{-1},$$

whenever $d(x_0,t_0) \le C_0 r$ and $Q_{2r}(x_0,t_0) \subset \Omega$.

Furthermore there exists a diffeomorphism $\psi \in C_p^\beta(\mathbb{R}^{n+1},\mathbb{R}^{n+1}) \cap C_x^\infty(\{d>0\},\{x_n>0\})$, such that

$$\psi(\Omega\cap Q_1) = Q_1 \cap \{x_n>0\}, \quad d(\psi^{-1}(x,t)) = x_n, \quad |D^k\psi| \le Cd^{\beta-k},$$

in $\{d>0\}$, for all $k>\beta$ and whenever $d(x_0,t_0) \le C_0 r$ and $Q_{2r}(x_0,t_0) \subset \Omega$, we have

$$\left[D^2\psi\right]_{C_t^{\frac{\beta-1}{2s}}(Q_r(x_0,t_0))} \le Cr^{-1}.$$

Proof. Let $f \in C_p^\beta(\{|x'| \le 1\} \times (-1,1))$ be such that $\Omega \cap Q_1 = \{x_n > f(x',t)\}$. We can extend f to full space $\mathbb{R}^{n-1}\times\mathbb{R}$, so that its norm does not increase. Let $L = 8||\nabla f||_{L^\infty}$ and $K = 16s8^{2s-1}||\partial_t f||_{L^\infty}$. Choose $\varphi \in C_c^\infty(B_1)$ with $\int \varphi dz = 1$, $\psi \in C_c^\infty(-1,1)$ with $\int_{-1}^1 \psi d\sigma = 1$, and define

$$F(x,t,\tau) = x_n - \int_{\mathbb{R}^{n+1}} f\left(x' - \frac{\tau}{L}z', t - \frac{\tau^{2s}}{K}\sigma\right)\varphi(z)\psi(\sigma)dzd\sigma.$$

Since φ and ψ are compactly supported, $F \in C_p^\beta(\mathbb{R}^{n+1}) \cap C_\tau^{2s}$. We compute

$$\begin{aligned}\partial_\tau F(x,t,\tau) =& \frac{1}{L}\int_{\mathbb{R}^{n+1}} \nabla f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\cdot z\varphi(z)\psi(\sigma)dzd\sigma +\\ &+ \frac{2s}{K}\tau^{2s-1}\int_{\mathbb{R}^{n+1}} \partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\sigma\varphi(z)\psi(\sigma)dzd\sigma,\\ \nabla F =& e_n - \int_{\mathbb{R}^{n+1}} \nabla f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\varphi(z)\psi(\sigma)dzd\sigma,\\ \partial_t F =& -\int_{\mathbb{R}^{n+1}} \partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\varphi(z)\psi(\sigma)dzd\sigma,\end{aligned} \tag{3.6.1}$$

so thanks to the choice of L and K, we have $|\partial_\tau F(x,t,\tau)| \leq \frac{1}{4}$, for $\tau < 8$. Therefore, in combination with $F(x,t,0) = x_n - f(x',t)$, we conclude that for every $(x,t) \in \{x_n - f(x,t) \in (0,3)\}$ there exists a unique $d = d(x,t) \in (0,8)$, so that

$$d(x,t) = F(x,t,d(x,t)), \tag{3.6.2}$$

as well as

$$|x_n - f(x,t) - d(x,t)| = |F(x,t,0) - F(x,t,d(x,t))| \leq \frac{1}{4} d(x,t),$$

which implies that d is comparable to $x_n - f(x,t)$. By implicit function theorem d is C^1 in both variables. Differentiating (3.6.2), we get

$$\nabla d(x,t) = \frac{\nabla F(x,t,d(x,t))}{1 - \partial_\tau F(x,t,d(x,t))}, \quad \partial_t d(x,t) = \frac{\partial_t F(x,t,d(x,t))}{1 - \partial_\tau F(x,t,d(x,t))}.$$

Notice also, that $\partial_n d \geq \frac{1}{2}$. Moreover since $F \in C^\infty(\{\tau > 0\})$ (it is a convolution with a C^∞ function), it also holds $d \in C^\infty(\{d > 0\})$. To get the expressions for higher order derivatives of d, we differentiate (3.6.2) and then insert the suitable derivatives of F. We show here only the computations for the second order derivatives, higher order ones are established analogously. Procedure is always the following. We perform an affine change of variables, to move the variables x, t to φ, ψ, then derive and finally change the coordinates back. The computations are long and technical, therefore we present only the results. We get

$$\begin{aligned}
\partial_j \partial_i F &= -\frac{L}{\tau} \int_{\mathbb{R}^{n+1}} \partial_i f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \partial_j \varphi(z) \psi(\sigma) dz d\sigma \\
&= -\frac{L}{\tau} \int_{\mathbb{R}^{n+1}} \left(\partial_i f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) - \partial_i f(x,t)\right) \partial_j \varphi(z) \psi(\sigma) dz d\sigma
\end{aligned}$$

where in the last equality we used that $\int \partial_j \varphi = 0$, since φ is compactly supported. Similarly we get

$$\begin{aligned}
\partial_j \partial_\tau F =& \frac{1}{\tau} \int_{\mathbb{R}^{n+1}} \nabla f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \cdot \partial_j (z\varphi(z)) \psi(\sigma) dz d\sigma \\
&+ 2s \frac{L}{K} \tau^{2s-2} \int_{\mathbb{R}^{n+1}} \partial_t f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \partial_j \varphi(z) \psi(\sigma) dz d\sigma \\
\partial_\tau \partial_\tau F =& -\frac{1}{\tau L} \int_{\mathbb{R}^{n+1}} \sum_{j,k} \partial_k f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \partial_j (z_k z_j \varphi(z)) \psi(\sigma) dz d\sigma \\
&- \frac{2s}{\tau L} \int_{\mathbb{R}^{n+1}} \sum_{k} \partial_k f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) z_k \varphi(z) (\sigma \psi(\sigma))' dz d\sigma \\
&- \frac{2s}{K} \tau^{2s-2} \int_{\mathbb{R}^{n+1}} \partial_t f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \sigma \nabla \cdot (z\varphi(z)) \psi(\sigma) dz d\sigma \\
&- \frac{(2s)^2}{K} \tau^{2s-2} \int_{\mathbb{R}^{n+1}} \partial_t f\left(x - \frac{\tau}{L} z, t - \frac{\tau^{2s}}{K}\sigma\right) \sigma^{2s-1} \varphi(z) (\sigma^{\frac{1+2s}{2s}} \psi(\sigma))' dz d\sigma.
\end{aligned}$$

Using that $f \in C_p^\beta$, we conclude that all the derivatives above are bounded with $C\tau^{\beta-2}$. We also compute

$$\begin{aligned}
\partial_{tt}F =& -\frac{K}{\tau^{2s}}\int_{\mathbb{R}^{n+1}}\partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\varphi(z)\psi'(\sigma)dzd\sigma\\
\nabla\partial_t F =& -\frac{L}{\tau}\int_{\mathbb{R}^{n+1}}\partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\nabla\varphi(z)\psi(\sigma)dzd\sigma\\
\partial_\tau\partial_t F =& \frac{1}{\tau}\int_{\mathbb{R}^{n+1}}\partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\nabla\cdot(z\varphi(z))\psi(\sigma)dzd\sigma\\
&+\frac{2s}{\tau}\int_{\mathbb{R}^{n+1}}\partial_t f\left(x-\frac{\tau}{L}z, t-\frac{\tau^{2s}}{K}\sigma\right)\varphi(z)(\sigma\psi(\sigma))'dzd\sigma.
\end{aligned}$$

Since again in every expression there are derivatives of compactly supported functions, we deduce $|\partial_{tt}F| \le C\tau^{\beta-4s}$ and $|\nabla\partial_t F|, |\partial_\tau\partial_t F| \le C\tau^{\beta-2s-1}$.

Finally, differentiating (3.6.2) twice, we extract

$$\begin{aligned}
D^2 d &= \frac{1}{1-\partial_\tau F}\left(D^2F + 2\nabla\partial_\tau F\nabla d^T + \partial_{\tau\tau}F\nabla d\nabla d^T\right)\\
\partial_{tt}d &= \frac{1}{1-\partial_\tau F}\left(\partial_{tt}F + 2\partial_t\partial_\tau F\partial_t d + \partial_{\tau\tau}F(\partial_t d)^2\right)\\
\nabla\partial_t d &= \frac{1}{1-\partial_\tau F}\left(\nabla\partial_t F + \nabla\partial_\tau F\partial_t d + \partial_t\partial_\tau F\nabla d + \partial_{\tau\tau}F\nabla d\partial d\right),
\end{aligned}$$

where all the arguments are either (x,t) or $(x,t,d(x,t))$. The previous computations yield

$$|D^2 d| \le Cd^{\beta-2}, \qquad |\partial_{tt}d| \le Cd^{\beta-4s}, \qquad |\nabla\partial_t d| \le Cd^{\beta-1-2s}.$$

These estimates imply that $d \in C_p^\beta(\{d \ge 0\})$. Finally using the estimates for $D^3 d$ and $\partial_t D^2 d$, we get

$$\left[D^2 d\right]_{C_p^{\beta-1}(Q_r(x_0,t_0))} \le Cr^{-1},$$

whenever $d(x_0,t_0) \le C_0 r$ and $Q_{2r}(x_0,t_0) \subset \Omega$. To extend d to the full set $\{x_n > f(x,t)\}$, we take a cut-off $\phi \in C_c^\infty([0,\infty))$, such that $\phi = 1$ in $[0,3)$, $\phi = 0$ in $[4,\infty)$ and define

$$\mathbf{d}(x,t) = \phi(d(x,t))d(x,t) + (1-\phi(d(x,t))x_n,$$

which satisfies all properties from the claim.

Once we get d, we define

$$\psi(x,t) = (x', d(x,t), t).$$

Then $|D_{x,t}\psi| = |\partial_n d| > \frac{3}{4}$, and the derivatives of ψ inherit estimates of derivatives of d. □

Lemma 3.6.2. *Let $s \in (\frac{1}{2},1)$, $a \in (0,2s)$ and $b \in (0,1)$. Let f, g, Q be functions on $Q_r(x_0,t_0)$ which satisfy $||f-Qg||_{L^\infty(Q_r)} \le Cr^{a+b}$, and $[f-Qg]_{C_p^{a+b}(Q_r(2re_n,0))} \le C$. Assume that g satisfies $||g^{-1}||_{L^\infty(Q_r(2re_n,0))} \le Cr^{-b}$, and $[g]_{C_p^a(Q_r(2re_n,0))} \le Cr^{b-a}$, for all $r \in (0,1)$.*

Then we have

$$\left[\frac{f}{g} - Q\right]_{C_p^a(Q_r(2re_n,t_0))} \le C.$$

Proof. We denote $v = f - Qg$. Note that by [57, Lemma B.2] v is $C_p^{\alpha+\beta}(\mathcal{C})$, where $\mathcal{C} = \cup_{r>0} Q_r(2re_n, 0)$. Hence also $\nabla v(0,0) = 0$ if $a+b > 1$, and then also $||\nabla v||_{Q_r \cap \mathcal{C}} \leq Cr^{a+b-1}$.

Assume first that $a < 1$. Then

$$\begin{aligned}\left[\frac{v}{g}\right]_{C_p^a(Q_r(2re_n,0))} &\leq [v]\,||g^{-1}|| + ||v||\left[g^{-1}\right] \\ &\leq Cr^b r^{-b} + Cr^{a+b} r^{-2b} r^{b-a} \\ &\leq C.\end{aligned}$$

Let now $a > 1$. We estimate

$$\begin{aligned}\left[g^{-1}\right]_{C_p^a} &= \left[g^{-1}\right]_{C_t^{\frac{a}{2s}}} + \left[\nabla(g^{-1})\right]_{C_p^{a-1}} \\ &\leq ||g^{-2}||\,[g]_{C_t^{\frac{a}{2s}}} + 2\left[g^{-1}\right]_{C_p^{a-1}} ||g^{-1}\nabla g|| + ||g^{-2}||\,[\nabla g]_{C_p^{a-1}} \\ &\leq Cr^{-2b} r^{b-a} + Cr^{-2b} r^{b-a+1} r^{-b} r^{b-1} + Cr^{-2b} r^{b-a} \\ &\leq Cr^{-a-b},\end{aligned}$$

where the estimate for $||\nabla g||$ follows from the one for $[\nabla g]_{C_p^{a-1}}$. This gives

$$\begin{aligned}\left[\frac{v}{g}\right]_{C_p^a(Q_r(2re_n,0))} &\leq [v]\,||g^{-1}|| + ||v||\left[g^{-1}\right] + [v]_{C_p^{a-1}} ||\nabla(g^{-1})|| + ||\nabla v||\left[g^{-1}\right]_{C_p^{a-1}} \\ &\leq Cr^b r^{-b} + Cr^{a+b} r^{-a-b} + Cr^{b+1} r^{-2b} r^{b-1} + Cr^{a+b-1} r^{-2b} r^{b-a+1} \\ &\leq C,\end{aligned}$$

which proves the claim. □

Lemma 3.6.3. *Let $s \in (0,1)$ and let L be an operator of the form (3.1.1). Let u be a solution of*

$$(\partial_t + L)u = f, \qquad \text{in } Q_1.$$

Then

$$[u]_{C_p^{2s-\varepsilon}(Q_{1/2})} \leq C\left(||f||_{L^\infty(Q_1)} + \sup_{R>1} R^{-2s+\varepsilon} ||u||_{L^\infty(B_R \times (-1,1))}\right),$$

where C depends only on n, s, ε and ellipticity constants.

If additionally $f \in C_p^\alpha(Q_1)$, for some $\alpha \in (0,1)$, so that $\frac{\alpha}{2s} \in (0,1)$, we have

$$[u]_{C_p^{\alpha+2s}(Q_{1/2})} \leq C\left([f]_{C_p^\alpha(Q_1)} + ||u||_{L^\infty(Q_1)} + \sup_{R>1} R^{-2s+\varepsilon}\,[u]_{C_p^\alpha(B_R \times (-1,1))}\right).$$

Moreover if the kernel K of the operator L satisfies $[K]_{C^\alpha(B_r^c)} \leq C_1 r^{-n-2s-\alpha}$, $r > 0$, then

$$\begin{aligned}[u]_{C_p^{\alpha+2s}(Q_{1/2})} \leq C\Big(&[f]_{C_p^\alpha(Q_1)} + \sup_{R>1} R^{-2s-\alpha+\varepsilon} ||u||_{L^\infty(B_R \times (-1,1))} \\ &+ \sup_{R>1} R^{2s-\varepsilon}\,[u]_{C_t^{\frac{\alpha}{2s}}(B_R \times (-1,1))}\Big).\end{aligned}$$

Proof. The proof the first estimate we take a smooth cut off function χ, applying the estimates on $u\chi$ and computing the error terms. Let $\chi \in C_c^\infty(B_1)$ so that $\chi \equiv 1$ in $B_{5/6}$. Then the function $\bar{u} = u\chi$ solves

$$(\partial_t + L)\bar{u} = f + \bar{f}$$

where $\bar{f} = -L(u(1-\chi))$. We can estimate

$$\begin{aligned}|\bar{f}(x,t)| \leq & \int_{B^c_{4/5}(-x)} |u(1-\chi)|(x+y,t)K(y)dy \\ \leq & CC_0 \int_{B^c_{4/5}(-x)} (1+|y|)^{2s-\varepsilon}|y|^{-n-2s}dy \\ \leq & CC_0,\end{aligned}$$

for $C_0 = \sup_{R>1} R^{-2s+\varepsilon}||u||_{L^\infty(B_R)\times(-1,1)}$. The first claim now follows from [32, Theorem 1.3].

For the second claim, we estimate

$$\begin{aligned}|\bar{f}(x,t) - \bar{f}(x',t')| = & \Bigg| \int_{B^c_{4/5}(-x)} u(1-\chi)(x+y,t)K(y)dy \\ & - \int_{B^c_{4/5}(-x')} u(1-\chi)(x'+y,t')K(y)dy \Bigg| \\ \leq & \int_{B^c_{1/20}} |u(x+y,t) - u(x'+y,t')|K(y)dy \\ & + ||u||_{L^\infty(Q_1)} \int_{B_1\setminus B_{1/20}} |\chi(x+y) - \chi(x'+y)|K(y)dy \\ \leq & (|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2s}}) \left(\int_{B^c_{1/20}} [u]_{C^\alpha(B_{|y|+1})} K(y)dy + C_\chi ||u||_{L^\infty(Q_1)} \right) \\ \leq & (|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2s}}) \left(\Lambda C_0 \int_{B^c_{1/20}} \frac{(|y|+1)^{2s-\varepsilon}}{|y|^{n+2s}} dy + C||u||_{L^\infty(Q_1)} \right) \\ \leq & (|x-x'|^\alpha + |t-t'|^{\frac{\alpha}{2s}}) \left(CC_0 + C||u||_{L^\infty(Q_1)} \right),\end{aligned}$$

where C_0 stands for $\sup_{R>1} R^{-2s+\varepsilon} [u_1]_{C^\alpha_p(B_R\times(-1,1))}$. In view of [32, Theorem 1.1], the second claim is proven.

For the moreover case we need to notice that

$$\int_{B^c_{4/5}(-x)} u(1-\chi)(x+y,t)K(y)dy - \int_{B^c_{4/5}(-x')} u(1-\chi)(x'+y,t')K(y)dy =$$
$$\int_{B^c_{4/5}} u(1-\chi)(y,t)K(y-x)dy - \int_{B^c_{4/5}} u(1-\chi)(y,t')K(y-x')dy =$$
$$\int_{B^c_{4/5}} \big(u(1-\chi)(y,t) - u(1-\chi)(y,t')\big)\, K(y-x')dy +$$
$$+ \int_{B^c_{4/5}} u(1-\chi)(y,t)\,\big(K(y-x) - K(y-x')\big)\, dy.$$

The first term can be bounded with

$$C \sup_{R>1} R^{-2s+\varepsilon} \, [u]_{C_t^{\frac{\alpha}{2s}}(B_R\times(-1,1))} |t-t'|^{\frac{\alpha}{2s}},$$

while the second one with

$$CC_1 \sup_{R>1} R^{-2s-\alpha+\varepsilon} ||u||_{L^\infty(R_R\times(-1,1))}|x-x'|^\alpha.$$

Using [32, Theorem 1.1] finishes the proof. □

Lemma 3.6.4. *Let $a, \alpha > 0$ such that $a+\alpha > 1+s$, $\alpha < s$, and $u_1, u_2, d \in C(Q_1)$, such that $u_2 \in C^s_p(Q_1)$, $u_2 \geq c_0 d^s$, for some $c_0 > 0$ and*

$$\left[u_1 - q_r u_2 - Q_r^{(1)} d^s\right]_{C_p^\alpha(Q_r)} \leq C_0 r^a,$$

for some $q_r \in \mathbb{R}$ and some 1-homogeneous polynomials $Q_r^{(1)}$.

Then there exist q_0 and $Q_0^{(1)}$, so that

$$\left[u_1 - q_0 u_2 - Q_0^{(1)} d^s\right]_{C_p^\alpha(Q_r)} \leq CC_0 r^a.$$

Proof. Writing $Q_r^{(1)}(x) = Q_r^{(1)} \cdot x$ and using rescaled [1, Lemma A.10], we compute

$$\begin{aligned}
|Q_r^{(1)} - Q_{2r}^{(1)}| \leq & Cr^{-1}||Q_r^{(1)} - Q_{2r}^{(1)}||_{L^\infty(Q_r\cap\{d>r/2\})} \\
\leq & Cr^{-1-s}||Q_r^{(1)}d^s - Q_{2r}^{(1)}d^s||_{L^\infty(Q_r)} \\
\leq & Cr^{-1-s+\alpha}\left[Q_r^{(1)}d^s - Q_{2r}^{(1)}d^s\right]_{C_p^\alpha(Q_r)} \\
\leq & Cr^{-1-s+\alpha}\bigg(\left[u_1 - q_r u_2 - Q_r^{(1)}d^s\right]_{C_p^\alpha(Q_r)} + \left[u_1 - q_{2r}u_2 - Q_{2r}^{(1)}d^s\right]_{C_p^\alpha(Q_r)} \\
& + [(q_r - q_{2r})u_2]_{C_p^\alpha(Q_r)}\bigg) \\
\leq & CC_0 r^{-1-s+\alpha+a} + Cr^{-1}|q_r - q_{2r}|.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}||(q_r-q_{2r})u_2-(Q_r^{(1)}-Q_{2r}^{(1)})d^s||_{L^\infty(Q_r)}\le& r^\alpha\left[(q_r-q_{2r})u_2-(Q_r^{(1)}-Q_{2r}^{(1)})d^s\right]_{C_p^\alpha\infty(Q_r)}\\ \le& C_0r^{\alpha+a},\end{aligned}$$

and hence also

$$||(q_r-q_{2r})u_2-(Q_r^{(1)}-Q_{2r}^{(1)})d^s||_{L^\infty(Q_r\cap\{d>r/2\})}\le C_0r^{\alpha+a}.$$

It follows from [1, Lemma A.11], that

$$|q_r-q_{2r}|\le CC_0r^{\alpha+a-s}.$$

Combining it with the first inequality, we also obtain

$$|Q_r^{(1)}-Q_{2r}^{(1)}|\le CC_0r^{\alpha+a-1-s}.$$

In the same way as in [1, Lemma 4.5] this implies the existence of the limits q_0 and $Q_0^{(1)}$, together with estimates

$$|Q_0^{(1)}-Q_r^{(1)}|\le CC_0r^{\alpha+a-1-s},\qquad |q_0-q_r|\le CC_0r^{\alpha+a-s},$$

and so

$$\begin{aligned}\left[u_1-q_0u_2-Q_0^{(1)}d^s\right]_{C_p^\alpha(Q_r)}\le&\left[u_1-q_ru_2-Q_r^{(1)}d^s\right]_{C_p^\alpha(Q_r)}\\ &+[(q_0-q_r)u_2]_{C_p^\alpha(Q_r)}+\left[(Q_0^{(1)}-Q_r^{(1)})d^s\right]_{C_p^\alpha(Q_r)}\\ \le& CC_0r^a.\end{aligned}$$

Therefore the claim is proved. □

Lemma 3.6.5. *Let $x\in B_{1/2}$ with $x_n>0$, $p\in(0,2s)$ and $q>\max\{1-p,0\}$. Then*

$$\int_{B_1}y_{n+}^{p-1}|x-y|^{-n+q}dy\le C_\varepsilon x_n^{-\varepsilon},$$

for any $\varepsilon>0$.

Proof. Denote $r=\frac{x_n}{2}$ and split the integral into $B_r(x)$ and the complement. In the integral with the pole we estimate y_n^{p-1} with r^{p-1} and then integrate the pole to obtain another Cr^q. Thanks to the assumptions this is bounded.

The remaining part is handled as follows. Choose $\varepsilon>0$ and estimate

$$\int_{B_1\setminus B_r(x)}y_{n+}^{p-1}|x-y|^{-n+q}dy\le\int_{B_1\setminus B_r(x)}y_{n+}^{p-1}|x-y|^{-n+1-p-\varepsilon}dy,$$

since $|x-y|\le 1$ and $q>1-p$. Rescaling the integral on the right-hand side we get

$$r^{-\varepsilon}\int_{B_{1/r}\setminus B_1(x/r)}y_{n+}^{p-1}\left|\frac{x}{r}-y\right|^{-n+1-p-\varepsilon}dy\le C_\varepsilon x_n^{-\varepsilon},$$

as stated, since the $n-$th coordinate of $\frac{x}{r}=2$. □

Chapter 4

Higher order parabolic boundary Harnack inequality

4.1 Introduction

We study the boundary behaviour of solutions of

$$\begin{cases} \partial_t u + Lu &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap Q_1, \end{cases} \tag{4.1.1}$$

where $\Omega \subset \mathbb{R}^{n+1}$ is an open set, $Q_1 = B_1 \times (-1, 1)$ and L is a second order operator of the form

$$Lu(x,t) = -\sum_{i,j=1}^{n} a_{ij}(x,t)\partial_{ij}u(x,t) + \sum_{i=1}^{n} b_i(x,t)\partial_i u(x,t). \tag{4.1.2}$$

The matrix $A(x,t) := [a_{i,j}(x,t)]_{ij}$ is assumed to be uniformly elliptic and the vector $b(x,t) = [b_i(x,t)]_i$ is bounded, that is, for every (x,t) it holds

$$0 < \lambda I \leq A(x,t) \leq \Lambda I, \qquad ||b||_{L^\infty} \leq \Lambda. \tag{4.1.3}$$

In the elliptic version of the problem, classical boundary Schauder estimates imply that solutions vanishing on the boundary of a $C^{k,\alpha}$ domain are of class $C^{k,\alpha}$ up to the boundary. This implies that the quotient of the solution with the distance function to the boundary is $C^{k-1,\alpha}$. This has been refined by De Silva and Savin [27], who showed that replacing the distance function with another solution gives that the quotient is $C^{k,\alpha}$ as well, under positivity assumption on the solution in the denominator. This is a higher order boundary Harnack estimate.

Such result was extended to the parabolic setting in [6], where Banerjee and Garofalo established a new parabolic higher order boundary Harnack estimate in $C^{k,\alpha}$ domains.

The goal of this paper is twofold. On the one hand, we give a new proof of the result in [6], which is based on blow-up and contradiction arguments. On the other hand, we extend the result to the case of parabolic C^1 domains, which was not covered in [6]. This is important when applying these estimates to obtain the higher regularity of free boundaries in parabolic obstacle problems, as we will see below.

We work in domains Ω which are allowed to be moving in time, and have to satisfy some parabolic regularity properties.

Definition 4.1.1. Let $\beta \geq 1$ and let C_p^β be the parabolic Hölder space defined in Definition 4.2.2. We say that $\Omega \subset \mathbb{R}^{n+1}$ is C_p^β in Q_1, if there is a function $F\colon Q_1' \to \mathbb{R}$, with $||F||_{C_p^\beta(Q_1')} \leq C_0$ and $F(0,0) = 0$, $\nabla_{x'}F(0,0) = 0$ so that

$$\Omega \cap Q_1 = \{(x', x_n, t);\ x_n > F(x', t)\}.$$

For a domain Ω as in the above definition we define the parabolic distance to the boundary as follows

$$d(x,t) = \inf_{(z,s)\in\partial\Omega\cap Q_1} \left(|x-z| + |t-s|^{\frac{1}{2}}\right), \quad (x,t) \in \Omega \cap Q_1.$$

Our first main result says that if we have two solutions of (4.1.1) in a C_p^1 domain, with a bounded right-hand side, then their quotient is nearly as smooth as the boundary. This is a parabolic boundary Harnack inequality, and reads as follows.

Theorem 4.1.2. *Let Ω be C_p^1 in Q_1, and let L be as in* (4.1.2)–(4.1.3)*, with $A \in C(\overline{\Omega})$ and $b \in L^\infty(\Omega)$. Let u_i be two solutions of* (4.1.4) *with $f_i \in L^\infty(\Omega \cap Q_1)$. Assume that $|u_2| \geq c_2 d$ with $c_2 > 0$ and $||f_2||_{L^\infty(\Omega\cap Q_1)} + ||u_2||_{L^\infty(\Omega\cap Q_1)} \leq C_2$.*

Then for any $\varepsilon \in (0,1)$ we have

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{1-\varepsilon}(\overline{\Omega}\cap Q_{1/2})} \leq C\left(||f_1||_{L^\infty(\Omega\cap Q_1)} + ||u_1||_{L^\infty(\Omega\cap Q_1)}\right),$$

with C depending only on $n, \varepsilon, c_2, C_2, \Omega$, the modulus of continuity of A and ellipticity constants.

It is established through expanding one solution with respect to the other one, combined with boundary regularity estimates for solutions. With a similar, but a bit more involved approach, we can also prove the higher order analogue of the result - the higher order parabolic boundary Harnack inequality.

Theorem 4.1.3. *Let $\beta > 1$, $\beta \notin \mathbb{N}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1 according to Definition 4.1.1. Let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3) *with the coefficients $A, b \in C_p^{\beta-1}(\overline{\Omega})$. For $i = 1, 2$ let u_i be a solution to*

$$\begin{cases} \partial_t u_i + L u_i = f_i & \text{in } \Omega \cap Q_1 \\ u_i = 0 & \text{on } \partial\Omega \cap Q_1, \end{cases} \tag{4.1.4}$$

with $f_i \in C_p^{\beta-1}(\Omega \cap Q_1)$. Assume that $|u_2| \geq c_2 d$ with $c_2 > 0$ and $||f_2||_{C_p^{\beta-1}(\Omega\cap Q_1)} + ||u_2||_{L^\infty(\Omega\cap Q_1)} \leq C_2$.

Then we have

$$\left|\left|\frac{u_1}{u_2}\right|\right|_{C_p^{\beta}(\overline{\Omega}\cap Q_{1/2})} \leq C\left(||f_1||_{C_p^{\beta-1}(\Omega\cap Q_1)} + ||u_1||_{L^\infty(\Omega\cap Q_1)}\right),$$

with C depending only on $n, \beta, c_2, C_2, \Omega$, ellipticity constants, $||A||_{C_p^{\beta-1}(\overline{\Omega})}$ and $||b||_{C_p^{\beta-1}(\overline{\Omega})}$.

Theorem 4.1.3 is already known under a bit stricter assumptions, see [6]. We provide a new, different proof, which relaxes the assumption on regularity of the domain from $C^{1,\alpha}$ in space and time, to $C_p^{1,\alpha}$.

One motivation for studying such equations comes from the parabolic obstacle problem, where we look for a function $v\colon Q_1 \to \mathbb{R}$ solving

$$\partial_t v - \Delta v = -f\chi_{\{v>0\}}, \quad v \geq 0, \tag{4.1.5}$$

for some function f depending only on x, together with some boundary data on $\partial_p Q_1 = \partial Q_1 \backslash (B_1 \times \{1\})$. Of particular interest is the so called *contact set* $K = \{v = 0\}$, and its boundary ∂K, called the *free boundary*. If we denote $\Omega = \{v > 0\}$, then in particular the solution v solves

$$\begin{cases} \partial_t v - \Delta v &= -f \quad \text{in } \Omega \cap Q_1 \\ v &= 0 \quad \text{on } \partial\Omega \cap Q_1. \end{cases}$$

Theorem 4.1.2 and Theorem 4.1.3 give a simple proof of the higher regularity of the free boundary in the parabolic obstacle problem, without making use of a hodograph transform as in [52, Theorem 3]. Analogous boundary Harnack inequalities give rise to the bootstrap argument, which yields the higher order regularity of the free boundaries in similar obstacle problems. Some examples are [27], where they establish it in the classical case, [51] for the case when the operator is the fractional Laplacian and [7] for the parabolic thin obstacle problem.

Corollary 4.1.4. *Let $v\colon Q_1 \to \mathbb{R}$ solve the parabolic obstacle problem* (4.1.5) *with $f \geq c_0 > 0$, $f \in C^\theta(B_1)$, for some $\theta > 1$. Assume that $(0,0) \in \partial\{v > 0\}$, that $\partial\{v > 0\}$ is C_p^1 in Q_1 in the sense of Definition 4.1.1. Then $\partial\{v > 0\}$ is $C_p^{\theta+1}$ in $Q_{1/2}$.*

In particular, if $f \in C^\infty(B_1)$, then the free boundary is C^∞ near the origin in space and time.

Remark 4.1.5. In the classical one-phase Stefan problem the function f in the right-hand side is constantly equal to 1 and moreover one has $\partial_t v \geq 0$. Then the initial $C^{1,\alpha}$ regularity of the free boundary near regular free boundary points[1] is established in [16]. For general right-hand side f in (4.1.5), we refer to [64, Theorem 1.7]. There the provided initial regularity is indeed C_p^1 only.

Let us stress, that one often does not have $C^{1,\alpha}$ initial regularity. Another example is the free boundary problem for fully non-linear parabolic equations, studied in [41]. There they establish the C^1 regularity of the free boundary in both space and time. This is why it is important to have Theorem 4.1.2 in C_p^1 domains in order to establish the higher regularity of free boundaries.

4.1.1 Strategy of the proofs

The main ingredient for proving Theorem 4.1.2 and Theorem 4.1.3 is establishing an expansion result of the form

$$|u_1(x,t) - p(x,t)u_2(x,t)| \leq C\left(|x|^{\beta+1} + |t|^{\frac{\beta+1}{2}}\right), \quad (x,t) \in Q_1,$$

[1]Regular points are those at which the blow-up is a 1-dimensional profile of the form $\frac{1}{2}(x \cdot e)_+^2$; see [64] for more details.

for some polynomial p of parabolic degree $\lfloor\beta\rfloor$. To get such an expansion, we perform a contradiction in combination with a blow-up argument. The contradiction is reached with Liouville theorem in half-space for the parabolic setting. We follow the ideas of an analogous result in the non-local elliptic setting [1]. In particular our proof is different from the one of De Silva-Savin in [27].

4.1.2 Notation

For a real number, we denote $\lfloor\cdot\rfloor$ its integer part and $\langle x\rangle = x - \lfloor x\rfloor$.

The ambient space is $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, where the first n coordinates we denote with $x = (x_1, \ldots, x_n)$ and the last one with t. Sometimes we furthermore split $x = (x', x_n)$, for $x' = (x_1, \ldots, x_{n-1})$. Accordingly we use the multi-index notation $\alpha \in \mathbb{N}_0^{n+1}$, $\alpha = (\alpha_1, \ldots, \alpha_n, \alpha_t)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n + \alpha_t$, and furthermore

$$\partial^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \circ \ldots \circ \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \circ \left(\frac{\partial}{\partial t}\right)^{\alpha_t}.$$

The gradient operator ∇ and Laplace operator Δ are taken only in x variables. When we want to use the full coordinate operators, we denote it with a subscript $\nabla_{(x,t)}$.

For $\Omega \subset \mathbb{R}^{n+1}$ we denote the time slits with $\Omega_t = \{x \in \mathbb{R}^n;\ (x,t) \in \Omega\}$ and the distance function to the boundary in space directions only with $d_t(x) = d_x(x,t) = \inf_{z\in\partial\Omega_t} |z - x|$. Moreover its parabolic boundary $\partial_p\Omega$ is the set $\{(x,t) \in \partial\Omega;\ \forall r > 0 :\ B_r(x) \times (t - r, t) \not\subset \Omega\}$.

We denote with $Q_r(x_0, t_0)$ the following parabolic cylinder of radius r centred at (x_0, t_0),

$$Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2).$$

When $(x_0, t_0) = (0, 0)$, we denote it simply Q_r. Sometimes we also denote

$$Q_r^+ = Q_r \cap \{x_n > 0\}.$$

Finally, C indicates an unspecified constant not depending on any of the relevant quantities, and whose value is allowed to change from line to line. We make use of sub-indices whenever we will want to underline the dependencies of the constant.

4.1.3 Organisation of the paper

In Section 4.2, we present the definitions of parabolic polynomials, parabolic derivatives and parabolic Hölder spaces, notion of viscosity solutions, and cite the results about them which we use. In Section 4.3 we establish basic results regarding the behaviour of solutions of (4.1.1) near the boundary. Well equipped in Section 4.4 we prove the boundary Harnack estimate in C^1_p domains and Theorem 4.1.2. In the following Section 4.5 we prove the higher order boundary Schauder estimates and Theorem 4.1.3. Finally, the proof of Corollary 4.1.4 is presented in Section 4.6. At the end there is an appendix, where we establish the results regarding the interior regularity of solutions of (4.1.1) needed for our specific setting. Moreover we there prove technical auxiliary results, to lighten the body of the paper.

4.2 Preliminaries

4.2.1 Parabolic Hölder spaces

We follow the definitions of the parabolic Hölder spaces from [61]. We start with defining parabolic derivatives and parabolic polynomial spaces.

Definition 4.2.1. For a multi-index $\alpha \in \mathbb{N}_0^{n+1}$, we denote α_x the first n components and α_t the last one and define

$$|\alpha|_p = |\alpha_x| + 2\alpha_t = \sum_{i=1}^{n} \alpha_i + 2\alpha_t.$$

Furthermore, we define the parabolic derivatives with

$$D_p^k = \{\partial^\alpha;\ |\alpha|_p = k\}.$$

Similarly, we define the parabolic polynomial spaces as follows:

$$\mathbf{P}_{k,p} = \left\{ \sum_{|\alpha|_p \leq k} c_\alpha (x,t)^\alpha;\ c_\alpha \in \mathbb{R} \right\}.$$

We say that k is the parabolic degree of polynomial p, if k is the least integer so that $p \in \mathbf{P}_{k,p}$. For a polynomial $p = \sum_\alpha c_\alpha (x,t)^\alpha$, we denote

$$||p|| = \sum_\alpha |c_\alpha|.$$

Next we define parabolic Hölder seminorms and spaces.

Definition 4.2.2. Let Ω be an open subset of $\mathbb{R}^{n+1}$. For $\alpha \in (0,1]$ we define the parabolic Hölder seminorm of order α as follows

$$[u]_{C_p^\alpha(\Omega)} = \sup_{(x,t),(y,s)\in\Omega} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}},$$

and

$$[u]_{C_t^\alpha(\Omega)} = \sup_{(x,t),(x,s)\in\Omega} \frac{|u(x,t) - u(x,s)|}{|t-s|^\alpha}.$$

If $\alpha \in (1,2]$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [u]_{C_t^{\frac{\alpha}{2}}(\Omega)}.$$

For bigger numbers $\alpha > 2$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [\partial_t u]_{C_p^{\alpha-2}(\Omega)}.$$

When $\alpha \notin \mathbb{N}$ we say that $u \in C_p^\alpha(\Omega)$, when $[u]_{C_p^\alpha(\Omega)} < \infty$, and define

$$||u||_{C_p^\alpha(\Omega)} = \sum_{k \leq \lfloor\alpha\rfloor} ||D_p^k u||_{L^\infty(\Omega)} + [u]_{C_p^\alpha(\Omega)}.$$

If $\alpha \in \mathbb{N}$, we say that $u \in C_p^\alpha(\Omega)$, if there exists a modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ – a continuous, increasing function with $\omega(0)=0$ – so that for all $(x,t),(y,s) \in \Omega$

$$|D_p^\alpha u(x,t) - D_p^\alpha u(y,s)| \le \omega(|x-y| + |t-s|^{\frac{1}{2}}),$$

and

$$|D_p^{\alpha-1} u(x,t) - D_p^{\alpha-1} u(x,s)| \le |t-s|^{\frac{1}{2}} \omega(|t-s|^{\frac{1}{2}}).$$

We set

$$||u||_{C_p^\alpha(\Omega)} = \sum_{k\le\alpha} ||D_p^k u||_{L^\infty(\Omega)}.$$

Moreover, we say $u \in C_p^{\alpha-1,1}(\Omega)$ if $[u]_{C_p^\alpha(\Omega)} < \infty$ and we set

$$||u||_{C_p^{\alpha-1,1}(\Omega)} = \sum_{k\le\alpha-1} ||D_p^k u||_{L^\infty(\Omega)} + [u]_{C_p^\alpha(\Omega)}.$$

Remark 4.2.3. Breaking down the definition of the parabolic Hölder seminorms, we get that

$$[u]_{C_p^\alpha(\Omega)} = \left[D_p^{\lfloor\alpha\rfloor} u\right]_{C_p^{\langle\alpha\rangle}(\Omega)} + \left[D_p^{\lfloor\alpha\rfloor-1} u\right]_{C_t^{\frac{\langle\alpha\rangle+1}{2}}(\Omega)}.$$

Notice that the seminorms are compatible with parabolic scaling. If $u_r(x,t) = u(rx, r^2t)$, then

$$[u_r]_{C_p^\alpha(Q_1)} = r^\alpha [u]_{C_p^\alpha(Q_r)}.$$

We want to remark as well, that for every power $\beta > 0$, we have

$$C^{-1}\left(|x| + |t|^{\frac{1}{2}}\right)^\beta \le |x|^\beta + |t|^{\frac{\beta}{2}} \le C\left(|x| + |t|^{\frac{1}{2}}\right)^\beta.$$

4.2.2 Viscosity solutions

For the notion of solutions of the equation (4.1.1) we use the viscosity solutions.

Definition 4.2.4. Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set. We say that $u \in C(\overline{\Omega})$ is a viscosity sub-solution of

$$\partial_t u + Lu = f$$

in Ω, if for any function $\phi \in C_p^2(\Omega)$ touching u from below at some point $(x_0,t_0) \in \Omega$ it holds

$$\partial_t \phi(x_0,t_0) + L\phi(x_0,t_0) \le f(x_0,t_0).$$

Analogously we define viscosity super-solutions; the opposite inequality has to hold for $C_p^2(\Omega)$ functions touching u from above.

We say that $u \in C(\overline{\Omega})$ is a viscosity solution of (4.1.1) if it is both viscosity sub-solution and viscosity super-solution.

One of the main reasons for using viscosity solutions is, because they are well behaved under uniform convergence. We often use the stability result [8, Theorem 1.1]. The existence and uniqueness result is established in [61, Theorem 5.15], while the interior regularity results are provided in [70, Theorem 2], when the right-hand side is bounded only, and in [61, Theorem 5.9], when the right-hand side is Hölder continuous.

4.3 Basic boundary regularity results

Since solutions of (4.1.1) satisfy the comparison principle (see for example [61, Lemma 2.1]), a first key to get estimates for solutions near the boundary is constructing suitable barriers.

Let us start with providing the existence of a so called *generalised distance* function. It is comparable to the parabolic distance to the boundary, but moreover satisfies some additional interior regularity properties, which are very important later on. The parabolic distance at a point $(x,t) \in \Omega$ is given as

$$\operatorname{dist}_p((x,t),\partial\Omega) = \inf_{(z,s)\in\partial\Omega} \left(|x-z| + |t-s|^{\frac{1}{2}} \right).$$

Lemma 4.3.1. *Let $\beta \in [1,2)$ and let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1 in the sense of Definition 4.1.1. Then there exists a function $d\colon \Omega \cap Q_1 \to \mathbb{R}$, satisfying*

$$C^{-1}\operatorname{dist}_p(\cdot,\partial\Omega) \le d \le C\operatorname{dist}_p(\cdot,\partial\Omega), \quad d \in C_p^\beta(Q_1)\cap C_p^2(\Omega\cap Q_1), \quad |\partial_t d| + |D^2 d| \le C d^{\beta-2},$$

$$\frac{2}{3} \le |\nabla d| \le C.$$

Moreover, when $\beta = 1$, we have

$$|\partial_t d| + |D^2 d| \le C d^{-1}\omega(d),$$

where ω is the modolus of continuity of ∇F. The constant C depends only on β, $||F||_{C_p^\beta(Q_1')}$ and n.

Proof. The construction is done in [61, Section IV.5]. The second part ($\beta = 1$) is established analogously as [62, Theorem 1.3]. □

With aid of the generalised distance we construct a barrier in C_p^1 and $C_p^{1,\alpha}$ domains. The barrier is a non-negative function, vanishing at the specific point of the boundary with positive right-hand side. When the boundary is C_p^1 only, the obtained barrier vanishes a bit slower than linearly.

Lemma 4.3.2. *Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^1 in Q_1 in the sense of Definition 4.1.1. Let L be an operator of the form (4.1.2) satisfying conditions (4.1.3). Let $\varepsilon \in (0,1)$. Then for every boundary point $(z,s) \in \partial\Omega \cap Q_{1/2}$ there exists a function $\psi_{(z,s)}$ satisfying the following properties:*

$$\begin{cases} (\partial_t + L)\phi_{(z,s)} \ge d^{-1-\varepsilon} + 1 & \text{in } \Omega\cap Q_1 \cap \{d<r\} \\ \phi_{(z,s)} \ge 1 & \text{in } \Omega \cap \partial_p(Q_1\cap\{d<r\}) \\ \phi_{(z,s)} \ge 0 & \text{in } \Omega\cap Q_1\cap\{d<r\} \\ \phi_{(z,s)}(z,s) = 0 & \\ \phi_{(z,s)}(z+\xi\nu_x,s) \le C d^{1-\varepsilon}(z+\xi\nu_x,s) & \text{for } \xi < r. \end{cases}$$

The constants C and $r>0$ depend only on $n, \varepsilon, \lambda, \Lambda$, $||F||_{C_p^1}$ and the modulus of continuity of ∇F.

Proof. Set first $\varphi = d^{1-\varepsilon}$. We compute

$$\begin{aligned}(\partial_t + L)\varphi &= \varepsilon(1-\varepsilon)\nabla d^T A \nabla d \cdot d^{-1-\varepsilon} + (1-\varepsilon)d^{-\varepsilon}(\partial_t + L)d \\ &\geq \varepsilon(1-\varepsilon)\frac{4\lambda}{9}d^{-1-\varepsilon} - C\Lambda(1-\varepsilon)d^{-1-\varepsilon}\omega(d) - C\Lambda(1-\varepsilon)d^{-\varepsilon},\end{aligned}$$

thanks to Lemma 4.3.1. If $(x,t) \in \{d < r\}$, for r so that $\max(\omega(r), r) < \varepsilon\frac{\lambda}{9C\Lambda}$, we get

$$(\partial_t + L)\varphi \geq \varepsilon(1-\varepsilon)\frac{2\lambda}{9}d^{-1-\varepsilon}.$$

The barrier can be taken as $\phi_{(z,s)}(x,t) = D\left(\varphi(x,t) + E(|x-z|^2 + |t-s|^2)\right)$, for small enough $E > 0$ and big enough D. □

In the case when the domain is $C_p^{1,\alpha}$, we can improve the barrier so that it grows linearly near the boundary.

Lemma 4.3.3. *Let $\beta \in (1,2)$ and let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1 in the sense of Definition 4.1.1. Let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3)*. Then for every boundary point $(z,s) \in \partial\Omega \cap Q_{1/2}$ there exists a function $\psi_{(z,s)}$ satisfying the following properties:*

$$\begin{cases} (\partial_t + L)\phi_{(z,s)} \geq d^{\beta-2} + 1 & \text{in } \Omega \cap Q_1 \cap \{d < r\} \\ \phi_{(z,s)} \geq 1 & \text{in } \Omega \cap \partial_p(Q_1 \cap \{d < r\}) \\ \phi_{(z,s)} \geq 0 & \text{in } \Omega \cap Q_1 \cap \{d < r\} \\ \phi_{(z,s)}(z,s) = 0 & \\ \phi_{(z,s)}(z + \xi\nu_x, s) \leq Cd(z + \xi\nu_x, s) & \text{for } \xi < r. \end{cases}$$

The constants C and $r > 0$ depend only on $n, \beta, \lambda, \Lambda$ and $||F||_{C_p^\beta}$.

Proof. Set $\varphi = d - Md^\beta$, for some positive constant M specified later. We compute

$$(\partial_t + L)\varphi = (\partial_t + L)d(1 - M\beta d^{\beta-1}) + M\beta(\beta-1)d^{\beta-2}\nabla d^T A \nabla d.$$

Hence we can estimate

$$\begin{aligned}(\partial_t + L)\varphi &\geq -\,|(\partial_t + L)d|(1 + M\beta d^{\beta-1}) + M\beta(\beta-1)d^{\beta-2}\nabla d^T A \nabla d \\ &\geq -\,Cd^{\beta-2}(1 + 2Md^{\beta-1}) + \frac{4M}{9}\beta(\beta-1)d^{\beta-2}\end{aligned}$$

Choosing M so that $\frac{4M}{9}\beta(\beta-1) > \max\{3C, 1\}$, and then r so that $2Md^{\beta-1} < 1$, we get that for $x \in \{d < r\} \cap \{d < 1\}$

$$(\partial_t + L)\varphi \geq d^{\beta-2} \geq \frac{1}{2}(d^{\beta-2} + 1).$$

We now set $\phi_{(z,s)}(x,t) = D\left(\varphi(x,t) + E((t-s)^2 + |z-x|^2)\right)$. Constant E has to be small enough so that $D^{-1}(\partial_t + L)\phi_{(z,s)} \geq \frac{1}{4}(d^{\beta-2} + 1)$, and then D is chosen big enough, so that $(\partial_t + L)\phi_{(z,s)} \geq d^{\beta-2} + 1$ and that $\phi_{(z,s)} \geq 1$ on $\Omega \cap \partial_p(\{d < r\} \cap Q_1)$. Since $\varphi(z + \xi\nu_x, s) \leq Cd(z + \xi\nu_x, s)$, provided that r is small enough, the same inequality holds for $\phi_{(z,s)}$ as well. □

The existence of such barriers straight forward imply that solutions to parabolic equations grow near the boundary at most linearly (almost linearly if the boundary is C^1_p).

Corollary 4.3.4. *Let $\beta \in [1,2)$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^β_p in Q_1 in the sense of Definition 4.1.1 and let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3)*. Suppose u is a solution to*

$$\begin{cases} \partial_t u + Lu &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap Q_1. \end{cases}$$

If $\beta = 1$,then for every $\varepsilon > 0$ have

$$|u| \le C\big(||fd||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big)d^{1-\varepsilon} \quad \text{in } \Omega \cap Q_{1/2},$$

with C depending only on $n, \varepsilon, \lambda, \Lambda$, and $||F||_{C^1_p}$.

When $\beta > 1$, we have

$$|u| \le C\big(||fd^{2-\beta}||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big)d \quad \text{in } \Omega \cap Q_{1/2},$$

with C depending only on $n, \beta, \lambda, \Lambda$, and $||F||_{C^\beta_p}$.

Proof. Cases $\beta = 1$ and $\beta > 1$ are treated in the same way. Let us only prove the statement when $\beta > 1$.

Dividing u with $||fd^{2-\beta}||_{L^\infty} + ||u||_{L^\infty}$, we can assume that $|u| \le 1$ and $|f| \le d^{\beta-2}$. Take $(x,t) \in \Omega \cap Q_{1/2}$. If $d(x,t) < r/2$ for the constant r and function $\phi_{(z,t)}$ from Lemma 4.3.3, we apply the comparison principle ([61, Lemma 2.1]) on $\phi_{(z,t)} \pm u$ and get that $|u(x,t)| \le Cd(x,t)$. Taking C bigger if necessary, the same holds also if $d_x(x,t) \ge r/2$. Note that thanks to Definition 4.1.1 the function d is also comparable to d_x. In case $\beta = 1$ Lemma 4.3.2 together with comparison principle gives the desired result. □

Having bounds on the growth near the boundary, combined with interior estimates quickly give Lipschitz bounds up to the boundary (almost Lipschitz in C^1_p domains).

Corollary 4.3.5. *For $\beta \in [1,2)$ let $\Omega \subset \mathbb{R}^{n+1}$ be C^β_p in Q_1 in the sense of Definition 4.1.1 and let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3)*, with $A \in C^0(\overline{\Omega})$. Let u be a solution to*

$$\begin{cases} \partial_t u + Lu &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap Q_1. \end{cases}$$

If $\beta = 1$ we have for every $\varepsilon > 0$

$$[u]_{C^{1-\varepsilon}_p(\Omega\cap Q_{1/2})} \le C\big(||fd^{2-\beta}||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big),$$

with C depending only on $n, \varepsilon, \lambda, \Lambda$, $||F||_{C^1_p}$, and the modulus of continuity of A.

If $\beta > 1$ we have

$$[u]_{C^1_p(\Omega\cap Q_{1/2})} \le C\big(||fd^{2-\beta}||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big),$$

with C depending only on $n, \beta, \lambda, \Lambda$, $||F||_{C^\beta_p}$, and the modulus of continuity of A.

Proof. Let first $\beta > 1$. Take any $Q_r(x_0,t_0) \subset Q_{2r}(x_0,t_0) \subset \Omega$ so that $d(x_0,t_0) \leq Cr$ with C independent on u, f, x_0, t_0. [2] Define $u_r(x,t) = u(x_0 + 2rx, t_0 + 4r^2 t)$. It solves

$$\partial_t u_r + Lu_r = 4r^2 f \quad \text{in } Q_1.$$

Hence by interior regularity results [70, Theorem 2]

$$\begin{aligned}[u_r]_{C^1_p(Q_{1/2})} &\leq C\big(||4r^2 f||_{L^\infty(Q_{2r(x_0,t_0)})} + ||u_r||_{L^\infty(Q_1)}\big)\\ &\leq Cr\big(||fd^{2-\beta}||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big),\end{aligned}$$

where we applied the growth control on u. Translating it back to u, we get

$$[u]_{C^1_p(Q_r)} \leq C\big(||fd^{2-\beta}||_{L^\infty(\Omega\cap Q_1)} + ||u||_{L^\infty(\Omega\cap Q_1)}\big).$$

Due to Lemma 4.8.2 the claim is proven.

When $\beta = 1$, the prove is analogous. The growth control of u is $|u| \leq Cd^{1-\varepsilon}$, which assures that we can bound $[u]_{C^{1-\varepsilon}_p(Q_r)}$ instead. □

Having boundary regularity estimates enables us to prove the following Liouville-type theorem in a half-space. Recall that $Q_R^+ = Q_R \cap \{x_n > 0\}$.

Proposition 4.3.6. *Assume that v solves*

$$\begin{cases} \partial_t v - \mathrm{tr}(AD^2 v) &= P \quad \text{on } \{x_n > 0\}\\ v &= 0 \quad \text{in } \{x_n = 0\}\\ ||v||_{L^\infty(Q_R^+)} &\leq C_0 R^\gamma \quad \forall R > 1,\end{cases}$$

for some constant, uniformly elliptic matrix A, $\gamma > 0$, $\gamma \notin \mathbb{N}$, and some polynomial $P \in \boldsymbol{P}_{\lfloor \gamma-2 \rfloor,p}$ (if $\gamma < 2$, then $P = 0$). Then $v = Qx_n$ for some $Q \in \boldsymbol{P}_{\lfloor \gamma-1 \rfloor,p}$.

Proof. Again define

$$v_R(x,t) := R^{-\gamma} v(Rx, R^2 t).$$

Then $||v_R||_{L^\infty(Q_1)} \leq C_0$, $(\partial_t - \mathrm{tr}(AD^2))v_R(x,t) = R^{2-\gamma}P(Rx, R^2 t)$, and $v_R = 0$ on $\{x_n = 0\}$. Hence applying Corollary 4.3.5, we get

$$[v_R]_{C^1_p(Q^+_{1/2})} \leq C(||P|| + C_0),$$

which translates to

$$[v]_{C^1_p(Q^+_{R/2})} \leq CR^{\gamma-1}(||P|| + C_0),$$

with C independent of R (same as in the previous proposition). Now we take arbitrary $h \in \mathbb{R}^n$ with $h_n = 0$ and $\tau \in \mathbb{R}$, and define

$$w_1(x,t) := \frac{1}{|h| + |\tau|^{\frac{1}{2}}}(v(x+h, t+\tau) - v(x,t)),$$

[2] The constant C can be taken as $\frac{\sqrt{S}-1}{S-1}$, where $S = [F]_{C_t^{\frac{1}{2}}}$.

so that $||w_1||_{L^\infty(Q_R^+)} \leq [v]_{C_p^1(Q_R^+)} \leq CR^{\gamma-1}$, as well as

$$\begin{cases} \partial_t w_1 - \mathrm{tr}(AD^2 w_1) & = & P_1 & \text{in } \{x_n > 0\} \\ w_1 & = & 0 & \text{in } x_n = 0, \end{cases}$$

for some polynomial $P_1 \in \mathbf{P}_{\lfloor \gamma-3 \rfloor, p}$. After $K = \lceil \gamma \rceil$ same steps we conclude that w_K is constant which implies that

$$v(x,t) = \sum_{|\alpha| \leq \gamma} v_\alpha(x_n)(x', t)^\alpha,$$

for some functions v_α depending only on x_n. Choose now some maximal multi-index α. Then $\partial^\alpha v = c_\alpha v_\alpha$ satisfies

$$\begin{cases} -a_{nn}\frac{d^2}{dx_n^2} v_\alpha & = & \partial^\alpha P & \text{in } \{x_n > 0\} \\ v_\alpha & = & 0 & \text{in } \{x_n = 0\}, \end{cases}$$

and hence v_α is a polynomial. Continuing in a similar manner and treating higher order coefficients as right-hand side, we conclude that v is a polynomial itself. Noticing that v satisfies suitable growth control and vanishes on $\{x_n = 0\}$ we conclude the wanted result. □

4.4 Boundary Harnack estimate in C_p^1 domains

In this section we prove Theorem 4.1.2. To establish it we first need to show the following expansion result, saying that solutions of (4.1.4) can be well approximated near the boundary by another non-trivial solution.

Proposition 4.4.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^1 in Q_1 in the sense of Definition 4.1.1, with $||F||_{C_p^1(Q_1')} \leq 1$. Let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3), *with $A \in C^0(\Omega)$. For $i = 1, 2$ let u_i be a solution to*

$$\begin{cases} \partial_t u_i + L u_i & = & f_i & \text{in } \Omega \cap Q_1 \\ u_i & = & 0 & \text{on } \partial\Omega \cap Q_1, \end{cases}$$

with $f_i \in L^\infty(\Omega \cap Q_1)$. Assume that $|u_2| \geq c_0 d$ with $c_0 > 0$, $||u_i||_{L^\infty(\Omega)} \leq 1$ and $||f_i||_{L^\infty(\Omega)} \leq 1$. Let $\varepsilon \in (0,1)$. Then for every $(z,s) \in \partial\Omega \cap Q_{1/2}$ there exists a constant $c_{(z,s)} \in \mathbb{R}$, so that

$$|u_1(x,t) - c_{(z,s)} u_2(x,t)| \leq C\big(|x-z|^{2-\varepsilon} + |t-s|^{\frac{2-\varepsilon}{2}}\big).$$

The constant C depends only on n, ε, c_0, the modulus of continuity of A and ellipticity constants.

Moreover for every $(x_0, t_0) \in \Omega \cap Q_{1/2}$, such that $d_t(x_0) = |x_0 - z| = c_\Omega r$, we have

$$\big[u_1 - c_{(z,t_0)} u_2\big]_{C_p^{2-\varepsilon}(Q_r(x_0,t_0))} \leq C. \tag{4.4.1}$$

Proof. With the same reasoning as in Proposition 4.5.1 we can assume that $(z,s)=(0,0)$. We prove the claim by contradiction. Suppose for $i=1,2$ and $k\in\mathbb{N}$, there exist Ω_k which are C^1_p in the sense of Definition 4.1.1 with $(0,0)\in\partial\Omega_k$ and $||F_k||_{C^1_p(Q_1')}\le 1$, functions $u_{i,k}$, $f_{i,k}$ with $||u_{i,k}||_{L^\infty}\le 1$, $u_{2,k}\ge c_0 d$, $||f||_{L^\infty}\le 1$ and operators $L_k=-\operatorname{tr}(A_kD^2)+b_k\nabla$, with the modulus of continuity of A_k independent of k, so that

$$\begin{cases} \partial_t u_{i,k}+L_k u_{i,k} &= f_{i,k} \quad \text{in } \Omega_k\cap Q_1\\ u_{i,k} &= 0 \quad\quad \text{on } \partial\Omega_k\cap Q_1,\end{cases}$$

but

$$\sup_k\sup_{r>0} r^{-2+\varepsilon}||u_{1,k}-c_ku_{2,k}||_{L^\infty(Q_r)}=\infty$$

for any choice of constants c_k. We extend functions $u_{i,k}$ with 0 outside of Ω_k to functions defined on whole Q_1. We define

$$c_{k,r}=\frac{\int_{Q_r}u_{1,k}u_{2,k}}{\int_{Q_r}u_{2,k}^2},$$

so that $\int_{Q_r}(u_{1,k}-c_{k,r}u_{2,k})u_{2,k}=0$. Furthermore we define

$$\theta(r)=\sup_k\sup_{\rho>r}\rho^{-2+\varepsilon}||u_{1,k}-c_{k,r}u_{2,k}||_{L^\infty(Q_\rho)}.$$

Thanks to Lemma 4.8.7 $\theta(r)\to\infty$ as $r\downarrow 0$. Choose a sequence r_m,k_m, with $r_m\le\frac{1}{m}$, so that

$$\frac{1}{r_m^{2-\varepsilon}\theta(r_m)}||u_{1,km}-c_{k_m,r_m}u_{2,k_m}||_{L^\infty(Q_{r_m})}\ge\frac{1}{2}.$$

We define the blow-up sequence

$$v_m(x,t):=\frac{1}{r_m^{2-\varepsilon}\theta(r_m)}\left(u_{1,k_m}-c_{k_m,r_m}u_{2,k_m}\right)(r_mx,r_m^2t).$$

Due to the choice of r_m,k_m and c_{k_m,r_m}, we have $1/2\le||v_m||_{L^\infty(Q_1)}\le 1$ and $\int_{Q_1}v_mu_{2,k_m}=0$.

We now turn our attention to constants $c_{k,\rho}$. Since $u_{2,k}>c_0d_k$, we can estimate

$$\begin{aligned}\rho|c_{k,\rho}-c_{k,2\rho}|\le& C||c_{k,\rho}u_{2,k}-c_{k,2\rho}u_{2,k}||_{L^\infty(Q_\rho\cap\{d_k>\rho/2\})}\\ \le& C||u_{1,k}-c_{k,\rho}u_{2,k}||_{L^\infty(Q_\rho)}+C||u_{1,k}-c_{k,2\rho}u_{2,k}||_{L^\infty(Q_{2\rho})}\\ \le& C\theta(\rho)\rho^{2-\varepsilon}+C\theta(2\rho)(2\rho)^{2-\varepsilon}\le C\theta(\rho)\rho^{2-\varepsilon},\end{aligned}$$

so that it holds

$$|c_{k,\rho}-c_{k,2\rho}|\le C\theta(\rho)\rho^{1-\varepsilon}.$$

Iterating the inequality above we get for any $j\in\mathbb{N}$

$$\begin{aligned}|c_{k,\rho}-c_{k,2^j\rho}|\le\sum_{i=0}^{j-1}|c_{k,2^i\rho}-c_{k,2^{i+1}\rho}|\le& C\sum_{i=0}^{j-1}\theta(2^i\rho)(2^i\rho)^{1-\varepsilon}\\ \le& C\theta(\rho)\rho^{1-\varepsilon}\sum_{i=0}^{j-1}\frac{\theta(2^i\rho)}{\theta(\rho)}2^{i(1-\varepsilon)}\\ \le& C\theta(\rho)(2^j\rho)^{1-\varepsilon}.\end{aligned}$$

It follows that for any $R>1$, we have

$$|c_{k,\rho}-c_{k,R\rho}|\leq C\theta(\rho)(R\rho)^{1-\varepsilon},$$

and therefore

$$||c_{k,\rho}u_{2,k}-c_{k,R\rho}u_{2,k}||_{L^\infty(Q_{R\rho})}\leq C\theta(\rho)(R\rho)^{2-\varepsilon}.$$

Hence

$$\begin{aligned}
&||v_m||_{L^\infty(Q_R)}=\frac{1}{r_m^{2-\varepsilon}\theta(r_m)}||u_{1,k_m}-c_{k_m,r_m}u_{2,k_m}||_{L^\infty(Q_{Rr_m})}\\
&\leq \frac{1}{r_m^{2-\varepsilon}\theta(r_m)}\Big(||u_{1,k_m}-c_{k_m,Rr_m}u_{2,k_m}||_{L^\infty(Q_{Rr_m})}+||c_{k_m,Rr_m}u_{2,k_m}-c_{k_m,r_m}u_{2,k_m}||_{L^\infty(Q_{Rr_m})}\Big)\\
&\leq \frac{1}{r_m^{2-\varepsilon}\theta(r_m)}\left(\theta(Rr_m)(Rr_m)^{2-\varepsilon}+C\theta(r_m)(Rr_m)^{2-\varepsilon}\right)\leq CR^{2-\varepsilon}.
\end{aligned}$$

Moreover for each $\rho>0$ we have

$$\frac{|c_{k,\rho}-c_{k,2^j\rho}|}{\theta(\rho)}\leq C\sum_{i=0}^{j}\frac{\theta(2^{j-i}\rho)}{\theta(\rho)}(2^{j-i}\rho)^{1-\varepsilon},$$

and choosing $j\in\mathbb{N}$ such that $2^j\rho\in[1,2)$, we deduce

$$\frac{|c_{k,\rho}-c_{k,2^j\rho}|}{\theta(\rho)}\leq C\sum_{i=0}^{j}\frac{\theta(2^{-i}\rho)}{\theta(\rho)}(2^{-i})^{1-\varepsilon}\longrightarrow 0\quad\text{as }\rho\downarrow 0.$$

Hence, since $c_{k,\rho}$ is bounded for $\rho\in[1,2)$, we get

$$\frac{|c_{k,\rho}|}{\theta(\rho)}\longrightarrow 0\quad\text{as }\rho\downarrow 0,$$

uniformly in k.

We compute

$$(\partial_t+\tilde{L}_{k_m})v_m(x,t)=\frac{r_m^\varepsilon}{\theta(r_m)}\left(f_{1,k_m}-c_{k_m r_m}f_{2,k_m}\right)(r_m x,r_m^2 t),$$

and hence

$$|(\partial_t+\tilde{L}_{k_m})v_m(x,t)|\leq\frac{r_m^\varepsilon(1+c_{k_m,r_m})}{\theta(r_m)}.$$

The right hand side converges to 0 uniformly in compact sets in $\{x_n>0\}$. Hence it is also bounded uniformly in m, and so by Lemma 4.3.5 we get uniform Hölder bounds $[v_m]_{C^{1-\varepsilon}(Q_R)}\leq C_R$. Hence passing to a subsequence v_m converges to v locally uniformly in $\mathbb{R}^{n+1}$, that by [8, Theorem 1.1] satisfies

$$\begin{cases}
(\partial_t+L)v &= 0 \quad \text{in } \{x_n>0\}\\
v &= 0 \quad \text{on } \{x_n\leq 0\},\\
||v||_{L^\infty(Q_R)} &\leq CR^{2-\varepsilon}\quad \text{for all } R\geq 1\\
||v||_{L^\infty(Q_1)} &\geq \frac{1}{2}\\
\int_{Q_1} v\cdot(x_n)_+ &= 0,
\end{cases}$$

where L is some constant coefficient 2-homogeneous uniformly elliptic operator. Hence Theorem 4.3.6 says that $v = c(x_n)_+$ for some $c \in \mathbb{R}$. Therefore the last two properties of v contradict each other.

To prove (4.4.1), we apply interior estimates [70, Theorem 2] on rescaled function $v_r(x,t) = (u_1 - c_{(z,t_0)}v_2)(x_0 + 2rx, t_0 + 4r^2t)$, and take into account the above proven growth control and the uniform boundedness of $|c_{(z,t_0)}|$, see Lemma 4.8.5. □

Combining the above result with estimates for u_2 yields Theorem 4.1.2.

Proof of Theorem 4.1.2. The proof goes along the same lines as [1, Theorem 1.3].

Choose a cylinder $Q_r(x_0,t_0)$, so that $Q_{2r}(x_0,t_0) \subset \Omega \cap Q_{3/4}$, and $d(x_0,t_0) \le Cr$, with C independent of x_0, t_0 and r. Let z be the closest point to x_0 in $\partial\Omega_{t_0}$. Take arbitrary (x,t) and (y,s) in $Q_r(x_0,t_0)$, denote $c = c_{(z,t_0)} \in \mathbb{R}$ from the previous proposition, and compute

$$\left|\frac{u_1}{u_2}(x,t) - \frac{u_1}{u_2}(y,s)\right| = \left|\left(\frac{u_1}{u_2}(x,t) - c\right) - \left(\frac{u_1}{u_2}(y,s) - c\right)\right| \le$$

$$\begin{aligned} &\le |(u_1 - cu_2)(x,t) - (u_1 - cu_2)(y,s)| \left|u_2^{-1}(x,t)\right| + \\ &\quad + |(u_1 - cu_2)(y,s)| \left|u_2^{-1}(x,t) - u_2^{-1}(y,s)\right|. \end{aligned} \tag{4.4.2}$$

By Proposition 4.4.1 and Lemma 4.8.2 we have that $u_1 - cu_2$ is $C_p^{2-\varepsilon}(\overline{\mathcal{C}})$, for a cone $\mathcal{C} = \cup_{d_{t_0}(x_0)=c_\Omega r=|x_0-z|} Q_r(x_0,t_0)$, with $D_p^k(u_1 - pu_2)(z,t_0) = 0$, for all $k < 2$. Hence by Lemma 4.8.1 we can estimate

$$|D_p^k(u_1 - pu_2)(x,t)| \le C(|x-z|^{2-\varepsilon-k} + |t-t_0|^{\frac{2-\varepsilon-k}{2}}), \quad (x,t) \in \mathcal{C}, k < 2.$$

We denote $v = u_1 - cu_2$ and continue estimating (4.4.2):

$$\begin{aligned} |v(x,t) - v(y,s)| \le& |v(x,t) - v(y,t)| + |v(y,t) - v(y,s)| \le \\ \le& ||D_p^1 v||_{L^\infty(Q_r(x_0,t_0))}|x-y| + [v]_{C^{2-\varepsilon}(Q_r(x_0,t_0))}|t-s|^{\frac{2-\varepsilon}{2}} \\ \le& Cr^{1-\varepsilon}r^{\varepsilon}|x-y|^{1-\varepsilon} + Cr^1|t-s|^{\frac{1-\varepsilon}{2}} \\ \le& Cr(|x-y|^{1-\varepsilon} + |t-s|^{\frac{1-\varepsilon}{2}}) \end{aligned}$$

Finally we also estimate

$$\begin{aligned} \left|u_2^{-1}(x,t) - u_2^{-1}(y,s)\right| &\le \left|u_2^{-1}(x,t)u_2^{-1}(y,s)\right| |u_2(x,t) - u_2(y,s)| \\ &\le Cr^{-2}\,[u_2]_{C^{1-\varepsilon}(\Omega\cap Q_{3/4})}\,(|x-y|^{1-\varepsilon} + |t-s|^{\frac{1-\varepsilon}{2}}) \\ &\le Cr^{-2+\varepsilon}(|x-y|^{1-2\varepsilon} + |t-s|^{\frac{1-2\varepsilon}{2}}), \end{aligned}$$

where we used Corollary 4.3.5 for u_2. Putting it all together we get

$$\left|\frac{u_1}{u_2}(x,t) - \frac{u_1}{u_2}(y,s)\right| \le C(|x-y|^{1-2\varepsilon} + |t-s|^{\frac{1-2\varepsilon}{2}}).$$

Up to choosing a different ε at the beginning, the claim is proven, thanks to Lemma 4.8.2. □

4.5 Higher order boundary Schauder and boundary Harnack estimates

We proceed with finding a finer, higher order description of solutions near the boundary. The first goal is to get Schauder type expansions of solutions at boundary points - approximations with polynomials multiplied with the distance function. Note that the boundary Schauder estimates are already known; see for example [61, Section IV.7] or [56, Section 10.3]. Still, the proof we present here is different and it is useful for the main result: the higher order parabolic boundary Harnack estimate.

Proposition 4.5.1. *Let $\beta > 1$, $\beta \notin \mathbb{N}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^{β}_p in Q_1 in the sense of Definition 4.1.1 with $||F||_{C^{\beta}_p(Q_1')} \leq 1$. Let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3)*, with $A, b \in C^{\beta-2}_p(\overline{\Omega})$, if $\beta > 2$ and $A \in C^0(\overline{\Omega})$, $b \in L^\infty(\Omega)$, if $\beta < 2$. Let u be a solution to*

$$\begin{cases} \partial_t u + Lu &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap Q_1, \end{cases}$$

with $||f||_{C^{\beta-2}_p(\Omega)} \leq 1$ if $\beta > 2$ and $|f| \leq d^{\beta-2}$ if $\beta < 2$. Assume that $||u||_{L^\infty(\Omega)} \leq 1$. Then for every $(z,s) \in \partial\Omega \cap Q_{1/2}$ there exists a polynomial $p_{(z,s)} \in \boldsymbol{P}_{\lfloor \beta-1 \rfloor, p}$, so that

$$|u(x,t) - p_{(z,s)}(x,t)d(x,t)| \leq C(|x-z|^{\beta} + |t-s|^{\frac{\beta}{2}}).$$

The constant C depends only on n, β, ellipticity constants and $||A||_{C^{\beta-2}_p(\overline{\Omega})}, ||b||_{C^{\beta-2}_p(\overline{\Omega})}$ if $\beta > 2$, and the modulus of continuity of A, if $\beta < 2$..

Proof. Thanks to assumption on Ω (translations and scaling preserves the assumptions), we can assume that $(z,s) = (0,0)$. We prove the claim with contradiction argument. Assume that the claim is false. Then there exist Ω_k which are C^{β}_p in Q_1 in the sense of Definition 4.1.1, with $||F_k||_{C^{\beta}_p(Q_1')} \leq 1$ and $(0,0) \in \partial\Omega_k$, and u_k, f_k, $L_k = \text{tr}(A_k D^2) + b_k \cdot \nabla$, with uniformly bounded $||A_k||_{C^{\beta-2}_p(\overline{\Omega}_k)}, ||b_k||_{C^{\beta-2}_p(\overline{\Omega}_k)}$ if $\beta > 2$, and the modulus of continuity of A_k independent of k, if $\beta < 2$, satisfying $\partial_t u_k + L_k u_k = f_k$ in $\Omega_k \cap Q_1$. Moreover, $u_k = 0$ on $\partial\Omega \cap Q_1$, $|u_k| \leq 1$, $||f||_{C^{\beta-2}_p} \leq 1$ (or $|f_k| \leq d_k^{\beta-2}$ if $\beta < 2$), L_k is (λ, Λ)-uniformly elliptic, but for every polynomial $p_k \in \mathbf{P}_{\lfloor \beta-1 \rfloor, p}$ we have

$$\sup_{r>0} r^{-\beta} ||u_k - p_k d_k||_{L^\infty(Q_r)} > k.$$

We extend the functions u_k and d_k with zero in $Q_1 \cap \Omega^c$ so that they are defined in the full cylinder Q_1, and denote them still u_k, d_k. Let $p_{k,\rho} d_k$ be the $L^2(Q_\rho)$ projection of u_k to space $\mathbf{P}_{\lfloor \beta-1 \rfloor, p} d_k$, so that we have

$$\int_{Q_\rho} (u_k - p_{k,\rho} d_k) p d_k = 0,$$

for every polynomial $p \in \mathbf{P}_{\lfloor \beta-1 \rfloor, p}$. Then define the monotone quantity

$$\theta(r) = \sup_k \sup_{\rho > r} \rho^{-\beta} ||u_k - p_{k,\rho} d_k||_{L^\infty(Q_\rho)}.$$

It follows from Lemma 4.8.4 that $\lim_{r\downarrow 0}\theta(r)=\infty$. Choose a sequence k_m, r_m with $r_m \leq 1/m$, so that

$$\frac{1}{2}\theta(r_m)\leq r^{-\beta}||u_{k_m}-p_{k_m,r_m}d_{k_m}||_{L^\infty(Q_{r_m})},$$

and define the blow-up sequence

$$v_m(x,t)=\frac{1}{r_m^\beta\theta(r_m)}\big(u_{k_m}(r_mx,r_m^2t)-p_{k_m,r_m}d_{k_m}(r_mx,r_m^2t)\big).$$

Notice that $||v_m||_{L^\infty(Q_1)}\geq 1/2$ and $\int_{Q_1}v_m p d_{k_m}(r_m\cdot,r_m^2\cdot)=0$ for every polynomial $p\in \mathbf{P}_{\lfloor\beta-1\rfloor,p}$.

Let us now turn our attention to the polynomials $p_{k,\rho}$. We write

$$p_{k,\rho}(x,t)=\sum_{|\alpha|_p\leq\lfloor\beta-1\rfloor}p_{k,\rho}^{(\alpha)}(x,t)^\alpha,\quad p_{k,\rho}^{(\alpha)}\in\mathbb{R}.$$

Using a rescaled version of [1, Lemma A.10] and that $d_k\geq c\rho$ in $Q_\rho\cap\{d_k>\rho/2\}$, we estimate for any α such that $|\alpha|_p\leq\lfloor\beta-1\rfloor$

$$\begin{aligned}\rho^{|\alpha|_p+1}|p_{k,\rho}^{(\alpha)}-p_{k,2\rho}^{(\alpha)}|\leq&C\rho||p_{k,\rho}-p_{k,2\rho}||_{L^\infty(Q_\rho\cap\{d_k>\rho/2\})}\\ \leq&C||p_{k,\rho}d_k-p_{k,2\rho}d_k||_{L^\infty(Q_\rho\cap\{d_k>\rho/2\})}\\ \leq&C||u_k-p_{k,\rho}d_k||_{L^\infty(Q_\rho)}+C||u_k-p_{k,2\rho}d_k||_{L^\infty(Q_{2\rho})}\\ \leq&C\theta(\rho)\rho^\beta+C\theta(2\rho)(2\rho)^\beta\leq C\theta(\rho)\rho^\beta,\end{aligned}$$

so that it holds

$$|p_{k,\rho}^{(\alpha)}-p_{k,2\rho}^{(\alpha)}|\leq C\theta(\rho)\rho^{\beta-1-|\alpha|_p}.$$

Iterating the inequality above we get for any $j\in\mathbb{N}$

$$\begin{aligned}|p_{k,\rho}^{(\alpha)}-p_{k,2^j\rho}^{(\alpha)}|\leq\sum_{i=0}^{j-1}|p_{k,2^i\rho}^{(\alpha)}-p_{k,2^{i+1}\rho}^{(\alpha)}|\leq&C\sum_{i=0}^{j-1}\theta(2^i\rho)(2^i\rho)^{\beta-1-|\alpha|_p}\\ \leq&C\theta(\rho)\rho^{\beta-1-|\alpha|_p}\sum_{i=0}^{j-1}\frac{\theta(2^i\rho)}{\theta(\rho)}2^{i(\beta-1-|\alpha|_p)}\\ \leq&C\theta(\rho)(2^j\rho)^{\beta-1-|\alpha|_p}.\end{aligned}$$

It follows that for any $R>1$, we have

$$|p_{k,\rho}^{(\alpha)}-p_{k,R\rho}^{(\alpha)}|\leq C\theta(\rho)(R\rho)^{\beta-1-|\alpha|_p},$$

and therefore

$$||p_{k,\rho}d_k-p_{k,R\rho}d_k||_{L^\infty(Q_{R\rho})}\leq C\theta(\rho)(R\rho)^\beta.$$

Hence

$$\begin{aligned}&||v_m||_{L^\infty(Q_R)}=\frac{1}{r_m^\beta\theta(r_m)}||u_{k_m}-p_{k_m,r_m}d_{k_m}||_{L^\infty(Q_{Rr_m})}\\ &\leq\frac{1}{r_m^\beta\theta(r_m)}\Big(||u_{k_m}-p_{k_m,Rr_m}d_{k_m}||_{L^\infty(Q_{Rr_m})}+||p_{k_m,Rr_m}d_{k_m}-p_{k_m,r_m}d_{k_m}||_{L^\infty(Q_{Rr_m})}\Big)\\ &\leq\frac{1}{r_m^\beta\theta(r_m)}\Big(\theta(Rr_m)(Rr_m)^\beta+C\theta(r_m)(Rr_m)^\beta\Big)\leq CR^\beta.\end{aligned}$$

Moreover for each $\rho > 0$ we have

$$\frac{|p_{k,\rho}^{(\alpha)} - p_{k,2^j\rho}^{(\alpha)}|}{\theta(\rho)} \le C \sum_{i=0}^{j} \frac{\theta(2^{j-i}\rho)}{\theta(\rho)} (2^{j-i}\rho)^{\beta-1-|\alpha|_p},$$

and choosing $j \in \mathbb{N}$ such that $2^j\rho \in [1,2)$, we deduce

$$\frac{|p_{k,\rho}^{(\alpha)} - p_{k,2^j\rho}^{(\alpha)}|}{\theta(\rho)} \le C \sum_{i=0}^{j} \frac{\theta(2^{-i}\rho)}{\theta(\rho)} (2^{-i})^{\beta-1-|\alpha|_p} \longrightarrow 0 \quad \text{as } \rho \downarrow 0.$$

Hence, since $p_{k,\rho}^{(\alpha)}$ is bounded for $\rho \in [1,2)$, we get

$$\frac{|p_{k,\rho}^{(\alpha)}|}{\theta(\rho)} \longrightarrow 0 \quad \text{as } \rho \downarrow 0.$$

We compute

$$\begin{aligned}(\partial_t + \tilde{L}_{k_m})v_m(x,t) &= \frac{r_m^{2-\beta}}{\theta(r_m)}\big(f_{k_m} - (\partial_t - \tilde{L}_{k_m})(p_{k_m,r_m} d_{k_m})\big)(r_m x, r_m^2 t) \\ &= \begin{cases} \frac{r_m^{2-\beta}}{\theta(r_m)} o_{m,\beta} & \text{if } \beta < 2 \\ P_m + \frac{1}{\theta(r_m)} o_{m,\beta} & \text{if } \beta > 2, \end{cases}\end{aligned}$$

where $P_m \in \mathbf{P}_{\lfloor \beta-2 \rfloor, p}$ is a suitable Taylor polynomial of quantities on the right-hand side. Note that f_{k_m} and $(\partial_t - \tilde{L}_{k_m})(p_{k_m,r_m} d_{k_m})$ are both $C_p^{\beta-2}$ functions, and they can be approximated with a polynomial up to order $|x|^{\beta-2} + |t|^{\frac{\beta-2}{2}}$ (see Lemma 4.8.1), if $\beta > 2$ and otherwise we bound the L^∞ norm. The reminder we denote with $o_{m,\beta}$, so

$$o_{m,\beta}(x,t) = \begin{cases} \big(f_{k_m} - (\partial_t - \tilde{L}_{k_m})(p_{k_m,r_m} d_{k_m})\big)(r_m x, r_m^2 t) & \text{if } \beta < 2 \\ \big(f_{k_m} - (\partial_t - \tilde{L}_{k_m})(p_{k_m,r_m} d_{k_m}) - P_m\big)(r_m x, r_m^2 t) & \text{if } \beta > 2, \end{cases}$$

In the case $\beta < 2$ we can estimate $\frac{r_m^{2-\beta}}{\theta(r_m)}|o_{m,\beta}| \le C\frac{|p_{k_m,r_m}|}{\theta(r_m)} d_m^{\beta-2}(x,0)$, with C independent of m, using the assumption on f_m, estimates on $|\partial_t d|$ and $|D^2 d_m|$ (see Lemma 4.3.1), while $\frac{p_{k_m,r_m}}{\theta(r_m)}$ converges to zero. In the case $\beta > 2$, we have $|\theta(r_m)^{-1} o_{m,\beta}| \le C\frac{||p_{k_m,r_m}||}{\theta(r_m)}(|x|^{\beta-2} + |t|^{\frac{\beta-2}{2}})$. Since $||v_m||_{L^\infty(Q_R)} \le CR^\beta$ independently of m and since $\frac{1}{\theta(r_m)} o_{m,\beta}$ are bounded uniformly in m, we get that P_m are uniformly bounded as well (see the end of the proof of Proposition 4.5.4). Therefore in both cases the term with $o_{m,\beta}$ converges to zero in $L^\infty_{\text{loc}}(\{x_n > 0\})$. Thanks to Corollary 4.3.5 we get uniform Lipschitz bounds for v_m on every Q_R. Passing to a subsequence, the convergence result [8, Theorem 1.1] assures the local uniform convergence of v_m to some function v defined in $\mathbb{R}^{n+1} \cap \{x_n > 0\}$, satisfying

$$\begin{cases} (\partial_t + L)v &= P & \text{in } x_n > 0 \\ v &= 0 & \text{in } x_n = 0, \\ ||v||_{L^\infty(Q_R^+)} &\le CR^\beta \\ ||v||_{L^\infty(Q_1^+)} &\ge \frac{1}{2} \\ \int_{Q_1} vpx_n &= 0 & \text{for every } p \in \mathbf{P}_{\lfloor \beta-1 \rfloor, p}, \end{cases}$$

where L is a constant coefficient, second order (λ, Λ)-elliptic operator and $P \in \mathbf{P}_{\lfloor \beta-2 \rfloor, p}$ if $\beta > 2$ and $P = 0$ if $\beta < 2$. Hence by Liouville theorem (Proposition 4.3.6) v has to be equal to qx_n for some polynomial q in $\mathbf{P}_{\lfloor \beta-1 \rfloor, p}$, which gives the contradiction. □

Remark 4.5.2. When $\beta \in (1, 2)$ the function d is taken from Lemma 4.3.1. When $\beta > 2$, we take the composition of the boundary flattening map $(\phi(x', x_n, t) = (x', x_n - F(x', t), t)$, with the n-th projection. Concretely, $d(x, t) = x_n - F(x', t)$, which is $C^\beta_p(Q_1)$. In both cases we get that $r^{-1}d(rx, r^2t)$ converges to x_n locally uniformly.

Additionally, the estimate $d(rx, r^2t) \leq Crd(x, rt) \leq C_1rd(x, 0)$, works because the boundary is flat enough.

The obtained description of solutions near the boundary implies the regularity up to the boundary, as follows.

Corollary 4.5.3. *Let $\beta > 1$, $\beta \notin \mathbb{N}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C^β_p in Q_1 in the sense of Definition 4.1.1. Let L be an operator of the form* (4.1.2) *satisfying conditions* (4.1.3), *with $A, b \in C^{\beta-2}_p(\overline{\Omega})$, if $\beta > 2$ and $A \in C^0(\overline{\Omega})$ if $\beta < 2$. Let u be a solution to*

$$\begin{cases} \partial_t u + Lu &= f \quad \text{in } \Omega \cap Q_1 \\ u &= 0 \quad \text{on } \partial\Omega \cap Q_1, \end{cases}$$

with $||f||_{C^{\beta-2}_p(\overline{\Omega} \cap Q_1)} \leq C_0$ if $\beta > 2$ and $|f| \leq C_0 d^{\beta-2}$ if $\beta < 2$.

Then

$$||u||_{C^\beta_p(\Omega \cap Q_{1/2})} \leq CC_0.$$

The constant C depends only on n, β, Ω, ellipticity constants and $||A||_{C^{\beta-2}_p(\overline{\Omega})}, ||b||_{C^{\beta-2}_p(\overline{\Omega})}$ if $\beta > 2$, and the modulus of continuity of A, if $\beta < 2$.

Proof. Dividing the equation with C_0 if necessary, we may assume that $C_0 = 1$. Thanks to Lemma 4.8.2, it is enough to prove

$$[u]_{C^\beta_p(Q_r(x_0,t_0))} \leq C,$$

whenever $d_{t_0}(x_0) = |x_0 - z| \leq Cr$, with C independent of x_0, t_0, r, and $Q_{2r}(x_0, t_0) \subset \Omega \cap Q_{1/2}$. To prove that, take $p_{(z,t_0)}$ from Proposition 4.5.1, and define

$$u_r(x, t) := (u - p_{(z,t_0)}d)(x_0 + 2rx, t_0 + (2r)^2t), \quad (x, t) \in Q_1.$$

Then in the case $\beta > 2$ use interior regularity estimates from Proposition 4.7.4, to get

$$[u_r]_{C^\beta_p(Q_{1/2})} \leq C \left(r^\beta ||f + (\partial_t + L)(p_{(z,t_0)}d)||_{C^{\beta-2}_p(\Omega \cap Q_1)} + ||u - p_{(z,t_0)}d||_{L^\infty(Q_{2r}(x_0,t_0))} \right)$$

and hence

$$\left[u - p_{(z,t_0)}d\right]_{C^\beta_p(Q_r(x_0,t_0))} \leq C,$$

thanks to Proposition 4.5.1, the fact that $p_{(z,t_0)}$ are uniformly bounded (see Lemma 4.8.5) and the fact that $d \in C^\beta_p(\overline{\Omega} \cap Q_1)$.

In case $\beta < 2$, use [70, Theorem 2] to get

$$[u_r]_{C^\beta_p(Q_{1/2})} \leq C \left(r^2 ||f + (\partial_t + L)(p_{(z,t_0)}d)||_{L^\infty(Q_{2r}(x_0,t_0))} + ||u - p_{(z,t_0)}d||_{L^\infty(Q_{2r}(x_0,t_0))} \right)$$

and hence due to similar reasons as before

$$\left[u - p_{(z,t_0)}d\right]_{C_p^\beta(Q_r(x_0,t_0))} \le C.$$

This implies that

$$[u]_{C_p^\beta(Q_r(x_0,t_0))} \le C,$$

thanks to regularity of d and boundedness of $p_{(z,t_0)}$, as wanted. □

We can now establish expansions of one solution with respect to the other. This result is the key ingredient to prove the boundary Harnack estimate. It is the higher order version of Proposition 4.4.1 Since the order exceeds 2, the constant in the expansion is replaced with a polynomial, which causes some difficulties in the proof.

Proposition 4.5.4. *Let $\beta > 1$, $\beta \notin \mathbb{N}$. Let $\Omega \subset \mathbb{R}^{n+1}$ be C_p^β in Q_1 in the sense of Definition 4.1.1, with $||F||_{C_p^\beta(Q_1')} \le 1$. Let L be an operator of the form (4.1.2) satisfying conditions (4.1.3), with $A, b \in C_p^{\beta-1}(\overline{\Omega})$. For $i = 1, 2$ let u_i be a solution to*

$$\begin{cases} \partial_t u_i + Lu_i &= f_i \quad \text{in } \Omega \cap Q_1 \\ u_i &= 0 \quad \text{on } \partial\Omega \cap Q_1, \end{cases}$$

with $f_i \in C_p^{\beta-1}(\overline{\Omega} \cap Q_1)$. Assume that $|u_2| \ge c_0 d$ with $c_0 > 0$, $||u_i||_{L^\infty(\Omega)} \le 1$ and $||f_i||_{C_p^{\beta-1}(\overline{\Omega})} \le 1$. Then for every $(z, s) \in \partial\Omega \cap Q_{1/2}$ exists a polynomial $p_{(z,s)} \in \boldsymbol{P}_{\lfloor\beta\rfloor,p}$, so that

$$|u_1(x,t) - p_{(z,s)}(x,t)u_2(x,t)| \le C\big(|x-z|^{\beta+1} + |t-s|^{\frac{\beta+1}{2}}\big).$$

The constant C depends only on n, β, c_0, $||A||_{C_p^{\beta-1}(\overline{\Omega})}, ||b||_{C_p^{\beta-1}(\overline{\Omega})}$ and ellipticity constants.

Moreover for every $(x_0, t_0) \in \Omega \cap Q_{1/2}$, such that $d_t(x_0) = |x_0 - z| = c_\Omega r$, we have

$$\left[u_1 - p_{(z,t_0)}u_2\right]_{C_p^{\beta+1}(Q_r(x_0,t_0))} \le C. \tag{4.5.1}$$

Proof. With the same reasoning as in Proposition 4.5.1 we can assume $(z, s) = (0, 0)$.

We write

$$p(x,t) = \sum_{|\alpha|_p \le \beta} p_\alpha \cdot (x,t)^\alpha = p^{(0)} + \sum_{1 \le |\alpha|_p \le \beta} p_\alpha (x,t)^\alpha = p^{(0)} + p^{(1)}(x,t),$$

for some constants p_α. In view of Proposition 4.5.1

$$u_2(x,t) = p_2(x,t)d(x,t) + v_2(x,t), \qquad p_2 \in \mathbf{P}_{\lfloor\beta-1\rfloor,p},\ |v_2(x,t)| \le C(|x|^\beta + |t|^{\frac{\beta}{2}}),$$

and then the claim is equivalent to

$$\left|u_1 - p^{(0)}u_2 - p^{(1)}p_2 d - p^{(1)}v_2\right|(x,t) \le C\big(|x-z|^{\beta+1} + |t-s|^{\frac{\beta+1}{2}}\big),$$

which is furthermore equivalent to

$$\left|u_1(x,t) - \tilde{p}^{(0)}u_2(x,t) - \tilde{p}^{(1)}(x,t)d(x,t)\right| \le C\big(|x-z|^{\beta+1} + |t-s|^{\frac{\beta+1}{2}}\big),$$

for a suitable polynomial $\tilde{p} \in \mathbf{P}_{\lfloor\beta\rfloor,p}$.

We prove that by contradiction. So suppose for $i = 1, 2$, and $k \in \mathbb{N}$ there exist Ω_k, which are C_p^β in Q_1 in the sense of Definition 4.1.1, with $0 \in \partial\Omega_k$ and $||F_k||_{C_p^\beta(Q_1')} \leq 1$, $u_{i,k}$, $f_{i,k}$ and (λ, Λ)-elliptic operator $L_k = -\operatorname{tr}(A_k D^2) + b_k \cdot \nabla$, with uniformly bounded norms $||A_k||_{C_p^{\beta-1}(\overline{\Omega}_k)}, ||b_k||_{C_p^{\beta-1}(\overline{\Omega}_k)}$, so that $||u_{i,k}||_{L^\infty} \leq 1$, $||f_{i,k}||_{C_p^{\beta-1}} \leq 1$, $u_{2,k} \geq c_0 d$ and

$$\begin{cases} \partial_t u_{i,k} + L_k u_{i,k} &= f_{i,k} \quad \text{in } \Omega_k \cap Q_1 \\ u_{i,k} &= 0 \quad \text{on } \partial\Omega_k \cap Q_1, \end{cases}$$

but

$$\sup_k \sup_{r>0} r^{-\beta-1} ||u_{1,k} - p_k^{(0)} u_{2,k} - p_k^{(1)} d||_{L^\infty(Q_r)} = \infty$$

for every choice of polynomials $p_k \in \mathbf{P}_{\lfloor\beta\rfloor,p}$. We extend $u_{i,k}$ and d_k to functions defined on whole Q_1 with zero outside of Ω_k. Define $p_{k,r}$ as the $L^2(Q_r)$ projection of $u_{1,k}$ on the space $\{q^{(0)} u_{2,k} + q^{(1)} d_k;\ q \in \mathbf{P}_{\lfloor\beta\rfloor,p}\}$, so that

$$\int_{Q_r} (u_{1,k} - p_{k,r}^{(0)} u_{2,k} - p_{k,r}^{(1)} d_k)(q^{(0)} u_{2,k} + q^{(1)} d_k) = 0$$

for every polynomial $q \in \mathbf{P}_{\lfloor\beta\rfloor,p}$. Furthermore, we define

$$\theta(r) = \sup_k \sup_{\rho>r} \rho^{-\beta-1} ||u_{1,k} - p_{k,\rho}^{(0)} u_{2,k} - p_{k,\rho}^{(1)} d_k||_{L^\infty(Q_\rho)}.$$

Lemma 4.8.7, ensures that $\theta(r) \to \infty$ as $r \downarrow 0$. Pick a sequence r_m, k_m, with $r_m \leq 1/m$, so that

$$\frac{1}{r_m^{\beta+1}\theta(r_m)} ||u_{1,k_m} - p_{r_m,k_m}^{(0)} u_{2,k_m} - p_{r_m,k_m}^{(1)} d_{k_m}||_{L^\infty(Q_{r_m})} \geq \frac{1}{2}.$$

We define the blow-up sequence

$$v_m(x,t) := \frac{1}{r_m^{\beta+1}\theta(r_m)} \left(u_{1,k_m} - p_{r_m,k_m}^{(0)} u_{2,k_m} - p_{r_m,k_m}^{(1)} d_{k_m} \right) (r_m x, r_m^2 t).$$

Note that $||v_m||_{L^\infty(Q_1)} \geq 1/2$, as well as $\int_{Q_1} v_m (q^{(0)} u_{2,k_m} + q^{(1)} d_{k_m}) = 0$ for every polynomial $q \in \mathbf{P}_{\lfloor\beta\rfloor,p}$. With same arguments as in Proposition 4.5.1, we get $||v_m||_{L^\infty(Q_R)} \leq CR^{\beta+1}$, and $\frac{|p_{r,k}^{(\alpha)}|}{\theta(r)} \to 0$ uniformly in k as $r \downarrow 0$. See also the proof of [1, Proposition 4.4].

Now we turn our attention to

$$(\partial_t + \tilde{L}_{k_m}) v_m(x,t) = \frac{1}{r_m^{\beta-1}\theta(r_m)} \left(f_{1,k_m} - p_{k_m,r_m}^{(0)} f_{2,k_m} - (\partial_t + \tilde{L}_{k_m})(p_{k_m,r_m}^{(1)} d_{k_m}) \right) (r_m x, r_m^2 t).$$

We want to approximate the right-hand side as well as possible with a polynomial. By assumption $f_{i,k_m} \in C_p^{\beta-1}$ and hence we can approximate it up to order $|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}}$, that is

$$|f_{i,m}(x,t) - P_{i,m}(x,t)| \leq C \left(|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}} \right),$$

for suitable polynomials $P_{i,m}$. Let us turn now to the term with the distance function

$$\begin{aligned} (\partial_t + \tilde{L}_{k_m})(p_{k_m,r_m}^{(1)} d_{k_m}) =& (\partial_t + \tilde{L}_{k_m}) p_{k_m,r_m}^{(1)} d_{k_m} + p_{k_m,r_m}^{(1)} (\partial_t + \tilde{L}_{k_m}) d_{k_m} \\ & - 2\nabla p_{k_m,r_m}^{(1)} A_{k_m} \nabla d_{k_m} \end{aligned}$$

Let us first consider the case $\beta > 2$. Then $d_{k_m} \in C_p^{\beta}$, and hence $\nabla d_{k_m} \in C_p^{\beta-1}$, and $(\partial_t + \tilde{L}_{k_m})d_{k_m} \in C_p^{\beta-2}$. Therefore d_{k_m} and ∇d_{k_m} can be approximated up to order $|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}}$, while the remaining term can only be approximated up to order $|x|^{\beta-2} + |t|^{\frac{\beta-2}{2}}$. But this term is multiplied with $p^{(1)}_{k_m,r_m}$, which can be estimated with $|x| + |t|$, and hence we get

$$\left|(\partial_t + \tilde{L}_{k_m})(p^{(1)}_{k_m,r_m} d_{k_m})(x,t) - P'_m(x,t)\right| \leq C\left(|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}}\right).$$

Thus we deduce

$$|(\partial_t + \tilde{L}_{k_m})v_m(x,t) - P_m(x,t)| \leq \frac{C||p_{k_m,r_m}||}{\theta(r_m)}\left(|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}}\right),$$

for a suitable polynomial $P_m \in \mathbf{P}_{\lfloor\beta-1\rfloor,p}$. Note that $\frac{||p_{k_m,r_m}||}{\theta(r_m)} \to 0$, as $m \to \infty$.

In the case $\beta \in (1,2)$, we have $d_{k_m} \in C^{\beta}$, and $|D^2 d_{k_m}| \leq C d^{\beta-2}$. Hence we get

$$|(\partial_t + \tilde{L}_{k_m})v_m(x,t) - P_m(x,t)| \leq \frac{C||p_{k_m,r_m}||}{\theta(r_m)}\left(|x|^{\beta-1} + |t|^{\frac{\beta-1}{2}} + (|x|+|t|)d_{k_m}^{\beta-2}\right).$$

In both cases we get local uniform convergence of the residue $o_m := (\partial_t + \tilde{L}_{k_m})v_m(x,t) - P_m(x,t)$ to zero. Denote $\Omega_m = \{(x,t);\ (r_m x, r_m^2 t) \in \Omega_{k_m}\}$ and split $v_m = v_{m,1} + v_{m,2}$, with

$$\begin{cases} (\partial_t + \tilde{L}_{k_m})v_{m,1} &= P_m \quad \text{in } \Omega_m \cap Q_1 \\ v_{m,1} &= 0 \quad \text{in } \partial_p(\Omega_m \cap Q_1), \end{cases}$$

and

$$\begin{cases} (\partial_t + \tilde{L}_{k_m})v_{m,2} &= o_m \quad \text{in } \Omega_m \cap Q_1 \\ v_{m,2} &= v_m \quad \text{in } \partial_p(\Omega_m \cap Q_1). \end{cases}$$

Note that the existence of $v_{m,1}, v_{m,2}$ is given by [61, Theorem 5.15]. Since v_m are bounded independently of m, we can get a bound $|v_{m,2}| \leq C d_m$, with C independent of m, using a barrier constructed in Lemma 4.3.3. Since Ω_m are closer and closer to $\{x_n > 0\}$, we can estimate $|v_{m,2}| \leq C$, independently of m. Hence we have uniform bounds for $||v_{m,1}||_{L^\infty(Q_1)}$ as well, which by Lemma 4.8.6 implies also the uniform boundedness of $||P_m||$. Then we apply Corollary 4.3.5 rescaled to any Q_R, to get uniform Lipschitz bounds for v_m. Together with the convergence result [8, Theorem 1.1] this ensures that a subsequence of the blow-up sequence converges to some function v satisfying

$$\begin{cases} (\partial_t + L)v &= P & \text{in } \{x_n > 0\} \\ v &= 0 & \text{on } \{x_n = 0\}, \\ ||v||_{L^\infty(Q_R)} &\leq CR^{\beta+1} & \text{for all } R \geq 1 \\ ||v||_{L^\infty(Q_1)} &\geq \frac{1}{2} & \\ \int_{Q_1} v p (x_n)_+ &= 0 & \text{for every } p \in \mathbf{P}_{\lfloor\beta\rfloor,p}, \end{cases}$$

where $P \in \mathbf{P}_{\lfloor\beta-1\rfloor,p}$. Proposition 4.3.6 then says that v equals $q x_n$, $q \in \mathbf{P}_{\lfloor\beta\rfloor,p}$, which gives the contradiction with the last two properties of v, as wanted.

Finally we prove also (4.5.1). If c_Ω is big enough, then for every point (x_0,t_0) with $d(x_0,t_0) = c_\Omega r$ we have $Q_{2r}(x_0,t_0) \subset \Omega$. It suffices applying Proposition 4.7.4 on the rescaled function $v_r(x,t) = (u_1 - p_{z,t_0}u_2)(x_0 + 2rx, t_0 + 4r^2 t)$, take into account the just proven bound and noticing that $||p_{(z,t_0)}||$ can be uniformly bounded thanks to Lemma 4.8.5. □

We now have all ingredients to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. The proof goes along the same lines as Theorem 4.1.2, just that the estimates with higher order parabolic seminorms are a bit more complicated.

Note that

$$\left[\frac{u_1}{u_2}\right]_{C_p^\beta(\Omega\cap Q_{1/2})} = \left[D_p^{\lfloor\beta\rfloor}\frac{u_1}{u_2}\right]_{C_p^{\langle\beta\rangle}(\Omega\cap Q_{1/2})} + \left[D_p^{\lfloor\beta\rfloor-1}\frac{u_1}{u_2}\right]_{C_t^{\frac{1+\beta}{2}}(\Omega\cap Q_{1/2})}.$$

We start with estimating the first term.

Choose $\gamma \in \mathbb{N}^{n+1}$, with $|\gamma|_p = \lfloor\beta\rfloor$ and a cylinder $Q_r(x_0,t_0)$, so that $Q_{2r}(x_0,t_0) \subset \Omega\cap Q_{3/4}$, and $d(x_0,t_0) \le Cr$, with C independent of x_0, t_0 and r. Let z be the closest point to x_0 in $\partial\Omega_{t_0}$. Take arbitrary (x,t) and (y,s) in $Q_r(x_0,t_0)$, denote $p = p_{(z,t_0)} \in \mathbf{P}_{\lfloor\beta\rfloor,p}$ and compute

$$\left|\partial^\gamma\frac{u_1}{u_2}(x,t) - \partial^\gamma\frac{u_1}{u_2}(y,s)\right| = \left|\partial^\gamma\left(\frac{u_1}{u_2}(x,t) - p(x,t)\right) - \partial^\gamma\left(\frac{u_1}{u_2}(y,s) - p(y,s)\right)\right| \le$$

$$\begin{aligned}\le \sum_{\alpha\le\gamma}\left|\partial^\alpha\left(u_1-pu_2\right)(x,t) - \partial^\alpha\left(u_1-pu_2\right)(y,s)\right|\left|\partial^{\gamma-\alpha}u_2^{-1}(x,t)\right| + \\ + \sum_{\alpha\le\gamma}\left|\partial^\alpha\left(u_1-pu_2\right)(y,s)\right|\left|\partial^{\gamma-\alpha}u_2^{-1}(x,t) - \partial^{\gamma-\alpha}u_2^{-1}(y,s)\right|.\end{aligned} \tag{4.5.2}$$

By Proposition 4.5.4 and Lemma 4.8.2 we have that $u_1 - pu_2$ is $C_p^{\beta+1}(\overline{\mathcal{C}})$, for a suitable cone $\mathcal{C} = \cup_{d_{t_0}(x_0)=c_\Omega r=|x_0-z|}Q_r(x_0,t_0)$, with $D_p^k(u_1-pu_2)(z,t_0) = 0$, for all $k < \beta+1$. Hence by Lemma 4.8.1 we can estimate

$$|D_p^k(u_1-pu_2)(x,t)| \le C(|x-z|^{\beta+1-k} + |t-t_0|^{\frac{\beta+1-k}{2}}), \quad (x,t)\in\mathcal{C}, k<\beta+1.$$

Next take $\alpha\in\mathbb{N}^{n+1}$ and compute

$$\partial^\alpha u_2^{-1} = \sum_{l\le|\alpha|}\frac{1}{u_2^{l+1}}\sum_{\alpha_1+\ldots+\alpha_l=\alpha}c_{\alpha_1,\ldots,\alpha_l}\partial^{\alpha_1}u_2\cdots\partial^{\alpha_l}u_2.$$

Since by Corollary 4.5.3 $u_2\in C_p^\beta(\Omega\cap Q_1)$ this implies

$$||\partial^\alpha u_2^{-1}||_{L^\infty(Q_r(x_0,t_0))} \le Cr^{-(|\alpha|+1)},$$

when $|\alpha|_p < \beta$, as well as

$$\left[\partial^\alpha u_2^{-1}\right]_{C_p^{\langle\beta\rangle}(Q_r(x_0,t_0))} \le Cr^{-(|\alpha|+1)}, \tag{4.5.3}$$

when $|\alpha|_p = \lfloor\beta\rfloor$, and

$$\left[\partial^\alpha u_2^{-1}\right]_{C_t^{\frac{\langle\beta\rangle+1}{2}}(Q_r(x_0,t_0))} \le Cr^{-(|\alpha|+1)}, \tag{4.5.4}$$

when $|\alpha|_p = \lfloor\beta\rfloor - 1$.

We are now equipped to estimate (4.5.2). Take $\alpha \leq \gamma$, with $|\alpha|_p < \beta$ and estimate

$$\begin{aligned}|\partial^\alpha v(x,t) - \partial^\alpha v(y,s)| \leq & |\partial^\alpha v(x,t) - \partial^\alpha v(y,t)| + |\partial^\alpha v(y,t) - \partial^\alpha v(y,s)| \leq \\ \leq & ||D_p^{|\alpha|_p+1} v|||x-y| + ||D_p^{|\alpha|_p+2} v|||t-s| \\ \leq & Cr^{\beta+1-|\alpha|_p-1} r^{1-\langle\beta\rangle}|x-y|^{\langle\beta\rangle} + r^{\beta+1-|\alpha|_p-2} r^{2(1-\frac{\langle\beta\rangle}{2})}|t-s|^{\frac{\langle\beta\rangle}{2}} \\ \leq & Cr^{\lfloor\beta\rfloor+1-|\alpha|_p}(|x-y|^{\langle\alpha\rangle} + |t-s|^{\frac{\langle\beta\rangle}{2}}).\end{aligned}$$

If $|\alpha|_p = \lfloor\beta\rfloor$, then we need to use the definition of $[\partial^\alpha v]_{C_t^{\frac{1+\langle\beta\rangle}{2}}}$, to get the same estimate. Similarly, if $|\gamma - \alpha|_p < \beta - 2$,

$$\begin{aligned}|\partial^{\gamma-\alpha} u_2^{-1}(x,t) - \partial^{\gamma-\alpha} u_2^{-1}(y,s)| \leq & ||D^{|\gamma-\alpha|+1} u_2^{-1}||(|x-y| + |t-s|) \\ \leq & Cr^{-|\gamma-\alpha|-2}(r^{1-\langle\beta\rangle}|x-y|^{\langle\beta\rangle} + r^{2(1-\frac{\langle\beta\rangle}{2})}|t-s|^{\frac{\langle\beta\rangle}{s}}) \\ \leq & Cr^{-|\gamma-\alpha|-1-\langle\beta\rangle}(|x-y|^{\langle\beta\rangle} + |t-s|^{\frac{\langle\beta\rangle}{2}}).\end{aligned}$$

When $|\gamma-\alpha|_p = \lfloor\beta\rfloor - 1$ we have to split the difference into the spatial part and time part and on the time part use (4.5.4). When $|\gamma - \alpha|_p = \lfloor\beta\rfloor$, we directly use (4.5.3) to get the same estimate in other cases. Plugging it all in expression (4.5.2), we get

$$(4.5.2) \leq \sum_{\alpha\leq\gamma} Cr^{\lfloor\beta\rfloor+1-|\alpha|_p} Cr^{-|\gamma-\alpha|-1}(|x-y|^{\langle\beta\rangle} + |t-s|^{\frac{\langle\beta\rangle}{2}}) +$$

$$+ \sum_{\alpha\leq\gamma} Cr^{\beta+1-|\alpha|_p} r^{-|\gamma-\alpha|-1-\langle\beta\rangle}(|x-y|^{\langle\beta\rangle} + |t-s|^{\frac{\langle\beta\rangle}{2}}).$$

It remains to notice that $|\gamma - \alpha| \leq |\gamma - \alpha|_p$, and hence $(4.5.2) \leq C(|x-y|^{\langle\beta\rangle} + |t-s|^{\frac{\langle\beta\rangle}{2}})$.

We argue similarly that $\left[D_p^{\lfloor\beta\rfloor-1}\frac{u_1}{u_2}\right]_{C_t^{\frac{1+\beta}{2}}(\Omega\cap Q_{1/2})} \leq C$. Choose $|\gamma|_p = \lfloor\beta\rfloor - 1$ remember the degree of p and compute

$$\left|\partial^\gamma \frac{u_1}{u_2}(x,t) - \partial^\gamma \frac{u_1}{u_2}(x,s)\right| = \left|\partial^\gamma\left(\frac{u_1}{u_2}(x,t) - p(x,t)\right) - \partial^\gamma\left(\frac{u_1}{u_2}(x,s) - p(x,s)\right)\right| \leq$$

$$\begin{aligned}
&\leq \sum_{\alpha\leq\gamma} \left|\partial^\alpha (u_1 - pu_2)(x,t) - \partial^\alpha (u_1 - pu_2)(x,s)\right| \left|\partial^{\gamma-\alpha} u_2^{-1}(x,t)\right| + \\
&\quad + \sum_{\alpha\leq\gamma} \left|\partial^\alpha (u_1 - pu_2)(x,s)\right| \left|\partial^{\gamma-\alpha} u_2^{-1}(x,t) - \partial^{\gamma-\alpha} u_2^{-1}(x,s)\right| \\
&\leq \sum_{\alpha\leq\gamma} ||D_p^{|\alpha|_p+2} v|| \cdot |t-s| \cdot ||D^{|\gamma-\alpha|} u_2^{-1}|| + \\
&\quad + \sum_{1\leq|\alpha|\leq|\gamma|} ||D_p^{|\alpha|_p} v|| \cdot ||D^{|\gamma-\alpha|+1} u_2^{-1}|| \cdot |t-s| \\
&\quad + ||v|| \cdot \left[\partial^\gamma u_2^{-1}\right]_{C_t^{\frac{1+\langle\beta\rangle}{2}}} \cdot |t-s|^{\frac{1+\langle\beta\rangle}{2}} \\
&\leq \sum_{\alpha\leq\gamma} C r^{\beta+1-|\alpha|_p-2} r^{2(\frac{1-\langle\beta\rangle}{2})} |t-s|^{\frac{1+\langle\beta\rangle}{2}} C r^{-|\gamma-\alpha|-1} \\
&\quad + \sum_{1\leq|\alpha|\leq|\gamma|} C r^{\beta+1-|\alpha|_p} C r^{-|\gamma-\alpha|-2} r^{2(\frac{1-\langle\beta\rangle}{2})} |t-s|^{\frac{1+\langle\beta\rangle}{2}} \\
&\quad + C r^{\beta+1} C r^{-|\gamma|-1} r^{2(\frac{1-\langle\beta\rangle}{2})} |t-s|^{\frac{1+\langle\beta\rangle}{2}} \\
&\leq C|t-s|^{\frac{1+\langle\beta\rangle}{2}}.
\end{aligned}$$

Therefore the claim is proven, thanks to Lemma 4.8.2.

The same steps prove the claim in case $\beta = 1$, but we use Proposition 4.4.1 instead of Proposition 4.5.4, and the C_p^β estimate for u_2 up to the boundary is replaced by $C_p^{1-\varepsilon}$ from Corollary 4.3.5. □

4.6 Higher regularity of free boundary in obstacle problem

Our parabolic higher order boundary Harnack inequality gives a simple way to prove that once the free boundary in the parabolic obstacle problems is C_p^1, it is in fact C^∞. The reason is that the normal to the free boundary can be expressed with the quotients of partial derivatives of the solution to obstacle problem, which are as smooth as the boundary. This gives the bootstrap argument, and does not require any use of a hodograph transform as in [52].

Let us start with preliminary results that we need in the proof of Corollary 4.1.4.

Lemma 4.6.1. *Let $v\colon Q_1 \to \mathbb{R}$ solve (4.1.5) with $f \in C^\theta(B_1)$, for some $\theta > 0$, with $f(0) = 1$. Assume that $(0,0) \in \partial\{v > 0\}$ is a regular free boundary point.*

Then $v \in C_p^{1,1}(Q_1)$, up to rotation of coordinates the free boundary is C_p^1 in Q_r for some $r > 0$ in the sense of Definition 4.1.1, $v_t \in C(Q_r)$ and $\partial_\nu v \geq c_1 d$, with $c_1 > 0$, where ν is the unit spatial normal vector of the free boundary at $(0,0)$.

Proof. The $C_p^{1,1}(Q_1)$ regularity of solutions of the obstacle problem follows from [63, Theorem 1.6]. The C_p^1 regularity of the free boundary is provided in [64, Theorem 1.7]. Denote the boundary defining map with F.

Let us now prove the continuity of the time derivative. This is well known when $f \equiv 1$ (see [16]). For completeness we prove it next in our setting. Since inside Ω the solution v

is $C_p^{\theta+2}$ and $v \equiv 0$ in Ω^c, we have to show that

$$\lim_{(x,t)\to(z,s)} \partial_t v(x,t) = 0,$$

for every free boundary point in Q_r. Choose a point $(x,t) \in \Omega \cap Q_r$, such that $(x,\tau) \in \Omega^c$ for $\tau \leq s$. Let $(x,s), (x', F(x',t), t)$ be free boundary points. Therefore by [63, Theorem 1.6] we have $v(x,t) \leq C|x_n - F(x',t)|^2$. But since $F \in C_p^1$, and $x_n = F(x',s)$, we have $v(x,t) \leq C|F(x't) - F(x',s)|^2 \leq C(|t-s|^{\frac{1}{2}}\omega(|t-s|^{\frac{1}{2}}))^2$, where ω is the modulus of continuity of ∇F. This gives $|u(x,t) - u(x,s)| = u(x,t) \leq |t-s|\eta(|t-s|)$, for some modulus of continuity η.

Finally, let us show that $\partial_\nu v \geq c_1 d$ where $c_1 > 0$, d is the regularised distance to the $\partial\Omega$ and ν is its spatial normal vector at $(0,0)$. Without loss of generality assume that $\nu = e_n$. Having the bound $||u||_{C_p^{1,1}(Q_1)} \leq C$, together with [63, Theorem 1.7] and Arzela-Ascoli theorem imply that v converges to its blow up at any free boundary point $(x_0,t_0) \in \partial\Omega \cap Q_r$ uniformly (independently of the free boundary point) in $C_p^{0,1}$, namely

$$\left|\left|\frac{1}{r^2} v(x_0 + rx, t_0 + r^2 t) - \frac{f(x_0)}{2}(x\nu_{x_0})_+^2\right|\right|_{C_p^{0,1}(Q_1)} \longrightarrow 0$$

as $r \to 0$, uniformly in (x_0,t_0). Hence reading it for the $n-$th derivative, this means that there exists a number $r_0 > 0$ so that for all $r < r_0$ we have

$$\left|\left|\frac{1}{r} v_n(x_0 + rx, t_0 + r^2 t) - f(x_0)(\nu_{x_0} e_n)(x\nu_{x_0})_+\right|\right|_{L^\infty(Q_1)} \leq \frac{1}{4}.$$

Restricting ourselves to a potentially smaller neighbourhood of $(0,0)$, so that $\nu_{x_0} e_n > \frac{3}{4}$, $|\nu_{x_0} - e_n| \leq \frac{1}{5}$, $|f(x_0) - 1| > \frac{1}{4}$ with a triangle inequality we deduce that for $(x,t_0) \in Q_r((x_0,t_0)) \cap \{(x - x_0)e_n > \frac{r}{2}\}$ we have

$$\partial_n v(x,t_0) \geq f(x_0)(\nu_{x_0} e_n)((x - x_0)\nu_{x_0}) - \frac{r}{4} \geq \frac{r}{4}.$$

But since $d_x(x,t_0) \leq |x - x_0| = r$, and the estimate is independent of (x_0,t_0), we get that $\partial_v \geq \frac{1}{4} d_x \geq \frac{1}{4C} d$, since the two distances are comparable. □

We are now equipped enough to prove Corollary 4.1.4.

Proof of Corollary 4.1.4. Denote $\Omega = \{v > 0\}$. By Lemma 4.6.1, rescaling and rotation if necessary Ω is C_p^1 in Q_1, the solution v is $C^{1,1}$ in space and C^1 in time. This implies that in Ω^c both $v_t := \partial_t v = 0$ and for $i = 1, \ldots, n$ also $v_i := \partial_i v = 0$. Without loss of generality assume that the normal vector to Ω_0 at 0 is e_n. Hence all the partial derivatives of v solve

$$\begin{cases} (\partial_t - \Delta)w &= \partial_e f \quad \text{in } \Omega \cap Q_1 \\ w &= 0 \quad \text{on } \partial\Omega \cap Q_1. \end{cases}$$

Remember that f is independent of time and by assumption $f \in C^\theta(B_1)$ and hence $\partial_e f \in C_p^{\theta-1}(Q_1)$. Moreover Lemma 4.6.1 says that $v_n \geq c_1 d$ with $c_1 > 0$ in a neighbourhood of 0. Hence we can apply Theorem 4.1.3, which gives that all quotients v_i/v_n and v_t/v_n are $C_p^{1-\varepsilon}(\overline{\Omega} \cap Q_{r_2})$, with bounds on the norms.

Now notice that every component the normal vector $\nu(x,t)$ to the level set $\{v = t\}$, $t > 0$ can be expressed as

$$\nu^i(x,t) = \frac{\partial_i v}{|\nabla_{(x,t)} v|}(x,t) = \frac{\partial_i v/\partial_n v}{\left(\sum_{j=1}^{n-1}(\partial_j v/\partial_n v)^2 + 1 + (\partial_t v/\partial_n v)^2\right)^{1/2}}.$$

Letting $t \downarrow 0$, we get that the normal vector is $C_p^{1-\varepsilon}(\partial\Omega \cap Q_{r/2})$. Hence the free boundary is $C_p^{2-\varepsilon}$.

The same reasoning gives that if the boundary is C_p^{β} it is $C_p^{\beta+1}$, as long as $\beta \leq \theta$ and hence the claim is proven. □

4.7 Interior regularity results

In this section we state the interior regularity results used in the body of the paper. Even though the results are not new, we provide the proofs since the claims are adapted to our specific setting. For overview of the theory of parabolic second order equations, we refer to [56, 61].

Our techniques rely on contradiction and blow up arguments. The contradiction at the end is usually provided by Liouville type results. We start with establishing such kind of result, saying that if a function solves an equation in the full space $\mathbb{R}^{n+1}$ with a polynomial in the right-hand side, then the function is a polynomial as well. The proof is based on using interior estimates iteratively on arbitrarily big cylinders.

Proposition 4.7.1. *Assume that v solves*

$$\begin{cases} \partial_t v - \operatorname{tr}(AD^2 v) &= P \quad \text{in } \mathbb{R}^{n+1} \\ ||v||_{L^\infty(Q_R)} &\leq C_0 R^\gamma \quad \forall R > 1, \end{cases}$$

for some constant, uniformly elliptic matrix A, $\gamma > 0$, $\gamma \notin \mathbb{N}$, and some polynomial P of parabolic order less than $\gamma - 2$ (if $\gamma < 2$, then $P = 0$). Then v is a polynomial of parabolic order less than γ.

Proof. Take $R > 1$ and define

$$v_R(x,t) := R^{-\gamma} v(Rx, R^2 t),$$

so that $(\partial_t v_R - \operatorname{tr}(AD^2 v_R))(x,t) = R^{-\gamma+2} P(Rx, R^2 t) = P_R(x,t)$. Note that P_R is again a polynomial of the same order and the coefficients reduce as R increases. Moreover, by assumption $||v_R||_{L^\infty(Q_1)} \leq C_0$. Interior regularity result [70, Theorem 2] gives

$$[v_R]_{C_p^{0,1}(Q_{1/2})} \leq C(||P_R|| + ||v_R||_{L^\infty(Q_1)}) \leq C(||P|| + C_0),$$

and hence

$$[v]_{C_p^{0,1}(Q_{R/2})} \leq C R^{\gamma-1}(||P|| + C_0).$$

Choose arbitrary $h \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, and define

$$w_1(x,t) = \frac{v(x+h, t+\tau) - v(x,t)}{|h| + |\tau|}.$$

Then w_1 solves

$$\partial_t w_1 - \mathrm{tr}(AD^2 w_1) = P_1,$$

for some polynomial $P_1 \in \mathbf{P}_{\lfloor \gamma-1 \rfloor, p}$ and it satisfies the growth control

$$||w_1||_{L^\infty(Q_R)} \leq C\,[v]_{C^{0,1}(Q_R)} \leq CR^{\gamma-1}.$$

Repeating the same procedure we conclude that

$$\left[w_{\lceil \gamma \rceil}\right]_{C^{0,1}(Q_{R/2})} \leq CR^{\gamma - \lceil \gamma \rceil},$$

which gives that $w_{\lceil \gamma \rceil}$ is constant. This implies that v is a polynomial, and the assumed growth assures that it is of parabolic order less than γ. □

Since we define the parabolic Hölder seminorms a bit differently than for example in [56], we prove the a priori Schauder estimates in our setting as well.

Proposition 4.7.2. *Assume $u \in C_p^{2+\alpha}(Q_1)$ solves the equation*

$$\partial_t u + Lu = f, \qquad \text{in } Q_1,$$

with $f \in C_p^\alpha(Q_1)$, $A \in C_p^\alpha(Q_1)$ and $b \in C_p^\alpha(Q_1)$. Then we have

$$||u||_{C_p^{2+\alpha}(Q_{1/2})} \leq C([f]_{C_p^\alpha(Q_1)} + ||u||_{L^\infty(Q_1)}),$$

where C depends only on n, α, ellipticity constants and Hölder norms of the coefficients.

Proof. Thanks to [33, Lemma 2.23] and interpolation inequality, it suffices to prove that for every $\delta > 0$ there exists C so that

$$[u]_{C_p^{2+\alpha}(Q_{1/2})} \leq \delta\,[u]_{C_p^{2+\alpha}(Q_1)} + C([f]_{C_p^\alpha(Q_1)} + ||u||_{C_p^2(Q_1)}).$$

Dividing the equation with a constant, we can assume that $[f]_{C_p^\alpha(Q_1)} + ||u||_{C_p^2(Q_1)} = 1$.

We argue with contradiction. Assume that there is $\delta > 0$ so that for every $k \in \mathbb{N}$ there exist u_k, f_k and L_k (λ, Λ)-uniformly elliptic operators so that $\partial_t u_k + L_k u_k = f_k$ in Q_1, but

$$[u_k]_{C_p^{2+\alpha}(Q_{1/2})} > \delta\,[u_k]_{C_p^{2+\alpha}(Q_1)} + k.$$

Choose points (x_k, t_k), and (y_k, s_k), so that

$$\begin{aligned} c(n)\,[u_k]_{C_p^{2+\alpha}(Q_{1/2})} \leq & \frac{|D^2 u_k(x_k,t_k) - D^2 u_k(y_k,s_k)|}{|x_k - y_k|^\alpha + |t_k - s_k|^{\frac{\alpha}{2}}} + \frac{|\partial_t u_k(x_k,t_k) - \partial_t u_k(y_k,s_k)|}{|x_k - y_k|^\alpha + |t_k - s_k|^{\frac{\alpha}{2}}} \\ & + \frac{|Du_k(x_k,t_k) - Du_k(x_k,s_k)|}{|t_k - s_k|^{\frac{1+\alpha}{2}}}. \end{aligned}$$

Define $\rho_k = |x_k - y_k| + |t_k - s_k|^{\frac{1}{2}}$. Then the above inequality implies

$$\begin{aligned} c(n)\,[u_k]_{C_p^{2+\alpha}(Q_{1/2})} \leq & \frac{|D^2 u_k(x_k,t_k) - D^2 u_k(y_k,s_k)|}{\rho_k^\alpha} + \frac{|\partial_t u_k(x_k,t_k) - \partial_t u_k(y_k,s_k)|}{\rho_k^\alpha} \\ & + \frac{|Du_k(x_k,t_k) - Du_k(x_k,s_k)|}{\rho_k^{1+\alpha}}. \end{aligned}$$

Applying triangle inequality and bounding the terms with $||u_k||_{C^2_p(Q_1)}$, we get

$$c(n)\,[u_k]_{C^{2+\alpha}_p(Q_{1/2})} \leq ||u_k||_{C^2_p(Q_1)}(\rho_k^{-\alpha}+\rho_k^{-1+\alpha}),$$

but thanks to the contradiction assumption,

$$c(n)\,[u_k]_{C^{2+\alpha}_p(Q_{1/2})} \leq \frac{[u_k]_{C^{2+\alpha}_p(Q_{1/2})}}{k}(\rho_k^{-\alpha}+\rho_k^{-1+\alpha}),$$

which assures that $\rho_k \to 0$ as $k \to \infty$.

Now we define the blow-up sequence

$$v_k(x,t) = \frac{1}{\rho_k^{2+\alpha}\,[u_k]_{C^{2+\alpha}_p(Q_1)}}\left(u_k(x_k+\rho_k x, t_k+\rho_k^2 t) - p_k(x,t)\right),$$

where p_k is a polynomial of parabolic order 2, so that $v_k(0,0) = Dv_k(0,0) = D^2 v_k(0,0) = \partial_t v_k(0,0) = 0$. Now take any $R < \frac{1}{2\rho_k}$. Since p_k is of lower order, it holds

$$[v_k]_{C^{2+\alpha}_p(Q_R)} = \frac{1}{[u_k]_{C^{2+\alpha}_p(Q_1)}}\,[u_k]_{C^{2+\alpha}_p(Q_{R\rho_k}(x_k,t_k))} \leq 1,$$

and hence also $||v_k(x,t)||_{L^\infty(Q_R)} \leq R^{2+\alpha}$ and in particular $[v_k]_{C^{2+\alpha}_p(Q_1)} \leq 1$. Moreover, by definition of (x_k,t_k) and ρ_k we have

$$[v_k]_{C^{2+\alpha}_p(Q_1)} \geq \delta c(n).$$

Furthermore

$$\begin{aligned}(\partial_t+\tilde{L}_k)v_k(x,t) =& \frac{1}{\rho_k^{\alpha}\,[u_k]_{C^{2+\alpha}_p(Q_1)}}\Bigg(f_k(x_k+\rho_k x, t_k+\rho_k^2 t) - \\ & - \partial_t u_k(x_k,t_k) - \sum_{i,j} a^k_{i,j}(x_k+\rho_k x, t_k+\rho_k^2 t)\partial_{ij}u_k(x_k,t_k) + \\ & + \sum_i b^k_i(x_k+\rho_k x, t_k+\rho_k^2 t)\partial_i u_k(x_k,t_k) + \\ & + \rho_k \sum_i b^k_i(x_k+\rho_k x, t_k+\rho_k^2 t)\sum_j \partial_{ij}u_k(x_k,t_k)x_j\Bigg).\end{aligned}$$

Taking into account the regularity of the coefficients, with adding and subtracting suitable terms, we get

$$\begin{aligned}(\partial_t+\tilde{L}_k)v_k(x,t) =& \frac{1}{\rho_k^{\alpha}\,[u_k]_{C^{2+\alpha}_p(Q_1)}}\Bigg(f_k(x_k+\rho_k x, t_k+\rho_k^2 t) - f_k(x_k,t_k) + \\ & + \sum_{i,j} o_{A,k}(x,t)\partial_{ij}u_k(x_k,t_k) + \sum_i o_{b,k}(x,t)\partial_i u_k(x_k,t_k) + \\ & + \rho_k \sum_i b^k_i(x_k+\rho_k x, t_k+\rho_k^2 t).\Bigg)\end{aligned}$$

and hence by contradiction assumption

$$|(\partial_t + \tilde{L}_k)v_k(x,t)| \leq \frac{C}{k}\left([f_k]_{C_p^\alpha} + ||A_k||_{C_p^\alpha} + ||b_k||_{C_p^\alpha} + \rho_k^{1-\alpha}\Lambda\right),$$

on any compact set inside $Q_{\frac{1}{2\rho_k}}$.

Passing to a subsequence, Arzela-Ascoli theorem provides that v_k converge in $C_p^2(\mathbb{R}^{n+1})$ on compact sets to some function v, which by [8, Theorem 1.1] solves

$$\begin{cases} (\partial_t - \mathrm{tr}(AD^2))v &= 0 \quad \text{in } \mathbb{R}^{n+1} \\ ||v||_{L^\infty(Q_R)} &\leq R^{2+\alpha} \quad \text{for all } R > 0 \\ v(0) = Dv(0) = D^2v(0) = \partial_t v(0) &= 0 \\ [v]_{C_p^{2+\alpha}(Q_1)} &\geq \delta c(n), \end{cases}$$

for some uniformly elliptic, constant matrix A. Hence by Liouville theorem (Proposition 4.7.1) v is a polynomial of parabolic order 2, which is a contradiction with last two properties of v. □

Next we establish the higher order a priori estimates.

Corollary 4.7.3. *Let $k \in \mathbb{N}$, $k \geq 2$, and $\alpha \in (0,1)$. Assume $u \in C_p^{k,\alpha}(Q_1)$ solves*

$$(\partial_t + L)u = f \qquad \text{in } Q_1,$$

for some $f \in C_p^{k-2,\alpha}(Q_1)$, $A, b \in C_p^{k-2,\alpha}(Q_1)$. Then

$$||u||_{C_p^{k+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}),$$

where C depends only on n, α, ellipticity constants and Hölder norms of the coefficients.

Proof. We prove the claim by induction. Proposition 4.7.2 proves the case $k = 2$. So let $k > 2$ and assume the claim is true for all $l < k$. Choose now $e \in \mathbb{S}^{n-1}$. Then by induction hypothesis

$$[\partial_e u]_{C_p^{k-1+\alpha}(Q_{1/2})} \leq C([\partial_e f]_{C_p^{k-3,\alpha}(Q_1)} + ||\partial_e u||_{L^\infty(Q_1)}),$$

which after interpolation inequality implies

$$[Du]_{C_p^{k-1+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}).$$

Hence also

$$\left[D^2u\right]_{C_p^{k-2+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}),$$

and so using that $\partial_t u = f - Lu$ we also get

$$[\partial_t u]_{C_p^{k-2+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}).$$

Combined, the inequalities render

$$[u]_{C_p^{k+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}),$$

and the claim is proven. □

We conclude the section by showing that solutions to equations with regular right-hand side are indeed as regular as the a priori estimates indicate.

Proposition 4.7.4. *Let $k \in \mathbb{N}$, $k \geq 2$, and $\alpha \in (0,1)$. Let $A, b, f \in C_p^{k-2,\alpha}(Q_1)$, and u a solution to*

$$(\partial_t + L)u = f, \quad \text{in } Q_1.$$

Then $u \in C_p^{k,\alpha}(Q_1)$, with

$$||u||_{C_p^{k+\alpha}(Q_{1/2})} \leq C([f]_{C_p^{k-2,\alpha}(Q_1)} + ||u||_{L^\infty(Q_1)}),$$

where C depends only on n, α, k, ellipticity constants and Hölder norms of the coefficients.

Proof. We only need to prove the regularity of u, since we have already established the estimates.

If $k = 2$, then the regularity is justified by [61, Theorem 5.9]. For $k > 2$, with the same argument as in [56, Theorem 8.12.1] we establish that $\nabla u \in C_p^{k-1,\alpha}(Q_1)$. Using the equation, this also gives that $\partial_t u \in C_p^{k-2,\alpha}(Q_1)$. □

4.8 Appendix: Technical tools and lemmas

Lemma 4.8.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain, such that Ω_0 is convex and $\{t; (t,x) \in \Omega\}$ is connected for every x. Let $\beta \notin \mathbb{N}$ and assume $u \in C_p^\beta(\overline{\Omega})$, with $D_p^k u(0,0) = 0$, for every $k \leq \lfloor \beta \rfloor$. Then for every $k \leq \beta$ and $r > 0$ we have*

$$||D_p^k u||_{L^\infty(\Omega \cap Q_r)} \leq C r^{\beta - k}.$$

Proof. We prove the claim with induction on β and k. If $k = \lfloor \beta \rfloor$, then it follows from the definition of C_p^β.

If $k = \lfloor \beta \rfloor - 1$, we estimate

$$|D_p^k u(x,t)| \leq |D_p^k u(x,t) - D_p^k u(x,0)| + |D^k u(x,0)| \leq C|t|^{\frac{1+\langle\beta\rangle}{2}} + \int_0^{|x|} ||D_p^{k+1} u||_{L^\infty(Q_s)} ds,$$

and hence by induction hypothesis

$$|D_p^k u(x,t)| \leq C r^{1+\langle\beta\rangle}.$$

If $k < \lfloor \beta \rfloor - 1$, we use $\partial_t D_p^k u \subset D_p^{k+2} u$ and estimate

$$|D_p^k u(x,t) - D_p^k u(x,0)| \leq \int_0^t ||D_p^{k+2} u||_{L^\infty(Q_s)} ds,$$

to obtain the same result. □

Lemma 4.8.2. *Let $\beta > 0$ and let $\Omega \subset \mathbb{R}^{n+1}$ be $C_p^{\max(\beta,1)}$ in Q_1 in the sense of Definition 4.1.1. Assume that a function $u\colon \Omega \cap Q_1 \to \mathbb{R}$ satisfies*

$$[u]_{C_p^\beta(Q_r(x_0,t_0))} \leq C_0,$$

whenever $Q_{2r}(x_0,t_0)\subset\Omega\cap Q_1$ *and* $d_{t_0}(x_0)\le C_1 r$*, with* C_1 *depending only on* Ω*.*

Then

$$[u]_{C_p^\beta(\Omega\cap Q_{1/2})}\le CC_0.$$

The constant C *depends only on* n,β *and* Ω*.*

Proof. Thanks to parabolic scaling and covering argument, we can assume $C_1=3$, and that $[u]_{C_p^\beta(Q_{\frac{3r}{2}}(x_0,t_0))}\le C_0$, or in other words, we have

$$[u]_{C_p^\beta(Q_r(x_0,t_0))}\le C_0,$$

whenever $d_{t_0}(x_0)=2r$.

Let us first prove, that $[u]_{C_p^\beta(\mathcal{C}_{(z,t_0)})}\le CC_0$, where

$$\mathcal{C}_{(z,t_0)}=\cup_{d_{t_0}(x_0)=2r=|x_0-z|}Q_r(x_0,t_0),$$

is a "parabolic cone" starting at a boundary point $(z,t_0)\in\partial\Omega$. First, notice that

$$\left[D_p^{\lfloor\beta\rfloor-1}u\right]_{C_t^{\frac{1+\langle\beta\rangle}{2}}(\mathcal{C}_{(z,t_0)})}\le C_0,$$

since the points we choose for comparing the function values have to lie in the same cylinder $Q_r(x_0,t_0)$, because the space coordinate coincide.

Now choose arbitrary (x,t) and (y,s) in $\mathcal{C}_{(z,t_0)}$. Without loss of generality assume $d(x,t)\le d(y,s)$. Denote x' and y' the central points of cylinders from $\mathcal{C}_{(z,t_0)}$, containing (x,t) and (y,s) respectively. We want to estimate

$$|D_p^{\lfloor\beta\rfloor}u(x,t)-D_p^{\lfloor\beta\rfloor}u(y,s)|\le CC_0(|x-y|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}}).$$

If $2d_{t_0}(x)>d_{t_0}(y)$, both points are contained in $Q_{\frac{3}{4}d_{t_0}(y)}(y,t_0)$, and hence another covering argument assures the above estimate. Otherwise we have $d_{t_0}(y)\le 2|x-y|$ as well as $|x-x'|\le\frac{d_{t_0}(x)}{2}$ and $|y-y'|\le\frac{d_{t_0}(y)}{2}$. For $i=0,1,\dots$ denote $x_i=x'+2^{-i}(y'-x')$ and $r_i=d_{t_0}(x_i)$, so that $(x_{i+1},t),(x_i,t)\in Q_{\frac{r_i}{2}}(x_i,t_0)$. Hence we have

$$\begin{aligned}|D_p^{\lfloor\beta\rfloor}u(x,t)-D_p^{\lfloor\beta\rfloor}u(y,s)|\le&|D_p^{\lfloor\beta\rfloor}u(y,s)-D_p^{\lfloor\beta\rfloor}u(y',t)|+\\&+\sum_{i=0}^{K}|D_p^{\lfloor\beta\rfloor}u(x_i,t)-D_p^{\lfloor\beta\rfloor}u(x_{i+1},t)|+\\&+|D_p^{\lfloor\beta\rfloor}u(x,t)-D_p^{\lfloor\beta\rfloor}u(x',t)|\\\le&C_0(|y-y'|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}})+\sum_{i=0}^{K}C_0|x_i-x_{i+1}|^{\langle\beta\rangle}\\&+C_0|x-x'|^{\langle\beta\rangle}\\\le&C_0\left(2|x-y|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}}+|x-y|^{\langle\beta\rangle}\sum_{i=0}^{\infty}2^{-i\langle\beta\rangle}\right)\\=&C_\beta C_0\left(|x-y|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}}\right).\end{aligned}$$

Let us now turn to the general case. Pick $(x,t),(y,s)\in\Omega\cap Q_{1/2}$. Denote $z_x\in\partial\Omega_t$, $z_y\in\partial\Omega_s$ so that $d_t(x)=|x-z_x|$ and $d_s(y)=|y-z_y|$. Now choose $x_1\in\Omega_t$, a central point of the cylinder from $\mathcal{C}_{(z_x,t)}$, so that $|x-x_1|\leq 2|t-s|^{\frac{1}{2}}$, and that $(x_1,s)\in\mathcal{C}_{(z_x,t)}$. If the boundary is flat enough, this is possible, meaning that we have to restrict ourselves to a small neighbourhood of the boundary before if necessary. Now take $z_{xy}\in\partial\Omega_s$, so that $d_s(x_1)=|x_1-z_{x,y}|$, and choose a point $y_1\in\Omega_s$, so that $(y_1,s)\in\mathcal{C}_{(z_{xy},s)}\cap\mathcal{C}_{(z_y,s)}$, but $|x_1-y_1|\leq 2\left(|x-y|+|t-s|^{\frac{1}{2}}\right)$. Note that $(x,t)\in\mathcal{C}_{(z_x,t)}$, as well as $(y,s)\in\mathcal{C}_{(z_y,s)}$, and hence

$$\begin{aligned}|D_p^{\lfloor\beta\rfloor}u(x,t)-D_p^{\lfloor\beta\rfloor}u(y,s)|\leq&|D_p^{\lfloor\beta\rfloor}u(x,t)-D_p^{\lfloor\beta\rfloor}u(x_1,s)|+|D_p^{\lfloor\beta\rfloor}u(x_1,s)-D_p^{\lfloor\beta\rfloor}u(y_1,s)|\\&+|D_p^{\lfloor\beta\rfloor}u(y_1,s)-D_p^{\lfloor\beta\rfloor}u(y,s)|\\\leq&C_\beta C_0\left(|x-x_1|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}}\right)+C_\beta C_0|x_1-y_1|^{\langle\beta\rangle}\\&+C_\beta C_0|y_1-y|^{\langle\beta\rangle}\\\leq&CC_0\left(|x-y|^{\langle\beta\rangle}+|t-s|^{\frac{\langle\beta\rangle}{2}}\right).\end{aligned}$$

Finally, choose two points $(x,t),(x,s)\in\Omega\cap Q_{1/2}$. Let $d_t(x)=|x-z_t|$ and $d_s(x)=|x-z_s|$. Pick x_1 and x_2 in $\Omega_t\cap\Omega_s$, so that x,x_1,x_2 lie on the same line, $|x-x_1=|x_1-x_2|$, $|x-x_1|\leq 2|t-s|^{\frac{1}{2}}$, and that $(x,t),(x_1,t),(x_2,t),(x_1,s),(x_2,s)\in\mathcal{C}_{(z_t,t)}$. If Ω is flat enough, this is possible, otherwise we restrict ourselves to the smaller neighbourhood of the boundary. Therefore we have

$$\begin{aligned}|D_p^{\lfloor\beta\rfloor-1}u(x,t)-D_p^{\lfloor\beta\rfloor-1}u(x,s)|\leq&|D_p^{\lfloor\beta\rfloor-1}u(x,t)-2D_p^{\lfloor\beta\rfloor-1}u(x_1,t)+D_p^{\lfloor\beta\rfloor-1}u(x_2,t)|\\&+|D_p^{\lfloor\beta\rfloor-1}u(x,s)-2D_p^{\lfloor\beta\rfloor-1}u(x_1,s)+D_p^{\lfloor\beta\rfloor-1}u(x_2,s)|\\&+2|D_p^{\lfloor\beta\rfloor-1}u(x_1,t)-D_p^{\lfloor\beta\rfloor-1}u(x_1,s)|\\&+|D_p^{\lfloor\beta\rfloor-1}u(x_2,t)-D_p^{\lfloor\beta\rfloor-1}u(x_2,s)|\\\leq&2CC_0|x-x_1|^{1+\langle\beta\rangle}+3CC_0|t-s|^{\frac{1+\langle\beta\rangle}{2}}\\\leq&CC_0|t-s|^{\frac{1+\langle\beta\rangle}{2}},\end{aligned}$$

and so the claim is proven. □

Remark 4.8.3. We need the boundary to be at least $C_p^{0,1}$, so that the cones $\mathcal{C}_{(z,s)}$ come up to the boundary.

Lemma 4.8.4. *Let $\beta>0$ $\beta\notin\mathbb{N}$, $\Omega\subset\mathbb{R}^{n+1}$, with $0\in\partial\Omega$, and let $u\in C(\overline{Q}_1)$. Assume that for every $r\in(0,1)$ we have a polynomial $p_r\in\boldsymbol{P}_{\lfloor\beta\rfloor,p}$ so that*

$$||u-p_rd||_{L^\infty(Q_r)}\leq C_0r^{\beta+1}.$$

Then there exists a polynomial $p_0\in\boldsymbol{P}_{\lfloor\beta\rfloor,p}$ which satisfies

$$||u-p_0d||_{L^\infty(Q_r)}\leq CC_0r^{\beta+1},\quad r\in(0,1),$$

where C depends only on n, and β.

Proof. The proof is the same as the one of [1, Lemma 4.3], just that every B_r is replaced with Q_r, and polynomials are of bounded parabolic degree. $\square$

Lemma 4.8.5. *Let $\Omega \subset \mathbb{R}^{n+1}$, and assume that for every point $z \in \partial\Omega \cap Q_1$ there exists a polynomial $p_z \in \boldsymbol{P}_K$, so that*

$$||u(x) - p_z(x)v(x)||_{L^\infty(B_1(z)\cap\Omega)} \leq C_0,$$

for some functions $u, v \in L^\infty(\Omega)$, with C_0 independent of z. Assume also that $v \geq d$.

Then we have $||p_z|| \leq CC_0$, for some C independent of z.

Proof. Find a non-empty, open set $B \subset \cap_{z\in\partial\Omega\cap B_1} B_1(z)$, that is away from the boundary $B \subset \{x;\ d(x) > r\}$, for some $r > 0$. Then we can estimate

$$\begin{aligned} ||p_z||_{L^\infty(B)} &\leq \frac{1}{r}||p_z v||_{L^\infty(B_1(z))} \leq \frac{1}{r}\left(||u - p_z v||_{L^\infty(B_1(z))} + ||u||_{L^\infty(B_1(z))}\right) \\ &\leq C(C_0 + ||u||_{L^\infty(\Omega)}). \end{aligned}$$

The claim follows from [1, Lemma A.10]. $\square$

Lemma 4.8.6. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded domain and let v be a solution to*

$$\begin{cases} (\partial_t + L)v &= P \quad \text{in } D \\ v &= 0 \quad \text{in } \partial_p D, \end{cases}$$

For some polynomial P. Then

$$||P||_{L^\infty(D)} \leq C||v||_{L^\infty(D)},$$

where C depends only on n, D, ellipticity constants and the degree of P.

Proof. If $P = 0$ we have nothing to prove. Otherwise, dividing the equation with a constant we can assume that $||P||_{L^\infty(D)} = 1$. Assume by contradiction, that there are sequences of (λ, Λ)-uniformly elliptic operators L_k, polynomials P_k of fixed degree, with $||P_k||_{L^\infty(D)} = 1$, and solutions v_k of

$$\begin{cases} (\partial_t + L_k)v_k &= P_k \quad \text{in } D \\ v_k &= 0 \quad \text{in } \partial_p D, \end{cases}$$

and that

$$\frac{1}{k} > ||v_k||_{L^\infty(D)}.$$

Since P_k are bounded in a finite dimensional vector space, we can get a subsequence converging uniformly in D to some polynomial P_0. Hence by convergence result [8, Theorem 1.1] the limit function v_0 should satisfy

$$\begin{cases} (\partial_t + L_0)v_0 &= P_0 \quad \text{in } D \\ v_0 &= 0 \quad \text{in } \partial_p D. \end{cases}$$

But $v_0 \equiv 0$, while $||P_0||_{L^\infty(D)} = 1$, which gives a contradiction. $\square$

Lemma 4.8.7. *Let $\beta > 0$ $\beta \notin \mathbb{N}$, $\Omega \subset \mathbb{R}^{n+1}$, with $0 \in \partial\Omega$, and let $u_1, u_2 \in C(\overline{Q}_1)$. Assume that for every $r \in (0,1)$ we have a polynomial $p_r \in \boldsymbol{P}_{\lfloor \beta \rfloor, p}$ so that*

$$||u_1 - p_r^{(0)} u_2 - p_r^{(1)} d||_{L^\infty(Q_r)} \leq C_0 r^{\beta+1}.$$

Furthermore assume $c_0 d \leq u_2 \leq C_0 d$, for some $c_0 > 0$. Then there exists a polynomial $p_0 \in \boldsymbol{P}_{\lfloor \beta \rfloor, p}$ which satisfies

$$||u_1 - p_0^{(0)} u_2 - p_0^{(1)} d||_{L^\infty(Q_r)} \leq C C_0 r^{\beta+1}, \quad r \in (0,1),$$

where C depends only on n, c_0 and β.

Proof. The proof is the same as the one of [1, Lemma 4.5], with Q_r instead of B_r. □

Chapter 5

Regularity theory for fully non-linear parabolic obstacle problems

5.1 Introduction

The parabolic obstacle problem is the following

$$\begin{cases} \partial_t u - \Delta u = -\chi_{\{u>0\}} & \text{in } Q_1 \subset \mathbb{R}^{n+1} \\ u \geq 0,\ \partial_t u \geq 0 & \text{in } Q_1, \end{cases}$$

where $\varphi\colon B_1 \to \mathbb{R}$ is a given function called the obstacle. This problem arises for example in the study of the phase transition in Stefan problem, see [36], or in the optimal stopping problem, see [31]. It is a free boundary problem, as we are not interested only in the study of the solution, but also in the study of the so called *free boundary* $\partial\{u > 0\}$.

The regularity of solutions to the obstacle problem follows from the theory of linear PDE – solutions are $C^{1,1}$ in space and C^1 in time and not more in general. More challenging problem is to determine the regularity of the free boundary, namely to answer the following question:

Is the free boundary C^∞, if the obstacle φ is C^∞?

The regularity theory for the free boundary was developed by Caffarelli in his groundbreaking paper [20]. There he shows that the free boundary can be split into *regular points* and *singular points*, that regular points form an open subset of the free boundary and that near regular points the free boundary is indeed C^∞. The set of singular points was later studied in [11, 64, 39]. The main result for the singular points states that they can locally be covered by a $C^1_x \cap C^{1/2}_t$ manifold of dimension $n-1$.

The goal of this paper is to study the fully non-linear version of the parabolic obstacle problem:

$$\begin{cases} \partial_t u - F(D^2 u, x) = f(x)\chi_{\{u>0\}} & \text{in } Q_1 \\ u \geq 0,\ \partial_t u \geq 0 & \text{in } Q_1, \end{cases} \tag{5.1.1}$$

where $F : \mathcal{S} \times B_1 \to \mathbb{R}$ and $f : B_1 \to \mathbb{R}$ are given functions and $\mathcal{S}$ denotes the linear space of $n \times n$ symmetric matrices. We assume that F satisfies the following conditions

$$\begin{cases} F \in C^\infty \\ F(\cdot, x) \text{ is convex for all } x \in B_1, \\ F \text{ is uniformly elliptic,} \\ F(O, \cdot) = 0 \quad \text{in } B_1. \end{cases} \tag{5.1.2}$$

We say that F is *uniformly elliptic* if there exist $0 < \lambda \leq \Lambda$ such that

$$\lambda \|N\| \leq F(M + N, x) - F(M, x) \leq \Lambda \|N\|,$$

for all $x \in B_1$, all $M \in \mathcal{S}$, and all $N \in \mathcal{S}$ satisfying $N \geq 0$; see [18, 34] for more information about fully non-linear uniformly elliptic operators. We also assume that

$$f \text{ is continuous and } f \leq -c_\circ, \tag{5.1.3}$$

for some $c_\circ > 0$.

In the stationary version of the problem, $F(D^2u, x) = \chi_{\{u>0\}}$, the optimal regularity of solutions and $C^{1,\alpha}$ regularity of the free boundary was proved by Lee in his PhD thesis [59]. Moreover the optimal regularity was studied in a more general setting by by Figalli and Shahgholian in [40], where they additionally prove that if the free boundary is Lipschitz, then it is C^1. Furthermore in [41] they extend the results to the parabolic setting, see also [49] by Indrei and Minne and [68] by Petrosyan and Shahgholian. The higher regularity of the free boundary in both elliptic and parabolic setting is provided by Kinderlehrer and Nirenberg [52]. Note that there was still a small gap between the initial regularity and the higher regularity results, since in [52] the solution is assumed to be C^2 in the positivity set up to the boundary, while in the above papers the solutions are only proved to be $C^{1,1}$.

The singular set has been studied in the elliptic case in [12] where Bonorino establishes that the singular set can be locally covered with a Lipschitz $(n-1)$-dimensional manifold, and in [77] where Savin and Yu improve the regularity of the covering manifold to $C^{1,\log^\varepsilon}$, see also [78]. For the parabolic case no results were known for singular points.

In this paper we study free boundaries for the fully non-linear parabolic obstacle problem described above. Our first main result finds the splitting of the free boundary into regular and singular points, and shows that near regular points the free boundary is indeed C^∞.

Theorem 5.1.1. *Let u be a solution of* (5.1.1) *with F, f satisfying* (5.1.2)-(5.1.3). *For every free boundary point $(x_0, t_0) \in \partial\{u > 0\}$ it holds*

(i) either

$$\lim_{r \downarrow 0} \frac{1}{r^2} u(x_0 + rx, t_0 + r^2 t) = c_0 (x \cdot e_0)_+^2,$$

for some $c_0 > 0$ and $e_0 \in \mathbb{S}^{n-1}$,

(ii) or

$$\lim_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0) \times \{t_0\}|}{|B_r(x_0)|} = 0,$$

and every blow-up at (x_0, t_0) *is a quadratic polynomial of the form* $x^T Ax$*, for some matrix* $A \geq 0$.[1]

The points where (i) *holds are called regular, they form an open subset of* $\partial\{u > 0\}$ *and the free boundary is* C^∞ *near them. Points where* (ii) *holds are called singular.*

We prove the theorem in two steps. First, we adapt the blow-up technique by Caffarelli to find the splitting into regular and singular points, and we deduce that near regular points non-tangential directional derivatives pointing towards interior are non-negative. From this, we are able to deduce that the free boundary has to be Lipschitz in time and C^1 in space. At this point, by proving that the solution is $C^2_x \cap C^1_t$ near regular points up to the free boundary, we are able to apply either higher order boundary Harnack estimates from [57] to deduce that the free boundary is actually C^∞, or the result from Kinderlehrer and Nirenberg [52].

We furthermore study the singular set, and establish that it can be covered with a Lipschitz manifold of dimension $n-1$, which is ε-flat in space, for any $\varepsilon > 0$. Note that the dimension of the ambient space is $n+1$.

Theorem 5.1.2. *Let* u *be a solution of* (5.1.1) *with* F *independent of* x *and satisfying* (5.1.2)-(5.1.3). *Assume that* $\partial_t u > 0$ *in* $\{u > 0\}$ *and let* $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ *be the set of all singular points. Then for any* $\varepsilon > 0$, Σ *can locally be covered by a Lipschitz manifold of dimension* $n-1$*, which is* ε*-flat in space.*[2]

This result is also proved in two steps. The first one is to prove that the free boundary can be written as a graph where time variable is expressed as a Lipschitz function depending on the space variables. The second one is the observation that near the free boundary the pure second derivatives in directions where a blow-up is strictly positive need to be non-negative. Near singular free boundary points this yields that the free boundary can be touched from at least two sides by C^1 "parabolas" which gives the result. Consequently, with results from [39] we deduce that for almost every time the singular set at that fixed time has to be of Hausdorff dimension $n-2$.

Corollary 5.1.3. *Let* u *be a solution of* (5.1.1) *with* F, f *satisfying* (5.1.2)-(5.1.3). *Assume that* $\partial_t u > 0$ *in* $\{u > 0\}$. *Then for almost every time* $t \in (-1, 1)$ *the singular set* Σ_t *at time* t *satisfies*

$$\dim_{\mathcal{H}}(\Sigma_t) \leq n-2,$$

where $\dim_{\mathcal{H}}(E)$ *denotes the Hausdorff dimension of a set* $E \subset \mathbb{R}^n$.

Theorem 5.1.2 is sharp in the sense of the dimension of the covering manifolds, as examples can be constructed where the singular set is indeed of dimension $n-1$. It remains an open problem to improve the regularity of the covering manifold and the generic regularity of the free boundary.

[1] See (5.6.2) for the definition of a blow-up.

[2] That means that the manifold can be expressed as a graph of a Lipschitz function whose Lipschitz semi-norm in space is smaller than ε.

Remark 5.1.4. The classical "parabolic obstacle problem" can be formulated in two equivalent ways:

$$\begin{cases} \partial_t v - \Delta v \geq 0 \\ \partial_t v - \Delta v = 0 \quad \text{in } \{v > \varphi\} \\ v \geq \varphi,\ \partial_t v \geq 0 \end{cases} \qquad \begin{cases} \partial_t w - \Delta w = g(x)\chi_{\{w>0\}} \\ w \geq 0,\ \partial_t w \geq 0, \end{cases}$$

where $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a smooth given obstacle and $g = \Delta\varphi$. The second is obtained by the first by setting $w := v - \varphi$, and viceversa.

The fully nonlinear analogues of the above problems are

$$\begin{cases} \partial_t v - G(D^2 v) \geq 0 \\ \partial_t v - G(D^2 v) = 0 \quad \text{in } \{v > \varphi\} \\ v \geq \varphi,\ \partial_t v \geq 0 \end{cases} \qquad \begin{cases} \partial_t w - G(D^2 w) = g(x)\chi_{\{w>0\}} \\ w \geq 0,\ \partial_t w \geq 0, \end{cases}$$

where $G : \mathcal{S} \to \mathbb{R}$ is assumed to satisfy (5.1.2). Since the diffusion is nonlinear, the two problems are *not* equivalent: the function $u := v - \varphi$ is not a solution to the second problem, but satisfies (5.1.1) with

$$F(M, x) := G(M + D^2\varphi(x)) - G(D^2\varphi(x)), \qquad f := G(D^2\varphi).$$

Notice that if G satisfies the assumptions in (5.1.2), then F satisfies (5.1.2), too (in particular, it is uniformly elliptic with the same ellipticity constants).

5.1.1 Structure of the paper

In Section 5.2 we present the notation used throughout the paper, as well as the definition of the parabolic Hölder spaces. In Section 5.3 we state preliminary results for non-linear equations that we need. Further on in Section 5.4 we prove the almost optimal $C_x^{1,1} \cap C_t^{0,1}$ regularity of solutions. In Section 5.5 we establish semi-convexity estimates and the continuity of the time derivative. Further on, the blow-ups are analysed in Section 5.6. Section 5.7 is devoted to the study of the free boundary near regular points and Section 5.8 to the study of singular points. At the end there is an appendix, where we prove some of the results from Section 5.3.

5.2 Notation and definitions

Below we present the notation that is used throughout the paper.

$B_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$
$\tilde{B}_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0|_\infty < r\}, \qquad |x|_\infty := \max\{|x_i| : i = 1, \dots, n\}$
$Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^2, t_0 + r^2)$
$Q_r^+(x_0, t_0) := B_r(x_0) \times (t_0, t_0 + r^2)$
$Q_r^-(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$
$D_r^+(x_0, t_0) := B_r(x_0) \times (t_0 + 3r^2, t_0 + 4r^2)$
$D_r^-(x_0, t_0) := B_r(x_0) \times (t_0 - 3r^2, t_0 - 2r^2)$
$\tilde{Q}_r(x_0, t_0) := \tilde{B}_r(x_0) \times (t_0 - r^2, t_0 + r^2)$
$\tilde{D}_r^+(x_0, t_0) := \tilde{B}_r(x_0) \times (t_0 + 3r^2, t_0 + 4r^2)$
$\tilde{D}_r^-(x_0, t_0) := \tilde{B}_r(x_0) \times (t_0 - 3r^2, t_0 - 2r^2)$
$d((x,t),(y,\tau)) := \sqrt{|x-y|^2 + |t-\tau|} \quad \text{or} \quad d((x,t),(y,\tau)) := \max\{|x-y|, |t-\tau|^{1/2}\}$
$d((x,t),A) := \inf_{(y,\tau)\in A} d((x,t),(y,\tau)), \quad A \subset \mathbb{R}^{n+1}$

$N_\delta(A) = \{(x,t) \in \mathbb{R}^{n+1} : \mathrm{dist}((x,t),A) < \delta\}), \quad A \subset \mathbb{R}^{n+1}$
$a_{ij}\partial_{ij}v := \sum_{i,j=1}^{n} a_{ij}\partial_{ij}v, \quad \{a_{ij}\}_{i,j=1}^n \ n \times n\text{-matrix}$

We define the following class of solutions, that simplifies the notation throughout the paper.

Definition 5.2.1. (Class of solutions) Let $r, K > 0$ and $(x_0, t_0) \in \mathbb{R}^{n+1}$. We say that $u \in \mathcal{P}_r(x_0, t_0; K)$ if

- $|u| + |f| \le K$ in $Q_r(x_0, t_0)$;
- u is a solution to (5.1.1) in $Q_r(x_0, t_0)$ with F, f satisfying (5.1.2)-(5.1.3).

When $(x_0, t_0) = (0, 0)$ we write $\mathcal{P}_r(K)$.

5.2.1 Parabolic Hölder spaces

We follow the definition of the parabolic Hölder spaces from [57]. For convenience we state the definition below.

Definition 5.2.2. Let Ω be an open subset of $\mathbb{R}^{n+1}$. For $\alpha \in (0, 1]$ we define the parabolic Hölder seminorm of order α as follows

$$[u]_{C_p^\alpha(\Omega)} = \sup_{(x,t),(y,s)\in\Omega} \frac{|u(x,t) - u(y,s)|}{|x-y|^\alpha + |t-s|^{\frac{\alpha}{2}}},$$

and

$$[u]_{C_t^\alpha(\Omega)} = \sup_{(x,t),(x,s)\in\Omega} \frac{|u(x,t) - u(x,s)|}{|t-s|^\alpha}.$$

If $\alpha \in (1,2]$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [u]_{C_t^{\frac{\alpha}{2}}(\Omega)}.$$

For bigger numbers $\alpha > 2$, we set

$$[u]_{C_p^\alpha(\Omega)} = [\nabla u]_{C_p^{\alpha-1}(\Omega)} + [\partial_t u]_{C_p^{\alpha-2}(\Omega)}.$$

When $\alpha \notin \mathbb{N}$ we say that $u \in C_p^\alpha(\Omega)$, when $[u]_{C_p^\alpha(\Omega)} < \infty$, and define

$$||u||_{C_p^\alpha(\Omega)} = \sum_{k\leq\lfloor\alpha\rfloor} ||D_p^k u||_{L^\infty(\Omega)} + [u]_{C_p^\alpha(\Omega)}.$$

If $\alpha \in \mathbb{N}$, we say that $u \in C_p^\alpha(\Omega)$, if there exists a modulus of continuity $\omega\colon [0,\infty) \to [0,\infty)$ – a continuous, increasing function with $\omega(0) = 0$ – so that for all $(x,t),(y,s) \in \Omega$

$$|D_p^\alpha u(x,t) - D_p^\alpha u(y,s)| \leq \omega(|x-y| + |t-s|^{\frac{1}{2}}),$$

and

$$|D_p^{\alpha-1} u(x,t) - D_p^{\alpha-1} u(x,s)| \leq |t-s|^{\frac{1}{2}}\omega(|t-s|^{\frac{1}{2}}).$$

We set

$$||u||_{C_p^\alpha(\Omega)} = \sum_{k\leq\alpha} ||D_p^k u||_{L^\infty(\Omega)}.$$

5.3 Preliminaries

In this section we present the tools for non-linear parabolic equations used throughout the paper.

We begin with the Harnack inequality for solutions of the fully non-linear parabolic equations, which says that values of non-negative solutions are comparable up to the error caused by the right-hand side. We denote M^+, M^- the Pucci extremal operators for elliptic constants λ and Λ. For the definition of Pucci extremal operators we refer to [34].

Theorem 5.3.1 (Harnack inequality [50, Theorem 4.32])**.** *Let $r > 0$, $(x_0,t_0) \in \mathbb{R}^{n+1}$ and let $u \geq 0$ satisfy*

$$\begin{cases} \partial_t u - M^+(D^2 u) & \leq & C_0 & \text{in } Q_r(x_0,t_0) \\ \partial_t u - M^-(D^2 u) & \geq & -C_0 & \text{in } Q_r(x_0,t_0). \end{cases} \tag{5.3.1}$$

Then there exists $C > 0$ depending only on n, λ and Λ such that

$$\sup_{D^-_{r/2}(x_0,t_0)} u \leq C\Big(\inf_{D^+_{r/2}(x_0,t_0)} u + r^2 C_0\Big).[3]$$

When considering non-negative supersolutions instead of solutions, we can control the infimum by the L^p average.

[3] The statement is slightly different as in [50], but the proof is exactly the same.

Theorem 5.3.2 (Weak Harnack inequality [50, Theorem 4.15])**.** *Let $r > 0$, $(x_0, t_0) \in \mathbb{R}^{n+1}$ and let $u \geq 0$ satisfy*

$$\partial_t u - M^-(D^2 u) \geq -C_0 \qquad \text{in } Q_r(x_0, t_0).$$

Then there exist $C > 0$ and $p \in (0, 1)$ depending only on N, λ and Λ such that

$$\left(\fint_{D^-_{r/2}(x_0,t_0)} u^p \right)^{\frac{1}{p}} \leq C \left(\inf_{D^+_{r/2}(x_0,t_0)} u + r^2 C_0 \right). \tag{5.3.2}$$

Sometimes this result is also called the half-Harnack inequality, as for subsolutions it holds that the supremum is controlled by the L^p average (see [50, Proposition 4.34]). We also need the following version of the weak Harnack inequality, that quantifies the growth of the constant, as the set from the L^p average approaches the boundary.

Lemma 5.3.3. *Let $\delta \in (0, \frac{1}{4})$, $r > 0$, $(x_0, t_0) \in \mathbb{R}^{n+1}$ and let $u \geq 0$ satisfy*

$$\partial_t u - M^-(D^2 u) \geq -C_0 \qquad \text{in } Q_r(x_0, t_0).$$

Then there exist $C, m > 0$, $p \in (0, 1)$ depending only on n, λ and Λ such that

$$\left(\fint_{Q_{\delta r}(x,t)} u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \left(\inf_{Q^+_{r/2}(x_0,t_0)} u + r^2 C_0 \right), \tag{5.3.3}$$

for all $(x, t) \in B_{(1-2\delta)r}(x_0) \times (t_0 - (1 - 4\delta^2)r^2, t_0 - \frac{3}{4}r^2)$.

We postpone the proof to the appendix, due to its technical nature.

Remark 5.3.4. We remark that the same statement remains valid when estimating the L^p norm of supersolutions in "parabolic cubes" $\tilde{Q}_{\delta r}(x, t)$: under the assumptions of Lemma 5.3.3, there holds

$$\left(\fint_{\tilde{Q}_{\delta r}(x,t)} u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \left(\inf_{Q^+_{r/2}(x_0,t_0)} u + r^2 C_0 \right), \tag{5.3.4}$$

for all $(x, t) \in B_{(1-2\delta)r}(x_0) \times (t_0 - (1 - 4\delta^2)r^2, t_0 - \frac{3}{4}r^2)$.

To see this, we fix $r = 1$, $(x_0, t_0) = (0, 0)$, $\delta \in (0, \frac{1}{4})$ and $(x, t) \in B_{1-2\delta} \times (-1+4\delta^2, -\frac{3}{4})$. We define

$$\varrho := \inf\{\rho > 0 : \tilde{Q}_\delta(x, t) \subset Q_\rho(x, t)\}.$$

By construction, $\varrho = c_n \delta$, for some $c_n > 0$ (depending only on n). Consequently, by (5.3.3)

$$\left(\fint_{\tilde{Q}_\delta(x,t)} u^p \right)^{\frac{1}{p}} \leq \left(\frac{|Q_\varrho|}{|\tilde{Q}_\delta|} \fint_{Q_\varrho(x,t)} u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \left(\inf_{Q^+_{1/2}} u + C_0 \right),$$

and (5.3.4) follows.

We also need a version of the weak Harnack inequality, where the average is not computed over a cylinder or a cube.

Corollary 5.3.5. *Let $u \geq 0$ satisfy*

$$\partial_t u - M^-(D^2 u) \geq -C_0 \qquad \text{in } Q_r(x_0, t_0).$$

Let $\delta \in (0, \frac{1}{4})$ and $C_0, m_0 > 0$. Then there exist $C, m > 0$, $p \in (0,1)$ depending only on n, λ, Λ, C_0 and m_0 such that

$$\left(\fint_A u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \left(\inf_{Q^+_{1/2}} u + C_0 \right), \tag{5.3.5}$$

for every open set $A \subset B_{1-2\delta} \times (-1 + 4\delta^2, -\frac{3}{4})$ satisfying $|A| \geq C_0 \delta^{m_0}$.

The proof is in the appendix.

Remark 5.3.6. Any solution u to (5.3.1) in $Q_r(x_0, t_0)$ enjoys some scaling properties, that we will repeatedly use in our proofs.

First, the function

$$\tilde{u}(x,t) = u(x_0 + rx, t_0 + r^2 t) \tag{5.3.6}$$

is a solution to

$$\partial_t v - \tilde{F}(D^2 v, x, t) = r^2 \tilde{f}(x,t) \quad \text{ in } Q_1,$$

where

$$\tilde{F}(M, x, t) := r^2 F(r^{-2} M, x_0 + rx, t_0 + r^2 t)$$
$$\tilde{f}(x,t) := f(x_0 + rx, t_0 + r^2 t).$$

One can verify that $\tilde{F}$ satisfies the conditions in (5.1.2) uniformly w.r.t. $r > 0$ and (x_0, t_0). Further,

$$\|\tilde{f}\|_{L^\infty(Q_1)} = \|f\|_{L^\infty(Q_r(x_0,t_0))}.$$

These properties allow us to prove our estimates in Q_1 and extend them in $Q_r(x_0, t_0)$ by means of the transformation (5.3.6).

Second, the *blow-up* family of u at (x_0, t_0) defined by

$$u_r^{(x_0,t_0)}(x,t) = \frac{u(x_0 + rx, t_0 + r^2 t)}{r^2} \tag{5.3.7}$$

solve

$$\partial_t v - F(D^2 v, x_0 + rx, t_0 + r^2 t) = f(x_0 + rx, t_0 + r^2 t) \quad \text{ in } Q_1. \tag{5.3.8}$$

5.4 $C^{1,1}_x \cap C^{0,1}_t$ regularity and non-degeneracy

In this section we establish the optimal $C^{1,1}_x$ spatial regularity of solutions to (5.1.1) and the almost optimal $C^{0,1}_t$ temporal regularity. The main result of this section is the following theorem.

Theorem 5.4.1. *Let $u \in \mathcal{P}_1(K)$. Then $u \in C^{1,1}_x \cap C^{0,1}_t(Q_{1/2})$ and, further, there exists $C > 0$ depending only on n, λ, Λ and K such that*

$$\|\partial_t u\|_{L^\infty(Q_{1/2})} + \|D^2 u\|_{L^\infty(Q_{1/2})} \leq C. \tag{5.4.1}$$

It is important to mention that the validity of the statement above is known, even in a more general framework: see [41, 68, 81] and also [40, 59] for the elliptic framework. Respect to these works, our approach heavily exploits the non-negativity and time-monotonicity of solutions, and it is equivalent to establish the optimal growth of solutions near the free boundary (see Lemma 5.4.3). Here we present two independent and new proofs: the first combines a special Harnack inequality with a comparison argument, while the second consists in a blow-up argument and has a perturbative flavour.

We begin with the following technical lemma.

Lemma 5.4.2. *Let $u \in \mathcal{P}_r^-(x_0,t_0;K)$ and let $\delta > 0$. Then there exists $C > 0$ depending only on n, λ, Λ and δ such that*

$$\sup_{P_r^\delta(x_0,t_0)} u \le C\big(u(x_0,t_0) + r^2\|f\|_{L^\infty(Q_r^-(x_0,t_0))}\big), \tag{5.4.2}$$

where

$$P_r^\delta(x_0,t_0) := \{(x,t) \in Q_{r/2}^-(x_0,t_0) : t - t_0 < -\delta|x-x_0|^2\}. \tag{5.4.3}$$

Proof. By Remark 5.3.6, it is enough to prove the statement for $r = 1$ and $(x_0,t_0) = (0,0)$. Let us set $P^\delta := P_1^\delta(0,0)$ and consider

$$D_{r/2}^+ := B_{r/2} \times (-\tfrac{1}{4}r^2, 0) \quad \text{and} \quad D_{r/2}^- := B_{r/2} \times (-\tfrac{7}{4}r^2, -\tfrac{3}{2}r^2),$$

for $r \in I := (0, \sqrt{7}/7)$ (with r in this range we automatically have $D_{r/2}^- \subset Q_{1/2}^-$). We first notice that applying the Harnack's inequality Theorem 5.3.1 at every scale $r \in I$, we obtain

$$\sup_{D_{r/2}^-} u \le C\bigg(\inf_{D_{r/2}^+} u + r^2\|f\|_{L^\infty(Q_1^-)}\bigg) \le C\big(u(0,0) + \|f\|_{L^\infty(Q_1^-)}\big), \qquad \forall r \in I.$$

By the arbitrariness of $r \in I$, it follows

$$\sup_{A_0} u \le C\big(u(0,0) + \|f\|_{L^\infty(Q_1^-)}\big), \qquad A_0 := P^7 = \big\{(x,t) \in Q_{1/2}^- : t < -7|x|^2\big\}. \tag{5.4.4}$$

We iterate this inequality as follows. Given a point $(y,\tau) \in Q_{1/2}^-$, we consider the set

$$\tilde{A}_{y,\tau} := \big\{(x,t) \in Q_{1/2}^- : t - \tau < -7|x-y|^2\big\}$$

and we define inductively the family

$$\begin{cases} R_0 := A_0 \\ R_k := \overline{A}_k \setminus A_{k-1}, \quad k \in \mathbb{N} \setminus \{0\} \end{cases} \qquad \text{where} \quad A_k := \bigcup_{(y,\tau) \in \partial A_{k-1}} \tilde{A}_{y,\tau}.$$

By construction $\{R_k\}_{k\in\mathbb{N}}$ is a partition of $Q_{1/2}^-$. Now, we fix $\delta \in (0,1)$, pick $k_\delta \in \mathbb{N}$ such that

$$P^\delta \subset \bigcup_{k=0}^{k_\delta} R_k$$

and, for an arbitrary $(x,t) \in P^\delta$, we take $k \in \{0,\dots,k_\delta\}$ such that $(x,t) \in R_k$. Consequently, by construction and (5.4.4), we have

$$u(x,t) \le \sup_{R_k} u \le C\big(\sup_{R_{k-1}} u + \|f\|_{L^\infty(Q_1^-)}\big) \le C^k \left(\sup_{P_0} u + \|f\|_{L^\infty(Q_1^-)} \sum_{j=0}^{k-1} C^{-j} \right)$$
$$\le C^{k+1}\big(u(0,0) + \tfrac{C}{C-1}\|f\|_{L^\infty(Q_1^-)}\big) \le 2C^{k_\delta+1}\big(u(0,0) + \|f\|_{L^\infty(Q_1^-)}\big).$$

The thesis follows by the arbitrariness of $(x,t) \in P^\delta$. □

Next we establish the optimal growth control of solutions near the free boundary.

Lemma 5.4.3. *(Optimal growth) Let $u \in \mathcal{P}_1(K)$. Then there exists $C > 0$ depending only on n, λ, Λ and K such that*

$$||u||_{L^\infty(Q_r(x_0,t_0))} \le Cr^2, \tag{5.4.5}$$

for every $(x_0,t_0) \in \{u=0\} \cap Q_{1/2}$ and every $r \in (0,\frac{1}{2})$.

First proof of Lemma 5.4.3. Let us fix $(x_0,t_0) \in \{u=0\} \cap Q_{1/2}$, and set $b := \frac{1}{2\Lambda n}$ and $\delta := \frac{b}{2}$. We first notice that by (5.4.2), there is $C_0 > 0$ depending on n, λ and Λ such that

$$\sup_{P_r^\delta(x_0,t_0)} u \le C_0\big(u(x_0,t_0) + r^2\|f\|_{L^\infty(Q_r^-(x_0,t_0))}\big) \le C_0 K r^2, \tag{5.4.6}$$

for every $r \in (0,\frac{1}{2})$, where $P_r^\delta(x_0,t_0)$ is defined in (5.4.3). In particular, it follows

$$\sup_{x \in \partial B_r(x_0),\, t = t_0 - \delta r^2} u(x,t) \le C_0 K r^2, \tag{5.4.7}$$

for every $r \in (0,\frac{1}{2})$.

Now, in view of (5.4.6), it is enough to focus on the set $Q_r(x_0,t_0) \setminus P_r^\delta(x_0,t_0)$. To prove the optimal growth on such set we proceed with a comparison argument as follows. Let us define

$$v(x,t) := a(t - t_0 + b|x-x_0|^2), \qquad a := \max\{2C_0, 8\} \cdot \tfrac{K}{b}.$$

By uniform ellipticity, the assumption $F(O,\cdot) \equiv 0$ and the definition of b, we see that

$$\partial_t v - F(D^2 v, x) = a - F(2nabI, x) \ge a(1 - 2\Lambda nb\|I\|) = 0 \quad \text{in } Q_1.$$

Further, by definition of v and δ, we have

$$v(x,t)|_{t=t_0-\delta|x-x_0|^2} = a(b-\delta)|x-x_0|^2 = \tfrac{ab}{2}|x-x_0|^2,$$

and so

$$v(x,t)|_{t=t_0-\delta|x-x_0|^2} = \tfrac{ab}{2} r^2 \quad \text{in } \partial B_r(x_0), \tag{5.4.8}$$

for every $r \in (0,\frac{1}{2})$. On the other hand,

$$v(x,t) \ge a(-\tfrac{\delta}{4} + \tfrac{b}{4}) = \tfrac{ab}{8} \quad \text{in } \partial_p Q_{1/2}(x_0,t_0) \setminus P_{1/2}^\delta(x_0,t_0).$$

At this point, combining (5.4.7) with (5.4.8) and using that $\frac{ab}{2} \geq C_0 K$ by definition of a, we obtain $v \geq u$ in $\partial P^{\delta}_{1/2}(x_0,t_0) \cap Q_{1/2}(x_0,t_0)$ while, since $\frac{ab}{8} \geq K$, we also have $v \geq u$ in $\partial_p Q_{1/2}(x_0,t_0) \setminus P^{\delta}_{1/2}(x_0,t_0)$. Consequently, by the comparison principle, it follows

$$u \leq v \quad \text{ in } Q_{1/2}(x_0,t_0) \setminus P^{\delta}_{1/2}(x_0,t_0),$$

and thus, since $v \leq Cr^2$ in $Q_r(x_0,t_0) \setminus P^{\delta}_r(x_0,t_0)$ for some $C > 0$ (depending only on n, λ, Λ and K), the bound in (5.4.5) is proved. □

Second proof of Lemma 5.4.3. We argue by contradiction, assuming the existence of sequences $\{F_k\}_{k\in\mathbb{N}}$ satisfying (5.1.2), $\{f_k\}_{k\in\mathbb{N}} \in L^\infty(B_1)$ and $\{u_k\}_{k\in\mathbb{N}} \in \mathcal{P}_1(K)$ solutions to

$$\begin{cases} \partial_t u_k - F_k(D^2 u_k, x) = f_k(x)\chi_{\{u_k>0\}} & \text{in } Q_1 \\ u_k, \partial_t u_k \geq 0 & \text{in } Q_1, \end{cases} \tag{5.4.9}$$

but

$$\sup_{r\in(0,1)} r^{-2} ||u_k||_{L^\infty(Q_r(x_k,t_k))} \geq k, \tag{5.4.10}$$

for some sequence $\{(x_k,t_k)\}_{k\in\mathbb{N}} \subset \{u_k = 0\} \cap Q_{1/2}$. To obtain a contradiction, we show that a suitable rescaled and renormalized subsequence of $\{u_k\}_{k\in\mathbb{N}}$ converge to a non-negative, non-trivial entire solution satisfying $u(0,0) = 0$ and growing at infinity less than a polynomial of degree 2, in contrast with the Liouville theorem.[4]

In this spirit, we consider the monotone non-increasing function $\theta : (0,1) \to \mathbb{R}_+$, defined as

$$\theta(r) = \sup_{k\in\mathbb{N}} \sup_{\rho\in(r,1)} \rho^{-2} ||u_k||_{L^\infty(Q_\rho(x_k,t_k))}.$$

By assumption, we have $|u_k| + |f_k| \leq K$ in Q_1 for every k. Combining this with (5.4.10), we easily deduce that $\theta(r) \leq K/r^2$ for every $r \in (0,1)$, with $\theta(r) \to \infty$ as $r \to 0$.

Now, due to (5.4.10) we have that for every $j \in \mathbb{N}$, there is $r'_j \in (0,1)$ such that $\theta(r'_j) > j$. Furthermore, by definition of θ, it is not difficult to see that there exist $k_j \in \mathbb{N}$, $r_j > r'_j$ and $(x_{k_j}, t_{k_j}) \in \{u_{k_j} = 0\} \cap Q_{1/2}$ such that

$$\theta(r'_j) \geq r_j^{-2} ||u_{k_j}||_{L^\infty(Q_{r_j}(x_{k_j},t_{k_j}))} \geq \tfrac{\theta(r'_j)}{2}.$$

So, using the definition of θ again, its monotonicity and $\theta(r'_j) > j$, we obtain

$$\begin{aligned} &\theta(r_j) \geq r_j^{-2} ||u_{k_j}||_{L^\infty(Q_{r_j}(x_{k_j},t_{k_j}))} \geq \tfrac{\theta(r_j)}{2} \\ &\theta(r_j) \geq \tfrac{j}{2}, \end{aligned} \tag{5.4.11}$$

which, in particular, implies that $r_j \to 0$ as $j \to \infty$. Then, we define the blow-up sequence

$$v_j(x,t) := \frac{1}{\theta(r_j) r_j^2} u_{k_j}(r_j x + x_{k_j}, r_j^2 t + t_{k_j}), \qquad (x,t) \in Q_{1/r_j},\ j \in \mathbb{N}.$$

[4]The Liouville theorem for entire parabolic fully nonlinear equations is an immediate consequence of the $C^{2,\alpha}$ estimates proved in [84, Theorem 4.13]

Notice that v_j is non-negative with $v_j(0,0) = 0$ and, by (5.4.11), uniformly non-degenerate:

$$||v_j||_{L^\infty(Q_1)} \geq \tfrac{1}{2}, \quad \forall j \in \mathbb{N}. \tag{5.4.12}$$

Moreover, thanks to (5.4.11) again and the monotonicity of θ, we have

$$||v_j||_{L^\infty(Q_R)} = \frac{1}{\theta(r_j)r_j^2}||u_{k_j}||_{L^\infty(Q_{Rr_j}(x_{k_j},t_{k_j}))} \leq \frac{\theta(Rr_j)(Rr_j)^2}{\theta(r_j)r_j^2} \leq R^2, \tag{5.4.13}$$

for all $1 \leq R < 1/r_j$ and, using the equation of u_{k_j}, we easily see that the function v_j satisfies

$$\partial_t v_j - \tilde{F}_j(D^2 v_j, x) = \tilde{f}_j(x)\chi_{\{v_j>0\}} \quad \text{ in } Q_{1/r_j},$$

where

$$\tilde{F}_j(M,x) := \frac{1}{\theta(r_j)}F_{k_j}(\theta(r_j)M, r_j x + x_{k_j})$$

$$\tilde{f}_j(x) := \frac{1}{\theta(r_j)}f_{k_j}(r_j x + x_{k_j}).$$

By definition, the sequence $\{\tilde{F}_j\}_{j\in\mathbb{N}}$ is made of functions satisfying (5.1.2), with ellipticity constants λ and Λ, while $\tilde{f}_j \to 0$ locally uniformly in $\mathbb{R}^{n+1}$ as $j \to +\infty$. This has two consequences:

- First, up to passing to a subsequence, we have $(x_{k_j}, t_{k_j}) \to (\tilde{x}, \tilde{t})$ and $\tilde{F}_j \to \tilde{F}$ locally uniformly, for some $(\tilde{x}, \tilde{t}) \in Q_{1/2}$ and some $\tilde{F}$ satisfying (5.1.2).
- Second, by [84, Theorem 4.8], for every fixed $R \geq 1$ we have

$$||v_j||_{C^{1,\alpha}(Q_R)} \leq C(R), \quad \forall j \in \mathbb{N}.$$

Combining these facts with (5.4.12) and (5.4.13), we deduce that, $v_j \to v$ locally uniformly in $\mathbb{R}^{n+1}$ (up to passing to another subsequence), for some continuous function v satisfying

$$\begin{cases} \partial_t v - \tilde{F}(D^2 v, \tilde{x}) = 0 \quad \text{ in } \mathbb{R}^{n+1} \\ v \geq 0, \ v(0,0) = 0 \\ ||v||_{L^\infty(Q_R)} \leq R^2, \ \forall R \geq 1 \\ ||v||_{L^\infty(Q_1)} \geq 1/2, \end{cases}$$

Now, we apply the Liouville theorem to conclude that v is a polynomial of degree at most 2 in space and 1 in time. Further, by the maximum principle [83, Corollary 3.20] applied in Q_R (for arbitrary $R > 0$), we deduce that $v = 0$ in $\mathbb{R}^n \times (-\infty, 0]$ which, in turn, implies that $v \equiv 0$. This contradicts the fact that $||v||_{L^\infty(Q_1)} \geq 1/2$ and gives us (5.4.5). □

Combining the growth control result with interior estimates (see [25, Theorem 1.1] or [84, Theorem 1.1]) yields the wanted bound for $\|\partial_t u\|_{L^\infty(Q_{1/2})} + \|D^2 u\|_{L^\infty(Q_{1/2})}$.

Proof of Theorem 5.4.1. Since $\partial_t u = \partial_{ij} u = 0$ in $\text{int}(\{u = 0\})$ for every $i, j \in \{1, \ldots, n\}$, it is enough to focus on points in $\{u > 0\}$. So, let us fix $(y, \tau) \in \{u > 0\} \cap Q_{1/2}$ and let

$$d := \sup\{r > 0 : Q_r(y,\tau) \subset \{u > 0\} \cap Q_{1/2}\}.$$

By Schauder estimates [84, Theorem 4.8, Theorem 4.12] and (5.4.5), we have

$$\begin{aligned}\|\partial_t u\|_{L^\infty(Q_{d/2}(y,\tau))} + \|D^2 u\|_{L^\infty(Q_{d/2}(y,\tau))} &\le \frac{C_0}{d^2}\big(\|u\|_{L^\infty(Q_d(y,\tau))} + d^2\|f\|_{L^\infty(Q_d(y,\tau))}\big)\\ &\le \frac{C_0}{d^2}\big(Cd^2 + d^2\|f\|_{L^\infty(Q_1)}\big)\\ &\le C_0(C+K),\end{aligned}$$

for some constants $C_0, C > 0$ depending only on n, λ, Λ and K. In particular,

$$|\partial_t u(y,\tau)| + |D^2 u(y,\tau)| \le C_0(C+1)$$

and thus (5.4.1) follows thanks to the arbitrariness of $(y,\tau) \in \{u > 0\} \cap Q_{1/2}$. □

We end the section with the following non-degeneracy property.

Lemma 5.4.4. *Let $u \in \mathcal{P}_1(K)$. Then there exists $c > 0$ depending only on n, Λ and $c_\circ$ such that*

$$||u||_{L^\infty(Q_r^-(x_0,t_0))} \ge cr^2, \tag{5.4.14}$$

for every $(x_0,t_0) \in \partial\{u > 0\} \cap Q_{1/2}$ and every $r \in (0, \frac{1}{2})$.

Proof. Let us fix $(x_0,t_0) \in \partial\{u > 0\} \cap Q_{1/2}$, $r \in (0, 1/2)$ and let

$$\{(x_k,t_k)\}_{k\in\mathbb{N}} \subset \{u > 0\} \cap Q_{1/2} \quad \text{such that} \quad (x_k,t_k) \to (x_0,t_0) \ \text{ as } k \to +\infty.$$

We set $c := c_\circ/(2\Lambda n + 1)$ and consider the sequence

$$v_k(x,t) := u(x,t) - u(x_k,t_k) - c(|x-x_k|^2 - (t-t_k)), \quad k \in \mathbb{N}.$$

Then

$$\begin{aligned}\partial_t v_k - F(D^2 v_k, x) &= \partial_t u - F(D^2 u - 2cI, x) + c\\ &= \partial_t u - F(D^2 u, x) + F(D^2 u, x) - F(D^2 u - 2cI, x) + c\\ &= f(x)\chi_{\{u>0\}} + F(D^2(u - c|x|^2) + 2cI, x) - F(D^2(u - c|x|^2), x) + c\\ &\le -c_\circ + c(2\Lambda n + 1) = 0 \quad \text{ in } \{u > 0\} \cap Q_r^-(x_k,t_k),\end{aligned}$$

for every $k \in \mathbb{N}$. Further by definition, we have

$$v_k(x_k,t_k) = 0 \quad \text{ and } \quad v_k < 0 \ \text{ in } \ \partial\{u > 0\} \cap Q_r^-(x_k,t_k)$$

for every $k \in \mathbb{N}$ and thus, by the maximum principle ([50, Proposition 4.34]) it follows

$$0 = v_k(x_k,t_k) \le \sup_{Q_r^-(x_k,t_k)} v_k = \sup_{\partial_p Q_r^-(x_k,t_k)} v_k = \sup_{\partial_p Q_r^-(x_k,t_k)} u - u(x_k,t_k) - cr^2.$$

In turn, this implies

$$\sup_{Q_r^-(x_k,t_k)} u \ge u(x_k,t_k) + cr^2,$$

for every $k \in \mathbb{N}$. Since $u(x_k,t_k) \to u(x_0,t_0) = 0$ as $k \to +\infty$ and c is independent of k, we obtain (5.4.14) by passing to the limit as $k \to +\infty$. □

Remark 5.4.5. The non-degeneracy property (5.4.14) gives us nontrivial information about the geometry of the free boundary: it excludes that free boundary points are parabolic interior for $\{u = 0\}$, in the sense that

$$\{u > 0\} \cap Q_r^-(x_0, t_0) = \emptyset, \quad \forall r \in (0, 1),$$

where $(x_0, t_0) \in \partial\{u > 0\}$ is fixed (see [21, Subsection 1.2]).

Remark 5.4.6. Let $u \in \mathcal{P}_1(K)$ and $(x_0, t_0) \in \partial\{u > 0\} \cap Q_{1/2}$. Combining the non-degeneracy estimate (5.4.14) with time-monotonicity $\partial_t u \geq 0$, we deduce that the function $u_{t_0} := u|_{t=t_0}$ satisfies

$$||u_{t_0}||_{L^\infty(B_r(x_0))} \geq cr^2,$$

where $c > 0$ is as in (5.4.14) and depends only on n, Λ and $c_\circ$.

5.5 Semi-convexity and C_t^1 estimates

The purpose of this section is to establish a semi-convexity estimate for solutions $u \in \mathcal{P}_1(K)$ and a log-continuity estimate for their time-derivatives $\partial_t u$, as stated in the following proposition. It is important to mention that for the semi-convexity estimates we require that the function F is independent of the variable x (which is enough for our purposes). Then the second derivatives of the solution become super-solutions to the linearised equation, thanks to convexity of F.

The main result states the following:

Proposition 5.5.1. *Let $u \in \mathcal{P}_1(K)$ with $(0, 0) \in \partial\{u > 0\}$. Then there exist $\varepsilon, C > 0$ depending only on n, λ, Λ and K such that*

$$\partial_t u \leq C\big|\log\big(|x| + \sqrt{|t|}\big)\big|^{-\varepsilon} \quad \text{in } Q_1. \tag{5.5.1}$$

Furthermore, if the function F in (5.1.2) is independent of x and

$$\|f\|_{C^{1,1}(B_1)} \leq K, \tag{5.5.2}$$

then

$$\partial_{ee} u \geq -C\big|\log\big(|x| + \sqrt{|t|}\big)\big|^{-\varepsilon} \quad \text{in } Q_1, \tag{5.5.3}$$

for every $e \in \mathbb{S}^{n-1}$.

This was already know for the Laplacian; see [17] and [20].

The proof of the above statement relies on the iterative use of lemmas established below, that exploit the Weak Harnack inequality from Lemma 5.3.3.

First we establish the auxiliary result for the second order spatial derivatives.

Lemma 5.5.2. *Let $u \in \mathcal{P}_1(K)$ with $(0, 0) \in \partial\{u > 0\}$. Assume that the function F in (5.1.2) does not depend on x and (5.5.2) holds true. Then there exist $a, b, q > 0$ depending only on n, λ, Λ and K such that if*

$$\partial_{ee} u \geq -\gamma \quad \text{in } Q_r,$$

for some $r \in (0, 1)$, $\gamma > 0$ and $e \in \mathbb{S}^{n-1}$, then

$$\partial_{ee} u \geq -\gamma + a\gamma^q - br^2 \quad \text{in } Q_{r/2}. \tag{5.5.4}$$

Proof. Let us fix $r, \gamma > 0$, $e \in \mathbb{S}^{n-1}$ and $(z,\tau) \in \{u > 0\} \cap Q_{r/2}$. We set

$$d := \sup\{\rho > 0 : Q_\rho(z,\tau) \subset \{u > 0\}\} < \tfrac{r}{2},$$

and we fix $(z_0, \tau_0) \in \partial\{u > 0\} \cap Q_d(z,\tau)$. Notice that (z_0,τ_0) is always in the bottom of $\partial_p Q_d(z,\tau)$ since $\partial_t u \geq 0$. For $h \in [0,d]$, we consider the points

$$(y,t) := (z_0 + \tfrac{h}{d}(z - z_0), \tau_0 + h^2),$$

with $(y,t) = (z_0,\tau_0)$ for $h = 0$, $(y,t) = (z,\tau)$ for $h = d$ and $|z_0 - y| = h$ (obviously, $t - \tau_0 = h^2$). Further, since $(\partial_e)^2 = (\partial_{-e})^2$, we may choose e such that $e \cdot (z - z_0) \geq 0$, i.e., e points "inwards" the ball $B_d(z)$. Notice that by Theorem 5.4.1 we have

$$u \leq Ch^2, \quad |\nabla u| \leq Ch \quad \text{ in } Q_{h/2}(y,t), \tag{5.5.5}$$

for every $h \in [0,d]$ and $C > 0$ depending only on n, λ, Λ and K.

Now, we define the set

$$A_h := \{(x',t') \in Q_{h/2}(y,t);\ x' \cdot e = 0\}.$$

Notice that by construction A_h is at least $\frac{h}{4}$ away from $\partial Q_d(z,\tau)$ (and $\partial\{u > 0\}$). For every $(x_0,t_0) \in A_h$, if $\tilde{x} := x_0 + \frac{1}{8}\sqrt{hd}\, e$, we have

$$0 \leq u(\tilde{x},t_0) = u(x_0,t_0) + \nabla u(x_0,t_0)\cdot(\tilde{x} - x_0) + \int_{x_0}^{\tilde{x}}\int_{x_0}^{x} \partial_{ee} u,$$

and thus, by (5.5.5) and the definition of $\tilde{x}$, we obtain

$$-C_0 h^{\frac{3}{2}} d^{\frac{1}{2}} \leq \int_{x_0}^{\tilde{x}}\int_{x_0}^{x} \partial_{ee} u,$$

for some $C_0 \geq 2C$ depending only on n, λ, Λ and K. This bound and the assumption $\partial_{ee} u + \gamma \geq 0$ in Q_r yield

$$\begin{aligned} \gamma - C_0 h^{\frac{1}{2}} d^{-\frac{1}{2}} &\leq \frac{128}{hd}\int_{x_0}^{\tilde{x}}\int_{x_0}^{x}(\partial_{ee}u + \gamma) \leq \frac{128}{hd}\int_{x_0}^{\tilde{x}}\int_{x_0}^{\tilde{x}}(\partial_{ee}u + \gamma) \\ &= \frac{16}{\sqrt{hd}}\int_{x_0}^{\tilde{x}}(\partial_{ee}u + \gamma) = 2 \fint_{x_0}^{\tilde{x}}(\partial_{ee}u + \gamma), \end{aligned}$$

and thus, choosing $h := \left(\frac{\gamma}{2C_0}\right)^2 d$, it follows

$$\fint_{x_0}^{\tilde{x}}(\partial_{ee}u + \gamma) \geq \tfrac{\gamma}{4}.$$

Notice that the choice of γ implies $|\tilde{x} - x_0| = \frac{C_0}{8\gamma}h \geq \frac{C_0}{8C}h \geq \frac{h}{4}$ (C as in (5.5.5)). Consequently, by the arbitrariness of $x_0 \in B_{h/2}(y)$, we conclude

$$\fint_{\mathcal{C}_h}(\partial_{ee}u + \gamma) \geq \tfrac{\gamma}{4}, \tag{5.5.6}$$

where $\mathcal{C}_h = \{(x' + s\frac{1}{8}\sqrt{h}de, t');\ (x',t') \in A_h, s \in (0,1)\}$ is a skew-cylinder.

In this last part, we set $v := \partial_{ee}u + \gamma$ and we exploit (5.5.6) to prove (5.5.4). Since F is independent of x, it is not difficult to check that

$$\partial_t v - DF(D^2u)D^2v = g + D^2(\partial_e u)D^2F(D^2u)D^2(\partial_e u) \quad \text{ in } \{u>0\} \cap Q_1,$$

where $g := \partial_{ee}f$ and thus, setting $a_{ij} := (DF(D^2u))_{ij}$ and recalling that $M \to F(M)$ is convex, we deduce

$$\begin{cases} \partial_t v - a_{ij}(x,t)\partial_{ij}v \geq g & \text{in } Q_d(z,\tau) \\ v \geq 0 & \text{in } Q_d(z,\tau). \end{cases} \tag{5.5.7}$$

Since the matrix $\{a_{ij}\}_{ij}$ is uniformly elliptic (with ellipticity constants λ and Λ), we may apply Corollary 5.3.5 to obtain

$$\left(\fint_{\mathcal{C}_h} v^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m}\big(v(z,\tau) + d^2\|g\|_{L^\infty(Q_1)}\big),$$

for some $C, m > 0$ and $p \in (0,1)$ depending only on n, λ and Λ, and $\delta := \frac{h}{8d}$. On the other hand, by optimal regularity and using that $p \in (0,1)$, we have

$$\tfrac{\gamma}{4} \leq \fint_{\mathcal{C}_h} v \leq \|v\|^{1-p}_{L^\infty(\tilde{Q}_{h/8}(y_0,t_0))} \fint_{\mathcal{C}_h} v^p \leq (2C)^{1-p} \fint_{\mathcal{C}_h} v^p, \tag{5.5.8}$$

where C is as in Theorem 5.4.1. Noticing that $\delta \sim \gamma^2$ and exploiting (5.5.2), we combine the last two inequalities to deduce

$$C\gamma^{\frac{1}{p}+2m} \leq v(z,\tau) + d^2K,$$

for some new $C > 0$ still depending only on n, λ, Λ and K. The thesis follows by choosing $a := C$, $q := \frac{1}{p} + 2m$, $b := K$ and using the arbitrariness of (z,τ) in $\{u>0\} \cap Q_{r/2}$. □

Analogous can be established also for the time derivative.

Lemma 5.5.3. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \partial\{u>0\}$. Then there exist $a, q > 0$ depending only on n, λ, Λ and K such that if*

$$\partial_t u \leq \gamma \quad \text{in } Q_r,$$

for some $r \in (0,1)$, $\gamma > 0$, then

$$\partial_t u \leq \gamma - a\gamma^q \quad \text{in } Q_{r/2}. \tag{5.5.9}$$

Proof. As in the proof of Lemma 5.5.2, we fix $r \in (0,1)$, $\gamma > 0$, $(z,\tau) \in \{u>0\} \cap Q_{r/2}$ and we define

$$d := \sup\{\rho > 0 : Q_\rho(z,\tau) \subset \{u>0\}\} < \tfrac{r}{2}.$$

Then we fix $(z_0,\tau_0) \in \partial\{u>0\} \cap Q_d(z,\tau)$ (belonging to the bottom of $Q_d(z,\tau)$) and we estimate $\partial_t u$ in a cylinder centred at (z_0,τ_0).

To do so, we fix $h \in (0, d)$ and, since $u(z_0, t) = 0$ for $t \le \tau_0$, we infer by (5.4.5)

$$\int_{\tau_0-3d^2}^{\tau_0-2d^2} \partial_t u(x,t)dt = u(x, \tau_0 - 2d^2) - u(x, \tau_0 - 3d^2) \le Ch^2, \qquad \forall x \in B_h(z_0),$$

where $C > 0$ is as in Lemma 5.4.3. Averaging over $D^-(z_0, \tau_0) := B_h(z_0) \times (\tau_0 - 3d^2, \tau_0 - 2d^2)$, it follows

$$\fint_{D^-(z_0,\tau_0)} \partial_t u(x,t)dt \le C\big(\tfrac{h}{d}\big)^2,$$

and thus, re-writing such inequality in terms of $v := \gamma - \partial_t u$ and choosing $h := (\frac{\gamma}{2C})^{\frac{1}{2}} d$, we obtain

$$\fint_{D^-(z_0,\tau_0)} v \ge \tfrac{\gamma}{2}.$$

If $p \in (0, 1)$ is as in Lemma 5.3.3, the same argument used in (5.5.8) shows that

$$C\gamma^{\frac{1}{p}} \le \bigg(\fint_{D^-(z_0,\tau_0)} v^p\bigg)^{\frac{1}{p}}, \tag{5.5.10}$$

for some new $C > 0$ depending only on n, λ, Λ and K.

On the other hand, notice that

$$\begin{cases} \partial_t v - a_{ij}(x,t)\partial_{ij} v = 0 & \text{in } \{u > 0\} \cap Q_r \\ 0 \le v \le \gamma & \text{in } Q_r, \end{cases}$$

while $v = \gamma$ in $\mathrm{int}(\{u = 0\}) \cap Q_r$. Consequently,

$$\begin{cases} \partial_t v - a_{ij}(x,t)\partial_{ij} v \ge 0 & \text{in } Q_r \\ v \ge 0 & \text{in } Q_r. \end{cases}$$

At this point, similar to the proof of Lemma 5.5.2, we may apply Lemma 5.3.3 with $\delta := (\frac{\gamma}{2C})^{\frac{1}{2}}$ and so, thanks to (5.5.10), we deduce

$$C\gamma^{\frac{1}{p}+\frac{m}{2}} \le v(z,\tau) = \gamma - \partial_t u(z,\tau),$$

for some new $C > 0$ and $m > 0$ depending only on n, λ, Λ and K. Thanks to the arbitrariness of $(z,\tau) \in \{u > 0\} \cap Q_{r/2}$, this yields (5.5.9) with $a := C$ and $q := \frac{1}{p} + \frac{m}{2}$. □

Iterating the established estimates yields the logarithmic decay near the free boundary.

Proof of Proposition 5.5.1. We first prove the existence of $C, \varepsilon > 0$ and $k_0 \in \mathbb{N}$ depending only on n, λ, Λ and K such that the sequence $m_k := -\inf_{Q_{2^{-k}}} \partial_{ee} u$ satisfy

$$m_k \le Ck^{-\varepsilon}, \quad \forall k \ge k_0. \tag{5.5.11}$$

Let a, b and q as in Lemma 5.5.2 and choose C, ε and k_0 such that

$$C \ge \big(\tfrac{2b}{a}\big)^{\frac{1}{q}}, \qquad \varepsilon \le \min\{\tfrac{1}{q}, b\}, \qquad Ck_0^{-\varepsilon} \le \big(\tfrac{1}{aq}\big)^{\frac{1}{q-1}}.$$

We proceed by induction on $k \geq k_0$. The case $k = k_0$ follows by optimal regularity (see (5.4.1)) and the definition of C. Now, assume (5.5.11) holds true for some $k > k_0$ and let us prove it for $k+1$. By (5.5.4) and the inductive assumption, we have

$$m_{k+1} \leq m_k - am_k^q + b2^{-2k} \leq Ck^{-\varepsilon} - aC^q k^{-q\varepsilon} + b2^{-2k},$$

where we have also used that the function $x \to x - ax^q$ is increasing in $\big(0, \sqrt[q-1]{1/(aq)}\big)$ and the definition of k_0. Further, since the function $x \to x^{-\varepsilon}$ is convex, we have

$$k^{-\varepsilon} - \varepsilon k^{-\varepsilon-1} \leq (k+1)^{-\varepsilon},$$

and thus

$$m_{k+1} \leq C(k+1)^{-\varepsilon} + b2^{-2k} + \varepsilon k^{-\varepsilon-1} - aC^q k^{-q\varepsilon}.$$

At this point, we infer

$$b2^{-2k} + \varepsilon k^{-\varepsilon-1} - aC^q k^{-q\varepsilon} = \big(b2^{-2k} - \tfrac{a}{2}C^q k^{-q\varepsilon}\big) + \big(\varepsilon k^{-\varepsilon-1} - \tfrac{a}{2}C^q k^{-q\varepsilon}\big) \leq 0,$$

by the definition of C and ε (notice that $\varepsilon < \frac{1}{q}$ implies $\varepsilon < \frac{1}{q-1}$), and (5.5.11) follows.

Now, we show that (5.5.11) yields (5.5.3). To see this, let us fix $(x,t) \in Q_1$ and let $k \in \mathbb{N}$ such that $(x,t) \in Q_{2^{-k}} \setminus Q_{2^{-k-1}}$, i.e., $2^{-k-1} \leq |x| + \sqrt{|t|} \leq 2^{-k}$. Thus if $k \geq k_0$, we have by (5.5.11)

$$-\partial_{ee} u(x,t) \leq Ck^{-\varepsilon} \leq C\big|\log\big(|x| + \sqrt{|t|}\big)\big|^{-\varepsilon},$$

up to taking a larger $C > 0$. If $k \leq k_0$ and $C > 0$ is as in (5.4.1), then

$$-\partial_{ee} u(x,t) \leq C \leq (Ck_0^{\varepsilon})k_0^{-\varepsilon} \leq (Ck_0^{\varepsilon})k^{-\varepsilon} \leq C\big|\log\big(|x| + \sqrt{|t|}\big)\big|^{-\varepsilon},$$

for a new constant $C > 0$, and (5.5.3) follows.

The proof of (5.5.1) is similar and exploits Lemma 5.5.3 instead of Lemma 5.5.2. □

5.6 Classification of blow-ups

In this section we classify blow-ups of solutions $u \in \mathcal{P}_1(K)$ at free boundary points $(x_0,t_0) \in \partial\{u > 0\}$ and we study the limit as $r \downarrow 0$ of the rescalings

$$u_r(x,t) := \frac{u(x_0 + rx, t_0 + r^2 t)}{r^2}, \tag{5.6.1}$$

introduced in (5.3.7). Each of such rescaling satisfies

$$\partial_t u_r - F(D^2 u_r, x_0 + rx) = f(x_0 + rx)\chi_{\{u_r > 0\}} \quad \text{in } Q_{1/r}(x_0,t_0),$$

according to (5.3.8). Consequently, by (5.4.1), (5.5.1) and the Arzelá-Ascoli theorem, there is a sequence $r_k \downarrow 0$ and a function $u_0 : \mathbb{R}^{n+1} \to \mathbb{R}$, called blow-up of u at (x_0,t_0), such that

$$u_{r_k} \to u_0 \quad \text{in } C_p^{1,\alpha} \quad \text{locally in } \mathbb{R}^{n+1}, \tag{5.6.2}$$

as $k \to \infty$, for every $\alpha \in (0,1)$. Further, writing (5.5.3) and (5.5.1) in terms of u_{r_k} and passing to the limit as $k \to \infty$, we immediately see that $\partial_t u_0 = 0$ and $\partial_{ee} u_0 \geq 0$ for every $e \in \mathbb{S}^{n-1}$. Finally, by stability of viscosity solutions under uniform limits (see [50, Proposition 3.11]), we deduce that

$$\begin{cases} u_0 \in C^{1,1}_{loc}(\mathbb{R}^n), \quad u_0 \not\equiv 0, \quad u_0 \geq 0 \\ 0 \in \partial\{u_0 > 0\} \\ u_0 \text{ is convex} \\ -F(D^2 u_0, x_0) = f(x_0)\chi_{\{u_0>0\}} \quad \text{in } \mathbb{R}^n. \end{cases} \tag{5.6.3}$$

Notice that the first two properties are direct consequences of optimal regularity (5.4.1) and non-degeneracy (5.4.14), respectively. In what follows, we will always assume that any blow-up of u at (x_0, t_0) satisfies problem (5.6.3).

There are two different behaviours of blow-ups – either the contact set $\{u_0 = 0\}$ has empty interior or not – which lead to a very different behaviour of the solution near free boundary points. We characterise this in the following definition.

Definition 5.6.1. Let $u \in \mathcal{P}_1(K)$ and $(x_0, t_0) \in \partial\{u > 0\}$. We say that (x_0, t_0) is a regular free boundary point, if there exists a blow-up at (x_0, t_0) whose contact set has non-empty interior. That is, there exist a sequence $r_k \downarrow 0$ and a solution u_0 to (5.6.3), such that (5.6.2) holds true and $\{u_0 = 0\}$ has non-empty interior. The set of regular free boundary points is denoted with $\mathrm{Reg}(u)$.

We denote with $\Sigma(u) := \partial\{u > 0\} \setminus \mathrm{Reg}(u)$ the set of singular free boundary points. By definition, if $(x_0, t_0) \in \Sigma(u)$, then any blow-up of u at (x_0, t_0) has contact set with empty interior.

Remark 5.6.2. Note that a singular point (x_0, t_0) we have

$$\lim_{r \to 0} \frac{|\{u = 0\} \cap B_r(x_0) \times \{t_0\}|}{|B_r(x_0)|} = 0,$$

which follows from [34, Lemma 5.29].

We first turn our attention towards the blow-ups near regular points. We proceed in the spirit of [14, Lemma 7]. We can use the convexity of the blow-up and the fact that the contact set has non-empty interior to show that the free boundary of the blow-up needs to be Lipschitz.

Lemma 5.6.3. *Let w be a solution to* (5.6.3) *and assume that $B_\rho(-\tau e_n) \subset \{w = 0\}$ for some $\rho > 0$, $\tau \in (0, 1)$. Then the following assertions hold true.*

(i) There exists $c_0 \in (0, 1)$ depending only on ρ, such that $\partial_\sigma w \geq 0$ in $B_{\rho/2}$, for all $\sigma \in \mathbb{S}^{n-1}$ with $\sigma_n > 1 - c_0$.

(ii) There exists a Lipschitz function g such that

$$\{w > 0\} \cap B_{\rho/2} = \{(x', x_n) \in B_{\rho/2} : x_n > g(x')\}.$$

Further, the Lipschitz norm of g is bounded by a constant depending only on ρ.

(iii) Let $c_0 \in (0, 1)$ and g as above, and define $d(x) := x_n - g(x')$, $x \in \{w > 0\} \cap B_\rho$. Then there exists $c > 0$ depending only on n, λ, Λ, $c_\circ$ as in (5.1.3) *and ρ such that*

$$\partial_\sigma w \geq cd \quad \text{in } B_{\rho/2} \cap \{w > 0\},$$

for all $\sigma \in \mathbb{S}^{n-1}$ with $\sigma_n > 1 - \frac{c_0}{2}$.

Proof. To prove (i) we notice that for every point $x \in B_{\rho/2}$ the line passing through x with direction $\sigma \in \mathbb{S}^{n-1}$ intersects $B_\rho(-\tau e_n)$, whenever $\sigma_n > 1 - c_0$ and c_0 is close enough to 1. Indeed, let $y := x - \sigma\tau$ and compute

$$|y - (-\tau e_n)| = |x + \tau(e_n - \sigma)| \le |x| + |e_n - \sigma| \le \rho,$$

where we have used that $\tau \in (0,1)$, $x \in B_{\rho/2}$ and we have chosen $\sigma \in \mathbb{S}^{n-1}$ such that $|e_n - \sigma| \le \frac{\rho}{2}$. In particular, it must be $\sigma_n > 1 - c_0$, for some $c_0 \in (0,1)$ depending only on ρ. We deduce that $\partial_\sigma w(x) \ge 0$ by convexity of w and the fact that $\partial_\sigma w(y) = 0$.

To show (ii), we consider the level sets $\{w = \varepsilon\}$, for $\varepsilon > 0$ small. First, we notice that

$$\partial_n w > 0 \quad \text{in } B_{\rho/2} \cap \{w = \varepsilon\}. \tag{5.6.4}$$

Indeed, let $v := \partial_n w$ and assume $v(x_0) = 0$ for some $x_0 \in B_{\rho/2} \cap \{w = \varepsilon\}$. Then, differentiating the equation of w and using part (i), we obtain

$$\begin{cases} a_{ij}(x)\partial_{ij} v = 0 & \text{in } B_r(x_0) \\ v \ge 0 & \text{in } B_r(x_0), \end{cases}$$

for some ball $B_r(x_0) \subset B_{\rho/2} \cap \{w > 0\}$ and some uniformly elliptic matrix $a_{ij} = a_{ij}(x)$ with ellipticity constants λ and Λ. It thus follows that x_0 is a minimum for v and so $v = 0$ in $B_r(x_0)$ by the strong maximum principle ([50, Proposition 4.34]). Consequently $w = \varepsilon$ in $B_r(x_0)$, which is impossible since the function f in the right-hand side of the equation of w is strictly negative by (5.1.3) and $F(O, \cdot) = 0$ by (5.1.2).

Now, in light of (5.6.4), we may apply the Implicit Function Theorem to deduce the existence of a function h such that $(x', x_n) \in B_{\rho/2} \cap \{w = \varepsilon\}$ if and only if $x_n = h(x', \varepsilon)$, with $\partial_\varepsilon h > 0$. At this point, by monotonicity, we set

$$g(x') := \inf_{\varepsilon > 0} h(x', \varepsilon) = \lim_{\varepsilon \to 0} h(x', \varepsilon), \qquad x' \in B_{\rho/2} \cap \{x_n = 0\},$$

and thus, by definition, $(x', x_n) \in B_{\rho/2} \cap \{w > 0\}$ if and only if $x_n > g(x')$.

To complete part (ii), we are left to prove that g is a Lipschitz function with Lipschitz norm depending only on ρ. To do so, given $x \in B_{\rho/2} \cap \partial\{w > 0\}$, we consider the cone $C_{x,\rho}$ with vertex at x and opening $\vartheta_{x,\rho} \in (0, \pi/2)$. The number $\vartheta_{x,\rho}$ is the smallest opening such that $B_\rho(-\tau e_n) \subset C_{x,\rho}$. By convexity, we know that the "lower" part of the cone $C^-_{x,\rho}$ is fully contained in $\{w = 0\}$, while the "upper" part $C^+_{x,\rho} \subset \{w > 0\}$. This implies that g is Lipschitz. To prove that the Lipschitz norm does not depend on the point, it is enough to notice that $\vartheta_{x,\rho} \ge \vartheta_\rho$ for all $x \in \overline{B}_{\rho/2}$, where

$$\vartheta_\rho := \inf\{\vartheta_{x,\rho} : x \in B_{\rho/2}\} > 0.$$

Let us turn to point (iii) and establish the inequality for $\sigma = e_n$. Let $x \in B_{\rho/2} \cap \{w > 0\}$ be fixed and denote $d := d(x) = x_n - g(x')$. By non-degeneracy (see (5.4.14)), it holds

$$\sup_{B_{c_0 d/2}(x', g(x'))} w \ge c \left(\frac{c_0}{2} d\right)^2,$$

for some $c > 0$ depending only on n, λ, Λ, $c_\circ$ as in (5.1.3) and ρ. Further, by part (i), w is non-decreasing in all the directions $\sigma = (\sigma', \sigma_n)$ satisfying $|\sigma'| \le c_0$ and so

$$w(x) \ge w(y) \ge cd^2$$

for some new $c > 0$ depending also on ρ, where y is any point in $\overline{B}_{c_0 d/2}(x', g(x'))$ where the above supremum is attained. Consequently, since $(x', g(x'))$ is a free boundary point, we have

$$\int_{g(x')}^{x_n} \partial_n w(x', \xi) d\xi = w(x) \ge cd^2,$$

and hence, by the mean value theorem, there must be a point y in the segment $(x', f(x'))$ and x such that

$$\partial_n w(y) \ge cd.$$

Exploiting the convexity of w again (or the monotonicity of $\partial_n w$), we deduce $\partial_n w(x) \ge \partial_n w(y) \ge cd$, and the case $\sigma = e_n$ follows. To deduce the claim for all $\sigma \in \mathbb{S}^{n-1}$ satisfying $\sigma_n > 1 - \frac{c_0}{2}$ as in the statement, it suffices to write $\sigma = ae_n + \nu$, where $a > \frac{c_0}{2}$ and $\nu \in \mathbb{S}^{n-1}$ with $\nu_n > 1 - c_0$, and exploit part (i) to deduce

$$\partial_\sigma w = a\partial_n w + \partial_\nu w \ge \tfrac{c_0}{2} cd + 0 = cd,$$

for some new $c > 0$ depending only on n, λ, Λ, $c_\circ$ as in (5.1.3) and ρ. □

With additional analysis of solutions to linear equations in Lipschitz domains (used on derivatives of the blow-up), we deduce that the blow-up near regular points need to be one-dimensional.

Lemma 5.6.4. *Let $x_0 \in \mathbb{R}^n$ and let w be a solution to* (5.6.3). *Assume that $\{w = 0\}$ contains a cone with non-empty interior and vertex at* 0. *Then, there are $e \in \mathbb{S}^{n-1}$ and $a > 0$ such that*

$$w(x) = a\,(e \cdot x)_+^2. \tag{5.6.5}$$

Furthermore, $\bar{a} \le a \le \bar{b}$ for some $0 < \bar{a} \le \bar{b}$ depending only on n, λ, Λ and $c_\circ$ as in (5.1.3).

Proof. Let $\mathcal{C} \subseteq \{w = 0\}$ be the cone having non-empty interior. Up to a rotation of the coordinate system, we may assume $\mathcal{C} = \{(x', x_n) \in \mathbb{R}^n : x_n < -\tan\vartheta|x'|\}$, for some $\vartheta \in [0, \pi/2)$. For any $\sigma \in \mathbb{S}^{n-1}$ with $-\sigma \in \mathcal{C}$ we can obtain that $v_\sigma := \partial_\sigma w \ge 0$ in $\mathbb{R}^n$, as in (*i*) from Lemma 5.6.3. Hence $\{w > 0\}$ must be Lipschitz as in (*ii*) from Lemma 5.6.3. Differentiating the equation of w, we deduce

$$\begin{cases} a_{ij}(x)\partial_{ij} v_\sigma = 0 & \text{in } \{w > 0\} \\ v_\sigma = 0 & \text{in } \partial\{w > 0\}, \end{cases}$$

for some uniformly elliptic matrix $\{a_{ij}\}_{ij}$ with ellipticity constants λ and Λ. Now, define $\mu(\sigma) = \sup\{\mu \ge 0;\ \partial_\sigma w - \mu\partial_n w \ge 0\}$ and conclude that

$$\partial_\sigma w = \mu(\sigma)\partial_n w \quad \text{in } \mathbb{R}^n.$$

Indeed, if there is a point where $\partial_\sigma w - \mu(\sigma)\partial_n w > 0$ we could apply the Boundary Harnack theorem in Lipschitz domains [29, Theorem 1.1] to $v_\sigma - \mu(\sigma)v_{e_n}$ and v_{e_n}, to get that

$$v_\sigma - \mu(\sigma)v_{e_n} \geq c_0 v_{e_n},$$

for some $c_0 > 0$, which contradicts the definition of $\mu(\sigma)$. Since σ can vary over an open subset of $\mathbb{S}^{n-1}$, we can chose a basis of $\mathbb{R}^n$ which is all in $\mathcal{C}$. Hence we conclude that every partial derivative is linearly dependant on $\partial_n w$, and so $\nabla w = b\partial_n w$ for some constant vector $b \in \mathbb{R}^n$. Consequently, the level sets of w are hyperplanes perpendicular to b, which says that w is one dimensional, so without loss of generality, $w = w(x_n)$. In particular, $\{w > 0\} = \{x_n > 0\}$. To complete the proof, we notice that

$$D^2 w = w'' M, \qquad F(w''M, x_0) = -f(x_0) \quad \text{ in } \mathbb{R}_+,$$

where $M = e_n e_n^T$. If we show that $w'' = a$ in $\mathbb{R}_+$ for some $a > 0$, our statement follows since $w(0) = w'(0) = 0$. To see this, we fix $h, k \geq 0$ and we notice that by uniform ellipticity

$$F(hM, x_0) - F(kM, x_0) = F(kM + (h-k)M, x_0) - F(kM, x_0) \geq \lambda |h-k| \|M\|,$$

that is, $h \neq k$ implies $F(hM, x_0) \neq F(kM, x_0)$. Consequently, since the r.h.s. of the equation of w is constant, w'' must be constant as well. Finally, by uniform ellipticity, we have

$$\Lambda a = \Lambda w'' \|M\| \geq F(w''M, x_0) = -f(x_0) \geq c_\circ,$$

and thus $a \geq \bar{a} := \frac{c_\circ}{\Lambda}$. Conversely,

$$K \geq -f(x_0) = F(w''M, x_0) \geq \lambda w'' \|M\| = \lambda a,$$

which yields $a \leq \bar{b} := \frac{K}{\lambda}$. □

On the other hand, when the contact set of the blow-up has empty interior, we can show that the blow-up solves the equation at all points. Then we conclude from the Liouville theorem that it has to be a quadratic polynomial.

Lemma 5.6.5. *Let u_0 be a solution to* (5.6.3). *Assume that $\{u_0 = 0\}$ has empty interior. Then*

$$u_0(x) = x^T A x, \tag{5.6.6}$$

for some $n \times n$ matrix $A \geq 0$.

Proof. Since u_0 is convex and its contact set has empty interior, it follows that $\{u_0 = 0\}$ is contained in a hyperplane and, in particular, it has zero Lebesgue measure. Hence, by [12, Theorem 2.7], u_0 satisfies $F(D^2 u_0) = f(0)$ in $\mathbb{R}^n$ in the classical sense while, by optimal growth, $\|u_0\|_{L^\infty(B_R)} \leq CR^2$, for all $R > 1$, where $C > 0$ is as in (5.4.5). Then, by the Liouville theorem for these operators, we deduce that u_0 has to be a a quadratic polynomial. Since $u_0(0) = |\nabla u_0(0)| = 0$, we conclude that $u_0(x) = x^T A x$ for some matrix A . Since $u_0 \geq 0$ also $A \geq 0$. □

5.7 Analysis of regular points

In this section we establish the C^∞ regularity of the regular part of the free boundary, as the following theorem asserts.

Theorem 5.7.1. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \mathrm{Reg}(u)$, and assume that $f \in C^\infty$. Then, there exists $\varrho_0 > 0$ and a C^∞ function g such that*

$$Q_{\varrho_0} \cap \{u > 0\} = \{(x', x_n, t) \in Q_{\varrho_0} : x_n > g(x', t)\},$$

up to a rotation of the spatial coordinates.

The key role in passing the information from the blow-up to the solution of the obstacle problem plays the so called "almost positivity" lemma.

Lemma 5.7.2. *Let $\varrho \in (0,1]$, $K > 0$ and let $\{u_r\}_{r\in(0,1)}$ be a family of solutions to*

$$\begin{cases} \partial_t v - F(D^2 v, rx) = f(rx)\chi_{\{v>0\}} & \text{in } Q_\varrho \\ v, \partial_t v \geq 0 & \text{in } Q_\varrho, \end{cases} \tag{5.7.1}$$

with F and f satisfying

$$\|f\|_{C^{0,1}(B_1)} + \|F\|_{C_x^{0,1}(\mathcal{S}\times B_1)} \leq K. \tag{5.7.2}$$

Then there exist $\varepsilon_0, r_0 \in (0,1)$ depending only on n, Λ, K, $c_\circ$ as in (5.1.3) and ϱ, such that if

$$\partial_\sigma u_r - \partial_t u_r - u_r \geq -\varepsilon \quad \text{in } Q_\varrho,$$

for some $r \in (0, r_0)$, $\varepsilon \in (0, \varepsilon_0)$ and $\sigma \in \mathbb{S}^{n-1}$, then

$$\partial_\sigma u_r - \partial_t u_r - u_r \geq 0 \quad \text{in } Q_{\varrho/2}.$$

Proof. By scaling (see Remark 5.3.6), we may assume $\varrho = 1$. The proof is based on a comparison argument as follows. Let us set $u := u_r$ and $a_{ij}(x,t) := (DF(D^2u, rx))_{ij}$. Since the function $M \to F(M, \cdot)$ is convex, we have

$$0 = F(O) \geq F(D^2 u(x,t), rx) - a_{ij}(x,t)\partial_{ij} u(x,t),$$

and hence

$$\partial_t u - a_{ij}(x,t)\partial_{ij} u \leq f(rx) \quad \text{in } Q_1 \cap \{u > 0\}. \tag{5.7.3}$$

Differentiating the equation in (5.7.1) along the direction σ, we obtain that $v := \partial_\sigma u$ satisfies

$$\partial_t v - a_{ij}(x,t)\partial_{ij} v = r\partial_\sigma F(D^2 u, rx) + r\partial_\sigma f(rx), \tag{5.7.4}$$

while, differentiating with respect to t and setting $\tilde{v} := \partial_t u$, we see that

$$\partial_t \tilde{v} - a_{ij}(x,t)\partial_{ij}\tilde{v} = 0. \tag{5.7.5}$$

Now, let $(x_0, t_0) \in Q_{1/2} \cap \{u > 0\}$ be arbitrarily fixed, set

$$r_0 := \min\{1, c_\circ/(2K)\},$$

and define

$$w = v - \tilde{v} - u + \tfrac{c_\circ}{4}\left[\tfrac{1}{2n\Lambda}|x-x_0|^2 - (t-t_0)\right].$$

Then, by (5.1.3) and combining (5.7.3), (5.7.4) and (5.7.5), we have

$$\begin{aligned}\partial_t w - a_{ij}(x,t)\partial_{ij} w &\geq r\left[\partial_\sigma F(D^2u, rx) + \partial_\sigma f(rx)\right] - f(rx) - \tfrac{c_\circ}{2}\\ &\geq -r\left(\|\nabla_x F\|_{L^\infty(\mathcal{S}\times B_1)} + \|\nabla f\|_{L^\infty(B_1)}\right) - f(rx) - \tfrac{c_\circ}{2}\\ &\geq -rK - f(rx) - \tfrac{c_\circ}{2}\\ &\geq -c_0 + f(rx) \geq 0 \qquad \text{in } Q_{1/2}\cap\{u>0\},\end{aligned}$$

for all $r \in (0, r_0)$ while, since $|\nabla u| = \partial_t u = u = 0$ on $\partial\{u>0\}$, it must be $w > 0$ in $Q^-_{1/4}(x_0,t_0)\cap\partial\{u>0\}$. Further, on $\partial_p Q^-_{1/4}(x_0,t_0)$, we have by assumption $w \geq -\varepsilon + \varepsilon_0$, where $\varepsilon_0 := \frac{c_\circ}{128n\Lambda}$ and thus the minimum principle ([50, Proposition 4.34]) yields

$$\inf_{Q^-_{1/4}(x_0,t_0)\cap\{u>0\}} w = \inf_{\partial_p(Q^-_{1/4}(x_0,t_0)\cap\{u>0\})} w \geq 0,$$

for all $\varepsilon \in (0,\varepsilon_0)$. The thesis follows by the arbitrariness of $(x_0,t_0) \in Q_{1/2}\cap\{u>0\}$. □

Applying this result to derivatives of the solution gives that near regular points the derivatives in directions close to the direction of growth of the blow-up must be non-negative. This furthermore yields the Lipschitz regularity of the free boundary.

Proposition 5.7.3. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \mathrm{Reg}(u)$, and assume that f and F satisfy* (5.7.2). *Then, there exists $\varrho_0 > 0$ and a Lipschitz function g such that*

$$Q_{\varrho_0}\cap\{u>0\} = \{(x', x_n, t) \in Q_{\varrho_0} : x_n > g(x',t)\},$$

up to a rotation of the spatial coordinates.

Proof. Since $(x_0,t_0) := (0,0)$ is a regular free boundary point, there exist a sequence $r_k \downarrow 0$ and solution u_0 to (5.6.3) such that the rescalings $u_k := u_{r_k}$ defined in (5.6.1) satisfy (5.6.2) as $k \to +\infty$ and $\{u_0 = 0\}$ has non-empty interior. Consequently, thanks to the invariance of the problem under spatial rotations, we may assume $B_\rho(-\tau e_n) \subset \{u_0 = 0\}$ for some $0 < \rho < \tau < 1$. Thus, by Lemma 5.6.3 (part (iii)) and (5.4.5), it follows

$$\partial_\sigma u_0 - u_0 \geq cd - Cd^2 \geq 0 \quad \text{ in } B_{\rho/2}\cap\{u_0>0\}\cap\{d < c/C\},$$

where $c > 0$, $\sigma \in \mathbb{S}^{n-1}$ and $d = d(x)$ are as in Lemma 5.6.3 and $C > 0$ as in Lemma 5.4.3 (in particular, $\sigma_n > 1 - c_0$, for some $c_0 \in (0,1)$ small depending on ρ). Combining this with (5.6.2), we deduce that, given any $\varepsilon > 0$, there is k_ε such that

$$\partial_\sigma u_k - u_k \geq -\tfrac{\varepsilon}{2} \quad \text{ in } Q_\varrho,$$

for all $k \geq k_\varepsilon$, where $\varrho > 0$ is taken small enough depending on ρ, c and C. Further, the rescaled version of (5.5.1) in Proposition 5.5.1 gives

$$\partial_\sigma u_k - \partial_t u_k - u_k \geq -\varepsilon \quad \text{ in } Q_\varrho,$$

up to taking k_ε larger. At this point, we fix $\varepsilon := \varepsilon_0/2$ and k_ε (eventually larger) such that $r_k \in (0, r_0/2)$ for all $k \geq k_\varepsilon$, where ε_0 and r_0 are as in Lemma 5.7.2. Since each u_k satisfies (5.7.1) in Q_ϱ with $r = r_k$, we conclude

$$\partial_\sigma u_k - \partial_t u_k - u_k \geq 0 \quad \text{in } Q_{\varrho/2},$$

by virtue of Lemma 5.7.2 and thus, scaling back to u, it follows

$$\partial_\sigma u - r_k \partial_t u \geq \tfrac{1}{r_k} u \quad \text{in } Q_{r_k \varrho/2}. \tag{5.7.6}$$

To complete the proof, we fix $k = k_\varepsilon$ (notice that k_ε depends only on n, Λ, K and $c_\circ$ as in (5.1.3)) and set $\varrho_0 := \frac{r_k \varrho}{4}$, $\nu := (\sigma, -r_k)$. The above inequality implies $\partial_\nu u \geq 0$ in Q_{2r}, with $\partial_\nu u > 0$ in $Q_{2r} \cap \{u > 0\}$: these facts, combined with the arbitrariness of $\sigma \in \mathbb{S}^{n-1}$ with $\sigma_n \in (1 - c_0, 1)$, allow to repeat the argument used in the proof of Lemma 5.6.3 part (ii) and to complete the proof of our statement. □

Once we get a uniform Lipschitzness of the free boundary near a regular point, we are able to get uniform rate of convergence to the blow-up in a neighbourhood of the free boundary point, which is an important step towards the C^1_x regularity of the free boundary. Moreover we conclude that the set of regular points is open inside the free boundary.

Lemma 5.7.4. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \partial\{u > 0\}$, $0 < \bar{a} < \bar{b}$ as in Lemma 5.6.4 and let $\{u_r\}_{r \in (0,1)}$ be the family of rescalings* (5.6.1)*. Then for every $\vartheta \in [0, \pi/2)$ and $\varepsilon \in (0,1)$, there exists $r = r(\vartheta, \varepsilon) \in (0,1)$ such that if*

$$Q_{1/r} \cap \{x_n \leq -\tan\vartheta\,(|x'| + \sqrt{|t|})\} \subset Q_{1/r} \cap \{u_r = 0\},$$

then there exist $e \in \mathbb{S}^{n-1}$ and $a \in [\bar{a}, \bar{b}]$ such that

$$\|u_r - a(e \cdot x)_+^2\|_{C^{1,\alpha}_x \cap C^1_t(Q_1)} \leq \varepsilon.$$

Proof. Assume by contradiction there are $\theta \in [0, \pi/2)$ and $\varepsilon \in (0,1)$ such that for every sequence $r_k \downarrow 0$ there holds

$$Q_{1/r_k} \cap \{x_n \leq -\tan\vartheta\,(|x'| + \sqrt{|t|})\} \subset Q_{1/r_k} \cap \{u_k = 0\},$$

but

$$\|u_k - a(e \cdot x)_+^2\|_{C^{1,\alpha}_x \cap C^1_t(Q_1)} \geq \varepsilon,$$

for every $e \in \mathbb{S}^{n-1}$ and $a \geq \bar{a}$, where we have set $u_k := u_{r_k}$. Passing to a subsequence, we may assume $u_k \to u_0$ as $k \to +\infty$ in the sense of (5.6.2), for some limit u_0 satisfying (5.6.3). Furthermore, since

$$u_k = 0 \quad \text{in } Q_{1/r_k} \cap \{x_n \leq -\tan\vartheta\,(|x'| + \sqrt{|t|})\}$$

for all $k \in \mathbb{N}$, we obtain $u_0 = 0$ in $\{x_n \leq -\tan\vartheta\,|x'|\}$, that is, $\{x_n \leq -\tan\vartheta\,|x'|\} \subset \{u_0 = 0\}$. We may thus apply Lemma 5.6.4 to deduce that u_0 is of the form (5.6.5), obtaining the desired contradiction. □

We next improve the spatial regularity of the free boundary to C^1_x near regular points and show that the second spatial derivatives are continuous up to the boundary there.

Proposition 5.7.5. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \mathrm{Reg}(u)$, and assume that f and F satisfy (5.7.2). Then, there exists $\varrho_0 > 0$ and a $C^1_{x'} \cap C^{0,1}_t$ function g such that*

$$Q_{\varrho_0} \cap \{u > 0\} = \{(x', x_n, t) \in Q_{\varrho_0} : x_n > g(x', t)\},$$

up to a rotation of the spatial coordinates. Furthermore, there exists a modulus of continuity $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\sup_{(x,t),(y,\tau) \in Q_r \cap \overline{\{u>0\}}} |D^2 u(x,t) - D^2 u(y,\tau)| \leq \omega(r)$$

for all $r \in (0, \frac{\varrho_0}{2})$.

Proof. To establish the first part of the statement, we show the existence of $\varrho, r_0 > 0$ and a modulus of continuity ω (that is, $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(r) \to 0$, as $r \downarrow 0$) such that for every $(x_0, t_0) \in Q_\varrho \cap \partial\{u > 0\}$, there is $e \in \mathbb{S}^{n-1}$ (depending on (x_0, t_0)) such that

$$Q_r(x_0, t_0) \cap \partial\{u > 0\} \subset \{(x,t) : |e \cdot (x - x_0)| \leq \omega(r) r\}, \tag{5.7.7}$$

for all $r \in (0, r_0)$. This will be achieved in a couple of steps as follows.

First, we show that there exists $\varrho > 0$ such that for every $\varepsilon \in (0,1)$, there is $r \in (0,1)$ such that, for every $(x_0, t_0) \in Q_\varrho \cap \partial\{u > 0\}$ there holds

$$\|u_r - a(e \cdot x)_+^2\|_{C^{1,\alpha}_x \cap C^1_t(Q_1)} \leq \varepsilon, \tag{5.7.8}$$

for some $a \in [\bar{a}, \bar{b}]$ and $e \in \mathbb{S}^{n-1}$ depending on (x_0, t_0), where $\bar{a}$ and $\bar{b}$ are as in Lemma 5.6.4. This claim follows by noticing that, in light of Proposition 5.7.3, we have

$$\{u > 0\} \cap Q_{2\varrho} = \{(x', x_n, t) \in Q_{2\varrho} : x_n > g(x', t)\},$$

up to a spatial rotation, for some $\varrho > 0$ and a Lipschitz function g (with Lipschitz norm depending only on ϱ). Consequently, there is $\vartheta \in [0, \pi/2)$ and $r = r(\vartheta, \varepsilon) \in (0,1)$ satisfying

$$Q_{1/r} \cap \{x_n \leq -\tan\vartheta\,(|x'| + \sqrt{|t|})\} \subset Q_{1/r} \cap \{u_r = 0\}.$$

Then (5.7.8) follows directly from Lemma 5.7.4.

Second, we exploit (5.7.8) to show

$$Q_r(x_0, t_0) \cap \partial\{u > 0\} \subset \{(x,t) : |e \cdot (x - x_0)| \leq C\sqrt{\varepsilon} r\}, \tag{5.7.9}$$

for all $(x_0, t_0) \in Q_\varrho \cap \partial\{u > 0\}$ and some $C > 0$ depending only on n, λ, Λ and $c_\circ$ as in (5.1.3). Once this inclusion is established, (5.7.7) easily follows: notice that ω is uniform in Q_ϱ since the constant C above is independent of (x_0, t_0), and r_0 is the biggest $r \in (0,1)$ for which (5.7.8) holds true (take for instance $\varepsilon = 1$).

To check (5.7.9), let us choose $C > 0$ such that

$$C^2 > \max\{1/\bar{a}, 1/c\},$$

where $c > 0$ is as in Lemma 5.4.4. Then, for every $(x,t) \in Q_r(x_0,t_0) \cap \{e \cdot (x - x_0) > C\sqrt{\varepsilon} r\}$, we have by rescaling (5.7.8)

$$u(x,t) \geq a[e \cdot (x - x_0)]^2 - \varepsilon r^2 \geq \bar{a} C^2 \varepsilon r^2 - \varepsilon r^2 = \varepsilon r^2 (\bar{a} C^2 - 1) > 0,$$

while, whenever $(x,t) \in Q_r(x_0,t_0) \cap \{e \cdot (x-x_0) < -C\sqrt{\varepsilon} r\}$ and $(x,t) \in \partial\{u>0\}$, we obtain by non-degeneracy and (5.7.8) again

$$cC^2\varepsilon r^2 \leq \sup_{Q_{C\sqrt{\varepsilon}r}(x,t)} u = \sup_{Q_{C\sqrt{\varepsilon}r}(x,t)} u - a[e\cdot(x-x_0)]_+^2 \leq \varepsilon r^2,$$

which is impossible and thus (5.7.9) follows.

At this point, it suffices to notice that (5.7.7) guarantees that at each point in Q_ϱ the graph of g can be touched from below and from above by the functions

$$x_n = \pm\,\omega\big(|x'| + \sqrt{|t|}\big)\big(|x'| + \sqrt{|t|}\big),$$

up to a rotation and a translation. Consequently, g is differentiable in a neighbourhood of $(0',0)$ and, since ω is uniform in Q_ϱ, it follows that $\nabla_{x'} g$ is continuous in a neighbourhood of $(0',0)$ with modulus of continuity 2ω. The fact that $g \in C_t^{0,1}$ is a direct consequence of Proposition 5.7.3 and the first part of our statement follows with $\varrho_0 := \varrho$.

Now, we show that D^2u is continuous in $Q_{\varrho_0/2} \cap \overline{\{u>0\}}$. As above, we proceed in some steps. Let us set

$$p^{(x_0,t_0)}(x) := a_0[e_0\cdot(x-x_0)]_+^2, \qquad\qquad q^{(x_0,t_0)}(x) := a_0[e_0\cdot(x-x_0)]^2,$$

where $a_0 \in [\bar{a},\bar{b}]$ and $e_0 \in \mathbb{S}^{n-1}$ stand for $a(x_0,t_0)$ and $e(x_0,t_0)$, respectively. By (5.7.8) and the first part of the proof, we have

$$\|u - p^{(x_0,t_0)}\|_{L^\infty(Q_r(x_0,t_0))} \leq \omega(r)r^2,$$

for all $r \in (0,r_0)$. Furthermore,

$$\|q^{(x_0,t_0)} - p^{(x_0,t_0)}\|_{L^\infty(Q_r(x_0,t_0)\cap\{u>0\})} \leq \omega(r)r^2,$$

for all $r \in (0,r_0)$, taking eventually r_0 smaller. To see this, let us assume $(x_0,t_0)=(0,0)$ and $e_0 = e_n$ (which is always the case up to a rotation and a translation) and set $p := p^{(x_0,t_0)}$, $q := q^{(x_0,t_0)}$. Then $p - q = 0$ in $x_n \geq 0$, whilst $p - q = -q$ in $x_n < 0$ and thus, for every $(x,t) \in Q_r \cap \{u>0\}$, we have

$$|p(x) - q(x)| \leq q(x) = a_0 x_n^2 \leq \bar{b} g(x',t)^2 \leq \bar{b}\omega^2(r)r^2 \leq \omega(r)r^2,$$

where g is the $C_x^1 \cap C_t^{0,1}$ function parametrizing $Q_r \cap \partial\{u>0\}$ as above and $\bar{b}>0$ is as in Lemma 5.6.5. Passing to the supremum, our claim follows.

Combining the two estimates above, we obtain

$$\|u - q^{(x_0,t_0)}\|_{L^\infty(Q_r(x_0,t_0)\cap\{u>0\})} \leq 2\omega(r)r^2, \tag{5.7.10}$$

for all $r \in (0,r_0)$ and, in a similar way, we also conclude

$$\|\nabla u - \nabla q^{(x_0,t_0)}\|_{L^\infty(Q_r(x_0,t_0)\cap\{u>0\})} \leq 2\bar{b}\,\omega(r)r. \tag{5.7.11}$$

In particular, (5.7.10) and (5.7.11) imply that $D^2u(x_0,t_0)$ exists and equals $D^2q^{(x_0,t_0)} := A_0$, for some suitable $n\times n$ matrix A_0 depending on a_0 and e_0 (and thus, on (x_0,t_0)). Actually, the following quantitative bound holds true:

$$\|D^2u - D^2q^{(x_0,t_0)}\|_{L^\infty(Q_r(x_0,t_0)\cap\{u>0\})} \leq C\omega(r), \tag{5.7.12}$$

for all $r \in (0, r_0)$ and some $C > 0$ depending only on n, λ and Λ.

To complete the proof, we notice that, since $p^{(x_0,t_0)}$ satisfies (5.6.3) and is $1D$, it immediately follows that $q^{(x_0,t_0)}$ satisfies

$$-F(D^2 q^{(x_0,t_0)}, x_0) = -F(A_0, x_0) = f(x_0).$$

Consequently, the function $v := u - q^{(x_0,t_0)}$ is a solution to

$$\partial_t v - \tilde{F}(D^2 v, x) = \tilde{f}(x) \quad \text{ in } Q_r(x_0,t_0) \cap \{u > 0\},$$

where $\tilde{F}(M,x) := F(M + A_0, x) - F(A_0, x)$ and $\tilde{f}(x) := f(x) - F(A_0, x)$. Notice that $|F(A_0,x) - F(A_0,x_0)| \le \tilde{\omega}(|x - x_0|)$, as well as $|f(x) - f(x_0)| \le \tilde{\omega}(|x - x_0|)$ for some modulus of continuity $\tilde{\omega}$. Since $\tilde{F}$ belongs to the class (5.1.2) and $\|f\|_{C^{0,1}} \le K$, we may combine the Schauder estimates [84, Theorem 4.8] with (5.7.10) to deduce

$$\begin{aligned} \|D^2 u - D^2 q^{(x_0,t_0)}\|_{L^\infty(Q_{r/4}(z,s))} &\le \frac{C}{r^2}\big(\|u - q^{(x_0,t_0)}\|_{L^\infty(Q_{r/2}(z,s))} + r^2 \|\tilde{f}\|_{L^\infty(Q_{r/2}(z,s))}\big) \\ &\le \frac{C}{r^2}\big(\omega(r) r^2 + K r^2 \tilde{\omega}(r)\big) \\ &\le (C + K)\max\{\omega(r), \tilde{\omega}(r)\}, \end{aligned} \tag{5.7.13}$$

where $Q_{r/2}(z,s) \subset Q_r(x_0,t_0) \cap \{u > 0\}$ and C depends only on n, λ and Λ. Applying this twice on $Q_{r/2}(z,s) \subset Q_r(x_0,t_0) \cap Q_r(y_0,\tau_0) \cap \{u > 0\}$, where $(y_0,\tau_0) \in \partial\{u > 0\}$, it follows

$$\|D^2 q^{(x_0,t_0)} - D^2 q^{(y_0,\tau_0)}\|_{L^\infty(Q_{r/4}(z,s))} \le C\omega(r),$$

for some new $C > 0$ depending only on n, λ, Λ and K. Since q is a polynomial of degree 2, this is equivalent to say

$$d((x_0,t_0),(y_0,\tau_0)) \le r \qquad \Rightarrow \qquad \|D^2 q^{(x_0,t_0)} - D^2 q^{(y_0,\tau_0)}\|_\infty \le C\omega(r), \tag{5.7.14}$$

for all $r > 0$, that is, the function $\partial\{u > 0\} \ni (x,t) \mapsto D^2 q^{(x,t)}$ is continuous.

To complete the proof, let us fix $(x,t),(y,\tau) \in \overline{\{u > 0\}}$, set $\rho := d((x,t),(y,\tau))$ and consider $(x_0,t_0),(y_0,\tau_0) \in \partial\{u > 0\}$ projections of (x,t) and (y,t) over $\partial\{u > 0\}$. Further, let us set

$$d_{x,t} := \sup\{r > 0 : Q_r(x,t) \subset \{u > 0\}\}, \qquad d_{y,\tau} := \sup\{r > 0 : Q_r(y,\tau) \subset \{u > 0\}\},$$

and $d := \min\{d_{x,t}, d_{y,\tau}\}$ (by symmetry we may assume $d = d_{x,t}$).

Let us first examine the case in which $4\rho \le d$: under such assumption, we may assume $Q_{2\rho}(x,t) \subset Q_d(x_0,t_0) \cap Q_d(y_0,\tau_0) \cap \{u > 0\}$ and thus, in light of (5.7.13), (5.7.14) and the definition of ρ, we deduce

$$\begin{aligned} |D^2 u(x,t) - D^2 u(y,\tau)| \le\ & \|D^2 u - D^2 q^{(x_0,t_0)}\|_{L^\infty(Q_\rho(x,t))} \\ &+ \|D^2 q^{(x_0,t_0)} - D^2 q^{(y_0,\tau_0)}\|_\infty \\ &+ \|D^2 u - D^2 q^{(y_0,\tau_0)}\|_{L^\infty(Q_\rho(y,\tau))} \le C\omega(|x - y| + \sqrt{|t - \tau|}), \end{aligned}$$

and the continuity of D^2u follows. On the other hand, when $4\rho \geq d$, we have $d_{y,t} \leq d + \rho$ and thus (5.7.12) and (5.7.14) yield

$$\begin{aligned} |D^2u(x,t) - D^2u(y,\tau)| &\leq \|D^2u - D^2q^{(x_0,t_0)}\|_{L^\infty(Q_d(x_0,t_0)\cap\{u>0\})} \\ &\quad + \|D^2q^{(x_0,t_0)} - D^2q^{(y_0,\tau_0)}\|_\infty \\ &\quad + \|D^2u - D^2q^{(y_0,\tau_0)}\|_{L^\infty(Q_{d+\rho}(y_0,\tau_0)\cap\{u>0\})} \\ &\leq 2C\omega(d) + C\omega(d+\rho) \leq C\omega(|x-y| + \sqrt{|t-\tau|}), \end{aligned}$$

for some new $C > 0$ and the proof is complete. □

At this point we can apply higher order boundary Harnack inequalities from [57] to complete the proof of Theorem 5.7.1. Alternatively we can apply the hodograph transform result from [52]. We obtain that near regular points the free boundary is C^∞ in space and time.

Proof of Theorem 5.7.1. The proof combines Proposition 5.7.5 and the higher order boundary Harnack inequalities established in [57] as follows.

By Proposition 5.7.5, we have

$$Q_{\varrho_0} \cap \{u > 0\} = \{(x', x_n, t) \in Q_{\varrho_0} : x_n > g(x', t)\},$$

up to a rotation of the spatial coordinates, for some $\varrho_0 > 0$ and some $C^1_{x'} \cap C^{0,1}_t$ function g. Further, as already obtained in (5.7.4) and (5.7.5), given any $\nu \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, the partial derivatives $v_\nu := \partial_\nu u$ satisfy

$$\begin{cases} \partial_t v_\nu - a_{ij}(x,t)\partial_{ij} v_\nu = f_\nu(x,t) & \text{in } Q_{\varrho_0} \cap \{u > 0\} \\ v_\nu = 0 & \text{in } Q_{\varrho_0} \cap \partial\{u > 0\}, \end{cases}$$

for some $f_\nu \in L^\infty(Q_{\varrho_0})$ and $a_{ij}(x,t) := (DF(D^2u, x))_{ij}$. The coefficients a_{ij} are continuous in $Q_{\varrho_0} \cap \overline{\{u > 0\}}$, by the second part of Proposition 5.7.5. On the other hand, by (5.7.6), we know that

$$\partial_\nu u \geq \tfrac{1}{r} u \quad \text{in } Q_{r\varrho_0/2}$$

whenever $\nu = (\sigma, \pm r)$, $\sigma \in \mathbb{S}^{n-1}$ is sufficiently close to e_n and $r > 0$ is small enough, up to taking ϱ_0 smaller: $\sigma_n \in (1 - c_0, 1)$ and $r \in (0, r_0/2)$, where $c_0 \in (0,1)$ is as in Lemma 5.6.3 and $r_0 \in (0,1)$ as in Proposition 5.7.3. In particular, $\partial_\nu u \geq 0$ in Q_{ϱ_0} up to taking ϱ_0 smaller and thus, a straightforward adaptation of the proof of Lemma 5.6.3 part (iii) shows that

$$\partial_n u \geq cd \quad \text{in } Q_{\varrho_0/2} \cap \{u > 0\}.$$

This allows us to apply the boundary Harnack principle (see [57, Theorem 1.2]) to the functions $v_{e_i} = \partial_i u$, $v_{e_{n+1}} := \partial_t u$ and $v_{e_n} = \partial_n u$ and deduce that

$$\frac{\partial_i u}{\partial_n u} \in C^\alpha_p(Q_{\varrho_0/4} \cap \overline{\{u > 0\}}), \tag{5.7.15}$$

for every $\alpha \in (0,1)$ and $i = 1, \ldots, n+1$. Consequently, the functions

$$\bar{\nu}_i = \frac{\partial_i u}{|\nabla_{(x,t)} u|} = \frac{\partial_i u/\partial_n u}{\left(\sum_{j=1}^{n-1}(\partial_j u/\partial_n u)^2 + 1 + (\partial_t u/\partial_n u)^2\right)^{1/2}}, \qquad i = 1, \ldots, n+1$$

can be C_p^α-extended up to $\overline{\{u>0\}}$ and so, by definition of $\bar\nu$ and g, this yields

$$g \in C_p^{1+\alpha},$$

for every $\alpha \in (0,1)$.

We can bootstrap this argument by means of the higher orders boundary Harnack inequalities [57, Theorem 1.3]. Assume that the free boundary is already C_p^β, for some $\beta > 1$. First by [57, Corollary 5.3] all the derivatives $\partial_\sigma u$ are C_p^β up to the boundary, hence D^2u is $C_p^{\beta-1}$. It follows that the coefficients of the equation for the derivatives are also $C_p^{\beta-1}$, which allows us to apply higher order boundary Harnack inequality [57, Theorem 1.3] to deduce that $\frac{\partial_i u}{\partial_n u}$, $i=1,\dots,n+1$ are C_p^β. Hence g is $C_p^{\beta+1}$. The claim follows. □

We now have all ingredients to prove Theorem 5.1.1

Proof of Theorem 5.1.1. The result is an immediate consequence of Theorem 5.7.1, Lemma 5.6.4, Lemma 5.6.5 and Remark 5.6.2.. □

5.8 Analysis of singular points

To derive the properties of the solution to the obstacle problem near singular points, we would like to exploit the fact that the second derivatives of the blow-up are positive. In this direction we prove a version of the almost positivity lemma stated below.

Lemma 5.8.1. *Let $\varrho \in (0,1)$, $K>0$ and let $\{u_r\}_{r\in(0,1)}$ be a family of solutions to*

$$\begin{cases} \partial_t v - F(D^2v) = f(rx)\chi_{\{v>0\}} & \text{in } Q_\varrho \\ v, \partial_t v \geq 0 & \text{in } Q_\varrho, \end{cases} \tag{5.8.1}$$

with F satisfying (5.1.2) *in B_1, f satisfying* (5.1.3) *in B_1. Let $a_{ij}(x,t) := (DF(D^2u_r))_{ij}$ and let $\{w_r\}_{r\in(0,1)}$ be a family of functions satisfying*

$$\partial_t v - a_{ij}(x,t)\partial_{ij} v \geq rg(rx) \quad \text{in } Q_\varrho \cap \{u_r > 0\},$$

for some bounded function g with $\|g\|_{L^\infty(B_\varrho)} \leq K$. Then there exist $\delta_0, \tilde\varepsilon_0 > 0$ depending only on n, $c_\circ$, Λ, K and ϱ such that, for every $\delta \in (0,\delta_0)$, $\varepsilon \in (0,1)$ and $\overline{C} > 0$ such that $\varepsilon/\overline{C} \leq \tilde\varepsilon_0$, there exists $r \in (0,1)$ such that if

- *$w_r \geq 0$ in $Q_\varrho \cap \partial\{u_r > 0\}$,*
- *$w_r \geq -\varepsilon$ in Q_ϱ,*
- *$w_r \geq \overline{C}$ in $Q_\varrho \cap N_\delta^c(\{u_r = 0\})$,*

then $w_r \geq 0$ in $Q_{\varrho/2}$.

Proof. By scaling we may assume $\varrho = 1$. Let us set $u := u_r$, $w := w_r$, $r \in (0,1)$ and define

$$v(x,t) := w(x,t) - \gamma\left[u(x,t) - \tfrac{c_\circ}{4}\left(\tfrac{1}{2n\Lambda}|x-x_0|^2 - (t-t_0)\right)\right]$$

for some $\gamma > 0$ and $(x_0, t_0) \in Q_{1/2} \cap \{u > 0\}$. Noticing that

$$\partial_t v \geq a_{ij}(x,t)\partial_{ij} w + rg(rx) - \gamma\big[F(D^2u) + f(rx) + \tfrac{c_\circ}{4}\big]$$
$$\partial_{ij} v = \partial_{ij} w - \gamma\big[\partial_{ij} u - \tfrac{c_\circ}{4n\Lambda}\delta_{ij}\big],$$

and recalling that the function $M \to F(M)$ is convex, it is not difficult to obtain

$$\begin{aligned}\partial_t v - a_{ij}(x,t)\partial_{ij} v &\geq \gamma\big[a_{ij}(x,t) - F(D^2u)\big] + rg(rx) - \gamma f(rx) - \gamma\tfrac{c_\circ}{4n\Lambda}\sum_{i=1}^n a_{ii} - \gamma\tfrac{c_\circ}{4}\\ &\geq 0 - r\|g\|_{L^\infty(B_1)} + \gamma c_\circ - \gamma\tfrac{c_\circ}{4} - \gamma\tfrac{c_\circ}{4}\\ &\geq -rK + \gamma\tfrac{c_\circ}{2} \geq 0 \quad \text{in } Q_1 \cap \{u > 0\},\end{aligned}$$

provided r is taken small enough depending on $c_\circ$, γ and K. By the minimum principle, it thus follows

$$\inf_{Q^-_{1/2}(x_0,t_0)\cap\{u>0\}} v = \inf_{\partial_p(Q^-_{1/2}(x_0,t_0)\cap\{u>0\})} v. \tag{5.8.2}$$

Now, let us take

$$C_0 := \tfrac{64n\Lambda}{c_\circ}, \qquad \gamma := C_0\varepsilon, \qquad \delta \leq \sqrt{\tfrac{1}{C_0 C}}, \qquad \tfrac{\overline{C}}{\varepsilon} \geq C_0 K,$$

where $C > 0$ is as in (5.4.5). Then, in $Q^-_{1/2}(x_0, t_0) \cap \partial\{u > 0\}$, we have $w \geq 0$, $u = 0$ and thus

$$v \geq \gamma\tfrac{c_\circ}{4}\left(\tfrac{1}{2n\Lambda}|x - x_0|^2 - (t - t_0)\right) \geq 0.$$

Further, in $\partial_p(Q^-_{1/2}(x_0, t_0)) \cap N_\delta(\{u = 0\})$, we have $w \geq -\varepsilon$ and, by optimal growth,

$$v \geq -\varepsilon - \gamma C\delta^2 + \gamma\tfrac{c_0}{32n\Lambda} \geq -\varepsilon - \varepsilon C_0 C\delta^2 + 2\varepsilon \geq \varepsilon(1 - C_0 C\delta^2) \geq 0,$$

thanks to the definitions of γ and δ. Finally, in $\partial_p(Q^-_{1/2}(x_0, t_0)) \cap N_\delta(\{u = 0\})^c$, there holds

$$v \geq \overline{C} - \gamma K = \varepsilon\big(\tfrac{\overline{C}}{\varepsilon} - C_0 K\big) \geq 0,$$

by the choice of $\frac{\overline{C}}{\varepsilon}$. The thesis follows by (5.8.2) and the arbitrariness of $(x_0, t_0) \in Q_{1/2} \cap \{u > 0\}$. □

5.8.1 Lipschitz regularity of the whole free boundary

We can use the almost positivity lemma to bound the gradient of the solution to the obstacle problem ∇u with its time derivative $\partial_t u$, locally near any free boundary point. Since the quotient $\nabla u/\partial_t u$ determines the normal vector to the free boundary, we conclude that the free boundary is locally a graph of a Lipschitz function over the time coordinate.

Lemma 5.8.2. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \partial\{u > 0\}$. Assume that $\|f\|_{C^{0,1}(B_1)} \leq K$ and that $\partial_t u > 0$ in $\{u > 0\}$. Then there exist $c > 0$ and $\varrho_0 > 0$ such that*

$$\partial_t u \geq c\,|\nabla u| \quad \text{in } Q_{\varrho_0}. \tag{5.8.3}$$

Proof. Let $\{u_r\}_{r\in(0,1)}$ be a family of rescalings of u as in (5.6.1), satisfying (5.8.1) in Q_1, and consider the functions

$$v := A\partial_t u_r \pm \partial_i u_r,$$

for $i = 1, \dots, n$ and some $A > 1$. Similar to the proof of Lemma 5.8.5, we easily see that v satisfies

$$\partial_t v - a_{ij}(x,t)\partial_{ij} v = r\partial_i f(rx) \quad \text{in } Q_1 \cap \{u_r > 0\},$$

where, as always, $a_{ij}(x,t) := (DF(D^2u_r))_{ij}$ and, in addition, $v = 0$ in $\partial\{u_r > 0\}$. Further, given a small $\delta > 0$ as in Lemma 5.8.1, we exploit Theorem 5.4.1 to see that

$$v \geq -C\delta := -\varepsilon \quad \text{in } Q_1 \cap N_\delta(\{u_r = 0\}),$$

where $C > 0$ depends only on n, λ, Λ and K. On the other hand, since $\partial_t u_r > 0$ in $\{u_r > 0\}$, we have that $\partial_t u_r \geq \bar{c}$ in $N_\delta(\{u_r = 0\})^c \cap Q_1$ for some constant $\bar{c} > 0$ depending on δ and r and hence, by Theorem 5.4.1 again, we may choose A large enough such that

$$v \geq A\bar{c} - C \geq \tfrac{A\bar{c}}{2} := \overline{C} \quad \text{in } Q_1 \cap N_\delta(\{u_r = 0\})^c.$$

Taking eventually δ smaller and A larger, we may assume both $\delta \leq \delta_0$ and $\varepsilon/\overline{C} \leq \tilde{\varepsilon}_0$ where δ_0 and $\tilde{\varepsilon}_0$ are as in Lemma 5.8.1 and, by the same lemma, we deduce the existence of r such that $v \geq 0$ in $Q_{1/2}$, which is equivalent to our statement with $c := 1/A$ and $\varrho_0 := r$. □

Below we prove the Lipschitz regularity of the free boundary near any point.

Corollary 5.8.3. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \partial\{u > 0\}$. Assume that $\|f\|_{C^{0,1}(B_1)} \leq K$ and that $\partial_t u > 0$ in $\{u > 0\}$. Then there exists a Lipschitz function $\tau : B_{1/2} \to \mathbb{R}$ such that*

$$Q_{1/2} \cap \{u > 0\} = \{(x,t) \in Q_{1/2} : t > \tau(x)\}. \tag{5.8.4}$$

Proof. Let us consider the level sets $\{u = \varepsilon\}$, for $\varepsilon > 0$. Since $\partial_t u > 0$ in $\{u > 0\}$, we may apply the Implicit Function theorem to deduce the existence of $\varrho_0, \varepsilon_0 > 0$ and a C^1 function $h : B_{\varrho_0} \times (0, \varepsilon_0) \to \mathbb{R}$ that locally parametrizes $\{u = \varepsilon\}$, that is, $u(x,t) = \varepsilon$ if and only if $t = h(x,\varepsilon)$. Furthermore, $\partial_\varepsilon h > 0$ and, by (5.8.3), we also have

$$|\partial_i h| = \frac{|\partial_i u|}{\partial_t u} \leq C, \qquad i = 1, \dots, n,$$

for some $C > 0$ independent of ε, up to taking ϱ_0 smaller. Consequently,

$$|h(x,\varepsilon) - h(y,\varepsilon)| \leq C|x - y|,$$

for some new C (still independent of ε) and all $x, y \in B_{\varrho_0}$. Consequently, setting $\tau(x) := \lim_{\varepsilon\to 0} h(x,\varepsilon)$, we may pass to the limit as $\varepsilon \to 0$ into the above inequality and conclude the proof of our statement. □

5.8.2 ε-flatness of the singular set

Since we want to apply the almost positivity lemma on the second spatial derivatives of the solution to the obstacle problem, we assume from now on that F in (5.1.2) is independent of x, so that they become supersolutions of the linearised equation, as in (5.5.7).

We begin with an auxiliary result, saying that when we move away from the free boundary, the convergence of the blow-up sequence is actually in C^2.

Lemma 5.8.4. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \Sigma(u)$. Let $\{u_r\}_{r\in(0,1)}$ be the family of rescalings defined in* (5.6.1) *and assume u_{r_k} is a blow-up sequence satisfying* (5.6.2)*. Then for every $\varepsilon, \delta > 0$, there exists $k_0 \in \mathbb{N}$ such that*

$$\|u_{r_k} - u_0\|_{C^2(Q_{1/2}\cap N_\delta(\{u_{r_k}=0\})^c)} \le \varepsilon,$$

for all $k \ge k_0$.

Proof. Similar to the proof of Proposition 5.7.5, since u_0 is a quadratic polynomial, $D^2u_0 := A$ is a constant matrix satisfying $-F(D^2u_0) = -F(A) = f(0)$. Now, let us set $u_k := u_{r_k}$ and $v_k := u_k - u_0$. Using the equations of u_k and u_0, we easily see that

$$\partial_t v_k - \tilde{F}(D^2 v_k) = \tilde{f}_k(x) \quad \text{in } Q_1 \cap \{u_k > 0\},$$

for all $k \in \mathbb{N}$, where $\tilde{F}(M) := F(M + A) - F(A)$ and $\tilde{f}_k(x) := f(r_k x) + F(A)$. Hence, interior estimates ([25, Theorem 1.1] or [84, Theorem 1.1]) yield

$$\|D^2 v_k\|_{L^\infty(Q_\delta(x_0,t_0))} \le \frac{C}{\delta^2}\left(\|v_k\|_{L^\infty(Q_{2\delta}(x_0,t_0))} + \delta^2\|\tilde{f}(r_k x)\|_{L^\infty(Q_{2\delta}(x_0,t_0))}\right),$$

for some $C > 0$ depending only on n, λ and Λ, and all $(x_0, t_0) \in Q_{1/2}$ and $\delta > 0$ such that $Q_{2\delta}(x_0, t_0) \subset Q_1 \cap \{u_k > 0\}$. Consequently, given any $\varepsilon \in (0, 1)$, we may combine the above estimate with the fact that both v_k and $\tilde{f}_k$ converge to zero locally uniformly in $\mathbb{R}^n$, to deduce the existence of $k_0 \in \mathbb{N}$ such that

$$\|D^2 v_k\|_{L^\infty(Q_\delta(x_0,t_0))} \le \varepsilon,$$

for all $k \ge k_0$. Exploiting the arbitrariness of $(x_0, t_0) \in Q_{1/2} \cap N_\delta(\{u_k = 0\})^c$, we complete the proof of our statement. □

From the above result we conclude that if the blow-up is positive in some direction, then the pure second derivative of the solution in this direction is also positive when we look away from the boundary. On the other hand, thanks to the semiconvexity estimate (5.5.3) the second derivative is not too negative and hence the almost positivity lemma applies. We deduce the following.

Lemma 5.8.5. *Let $u \in \mathcal{P}_1(K)$ with F as in* (5.1.2) *and independent of x. Let $(0,0) \in \Sigma(u)$ and f satisfying* (5.5.2)*. Assume also*

$$u_0(x) = \sum_{j=1}^{n} \lambda_j x_j^2, \qquad \lambda_j \ge 0, \quad \lambda_n > 0, \tag{5.8.5}$$

is a blow-up of u at $(0,0)$. Then for every $\vartheta \in [0, \pi/2)$, there exist $\varrho_0, c > 0$ depending on ϑ, λ_n, such that

$$\partial_{ee} u \ge c\,|\nabla u| \quad \text{ in } Q_{\varrho_0},$$

for every $e \in \mathbb{S}^{n-1}$ satisfying $e_n \ge \cos\vartheta$.

Proof. Let us fix $\vartheta \in [0,\pi/2)$, $\lambda_n > 0$ and let $u_k := u_{r_k}$ be a blow-up sequence converging to u_0. The main idea is to apply Lemma 5.8.1 to the functions

$$v_{k,i} := \partial_{ee} u_k \pm c\partial_i u_k, \quad k \in \mathbb{N}, \quad i = 1,\dots,n.$$

Indeed, following the argument of the proof of Lemma 5.5.2, it is not difficult to check that each $v := v_{k,i}$ satisfies

$$\partial_t v - a_{ij}(x,t)\partial_{ij} v \geq r_k g(r_k x) \quad \text{in } Q_1 \cap \{u_k > 0\},$$

where $g(x) := r\partial_{ee} f(x) \mp \partial_i f(x)$ satisfies $\|g\|_{L^\infty(B_1)} \leq 2K$ by assumption. Furthermore, we have $v \geq 0$ in $Q_1 \cap \partial\{u_k > 0\}$ (this is an immediate consequence of the fact that $u_k \geq 0$ and $|\nabla u_k| = 0$ on $\partial\{u_k > 0\}$).

Now, let $e \in \mathbb{S}^{n-1}$ with $e_n \geq \cos\vartheta$. Since $\partial_{nn} u_0 = 2\lambda_n > 0$ and $\lambda_j \geq 0$ for every $j = 1,\dots,n-1$, it is not difficult to see that $\partial_{ee} u_0 \geq \overline{C}$ for some $\overline{C} > 0$ depending on ϑ and λ_n. Consequently, in view of Lemma 5.8.4, for every $\delta > 0$, there holds

$$\partial_{ee} u_k \geq \tfrac{\overline{C}}{2} \quad \text{in } N_\delta(\{u_k = 0\})^c,$$

for all $k \geq k_0$ and some k_0 depending on ϑ, λ_n and δ. If $C_0 > 0$ is such that $|\nabla u_k| \leq C_0$ in Q_1 ($k \geq k_0$), we may choose $c < \frac{\overline{C}}{4C_0}$ to obtain

$$v \geq C_* := \tfrac{\overline{C}}{4} \quad \text{in } Q_1 \cap N_\delta(\{u_k = 0\})^c.$$

Finally, by Proposition 5.5.1 and Lemma 5.4.3, we have that

$$v \geq -C|\log\delta|^{-\varepsilon} - C\delta \quad \text{in } N_\delta(\{u_k = 0\}),$$

for some new $C > 0$ and $\varepsilon \in (0,1)$ depending only on n, λ, Λ and K. Choosing δ small enough, the assumptions of Lemma 5.8.1 are fulfilled for each $k \geq k_0$, and so

$$v \geq 0 \quad \text{in } Q_{1/2},$$

that is, taking $k = k_0$,

$$\partial_{ee} u \geq \tfrac{c}{r_k}|\partial_i u| \quad \text{in } Q_{r_k/2},$$

for all $i = 1,\dots,n$, which readily implies our claim with $\varrho_0 := r_{k_0}$. □

Once we get the non-negativity of pure second order derivatives in a neighbourhood of a singular point, we can establish that near any other singular free boundary point in that neighbourhood, the solution has to be positive in these directions starting from the singular point.

Corollary 5.8.6. *Let $u \in \mathcal{P}_1(K)$ with F as in* (5.1.2) *and independent of x. Let $(0,0) \in \Sigma(u)$ and f satisfying* (5.5.2). *Assume also*

$$u_0(x) = \sum_{j=1}^{n} \lambda_j x_j^2, \qquad \lambda_j \geq 0, \quad \lambda_n > 0,$$

is a blow-up of u at $(0,0)$. Then for every $\vartheta \in [0,\pi/2)$, there exists $\varrho_0 > 0$ depending on ϑ, λ_n, such that for every $(x_0,t_0) \in Q_{\varrho_0} \cap \Sigma(u)$, we have

$$u > 0 \quad \textit{in } \{(x,t_0) \in B_{\varrho_0} : |(x-x_0)_n| > \cos\vartheta\, \|x - x_0\|\}.$$

Proof. Let us fix $\vartheta \in (0, \pi/2)$ and $\theta \in (0, \vartheta)$. By Lemma 5.8.5, there are $\varrho_0, c > 0$ depending on λ_n and θ, such that

$$\partial_{ee} u \geq c\,|\nabla u| \quad \text{ in } Q_{\varrho_0}, \tag{5.8.6}$$

for every $e \in \mathbb{S}^{n-1}$ with $e_n > \cos\theta$.

Now, by contradiction, we assume there is $(x_0, t_0) \in Q_{\varrho_0} \cap \Sigma(u)$ and $(x, t_0) \in \{(x, t_0) : |(x - x_0)_n| > \cos\theta \, \|x - x_0\|\}$ such that $u(x, t_0) = 0$, and we proceed with a delicate geometrical construction as follows.

Let $e := (x - x_0)/\|x - x_0\|$. By (5.8.6), we have $\partial_{ee} u \geq 0$ in Q_{ϱ_0} and thus $u = 0$ on the segment $[x_0, x]$. Now, let us fix $x_1, x_2 \in \mathrm{int}([x_0, x])$ satisfying $|x_1 - x_2| = \ell > 0$, and let us choose a system of coordinates $y = (y', y_n)$ such that x_1 coincides with the new origin and e is the new n^{th} unit vector.

Let $\bar{e} \in \mathbb{S}^{n-1}$ be perpendicular to e and $z = (1 - s)x_1 + sx_2 + r\bar{e}$, where $s \in [0, 1]$ and $r > 0$. Then if $r \in (0, r_0)$ and $r_0 > 0$ is small enough (depending on θ, x_1, x_2), we have $\partial_{e_i e_i} u \geq 0$, $i = 1, 2$, where

$$e_1 := \tfrac{x_0 - z}{\|x_0 - z\|}, \qquad e_2 := \tfrac{z - x}{\|z - x\|},$$

and thus we may combine this observation with $u(x_0) = u(x) = 0$ to deduce

$$\partial_{e_1} u(z) \geq 0 \quad \text{ and } \quad \partial_{e_2} u(z) \leq 0.$$

Consequently, writing $e_i = \alpha_i(r)e + \beta_i(r)\bar{e}$ $(i = 1, 2)$ for some $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}$ satisfying $\alpha_i(r) \in (1/2, 1)$ in $(0, r_0)$ and $\beta_i(r) \leq Cr$ in $(0, r_0)$ for some constant $C > 0$ depending on θ and x_1, we obtain

$$\partial_e u(z) \geq -\tfrac{|\beta_1(r)|}{\alpha_1(r)} |\partial_{\bar{e}} u(z)| \geq -Cr^2,$$

for some new $C > 0$. Repeating the argument with e_2, it follows

$$|\partial_e u(z)| \leq Cr^2.$$

As a consequence, if $y' \in B'_r$ and $p_1 := x_1 + (y', 0)$, $p_2 := x_2 + (y', 0)$, we deduce

$$\int_{p_1}^{p_2} \partial_{ee} u = \partial_e u(x_2 + y') - \partial_e u(x_1 + y') \leq C|y'|^2,$$

which, combined with (5.8.6), yields

$$\int_{p_1}^{p_2} |\nabla u| \leq C|y'|^2, \tag{5.8.7}$$

for some new $C > 0$. Now, we consider the cylinder

$$\mathcal{C}_r := \{(1 - s)x_1 + sx_2 + (y', 0) : s \in (0, 1),\, y' \in B'_r\},$$

with $r \in (0, r_0)$ and we show that

$$⨍_{\mathcal{C}_r} u \leq Cr^3. \tag{5.8.8}$$

Indeed, if $\nu(y') := y'/|y'|$, we may exploit (5.8.7) and that $r \le r_0$ to estimate

$$\begin{aligned}
\fint_{\mathcal{C}_r} u(y)dy =& \frac{c_n}{\ell r^{n-1}} \int_{\mathcal{C}_r} \int_0^{|y'|} \nabla u(s\nu(y'), y_n) \cdot \nu(y')\, dsdy \\
\le& \frac{c_n}{\ell r^{n-1}} \int_{B'_r} \int_0^{|y'|} \int_{p_1}^{p_2} |\nabla u|(s\nu(y'), y_n)\, dy_n ds dy' \\
\le& \frac{c_n C}{\ell r^{n-1}} \int_{B'_r} \int_0^{|y'|} s^2 ds dy' \le \frac{c_n C}{\ell r^{n-1}} \int_{B'_r} |y'|^3 dy' \le C\frac{r^3}{\ell} \le Cr^3,
\end{aligned}$$

for some new $C > 0$. Since we can take disjoint balls of radius r inside $\mathcal{C}_r$ so that the total volume of the balls is comparable to the volume of $\mathcal{C}_r$, we conclude from the above estimate that for every $r \in (0, r_0)$ there is y_r so that $\fint_{B_r(y_r)} u \le Cr^3$.

Finally let u show that this is in contradiction with the non-degeneracy estimate (5.4.14). Since we can take disjoint balls of radius r inside $\mathcal{C}_r$ so that the total volume of the balls is comparable to the volume of $\mathcal{C}_r$, we conclude from (5.8.8) that for every $r \in (0, r_0)$ there is y_r so that $\fint_{B_r(y_r)} u \le Cr^3$. Notice that since $\partial_t u \ge 0$, $u(\cdot, t_0)$ in particular solves

$$F(D^2 u(x, t_0), x) \ge 0.$$

Hence by the maximum principle [50, Proposition 4.34], we have that

$$\sup_{B_r(y_0)} u \le C \fint_{B_{2r}(y_0)} u$$

for any y_0 and $r > 0$. But then for y_r at scale r we have

$$cr^2 \le \sup_{Q_r^-(y_r, t_0)} u \le \sup_{B_r(y_r)} u(\cdot, t_0) \le Cr^3,$$

which leads to contradiction. □

The above result says that the free boundary can be touched at singular points with two-sided cones of arbitrarily large opening, looking at the time slice of that singular point. Combining this property with non-negativity of the time derivative and the fact that the free boundary is Lipschitz in time, we can derive that the same is true also for the projection of the singular set

$$\mathrm{pr}(\Sigma) = \{x \in B_1;\ (x, t) \in \Sigma \text{ for some } t \in (-1, 1)\}.$$

For simplicity we denote the two-sided cone centred at x_0 of opening θ in direction e_0 with $\mathcal{C}(x_0, e_0, \theta) := \{x \in \mathbb{R}^n;\ \frac{\sqrt{|x-x_0|^2 - ((x-x_0)\cdot e_0)^2}}{|(x-x_0)\cdot e_0|} < \tan\theta\}$.

Lemma 5.8.7. *Let $u \in \mathcal{P}_1(K)$ with F independent of x. Let $(0,0) \in \Sigma(u)$ and f satisfying* (5.5.2)*. Then there exists $e_0 \in \mathbb{S}^{n-1}$ such that for every $\vartheta \in (0, \pi/2)$, there exists $\varrho_0 > 0$ such that*

$$B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u)) \cap \mathcal{C}(x_0, e_0, \vartheta) = \emptyset \tag{5.8.9}$$

for all $x_0 \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u))$.

Proof. Let u_0 be a blow-up of u at $(0,0) \in \Sigma(u)$. Then, up to a rotation of the coordinates system, we may assume that u_0 is of the form (5.8.5). Consequently, by Corollary 5.8.6, we deduce the existence of $\rho_0 > 0$ such that $u > 0$ in $\mathcal{C}(x_0, e_n, \theta) \times \{t_0\}$, for any other singular point $(x_0, t_0) \in Q_{\rho_0}$. Furthermore, since $\partial\{u > 0\}$ can be written as the graph of a Lipschitz function $\tau = \tau(x)$ by Corollary 5.8.3, the full free boundary in $B_{\varrho_0} \times \mathbb{R}$ is actually contained in Q_{ρ_0}, if $\varrho_0 := \frac{1}{L}\rho_0^2$, where L is the Lipschitz constant of the function τ.

Now let $x_0 \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u))$ and assume by contradiction there exists $y_0 \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u)) \cap \mathcal{C}(x_0, e_n, \vartheta)$. The discussion above yields the existence of t_0 and s_0 such that $(x_0, t_0), (y_0, s_0) \in Q_{\varrho_0} \cap \Sigma(u)$ (notice that by symmetry we may assume $t_0 < s_0$). By Corollary 5.8.6, there holds $u > 0$ in $\mathcal{C}(x_0, e_n, \vartheta) \times \{t_0\}$. However, by time-monotonicity, the same is true in $\mathcal{C}(x_0, e_n, \vartheta) \times \{s_0\}$ which, in particular, contains (y_0, s_0). This contradicts the fact that (y_0, s_0) is a free boundary point. □

As the cone can be taken arbitrarily close to the half-space, this implies that the projection of the singular set is ε-flat for any $\varepsilon > 0$. The precise formulation of the result is stated next.

Corollary 5.8.8. *Let $u \in \mathcal{P}_1(K)$ with $(0,0) \in \Sigma(u)$ and f satisfying* (5.5.2)*. Then for every $\varepsilon > 0$, there exists $\varrho_0 > 0$ and a Lipschitz function $G : \mathbb{R}^{n-1} \to \mathbb{R}$ with $[G]_{C^{0,1}(\mathbb{R}^{n-1})} \le \varepsilon$ such that*

$$B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u)) \subset B_{\varrho_0} \cap \mathrm{graph}(G),$$

up to a rotation of the coordinates system.

Proof. Let us fix $\varepsilon > 0$ and set $\vartheta := \arctan(1/\varepsilon)$. Then, up to a rotation, Lemma 5.8.7 yields the existence of $\varrho_0 > 0$ such that

$$B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u)) \cap \mathcal{C}(x_0, e_n, \theta) = \emptyset,$$

for any other $x_0 \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma)(u)$. In particular, we deduce that for every $x' \in B'_{\varrho_0}$ there is at most one x_n such that $(x', x_n) \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma(u))$. Now, set

$$S = \{x' : \text{ there is } x_n \text{ such that } (x', x_n) \in B_{\varrho_0} \cap \mathrm{pr}(\Sigma)(u)\},$$

and define $G(x') = x_n$ on S. By the above property, G is Lipschitz continuous with $[G]_{C^{0,1}(B'_{\varrho_0})} \le \varepsilon$ and thus, by Kirszbraun's theorem, G may be extended to $\mathbb{R}^{n-1}$ without increasing its Lipschitz seminorm. It thus follows that $B_{\varrho_0} \cap \mathrm{pr}(\Sigma)$ is covered by the graph of (the extension of) G and our statement follows. □

In particular as the free boundary is Lipschitz, this implies Theorem 5.1.2.

Proof of Theorem 5.1.2. Thanks to Corollary 5.8.3 the free boundary can be written as the graph over the time coordinate of a Lipschitz function τ as in (5.8.4). But as for any $\varepsilon > 0$ the projection of the singular set can be locally covered by a graph of a function G with $[G]_{C^{0,1}(\mathbb{R}^{n-1})} \le \varepsilon$, see Corollary 5.8.8, the full singular set is locally covered by

$$\Sigma \cap Q_r \subset \{(x', G(x'), \tau(x', G(x')))\}.$$

The claim follows. □

Once we establish that the singular set can be covered by a Lipschitz, $(n-1)$-dimensional manifold, it follows from the geometric measure theoretic results, that the singular set cannot be too large at most times.

Proof of Corollary 5.1.3. Because the projection of the singular set is locally contained in a Lipschitz manifold of dimension $n-1$, it is in particular of Hausdorff dimension $n-1$. Moreover, since the full singular set is Lipschitz, it can be touched from above by cones. Hence [39, Corollary 7.8] applies and yields the result. □

5.9 Appendix

Proof of Lemma 5.3.3. By scaling, we may assume $r = 1$. Let us fix $\delta \in (0, \frac{1}{4})$ and $(x,t) \in B_{1-2\delta} \times (-1+4\delta^2, -\frac{3}{4})$. By Remark 5.3.6, it is enough to prove

$$\left(\fint_{Q_\delta^+(x,t)} u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \big(u(0,1) + ||f||_{L^\infty(Q_1)} \big),$$

and recover (5.3.3) by translation. The idea is to recursively apply Theorem 5.3.2 on cylinders $D^-_{\delta_k}(x_k,t_k)$ and $D^+_{\delta_k}(x_k,t_k)$, where the sequences $\{\delta_k\}_{k\in\mathbb{N}}$ and $\{(x_k,t_k)\}_{k\in\mathbb{N}} \subset Q_1$ will be suitably chosen.

We set $\delta_0 := \delta$ and we choose $(x_0,t_0) \in Q_1$ such that $D^-_{\delta_0}(x_0,t_0) = Q^+_{\delta_0}(x,t)$ (that is, $x_0 = x$ and $t_0 = t + 3\delta_0^2$). Further, we define

$$\delta_1 := 2\delta_0, \quad x_1 := \big(1 - \tfrac{2}{|x_0|}\delta_0\big)x_0, \quad t_1 := t_0 + 3\delta_1^2 + 3\delta_0^2.$$

Notice that x_1 belongs to the segment joining x_0 and the origin, with $|x_1 - x_0| = \delta_1$. Now, we firstly apply (5.3.2) with $r = 2\delta_0$ to obtain

$$\left(\fint_{D^-_{\delta_0}(x_0,t_0)} u^p \right)^{\frac{1}{p}} \leq C \left(\inf_{D^+_{\delta_0}(x_0,t_0)} u + \delta_0^2 ||f||_{L^\infty(Q_1)} \right), \tag{5.9.1}$$

for some $C > 0$ depending only on n, λ and Λ. Second, we notice that there exists $\bar{x}_0 \in B_{\delta_0}(x_0)$ such that $B_{\delta_0/2}(\bar{x}_0) \subset B_{\delta_0}(x_0) \cap B_{\delta_1}(x_1)$, by definition of δ_1 and x_1. This implies that the set $\hat{Q}_{0,1} := D^+_{\delta_0}(x_0,t_0) \cap D^-_{\delta_1}(x_1,t_1)$ satisfies

$$|\hat{Q}_{0,1}| \geq |B_{\delta_0/2}| \cdot \delta_0^2 = 2^{-n}\omega_n \delta_0^{n+2}.$$

Consequently,

$$\inf_{D^+_{\delta_0}(x_0,t_0)} u \leq \inf_{\hat{Q}_{0,1}} u \leq \left(\frac{|D^-_{\delta_1}|}{|\hat{Q}_{0,1}|} \fint_{D^-_{\delta_1}(x_1,t_1)} u^p \right)^{\frac{1}{p}} \leq 2^{\frac{2n+2}{p}} \left(\fint_{D^-_{\delta_1}(x_1,t_1)} u^p \right)^{\frac{1}{p}}.$$

Combining the above inequality with (5.9.1), it follows

$$\left(\fint_{D^-_{\delta_0}(x_0,t_0)} u^p \right)^{\frac{1}{p}} \leq C \left\{ \left(\fint_{D^-_{\delta_1}(x_1,t_1)} u^p \right)^{\frac{1}{p}} + \delta_0^2 ||f||_{L^\infty(Q_1)} \right\}, \tag{5.9.2}$$

for some new $C > 0$ (depending only on n, λ and Λ).

Then, we iterate this procedure. Set

$$\delta_{k+1} := 2\delta_k, \quad x_{k+1} := \big(1 - \tfrac{2}{|x_k|}\delta_k\big)x_k, \quad t_{k+1} := t_k + 3\delta_{k+1}^2 + 3\delta_k^2.$$

On the lines of the argument above, it is not difficult to find

$$\Big(\fint_{D^-_{\delta_k}(x_k,t_k)} u^p\Big)^{\frac{1}{p}} \le C\Big\{\Big(\fint_{D^-_{\delta_k}(x_{k+1},t_{k+1})} u^p\Big)^{\frac{1}{p}} + \delta_k^2||f||_{L^\infty(Q_1)}\Big\},$$

for every $k \in \mathbb{N}$, where C is as in (5.9.2). The iteration stops at an index $k = N-1$ for which either

$$\begin{aligned}\Big(\fint_{D^-_{\delta_0}(x_0,t_0)} u^p\Big)^{\frac{1}{p}} &\le C^{N-1}\Big\{\Big(\fint_{D^-_{\delta_{N-1}}(x_{N-1},t_{N-1})} u^p\Big)^{\frac{1}{p}} + 2^N||f||_{L^\infty(Q_1)}\sum_{k=0}^{N-2}(2C)^{-j}\Big\} \\ &\le C^N\Big\{\Big(\fint_{D^-_{\delta_{N-1}}(x_{N-1},t_{N-1})} u^p\Big)^{\frac{1}{p}} + 2^N||f||_{L^\infty(Q_1)}\Big\},\end{aligned} \tag{5.9.3}$$

and $D^+_{\delta_{N-1}}(x_{N-1},t_{N-1}) \not\subseteq Q_1$, or

$$\Big(\fint_{D^-_{\delta_0}(x_0,t_0)} u^p\Big)^{\frac{1}{p}} \le C^N\Big\{\inf_{D^+_{\delta_{N-1}}(x_{N-1},t_{N-1})} u + 2^N||f||_{L^\infty(Q_1)}\Big\},$$

and $D^-_{2\delta_N}(x_N,t_N) \not\subseteq Q_1$. In both cases, it is not difficult to check that $2^N\delta_0 \sim 1$, in the sense that $a_0 \le 2^N\delta_0 \le b_0$, where a_0 and b_0 are two positive numerical constants (for instance, one could easily check that in the first scenario we have $t_N - t_0 = 5\delta_0^2(2^{2N}-1)$).

Now, let us assume that (5.9.3) holds true with $D^+_{\delta_{N-1}}(x_{N-1},t_{N-1}) \not\subseteq Q_1$ (the other case can treated similarly). Then we refine the definition of δ_k and (x_k,t_k), by setting

$$\delta'_{k+1} := \varrho\delta'_k, \quad x'_{k+1} := \big(1 - \tfrac{\varrho}{|x_k|}\delta'_k\big)x'_k, \quad t'_{k+1} := t'_k + 3(\delta'_{k+1})^2 + 3(\delta'_k)^2,$$

where $\varrho \in (1,2]$, $\delta'_0 := \delta_0$ and $(x'_0,t'_0) := (x_0,t_0)$. We then repeat the iteration above up to the step $N-1$. At this point, by continuity, we may choose $\varrho := \varrho_0 \in (1,2]$ such that

$$(0,1) \in \overline{D^+_{\delta'_{N-1}}(x'_{N-1},t'_{N-1})} \subseteq \overline{Q_1}$$

and so, applying once more (5.3.2) to the r.h.s. of (5.9.3), it follows

$$\begin{aligned}\Big(\fint_{D^-_{\delta_0}(x_0,t_0)} u^p\Big)^{\frac{1}{p}} &\le C^{N+1}\Big\{\inf_{D^+_{\delta'_{N-1}}(x'_{N-1},t'_{N-1})} u + 2^N||f||_{L^\infty(Q_1)}\Big\} \\ &\le C^{N+1}\big(u(0,1) + 2^N||f||_{L^\infty(Q_1)}\big).\end{aligned}$$

Recalling that $2^N\delta_0 \sim 1$ (that is, $N \sim |\log_2(\delta_0)|$), we immediately see that $C^{N+1} \le C\delta_0^{-m}$ for some $C, m > 0$ depending only on n, λ and Λ, and the thesis follows. □

Proof of Corollary 5.3.5. Let us fix $\delta \in (0, 1/4)$ and cover $B_1 \times (-1, -3/4)$ with a finite number of disjoint "parabolic cubes" $\tilde{Q}_\delta(x_k, t_k)$, $k \in \{1, \dots, k_\delta\}$ for some $k_\delta \in \mathbb{N}$, such that

$$|\cup_k \tilde{Q}_\delta(x_k, t_k)| \leq |Q_1|. \tag{5.9.4}$$

We also consider a family of cylinders $Q_r(x_k, t_k)$, where the radius r is defined as follows:

$$r := \inf\{\varrho > 0 : \tilde{Q}_\delta(x_k, t_k) \subset Q_\varrho(x_k, t_k)\}.$$

By construction, $r = c_n \delta$, for some $c_n > 0$ (depending only on n). Consequently, by (5.3.3)

$$\left(\fint_{\tilde{Q}_\delta(x_k,t_k)} u^p \right)^{\frac{1}{p}} \leq \left(\frac{|Q_r|}{|\tilde{Q}_\delta|} \fint_{Q_r(x_k,t_k)} u^p \right)^{\frac{1}{p}} \leq \frac{C}{\delta^m} \left(\inf_{Q^+_{1/2}} u + ||f||_{L^\infty(Q_1)} \right),$$

for every $k \in \{1, \dots, k_\delta\}$, where C, m and p depend only on n, λ and Λ. Summing on k, using Jensen's inequality ($p \in (0, 1)$) and recalling (5.9.4), we obtain

$$\begin{aligned}
\frac{C}{\delta^m} \left(\inf_{Q^-_{1/2}} u + ||f||_{L^\infty(Q_1)} \right) &\geq \frac{1}{k_\delta} \sum_{k=1}^{k_\delta} \left(\fint_{\tilde{Q}_\delta(x_k,t_k)} u^p \right)^{\frac{1}{p}} \geq \left(\sum_{k=1}^{k_\delta} \frac{1}{k_\delta |\tilde{Q}_\delta|} \int_{\tilde{Q}_\delta(x_k,t_k)} u^p \right)^{\frac{1}{p}} \\
&= |\cup_k \tilde{Q}_\delta(x_k, t_k)|^{-\frac{1}{p}} \left(\int_{\cup_k \tilde{Q}_\delta(x_k,t_k)} u^p \right)^{\frac{1}{p}} \geq \left(\frac{|A|}{|Q_1|} \right)^{\frac{1}{p}} \left(\fint_A u^p \right)^{\frac{1}{p}} \\
&\geq \left(\frac{C_0}{|Q_1| \delta^{m_0}} \right)^{\frac{1}{p}} \left(\fint_A u^p \right)^{\frac{1}{p}},
\end{aligned}$$

which yields (5.3.5).

$\square$

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